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ON EQUIVARIANT SHEAF COHOMOLOGY AND ELEMENTARY C^*-BUNDLES

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In what follows the equivariant sheaf cohomology of Grothendieck (see [14], Chapter V) is applied to the study of an appropriate class of groupoids. This cohomology is given as the sequence of right derived functors of the invariant section functor (defined on equivariant sheaves). Extensions of groupoids, here called twists (in order that no confusion with the bifunctor Ext will result) are classified by a functor which is distinct from yet closely related to the second cohomology. This situation is clarified by the existence of a long exact sequence relating equivariant sheaf cohomology to ordinary sheaf cohomology in which the twist functor occurs (cf. [20]). Using this machinery we show that actions of a groupoid on elementary C^*-bundles fibred over its unit space are classified up to Morita equivalence by an invariant taking values in the second (equivariant) cohomology with the sheaf of germs of continuous circle-valued functions as coefficients: this invariant includes the Dixmier-Douady invariant and the Mackey obstruction as special cases. Renault has conjectured that this classifier may be identified with $\mathrm{Ext}(\Gamma, \mathbf{T})$ (see [36]) for a more general class of groupoids.

The class of groupoids studied in this paper are termed sheaf groupoids and defined to be topological groupoids for which the source map is a local homeomorphism. In addition to the Grothendieck cohomology, denoted $H^*(\Gamma, \cdot)$ (where Γ is a sheaf groupoid),there is available a continuous cocycle cohomology of topological groupoids (see [17], [22]). This is the subject of our first addendum (§5); as expected the continuous cocycle cohomology, noted $H_c^*(\Gamma, \cdot)$, may be characterized as derived functors with respect to a relative notion of exactness. In the second addendum we collect an assortment of facts relating to elementary C^*-bundles needed in §4. Since there is a bijective correspondence between continuous trace algebras and elementary C^*-bundles, this addendum serves to translate known results concerning continuous trace algebras into the language of bundles (see [7], Chapter 10). The third addendum deals with the computation of the cohomology of certain inductive limit groupoids in terms of a $\varprojlim^1$ sequence.

The main body of the text begins with a preliminary treatment of sheaf groupoids and the relevant coefficient category (§0). If Γ is a sheaf groupoid, a Γ-sheaf is defined to be a sheaf over Γ^0 equipped with an action of Γ (see [15]). The category of Γ-sheaves, noted $\mathscr{S}(\Gamma)$, is easily seen to be an abelian category. A homomorphism of sheaf groupoids gives rise to a pull-back construction (just as with ordinary sheaves) which yields an exact functor between coefficient categories.

In order to invoke the calculus of derived functors, one must show that $\mathscr{S}(\Gamma)$ has enough injectives. This is done in §1 by defining a functor $J: \mathscr{S}(\Gamma^0) \to \mathscr{S}(\Gamma)$ which is a right adjoint for the functor which forgets the action of Γ; since J takes injectives to injectives $\mathscr{S}(\Gamma)$ is seen to have enough injectives. The cohomology functors $H^*(\Gamma, \cdot)$ are then defined to be the right derived functors of the invariant section functor $S_\Gamma \cong \mathrm{Hom}_\Gamma(\underset{\sim}{\mathbf{Z}}, \cdot)$ (where $\underset{\sim}{\mathbf{Z}}$ is the constant sheaf with fiber $\mathbf{Z}$ and trivial action): alternatively, $H^*(\Gamma, \cdot) \cong \mathrm{Ext}^*_\Gamma(\underset{\sim}{\mathbf{Z}}, \cdot)$. The contravariant dependence of the cohomology of Γ is then easily verified.

Given a Γ-sheaf A, the group of twists $T_\Gamma(A)$ is defined (§2) to be isomorphism classes of groupoid extensions:

$$A \to \Sigma \to \Gamma$$

compatible with the action of Γ on A with addition given by the Baer sum. We show that T_Γ is a half-exact functor and that $T_\Gamma(Q) = 0$ if Q is injective.

