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COMBINATORIAL PROPERTIES OF GROUPS AND SIMPLE C^*-ALGEBRAS WITH A UNIQUE TRACE

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Let G be a discrete group and $C_r^*(G)$ be its reduced C^*-algebra. The problem of finding conditions which imply that $C_r^*(G)$ is simple with unique trace has already been considered by several authors. In [6] Powers has shown that the reduced C^*-algebra $C_r^*(F_2)$ of the free group with two generators is simple with unique trace. Then, the class of these groups was essentially extended by Paschke and Salinas ([5]), Akemann and Lee ([1]), Bédos, de la Harpe and Jhabvala ([2], [3]). De la Harpe has introduced ([2]) the Powers property and has shown by geometrical means important classes consisting of Powers groups. The reduced C^*-algebra of a Powers group is simple with unique trace but it is still unknown whether the class of these groups is stable by extensions, and whether the direct product of two Powers groups is a Powers group.

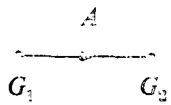
The main idea in this paper is to make a slight modification in the definition of Powers groups. The so called weak Powers groups preserve the good features of Powers groups and have good extension properties. More exactly, we check in Section 1 that the direct product of two weak Powers groups is still a weak Powers group and that any extension of a Powers group by a weak Powers group is a weak Powers group.

In Section 2 we prove that for any extension G of a weak Powers group by a weak Powers group, the reduced C^*-algebra $C_r^*(G)$ is simple with unique trace. Moreover, we extend the main results obtained by de la Harpe and Skandalis in [4], replacing the Powers property by the weak Powers property and the usual reduced cross-product $A \rtimes_{\alpha} G$ of G and of the unital C^*-algebra A relatively to an action $\alpha\colon G \to \mathrm{Aut}(A)$ by the reduced cross-product $A \rtimes_{\alpha,c} G$, where c is a 2-cocycle on G with values in the unitary group of the centre of A. Using some arguments from [4] and a simple combinatorial trick we prove the following assertions:

1) If $A \rtimes_{\alpha,c} G$ has a trace τ, then there exists a G-invariant trace σ on A with $\tau = \sigma e$ (e is the canonical conditional expectation $e\colon A \rtimes_{\alpha,c} G \to A$).

2) If A is G-simple, then $A \rtimes_{\alpha,c} G$ is simple.

Bédos and de la Harpe have studied some fundamental groups of graphs of groups. When the graph of groups is a segment



respectively a loop G_1  A one obtains the free product with amalgamation $G = G_1 *_A G_2$ and respectively the Higman-Neumann-Neumann extension $G = \mathrm{HNN}(G_1, A, \theta)$. For both cases it was proved in [2] that if for any finite subset $F \subset G \setminus \{1\}$, there exists $g \in G$ such that $gFg^{-1} \cap A = \emptyset$, then G is a Powers group (for the HNN-extension it is also necessary to suppose that the index of A in G_1 is at least three). In Section 3 we consider a class consisting of fundamental groups of graphs of groups. Using an exercise from [7] and the results from Section 1, it is easy to check that the fundamental group of a connected, non-empty graph of groups, which has in every vertice non-trivial groups, which contains an edge y with $G_y = \{1\}$ and has the homotopy type of a bouquet consisting of n-circles ($n \in \mathbf{N} \cup \{\infty\}$, $n \geqslant 2$), is a weak Powers group.

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1. WEAK POWERS PROPERTY

All groups which appear in this paper will be discrete groups. For any set $M \subset G$ and $f \in G$ we denote $\langle f \rangle_M = \{xfx^{-1} \mid x \in M\}$.

DEFINITION 1.1. ([2]). A *Powers group* is a group G having the following property: given any non-empty finite subset $F \subset G \setminus \{1\}$ and any integer $N \geqslant 1$ there exist a partition $G = A \sqcup B$ and elements $g_1, \ldots, g_N$ in G such that

(i) $fA \cap A = \emptyset$, for all $f \in F$;

(ii) $g_j B \cap g_k B = \emptyset$, for $j, k = 1, \ldots, N$, $j \neq k$.

It is easy to see that the free groups with n generators F_n, where $n \in \mathbf{N} \cup \{\infty\}$, $n \geqslant 2$ are Powers groups. The papers [1, 2, 3, 5, 6] describe several classes of Powers groups.

DEFINITION 1.2. A *weak Powers group* is a group G having the following property: given any non-empty finite subset $F \subset G \setminus \{1\}$, which is *included into a con-*

jugacy class, and any integer $N \geq 1$, there exist a partition $G = A \amalg B$ and elements $g_1, \ldots, g_N$ in G such that

(i) $fA \cap A = \emptyset$, for all $f \in F$;

(ii) $g_j B \cap g_k B = \emptyset$, for $j, k = 1, \ldots, N$, $j \neq k$.

It is clear that any Powers group is a weak Powers group. It is also clear that in both definitions (i) is true for any $f \in F \cup F^{-1}$.

The proof of [2, Proposition 1] shows that the weak Powers groups have the following elementary properties:

PROPOSITION 1.3. *Let G be a weak Powers group. Then:*

(i) *Any conjugacy class in G other than* $\{1\}$ *is infinite;*

(ii) *G is not amenable;*

(iii) *Any subgroup G′ of G of finite index is a weak Powers group.*

Hence all observations from [2, Section 1] are still true for weak Powers groups.

In [2] de la Harpe asks whether for G_1, G_2 Powers groups, $G_1 \times G_2$ is still a Powers group. We show by a simple argument that the similar problem for weak Powers groups has an affirmative answer.

PROPOSITION 1.4. *If G_1 and G_2 are weak Powers groups, then their direct product $G = G_1 \times G_2$ is also a weak Powers group.*

Proof. Let $f = (f_1, f_2)$ be any element of $G \setminus \{1\}$. We may assume $f_1 \neq 1$. Let $N \geq 1$ be an integer and F be a non empty finite subset of $G \setminus \{1\}$ included into the conjugacy class of some element $f \in G$. Thus, $F = \langle f \rangle_M$, where M is a finite subset of G. Then $M_1 = \mathrm{pr}_1 M$ is a finite subset of the weak Powers group G_1, so $G_1 = A_1 \amalg B_1$ and there exist $g_1', \ldots, g_N' \in G_1$ such that

$$g' A_1 \cap A_1 = \emptyset, \quad \text{for all } g' \in \langle f_1 \rangle_{M_1};$$

$$g_j' B_1 \cap g_k' B_1 = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

Then $A = A_1 \times G_2$, $B = B_1 \times G_2$, $g_j = (g_j', 1) \in G$, $j = 1, \ldots, N$, will fulfil conditions (i), (ii) from Definition 1.2 proving that G is indeed a weak Powers group. ▨

PROPOSITION 1.5. *If $1 \to G' \to G \to G'' \to 1$ is an exact sequence of groups, with G' Powers group and G'' weak Powers group, then G is a weak Powers group.*

Proof. G' is identified with a normal subgroup of G, and G'' with the quotient group G/G'. Let $\pi: G \to G/G'$ be the quotient map and choose $\{\gamma_i\}_{i \in I}$ a complete system of representants for G modulo G'. Let $f \in G \setminus \{1\}$, $M \subset G$ a finite set and an integer $N \geq 1$.

Assume first that $f \in G' \setminus \{1\}$. It is clear that $\langle f \rangle_M \subset G' \setminus \{1\}$. G' is a Powers group, hence there exist a partition $G' = A_1 \amalg B_1$ and $h_1, \ldots, h_N \in G'$ such that

$$hA_1 \cap A_1 = \emptyset, \quad \text{for all } h \in \langle f \rangle_M ;$$

$$h_j B_1 \cap h_k B_1 = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

Now if $A = \coprod_{i \in I} A_1 \gamma_i$, $B = \coprod_{i \in I} B_1 \gamma_i$, then $G = A \amalg B$ and it is easy to verify that

$$hA \cap A = \emptyset, \quad \text{for all } h \in \langle f \rangle_M ;$$

$$h_j B \cap h_k B = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

Now assume that $f \in G \setminus G'$. In this case, there are unique $i_0 \in I$, $h \in G'$ such that $f = h\gamma_{i_0}$ and clearly $\gamma_{i_0} \notin G'$. Since G/G' is a weak Powers group it follows that $G/G' = A' \amalg B'$ and there are $y_j \in G/G'$, $j = 1, \ldots, N$ such that

$$yA' \cap A' = \emptyset, \quad \text{for all } y \in \langle \pi(\gamma_{i_0}) \rangle_{\pi(M)} ;$$

$$y_j B' \cap y_k B' = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

If $A = \pi^{-1}(A')$ and $B = \pi^{-1}(B')$, it is clear that $G = A \amalg B$. For $g_j \in \pi^{-1}(y_j)$, $j = 1, \ldots, N$ it is easy to check that

$$\gamma A \cap A = \emptyset, \quad \text{for all } \gamma \in \langle f \rangle_M ;$$

$$g_j B \cap g_k B = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k. \qquad \blacksquare$$