The twist group is seen to be the obstruction to lifting one-cocycles (§3); more precisely, T_Γ is the first derived functor of Z_Γ (where $Z_\Gamma(A)$ is the group of continuous A-valued one-cocycles). Letting S denote the section functor, it is easy to see that:

$$0 \to S_\Gamma(Q) \to S(Q) \to Z_\Gamma(Q) \to 0$$

is exact if Q is injective. Applying this triple to an injective resolution of a given Γ-sheaf yields a short exact sequence of complexes which by standard homological algebra gives rise to a long exact sequence relating $H^*(\Gamma, \cdot)$, $H^*(\Gamma^0, \cdot)$ and the right derived functors of Z_Γ. A fragment of this sequence was obtained in [20] for the groupoid $R(\psi)$ where ψ is a local homeomorphism (and trivial $R(\psi)$-sheaf $\underset{\sim}{\mathbf{T}}$).

In the final section (§4), actions of a sheaf groupoid Γ on elementary C^*-bundles fibred over the unit space Γ^0 are studied. If the elementary C^*-bundle is trivial (viz. $\cong \mathscr{K} \times \Gamma^0$ where $\mathscr{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space), an element of $T_\Gamma(\underset{\sim}{\mathbf{T}})$ results as the obstruction to lifting an action to a unitary representation of Γ (cf. [40], where an analogous result for Borel equivalence relations was obtained); the obstruction is a complete invariant for exterior (unitary cocyle) equivalence. A complete invariant for Morita equivalence (for such actions) is then obtained by taking the image of the obstruction under the map

$$T_\Gamma(\underset{\sim}{\mathbf{T}}) \to H^2(\Gamma, \underset{\sim}{\mathbf{T}}).$$

The general case is reduced to the above via stabilization and functoriality in Γ. We remark that this invariant is additive on tensor products and that all values in $H^2(\Gamma, \mathbf{T})$ occur.

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0. PRELIMINARIES

We view a groupoid Γ as a small category with inverses. The distinguished subset of units is noted Γ^0; the range and source maps are denoted by r and s while inversion is denoted $\gamma \to \gamma^*$. Composition is defined on the set $\Gamma^2 = \{(\gamma, \gamma') \in \Gamma \times \Gamma : s(\gamma) = r(\gamma')\}$ and is written $(\gamma, \gamma') \in \Gamma^2 \to \gamma\gamma' \in \Gamma$. A groupoid endowed with a topology for which the above structure maps are continuous is referred to as a topological groupoid. The reader is referred to [35] for a detailed treatment.

1. DEFINITION. A topological groupoid is called a *sheaf groupoid* if the source map $s: \Gamma \to \Gamma^0$ is a local homeomorphism.

It follows that $r: \Gamma \to \Gamma^0$ is a local homeomorphism as well and that Γ^0 is open in Γ.

2. REMARK. In dealing with applications to C^*-algebras (§4) we will assume that Γ is locally compact, Hausdorff, and second countable (hence paracompact).

3. DEFINITION. A continuous map $\varphi: \Lambda \to \Gamma$ where Λ and Γ are sheaf groupoids is said to be a *homomorphism* if:

$$(\lambda_1, \lambda_2) \in \Lambda^2 \Rightarrow (\varphi(\lambda_1), \varphi(\lambda_2)) \in \Gamma^2 \text{ and } \varphi(\lambda_1\lambda_2) = \varphi(\lambda_1)\varphi(\lambda_2).$$

It follows that $\varphi(\Lambda^0) \subset \Gamma^0$.

Thus the category of sheaf groupoids contains both the category of topological spaces (with continuous maps) and the category of discrete groups (with homomorphisms). We consider some examples.

4. EXAMPLE. i) Let G be a discrete group and X a topological space. An action of G on X is given by a homomorphism $\alpha: G \to \mathrm{Homeo}(X)$. One may associate a groupoid Γ_α to this action as follows. Put $\Gamma_\alpha = G \times X$, $\Gamma_\alpha^0 = \{e\} \times X \cong X$, $s(g, x) = x$, $r(g, x) = \alpha_g(x)$, $(g, x)^* = (g^{-1}, \alpha_g^{-1}(x))$ and composition is defined for (g, x), (g', x') if $x = \alpha_{g'}(x')$ in which case set $(g, x)(g', x') = (gg', x')$.

ii) If $\psi: X \to Z$ is a local homeomorphism put $R(\psi) = X * X = \{(x, x') \in X \times X : \psi(x) = \psi(x')\}$. One checks that $R(\psi)$ is a groupoid with $R(\psi)^0 \cong X$ $(x \to$

$\to (x, x)$) $s(x, x') = x'$, $r(x, x') = x$, $(x, x')^* = (x', x)$, $(x, x')(x', x'') = (x, x'')$. Such groupoids will be called *preliminary*.