2. SIMPLICITY AND UNIQUE TRACE PROPERTY

Throughout this section A is a unital C^*-subalgebra of $B(H)$ for some Hilbert space H, G a discrete group and $\alpha \colon G \to \mathrm{Aut}(A)$ an action. Let $Z_u(A)$ be the unitary group of the centre of A. A (normalised) 2-cocycle on G with values in $Z_u(A)$ is a map $c \colon G \times G \to Z_u(A)$ such that

$$c(g_1, g_2)c(g_1g_2, g_3) = \alpha_{g_1}(c(g_2, g_3))c(g_1, g_2g_3), \quad \text{for } g_1, g_2, g_3 \in G;$$

$$c(g, g^{-1}) = 1, \quad \text{for } g \in G;$$

$$c(1, g) = c(g, 1) = 1, \quad \text{for } g \in G.$$

The set of this cocycles is denoted by $Z^2(G, Z_u(A))$. Define $\pi_\alpha : A \to B(\ell^2(G, H))$ and $\Lambda_c : G \to U(\ell^2(G, H))$ by

$$(\pi_\alpha(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g), \quad \text{for } x \in A,\ \xi \in \ell^2(G, H),\ g \in G;$$

$$(\Lambda_c(g)\xi)(g_1) = c(g_1^{-1}, g)\xi(g^{-1}g_1), \quad \text{for } g,\ g_1 \in G,\ \xi \in \ell^2(G, H).$$

Then the C^*-subalgebra of $B(\ell^2(G; H))$ generated by $\pi_\alpha(A)$ and $\Lambda_c(G)$ is called the (reduced) cross-product of G and A by c relatively to α and is denoted by $A \rtimes_{\alpha,c} G$ (see [9]). For $A = \mathbf{C}$ and $c \in Z^2(G, \mathbf{T})$, we obtain the (reduced) c-C^*-algebra $C_r^*(G, c)$. If in addition c is a trivial 2-cocycle, then we get the usual reduced C^*-algebra $C_r^*(G)$. We denote $u_g = \Lambda_c(g)$, $g \in G$ and we write simply x instead of $\pi_\alpha(x)$, $x \in A$. The u_g are unitaries and

$$u_g a u_g^* = \alpha_g(a), \quad \text{for } g \in G,\ a \in A;$$

$$u_g^* = u_{g^{-1}}, \quad \text{for } g \in G;$$

$$u_1 = 1$$

$$u_{g_1} u_{g_2} = c(g_1, g_2) u_{g_1 g_2}, \quad \text{for } g_1,\ g_2 \in G.$$

For further reference we remark that

(1) $$u_{g_1} \dots u_{g_n} = c_l(g_1, \dots, g_n) u_{g_1 \dots g_n}, \quad \text{for } g_1, \dots, g_n \in G,\ n \geqslant 2;$$

(2) $$u_{g_1} \dots u_{g_n} = u_{g_1 \dots g_n} c_r(g_1, \dots, g_n), \quad \text{for } g_1, \dots, g_n \in G,\ n \geqslant 2,$$

where

$$c_l(g_1, \dots, g_n) = c(g_1, g_2) c(g_1 g_2, g_3) \dots c(g_1 \dots g_{n-1}, g_n) \in Z_u(A);$$

$$c_r(g_1, \dots, g_n) = \alpha_{(g_1 \dots g_n)^{-1}}(c_l(g_1, \dots, g_n)) \in Z_u(A).$$

For the rest of this paper, all sums $x = \sum_g a_g u_g$, $a_g \in A$ will be finite. For such an element of the C^*-algebra $B = A \rtimes_{\alpha,c} G$, we denote $\operatorname{supp} x = \{g \in G \mid a_g \neq 0\}$.

The canonical conditional expectation $e : B \to A$ is determined by $e(\sum_g a_g u_g) = a_1$ and we have

(3) $$e(u_g x u_g^*) = \alpha_g(e(x)), \quad \text{for } g \in G,\ x \in B.$$

If A has a G-invariant trace τ_0, then B has a canonical trace τ, defined by $\tau(x) = \tau_0(e(x))$, $x \in B$.

LEMMA 2.1. *Let D be a subset of G, $g \in G$ and denote by p and p_g the projections of $\ell^2(G, H)$ onto $\ell^2(D, H)$ and onto $\ell^2(gD, H)$. Then $u_g p u_g^* = p_g$.*

Proof. The partition $G = D \sqcup E$ gives raise to the orthogonal decomposition $\ell^2(G, H) = \ell^2(D, H) \oplus \ell^2(E, H)$. Let $\xi \in \ell^2(G, H)$. $\xi_1 = p\xi$, $\xi_2 = (1-p)\xi$. By the very definition of u_g we have

$$(u_g p\xi)(\gamma) = (u_g \xi_1)(\gamma) = c(\gamma^{-1}, g)\xi_1(g^{-1}\gamma);$$

$$(p_g u_g \xi)(\gamma) = p_g(\gamma \to c(\gamma^{-1}, g)\xi(g^{-1}\gamma))(\gamma).$$