(iii) Let $q: M \to X$ be a sheaf of abelian groups (as in [16]) over X. Then M may be regarded as groupoid with $M^0 \cong X$ identified with the image of the zero section. One has $q = r = s$; $m^* = -m$ and if $q(m) = q(m')$ write $mm' = m + m'$. Write $q^{-1}(x) = M_x$. We shall usually omit explicit reference to the projection q when no confusion results. If $U \subset X$ is open, let $S(U, M)$ denote the group of continuous sections over U: write $S(M) = S(X, M)$.

5. OPERATIONS. There are several ways of constructing new groupoids from old ones. What follows should not be regarded as a complete list.

i) If Γ_0 and Γ_1 are groupoids one may form their product $\Gamma_0 \times \Gamma_1$ (with $(\Gamma_0 \times \Gamma_1)^0 = \Gamma_0^0 \times \Gamma_1^0$) or their (disjoint) sum $\Gamma_0 \amalg \Gamma_1$ (with $(\Gamma_0 \amalg \Gamma_1)^0 = \Gamma_0^0 \amalg \Gamma_1^0$).

ii) If Γ is a groupoid and $U \subset \Gamma^0$ is open, set $\Gamma_U = \{\gamma \in \Gamma : s(\gamma), r(\gamma) \in U\}$; Γ_U is called the *reduction* of Γ to U (note $(\Gamma_U)^0 = U$).

If U is full, viz. $r(s^{-1}(U)) = \Gamma^0$, Γ_U is called a *full reduction*.

iii) Let Γ be a groupoid and $\psi: X \to \Gamma^0$ a local homeomorphism. One forms the induced groupoid Γ^ψ with unit space X as follows. Set $\Gamma^\psi = X * \Gamma * X = \{(x, \gamma, x') \in X \times \Gamma \times X : \psi(x) = r(\gamma), \psi(x') = s(\gamma)\}$; the unit space of Γ^ψ is identified with X by $x \to (x, \psi(x), x)$. The structure maps are given by: $s(x, \gamma, x') = x'$, $(x, \gamma, x')^* = (x', \gamma^*, x)$ and

$$(x, \gamma, x')(x', \gamma', x'') = (x, \gamma\gamma', x'').$$

There is a homomorphism $\pi_\psi : \Gamma^\psi \to \Gamma$ given by $(x, \gamma, x') \to \gamma$. Note that $R(\psi) \cong \mathbf{Z}^\psi$.

iv) Given groupoid homomorphisms $\varphi_i : \Lambda_i \to \Gamma$ for $i = 0,1$ one may form the fiber-product over Γ, $\Lambda_0 \underset{\Gamma}{*} \Lambda_1 = \{(\lambda_0, \lambda_1) \in \Lambda_0 \times \Lambda_1 : \varphi_0(\lambda_0) = \varphi_1(\lambda_1)\}$ with unit space $(\Lambda_0 \underset{\Gamma}{*} \Lambda_1)^0 = \Lambda_0^0 * \Lambda_1^0$ (groupoid structure is inherited from $\Lambda_0 \times \Lambda_1$).

v) Let $\Sigma \subset \Gamma$ be an open subgroupoid with $\Sigma^0 = \Gamma^0$: Σ is said to be normal if the following holds:

a) $r(\sigma) = s(\sigma)$ for all $\sigma \in \Sigma$,

b) if $\sigma \in \Sigma$, $\gamma \in \Gamma$, with $r(\sigma) = s(\gamma)$ then $\gamma\sigma\gamma^* \in \Sigma$.

If Σ is normal one forms the quotient groupoid with unit space identified with Γ^0 as follows. Set $\gamma \sim \gamma'$ if there is $\sigma \in \Sigma$ such that $\gamma' = \gamma\sigma$; condition (b) implies that involution and composition respect this equivalence relation (thus if $\gamma_0 \sim \gamma_0'$ and $\gamma_1 \sim \gamma_1'$ then $\gamma_0\gamma_1 \sim \gamma_0'\gamma_1'$). The quotient of Γ by this equivalence relation is denoted Γ/Σ and is easily seen to be a sheaf groupoid with groupoid structure induced by the quotient map $\Gamma \to \Gamma/\Sigma$.

We proceed now to a discussion of the coefficient category associated to a sheaf groupoid Γ (cf. [15]).

6. DEFINITION. Let A be a sheaf over Γ^0; an *action* of Γ on A is given by a continuous map $\alpha : \Gamma * A \to A$, where $\Gamma * A = \{(\gamma, a) : a \in A_{s(\gamma)}\}$, satisfying the conditions:

i) $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$ is an isomorphism,

ii) if $(\gamma, \gamma') \in \Gamma^2$ then $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$.

A sheaf equipped with such an action is called a *Γ-sheaf*, It is often convenient to avoid explicit mention of α — we write $(\gamma, a) \to \gamma a$ — and by standard abuse of language to refer to A as a Γ-sheaf. Given Γ-sheaves A and A', a sheaf morphism $f: A \to A'$ is said to be a Γ-morphism if $f(\gamma a) = \gamma f(a)$ for all $(\gamma, a) \in \Gamma * A$. Let $\mathrm{Hom}_\Gamma(A, A')$ denote the set of Γ-morphisms from A to A'. Let $\mathcal{S}(\Gamma)$ denote the category of Γ-sheaves with Γ-morphisms.

7. PROPOSITION. *$\mathcal{S}(\Gamma)$ is an abelian category.*

Proof. It is immediate that $\mathrm{Hom}_\Gamma(A, A')$ is an abelian group and that direct sums exist (write $A \oplus B$). Further if $f \in \mathrm{Hom}_\Gamma(A, A')$ then $\ker f$ is a subsheaf of A which is invariant under the action of Γ (if $f(a) = 0$ then $\gamma f(a) = f(\gamma a) = 0$). Hence kernels and similarly cokernels exist in $\mathcal{S}(\Gamma)$. Moreover, the induced map $\bar{f} : A/\ker f \to \mathrm{Im} f$ is an isomorphism. ▨

REMARKS. A sequence of Γ-sheaves is exact iff it is exact as a sequence of sheaves. If G is a (discrete) abelian group let $\underset{\sim}{G} = G \times \Gamma^0$ be the constant sheaf with fiber G. If H is a locally compact abelian group let $\underset{\sim}{H}$ denote the sheaf of germs of continuous H-valued functions on Γ^0. In either case we may regard these as Γ-sheaves with trivial action. If A is a Γ-sheaf $\mathrm{Hom}_\Gamma(\underset{\sim}{Z}, A)$ may be identified with the group of (continuous) invariant sections $S_\Gamma(A)$ of A, that is, $\mathrm{Hom}_\Gamma(\underset{\sim}{Z}, A) \cong \{f : \Gamma^0 \to A : f(x) \in A_x \text{ and } f(r(\gamma)) = \gamma f(s(\gamma)) \text{ for all } \gamma \in \Gamma\}$. Observe that $S_\Gamma : \mathcal{S}(\Gamma) \to \mathcal{Ab}$ is a left-exact covariant functor from the category of Γ-sheaves to abelian groups.

Let $\varphi : \Lambda \to \Gamma$ be a homomorphism of sheaf groupoids. As with sheaves one may define the notion of pull-back as a functor $\varphi^* : \mathcal{S}(\Gamma) \to \mathcal{S}(\Lambda)$. If A is a Γ-sheaf, set $\varphi^*(A) = \Lambda^0 * A = \{(z, a) \in \Lambda^0 \times A : a \in A_{\varphi(z)}\}$ and observe that $\Lambda * \varphi^*(A) \cong \Lambda * A = \{(\lambda, a) : a \in A_{\varphi(s(\lambda))}\}$. The action, $\Lambda * \varphi^*(A) \to \varphi^*(A)$, is given by $(\lambda, a) \to \lambda a = (r(\lambda), \varphi(\lambda)a)$. If $f \in \mathrm{Hom}_\Gamma(A, A')$ one defines $\varphi^*(f) : \varphi^*(A) \to \varphi^*(A')$ as in the case of sheaves and checks that $\varphi^*(f) \in \mathrm{Hom}_\Lambda(A, A')$.