If $\gamma \in gD$, then $\xi_2(g^{-1}\gamma) = 0$. From the above formulas it follows that

$$(p_g u_g \xi)(\gamma) = c(\gamma^{-1}, g)\xi(g^{-1}\gamma) = c(\gamma^{-1}, g)\xi_1(g^{-1}\gamma) = (u_g p\xi)(\gamma).$$

If $\gamma \notin gD$, then $\xi_1(g^{-1}\gamma) = 0$, so $(p_g u_g \xi)(\gamma) = p_g(\gamma \to c(\gamma^{-1}, g)\xi(g^{-1}\gamma))(\gamma) = 0$ and $(u_g p\xi)(\gamma) = 0$. Therefore $u_g p = p_g u_g$. ∎

We now extend the results from [4] for weak Powers groups and for $A \times_{\alpha,c} G$, instead of $A \times_\alpha G$. The next lemma is an important step in the proof.

LEMMA 2.2. *If G is a weak Powers group, then for any self-adjoint element $x = \sum_{g \in F} a_g u_g$ in B, with $e(x) = 0$ and for any $\varepsilon > 0$, there exist an integer $n \geq 1$, $g_1, \ldots, g_n \in G$, $c_1, \ldots, c_n \in Z_u(A)$ such that*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u_{g_k} c_k x c_k^* u_{g_k}^* \right\| \leq \varepsilon.$$

Proof. First note that for $a \in A$, $f, g \in G$

$$u_g(au_f)u_g^* = \alpha_g(a)u_g u_f u_{g^{-1}} = \alpha_g(a)c(g, f)u_{gf}u_{g^{-1}} =$$

$$= \alpha_g(a)c(g,f)c(gf,g^{-1})u_{gfg^{-1}} =$$

$$= \alpha_g(a)\alpha_g(c(f,g^{-1}))c(g,fg^{-1})u_{gfg^{-1}} =$$

$$= \alpha_g(ac(f, g^{-1}))c(g,fg^{-1})u_{gfg^{-1}}.$$

Let $y = au_f + \alpha_{f^{-1}}(a^*)u_f^*$, $f \in G \setminus \{1\}$, $a \in A$. Then $y = y^*$ and $e(y) = 0$. Let $M \subset G$ be a finite subset which consists of s elements and $z = \frac{1}{s} \sum_{g \in M} u_g y u_g^*$. The above computation shows that $\operatorname{supp} z \subset \langle f \rangle_M \cup \langle f^{-1} \rangle_M$ and, since G is a weak

Powers group, we can conclude that there exist $G = D \amalg E$ and $g_1, g_2, g_3 \in G$ such that

$$fD \cap D = \emptyset, \quad \text{for all } f \in \operatorname{supp} z; \tag{4}$$

$$g_jE \cap g_kE = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k. \tag{5}$$

If p is the orthogonal projection of $\ell^2(G, H)$ onto $\ell^2(D, H)$, then (4) implies $pzp = 0$. Using (5) and Lemma 2.1 we obtain the projections $u_{g_j}(1-p)u^*_{g_j}$, $j = 1, 2, 3$. Quoting [4, Lemma 1] we get

$$\left\| \frac{1}{3} \sum_{k=1}^{3} u_{g_k} z u^*_{g_k} \right\| \leqslant c\|z\|,$$

where $c = 0.995$, and $\operatorname{supp} z' \subset \langle f \rangle_{g_1M \cup g_2M \cup g_3M} \cup \langle f^{-1} \rangle_{g_1M \cup g_2M \cup g_3M}$, where $z' = \frac{1}{3} \sum_{k=1}^{3} u_{g_k} z u^*_{g_k}$. Using repeatedly this remark and (2) we see that for any $\varepsilon > 0$, there exist an integer $r \geqslant 1$, $g_1, \ldots, g_r \in G$, $c_1, \ldots, c_r \in Z_u(A)$ such that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} u_{g_j} c_j y c_j^* u^*_{g_j} \right\| \leqslant \varepsilon.$$

Now, let x be as in the statement. Then, there exists an integer $m \geqslant 1$ such that $x = \sum_{i=1}^{m} x_i$, where $x_i = a_{g_i} u_{g_i} + \alpha_{g_i^{-1}}(a^*_{g_i}) u_{g_i^{-1}}$, $g_i \neq 1$, for $i = 1, \ldots, m$. From the first part of the proof, there exist $g_{1,1}, \ldots, g_{1,n_1} \in G$, $c_{1,1}, \ldots, c_{1,n_1} \in Z_u(A)$ such that

$$\left\| \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1,j_1}} c_{1,j_1} x_1 c^*_{1,j_1} u^*_{g_{1,j_1}} \right\| \leqslant \frac{\varepsilon}{m}. \tag{6}$$

Denote $\tilde{x}_2 = \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1,j_1}} c_{1,j_1} x_2 c^*_{1,j_1} u^*_{g_{1,j_1}}$. As for (6), there exist $g_{2,1}, \ldots, g_{2,n_2} \in G$, $c_{2,1}, \ldots, c_{2,n_2} \in Z_u(A)$ such that

$$\left\| \frac{1}{n_2} \sum_{j_2=1}^{n_2} u_{g_{2,j_2}} c_{2,j_2} x_2 c^*_{2,j_2} u^*_{g_{2,j_2}} \right\| \leqslant \frac{\varepsilon}{m}.$$

By induction, denote $\tilde{x}_{k+1} = \frac{1}{n_k} \sum_{j_k=1}^{n_k} u_{g_{k,j_k}} c_{k,j_k} x_{k+1} c_{k,j_k}^* u_{g_{k,j_k}}^*$. Then, there exist $g_{k+1,1}, \ldots, g_{k+1,n_{k+1}} \in G$, $c_{k+1,1}, \ldots, c_{k+1,n_{k+1}} \in Z_u(A)$ such that

$$\left\| \frac{1}{n_{k+1}} \sum_{j_{k+1}=1}^{n_{k+1}} u_{g_{k+1,j_{k+1}}} c_{k+1,j_{k+1}} \tilde{x}_{k+1} c_{k+1,j_{k+1}}^* u_{g_{k+1,j_{k+1}}}^* \right\| \leqslant \frac{\varepsilon}{m},$$

for $k = 1, \ldots, m-1$. One has

$$\left\| \frac{1}{n_1 \ldots n_m} \sum_{j_1=1}^{n_1} \ldots \sum_{j_m=1}^{n_m} u_{g_{m,j_m}} c_{m,j_m} \ldots u_{g_{1,j_1}} c_{1,j_1} (x_1 + \ldots + x_m) c_{1,j_1}^* u_{g_{1,j_1}}^* \ldots c_{m,j_m}^* u_{g_{m,j_m}}^* \right\| \leqslant$$

$$\leqslant \left\| \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1,j_1}} c_{1,j_1} x_1 c_{1,j_1}^* u_{g_{1,j_1}}^* \right\| + \left\| \frac{1}{n_2} \sum_{j_2=1}^{n_2} u_{g_{2,j_2}} c_{2,j_2} \tilde{x}_2 c_{2,j_2}^* u_{g_{2,j_2}}^* \right\| + \ldots$$

$$\ldots + \left\| \frac{1}{n_m} \sum_{j_m=1}^{n_m} u_{g_{m,j_m}} c_{m,j_m} \tilde{x}_m c_{m,j_m}^* u_{g_{m,j_m}}^* \right\| \leqslant m \cdot \frac{\varepsilon}{m} = \varepsilon.$$

Using (1) and $u_g a = \alpha_g(a) u_g$, for $g \in G$, $a \in A$, we obtain $n = n_1 \ldots n_m \in \mathbf{N}$ $g_1, \ldots, g_n \in G$, $c_1, \ldots, c_n \in Z_u(A)$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u_{g_k} c_k x c_k^* u_{g_k}^* \right\| \leqslant \varepsilon .$$ ▨

PROPOSITION 2.3. *If G is a weak Powers group then, for any trace τ on $B = A \rtimes_{\alpha,c} G$, there exists a G-invariant trace σ on A with $\tau = \sigma e$.*

Proof. Let $x = \sum_g a_g u_g \in B$. Lemma 2.2 implies that the closed convex hull of

$$\{u_g c_g (x - e(x)) c_g^* u_g^* \mid g \in G,\ c_g \in Z_u(A)\}$$

contains 0. Consequently $\tau(x - e(x)) = 0$ and the assertion follows. ▨

COROLLARY 2.4. *If G is a weak Powers group and if there exists a unique G-invariant trace on A, then there exists a unique trace on $A \rtimes_{\alpha,c} G$.*

The unital C^*-algebra A is called *G-simple* if any $\alpha(G)$-invariant closed two-sided ideal in A is either $\{0\}$ or A. We shall also use the following assertion which is Lemma 9 from [4].

LEMMA 2.5. *Assume that A is G-simple. Let $x \in A$ with $x \geqslant 0$ and $x \neq 0$. There exist $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in A$ such that*

$$\sum_{j=1}^{n} z_j \alpha_{g_j}(x) z_j^* \geqslant 1.$$

PROPOSITION 2.6. *If G is a weak Powers group and A a G-simple C^*-algebra, then $A \rtimes_{\alpha,c} G$ is simple.*