8. PROPOSITION. *$\varphi^* : \mathcal{S}(\Gamma) \to \mathcal{S}(\Lambda)$ is an exact covariant functor for which $\varphi^*(\underset{\sim}{Z}) \cong \underset{\sim}{Z}$; if $\theta : \Sigma \to \Lambda$ is another homomorphism then $(\varphi\theta)^* = \theta^*\varphi^*$.*

Proof. A sequence of sheaves is exact iff it is exact at each fiber. ▨

9. THEOREM. *The functor $\pi_\psi^* : \mathcal{S}(\Gamma) \to \mathcal{S}(\Gamma^\psi)$ yields an equivalence of categories.*

Proof. It suffices to construct a functor $F^\psi : \mathcal{S}(\Gamma^\psi) \to \mathcal{S}(\Gamma)$ such that $F^\psi \pi_\psi^*$ is naturally isomorphic to the identity on $\mathcal{S}(\Gamma)$ and $\pi_\psi^* F^\psi$ is naturally isomorphic to the identity on $\mathscr{S}(\Gamma^{\psi})$. First identify $R(\psi)$ with its image in Γ^{ψ} by $j(x, x') = (x, \psi(x), x')$. Given a Γ^{ψ}-sheaf A, consider the equivalence relation on A defined by the action restricted to $R(\psi)$, that is, set $a \in A_x \sim a' \in A_{x'}$ if $\psi(x) = \psi(x')$ and $a = j(x, x')a'$. Note that the quotient of A by this equivalence relation becomes a sheaf over Γ^0 — denote this sheaf $F^{\psi}(A)$ and the image of $a \in A$ by $[a] \in F^{\psi}(A)$. If $a \in A_{x'}$ and $\gamma \in \Gamma$ with $s(\gamma) = \psi(x')$ choose $x \in \psi^{-1}(r(\gamma))$ and set $\gamma[a] = [(x, \gamma, x')a]$. This defines an action of Γ on $F^{\psi}(A)$ unambiguously. Clearly $A \cong \pi_{\psi}^{*}(F^{\psi}(A))$ and, conversely, given a Γ-sheaf B one has $B \cong F^{\psi}(\pi_{\psi}^{*}(B))$.

10. Corollary. *The functors S_{Γ} and $S_{\Gamma^{\psi}}\pi_{\psi}^{*}$ are naturally isomorphic.*

Proof. One has $S_{\Gamma}(A) \cong \mathrm{Hom}_{\Gamma}(\underset{\sim}{\mathbf{Z}}, A) \cong \mathrm{Hom}_{\Gamma^{\psi}}(\pi_{\psi}^{*}(\underset{\sim}{\mathbf{Z}}), \pi_{\psi}^{*}(A)) \cong S_{\Gamma^{\psi}}(\pi_{\psi}^{*}(A))$ for each Γ-sheaf A. ∎

Following Grothendieck (see [14]) we propose to define the cohomology of a sheaf groupoid Γ as the right derived functors of the invariant section functor $S_{\Gamma} : \mathscr{S}(\Gamma) \to \mathscr{A}\ell$.

11. Definition. A *cohomology theory* for Γ is a sequence of covariant functors $H^{*}(\Gamma, \cdot) : \mathscr{S}(\Gamma) \to \mathscr{A}\ell$ satisfying the conditions:

i) $H^0(\Gamma, \cdot) \cong S_{\Gamma}$;

ii) $H^n(\Gamma, Q) = 0$ for $n > 0$ and Q injective;

iii) Given a short exact sequence of Γ-sheaves, $0 \to A \to B \to C \to 0$, there are natural connecting maps $\delta^n : H^n(\Gamma, C) \to H^{n+1}(\Gamma, A)$ which make the following a long exact sequence

$$0 \to H^0(\Gamma, A) \to H^0(\Gamma, B) \to H^0(\Gamma, C) \xrightarrow{\delta^0} H^1(\Gamma, A) \to \ldots \to H^n(\Gamma, C) \xrightarrow{\delta^n} H^{n+1}(\Gamma, A) \to \ldots.$$

In order to make this a useful definition one must show that $\mathscr{S}(\Gamma)$ has enough injectives, that is, each Γ-sheaf A may be embedded in an injective. One may then "compute" $H^{*}(\Gamma, A)$ by applying S_{Γ} to an injective resolution of A. Grothendieck offers criteria for the existence of enough injectives but we prefer to deal with the matter in a more concrete manner as it will illuminate the relationship between $\mathscr{S}(\Gamma)$ and $\mathscr{S}(\Gamma^0)$. Having done this the contravariant dependence of cohomology on Γ will be easy to verify.