Proof. Let $I \subset B$ be a two-sided ideal and assume that $x \in I$, $x \neq 0$. One may assume $x \geqslant 0$, hence $e(x) \geqslant 0$. By Lemma 2.5 there exist $g_1, \ldots, g_n \in G$, $a_1, \ldots, a_n \in A$ such that

$$\sum_{j=1}^{n} a_j \alpha_{g_j}(e(x)) a_j^* \geqslant 1.$$

Denote $x' = \sum_{j=1}^{n} a_j u_{g_j} x u_{g_j}^* a_j^* \in I$. Then $x' \geqslant 0$ and we obtain

$$e(x') = \sum_{j=1}^{n} e(a_j u_{g_j} x u_{g_j}^* a_j^*) = \sum_{j=1}^{n} a_j e(u_{g_j} x u_{g_j}^*) a_j^*.$$

From the above calculus and (3) we conclude

$$e(x') = \sum_{j=1}^{n} a_j \alpha_{g_j}(e(x)) a_j^* \geqslant 1.$$

Let $y = \sum_{g \in F} a_g u_g \in B$, with $F \subset G$ finite set, such that $y = y^*$ and $\|x' - y\| \leqslant 1/6$. Then $\|e(x') - e(y)\| \leqslant 1/6$, hence $e(y) \geqslant e(x') - (1/6) \geqslant 5/6$. By Lemma 2.2 there exist $h_1, \ldots, h_N \in G$, $c_1, \ldots, c_N \in Z_u(A)$ such that

$$\left\| \frac{1}{N} \sum_{j=1}^{N} u_{h_j} c_j (y - e(y)) c_j^* u_{h_j}^* \right\| \leqslant \frac{1}{6}.$$

Let $r = \frac{1}{N} \sum_{j=1}^{N} u_{h_j} c_j y c_j^* u_{h_j}^*$ and $r_1 = \frac{1}{N} \sum_{j=1}^{N} u_{h_j} e(y) u_{h_j}^* \geqslant 5/6$. It follows that $\|r - r_1\| \leqslant 1/6$, hence $r \geqslant r_1 - (1/6) \geqslant (5/6) - (1/6) = 2/3$. Denoting $z = \frac{1}{N} \sum_{j=1}^{N} u_{h_j} c_j x' c_j^* u_{h_j}^*$ we have $z \in I$, $z \geqslant 0$ and $\|r - z\| \leqslant \|x' - y\| \leqslant 1/6$, hence $z \geqslant r - (1/6) \geqslant (2/3) - (1/6) = 1/2$ and z is an invertible element. ▨

COROLLARY 2.7. *If G is a weak Powers group and A a simple unital C^*-algebra, then $A \times_{\alpha,c}^r G$ is simple.*

COROLLARY 2.8. *If G is a weak Powers group and $c \in Z^2(G, \mathbf{T})$, then $C_r^*(G, c)$ is simple with a unique trace.*

COROLLARY 2.9. *If G is a weak Powers group, then $C_r^*(G)$ is simple with a unique trace.*

This corollary and Proposition 1.4 show that if G_1, G_2 are weak Powers groups, then $C_r^*(G_1 \times G_2)$ is simple with unique trace. Actually, it is easy to see that the C^*-tensor product $C_r^*(G_1) \otimes C_r^*(G_2)$ is isomorphic with $C_r^*(G_1 \times G_2)$. Using this and [8, Corollary 4.21] it follows that if $C_r^*(G_i)$, $i = 1, 2$ are simple with unique trace, then $C_r^*(G_1 \times G_2)$ is simple with a unique trace.

PROPOSITION 2.10. *Let G be a discrete group and G' a normal subgroup of G. If G' and G/G' are weak Powers groups, then $C_r^*(G)$ is simple with a unique trace.*

Proof. Let $\varepsilon > 0$ and $Y = Y^* \in \mathbf{C}[G]$ with $\tau(Y) = 0$ ($\mathbf{C}[G]$ is the group algebra of G). Then $Y = \sum_{j=1}^{r} \lambda_j u_{g_j} + \overline{\lambda}_j u_{g_j}^*$, where $1 \notin \{g_1, \ldots, g_r\}$. Denote $Y_j = \lambda_j u_{g_j} + \overline{\lambda}_j u_{g_j}^*$, for $j = 1, \ldots, r$. One may assume that $g_1, \ldots, g_p \in G'$ and $g_{p+1}, \ldots, g_r \in G \setminus G'$. Since G' is a weak Powers group, Lemma 2.2 implies that there exist $h_1, \ldots, h_n \in G'$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u_{h_k} \tilde{Y} u_{h_k}^* \right\| \leqslant \frac{p\varepsilon}{r}, \tag{7}$$

where $\tilde{Y} = Y_1 + \ldots + Y_p$. Let $\tilde{Y}_{p+1} = \frac{1}{n} \sum_{k=1}^{n} u_{h_k} Y_{p+1} u_{h_k}^*$. Then $\operatorname{supp} \tilde{Y}_{p+1}$ is included in the union of the conjugacy class of $g_{p+1} \in G \setminus G'$ and its inverse. Hence $\operatorname{supp} \tilde{Y}_{p+1}$ is included in $G \setminus G'$. As in the second case of the proof of Proposition 1.5 we see that for any $f \in G \setminus G'$, any finite set $M \subset G$ and any integer $N \geqslant 1$, there exist $G = A \amalg B$ and $g_1, \ldots, g_N \in G$ such that

$$\gamma A \cap A = \emptyset, \quad \text{for all } \gamma \in \langle f \rangle_M;$$

$$g_j B \cap g_k B = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ \ j \neq k.$$

As in Lemma 2.2 we obtain $g_{1,1}, \ldots g_{1,n_1} \in G$ with

$$\left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} u_{g_{1,k_1}} \tilde{Y}_{p+1} u_{g_{1,k_1}}^* \right\| \leqslant \frac{\varepsilon}{r}. \tag{8}$$

By (7) and (8) we get

$$\left\| \frac{1}{nn_1} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} u_{g_{1,k_1}} u_{h_i}(\tilde{Y} + Y_{p+1}) u_{h_i}^* u_{g_{1,k_1}}^* \right\| \leqslant$$

$$\leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} u_{h_i} \tilde{Y} u_{h_i}^* \right\| + \left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} u_{g_{1,k_1}} \tilde{Y}_{p+1} u_{g_{1,k_1}}^* \right\| \leqslant$$

$$\leqslant \frac{p\varepsilon}{r} + \frac{\varepsilon}{r} = \frac{(p+1)\varepsilon}{r}.$$

Take now $\tilde{Y}_{p+2} = \frac{1}{nn_1} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} u_{g_{1,k_1}} u_{h_i} Y_{p+2} u_{h_i}^* u_{g_{1,k_1}}^*$. By induction, as in Lemma 2.2 we obtain:

$$\left\| \frac{1}{nn_1 \dots n_{r-p}} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} \dots \sum_{k_{r-p}=1}^{n_{r-p}} u_{g_{r-p,k_{r-p}}} \dots u_{g_{1,k_1}} Y u_{g_{1,k_1}}^* \dots u_{g_{r-p,k_{r-p}}}^* \right\| \leqslant \varepsilon.$$

Hence, there exist $m = nn_1 \dots n_{r-p} \in \mathbf{N}$ and $g_1, \dots, g_m \in G$ such that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} u_{g_k} Y u_{g_k}^* \right\| \leqslant \varepsilon. \tag{9}$$

By approximation we obtain for every $y = y^* \in C_r^*(G)$ with $\tau(y) = 0$ and for every $\varepsilon > 0$, an integer $m \geqslant 1$ and $g_1, \dots, g_m \in G$ such that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} u_{g_k} y u_{g_k}^* \right\| \leqslant \varepsilon. \tag{10}$$

The last part of the argument is standard ([2], [4]). Let $I \subset C_r^*(G)$ be a non-zero two-sided ideal in A. Choose $y \neq 0$ in I. One may assume $y \geqslant 0$ and $\tau(y) = 1$. By (10) we obtain

$$\left\| \frac{1}{N} \sum_{k=1}^{N} u_{g_k} y u_{g_k}^* - 1 \right\| \leqslant \varepsilon.$$

For $\varepsilon < 1$, the element $z = \frac{1}{N} \sum_{k=1}^{N} u_{g_k} y u_{g_k}^* \in I$ is invertible hence $I = C_r^*(G)$. Let τ' be a trace on $C_r^*(G)$. Taking ε arbitrarily small it is clear that $\tau'(y) = 1 = \tau(y)$ for any $y \in C_r^*(G)$, $y \geqslant 0$, $\tau(y) = 1$. Hence $\tau' = \tau$. ▨

The above considerations suggest the following questions:

1) Does there exist a weak Powers group which is not a Powers group? The answer is unknown to us even for some semidirect products of F_2 by F_2.

2) Let G_1 and G_2 be discrete groups such that $C_r^*(G_1)$ and $C_r^*(G_2)$ are simple with unique trace and $1 \to G_1 \to G \to G_2 \to 1$ be a short exact sequence of groups. Is it true that $C_r^*(G)$ is necessary simple with a unique trace?

If G_1 and G_2 are weak Powers groups, the answer is affirmative by Proposition 2.10. When G is compatible with the action of G_2 on G_1, then $C_r^*(G)$ is the cross-product of $C_r^*(G_2)$ by G_1 (see [9]), hence in this case the answer is affirmative whenever G_1 is a weak Powers group (2.3, 2.6, 2.9).