In the case that Grothendieck deals with, $\Gamma \cong G \times X$ (see Example 4i), he writes $H^{*}(X, G; \cdot)$ for our $H^{*}(\Gamma, \cdot)$.

1. INJECTIVES

Given a sheaf groupoid Γ we show that the functor $S(\Gamma) \to \mathscr{S}(\Gamma^0)$, write $A \to A^0$, given by disregarding the action has a right adjoint J (so that $\mathrm{Hom}(B^0, A) \cong \mathrm{Hom}_{\Gamma}(B, J(A))$) which takes injectives to injectives. This is used to show that

$\mathscr{S}(\Gamma)$ has enough injectves — the cohomology of Γ will then be realized as the cohomology of the complex obtained by applying the invariant section functor to an injective resolution.

1. Given a sheaf A over Γ^0, set $J(A) = s_*(r^*(A))$ where s_* denotes the push-forward; the associated presheaf of sections is given by

$$S(U,\ J(A)) = S(s^{-1}(U),\ r^*(A)) = \{f : s^{-1}(U) \to A\ :\ f(\gamma) \in A_{r(\gamma)}\}.$$

Elements of $J(A)$ are regarded as equivalence classes of such functions (note that $J(A)_x = \lim\limits_{x \in U} S(s^{-1}(U), r^*(A))$ for each $x \in \Gamma^0$). To define the action one must make sense of the formula:

$$(*) \qquad (\gamma f)(\gamma') = f(\gamma'\gamma) \quad \text{where } \gamma \in \Gamma \text{ and } f \in J(A)_{s(\gamma)}.$$

This is done by extending the homeomorphism $s^{-1}(r(\gamma)) \to s^{-1}(s(\gamma))$ given by $\gamma' \to \gamma'\gamma$ to a suitable neighbourhood. Since $s, r : \Gamma \to \Gamma^0$ are local homeomorphisms there is for each $\gamma \in \Gamma$ an open set $r(\gamma) \in U_\gamma \subset \Gamma^0$ and a continuous map $\rho_\gamma : U_\gamma \to \Gamma$ satisfying the conditions:

i) $\rho_\gamma(r(\gamma)) = \gamma$,

ii) $r(\rho_\gamma(x)) = x$ for all $x \in U_\gamma$,

iii) $s \circ \rho_\gamma : U_\gamma \to \Gamma^0$ is injective.

Given another choice of neighbourhood U'_γ and continuous map $\rho'_\gamma : U'_\gamma \to \Gamma$ satisfying these conditions, one has $\rho_\gamma| U_\gamma \cap U'_\gamma = \rho'_\gamma| U_\gamma \cap U'_\gamma$. Set $V_\gamma = s(\rho_\gamma(U_\gamma))$ and note that the map $s^{-1}(U_\gamma) \to S^{-1}(V_\gamma)$ defined by $\gamma' \to \gamma'\rho_\gamma(s(\gamma'))$ is a homeomorphism and that $r(\gamma') = r(\gamma'\rho_\gamma(s(\gamma')))$, for each $\gamma' \in s^{-1}(U_\gamma)$. Hence, the formula:

$$(\alpha_\gamma f)(\gamma') = f(\gamma'\rho_\gamma(s(\gamma')))$$

gives an isomorphism $\alpha_\gamma : S(s^{-1}(U_\gamma),\ r^*(A)) \to S(s^{-1}(V_\gamma),\ r^*(A))$ which is compatible with restriction maps. This then provides us with the desired action.

Given a Γ-sheaf B and a sheaf morphism $k : B^0 \to A$, we define a Γ-morphism $\tilde{k} : B \to J(A)$ by $k(b)(\gamma) = k(\gamma b)$. This yields a homomorphism $\Phi^A_B : \mathrm{Hom}(B^0, A) \to$ $\to \mathrm{Hom}_\Gamma(B, J(A))$ (where Hom denotes sheaf morphisms by $\Phi^A_B(k) = \tilde{k}$.