3. AN EXAMPLE

In this section we present a non-trivial class of weak Powers groups related to some question of P. de la Harpe ([2]). Indeed, we show a class of fundamental groups of graphs of groups which are weak Powers groups. These groups are semidirect products of a free group and the fundamental group of the universal cover of the graph relative to a maximal tree and in view of Proposition 1.5 they are weak Powers groups. For such groups G, the results of [4] imply only the fact that $C_r^*(G)$ is simple with unique trace without pointing the combinatorial properties of G.

The following lemma is implicitly contained in [2] and [4].

LEMMA 3.1. *If $\{G_\lambda\}_{\lambda\in A}$ is an increasing family of Powers groups (weak Powers groups), then $G = \bigcup_{\lambda\in A} G_\lambda$ is a Powers group (weak Powers group).*

Proof. Assume G_λ Powers groups. Let $N \geqslant 1$ and $F \subset G\setminus\{1\}$ be a finite set. There exists $\lambda_0 \in A$ such that $F \subset G_{\lambda_0}\setminus\{1\}$, hence $G_{\lambda_0} = A_{\lambda_0} \amalg B_{\lambda_0}$ and there are $g_1, \ldots, g_N \in G_{\lambda_0}$ which verify

$$fA_{\lambda_0} \cap A_{\lambda_0} = \emptyset, \quad \text{for all } f \in F;$$

$$g_jB_{\lambda_0} \cap g_kB_{\lambda_0} = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

Let $\{\gamma_i\}_{i\in I}$ be a complete system of right representants of G modulo G_{λ_0}. If $A = \coprod_{i\in I} A_{\lambda_0}\gamma_i$, $B = \coprod_{i\in I} B_{\lambda_0}\gamma_i$ then it is clear that

$$fA \cap A = \emptyset, \quad \text{for all } f \in F;$$

$$g_jB \cap g_kB = \emptyset, \quad \text{for } j, k = 1, \ldots, N, \ j \neq k.$$

A similar argument holds for the second part of the statement. ▨

For definitions and results about graphs and fundamental groups of graphs of groups we refer to Serre's monography [7].

LEMMA 3.2. *Let (G, T) be an infinite tree of groups such that G_Q has at least two elements for each $Q \in \operatorname{vert} T$. Let $P \in \operatorname{vert} T$ be fixed and*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \{P\}$$

the inverse system associated to P ([7, I.2.2]). If there are infinitely many indices k such that between X_k and X_{k+1} there exists an edge y with $G_y = \{1\}$, then the direct limit $G_T = \varinjlim(G, T)$ is a Powers group.

Proof. It is known by [5] that for G_1, G_2 groups, where G_1 has at least three elements and G_2 has at least two elements, the free product $G_1 * G_2$ is a Powers group. Hence G_T is the union of an increasing family of Powers group. Now, Lemma 3.1 shows that G_T is a Powers group. ▨

The next lemma is an exercise in [7, I.5.1].

LEMMA 3.3. *Let (G, Y) be a non-empty connected graph of groups, and let T be a maximal tree of Y. Let $(\tilde{Y}, T)$ be the universal cover of Y relative to T; the graph $\tilde{Y}$ is a tree, on which the group $\pi_1(Y, T)$ acts freely. If $Q \in \operatorname{vert} \tilde{Y}$ projects to $P \in \operatorname{vert} Y$, we put $G_Q = G_p$; we define similarly G_y for $y \in \operatorname{edge} \tilde{Y}$ as well as $G_y \to G_{t(y)}$; the result is a tree of groups $(G, \tilde{Y})$ on which $\pi_1(Y, T)$ acts in a natural way.*

Then $\pi_1(G, Y, T)$ is canonically isomorphic to the semidirect product of $\pi_1(Y, T)$ and the group $\pi_1(G, \tilde{Y}, \tilde{Y}) = \varinjlim(G, \tilde{Y})$.

PROPOSITION 3.4. *Let (G, Y) be a non-empty connected graph of groups and let T be a maximal tree of Y. Assume that:*

(i) *The group G_Q has at least two elements for every $Q \in \operatorname{vert} Y$;*

(ii) *There exists $y \in \operatorname{edge} Y$ with $G_y = \{1\}$;*

(iii) *The fundamental group $\pi_1(Y, T)$ has at least two generators.*

Then the fundamental group $\pi_1(G, Y, T)$ is a weak Powers group.

Proof. By Lemma 3.3, $\pi_1(G, Y, T)$ is the semidirect product of $\pi_1(Y, T)$ and $\pi_1(G, \tilde{Y}, \tilde{Y})$. The group $\pi_1(Y, T)$ is isomorphic with the free group with n generators ($n \in \mathbf{N} \cup \{\infty\}$) and $\pi_1(G, \tilde{Y}, \tilde{Y})$ is a Powers group (by Lemma 3.2). Hence $\pi_1(G, Y, T)$ is the semidirect product of two Powers groups and so, by Proposition 1.5, $\pi_1(G, Y, T)$ is a weak Powers group. ▨

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