2. Lemma. *Φ^A_B is a natural isomorphism.*

Proof. Let $e_A : J(A)^0 \to A$ be given by $e_A(f) = f(x)$ where $f \in J(A)^0_x$. For all $k \in \mathrm{Hom}(B^0, A)$, $k = e_A \Phi^A_B(k)^0$ and for all $h \in \mathrm{Hom}_\Gamma(B, J(A))$, $h = \Phi^A_B(e_A h^0)$. ▨

3. Definition. A Γ-sheaf D is called a *relative injective* if given any injective Γ-morphism $j : B \to C$ for which $j(B)^0$ is a direct summand of C^0, the induced map $j^* : \mathrm{Hom}_\Gamma(C, D) \to \mathrm{Hom}_\Gamma(B, D)$ is surjective.

4. Proposition. *For any sheaf A over Γ^0, $J(A)$ is a relative injective.*

Proof. Given a diagram

$$\begin{array}{ccc} B & \underset{p}{\overset{j}{\rightleftarrows}} & C \\ {\scriptstyle h}\downarrow & & \\ J(A) & & \end{array}$$

with $p \in \mathrm{Hom}(C^0, B^0)$ such that $pj^0 = \mathrm{id}_{B^0}$. Set $k = \Phi_C^A(e_A h^0 p)$ and note that $h = kj$. ∎

Relative injectives are relevant to the continuous cocycle cohomology (see Addendum 1). It is well-known that the category of sheaves over a given space has enough injectives. Indeed, given a sheaf $A \to X$, let $A_x \to D^x$ denote the embedding of the fiber over x into its divisible hull; let $D(A)$ be the sheaf over X given by the presheaf:

$$S(U, D(A)) = \prod_{x \in U} D^x \quad \text{for } U \subset X \text{ open.}$$

$D(A)$ is clearly injective and one has $i_A : A \to D(A)$. To show that $\mathscr{S}(\Gamma)$ has enough injectives, it suffices to verify that J takes injectives to injectives.

5. Proposition. *If D is an injective in $\mathscr{S}(\Gamma^0)$, then $J(D)$ is an injective in $\mathscr{S}(\Gamma)$.*

Proof. Suppose $j : A \to B$ is an embedding of Γ-sheaves. Consider the commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}(B^0, D) & \xrightarrow[(j^0)^*]{} & \mathrm{Hom}(A^0, D) \\ {\scriptstyle \Phi_B^D}\downarrow & & \downarrow{\scriptstyle \Phi_A^D} \\ \mathrm{Hom}_\Gamma(B, J(D)) & \xrightarrow[j^*]{} & \mathrm{Hom}_\Gamma(A, J(D)) \end{array}$$

where the vertical arrows are isomorphisms. Since D is an injective, $(j^0)^*$ is surjective. Hence, j^* is surjective and $J(D)$ is an injective. ∎

6. Corollary. *$\mathscr{S}(\Gamma)$ has enough injectives.*

Proof. For each Γ-sheaf A, let $Q(A) = J(D(A^0))$; the embedding $A \to Q(A)$ is given by $\Phi_A^{D(A^0)}(i_A)$. ∎

Given an abelian category $\mathscr{S}$ with enough injectives and a left-exact covariant functor $T: \mathscr{S} \to \mathscr{S}'$ where $\mathscr{S}'$ is another abelian category one defines the right derived functions $R^n T : \mathscr{S} \to \mathscr{S}'$ using injective resolutions (cf. [14, 2.3]). Given an object

A in $\mathscr{S}$, form an injective resolution of A, $A \to Q_0 \to Q_1 \to Q_2 \to \dots$ and let $R^*T(A)$ be the cohomology of the complex $T(Q_0) \to T(Q_1) \to \dots$.

7. FACT. This sequence of functors is characterized up to natural isomorphism by the properties:

i) $R^0T \cong T$,

ii) $R^nT(Q) = 0$ if Q is an injective and $n > 0$,

iii) For each short exact sequence $E: 0 \to A \to B \to C \to 0$ there are natural connecting maps $\delta^n_E: R^nT(C) \to R^{n+1}T(A)$ such that the sequence:

$$0 \to R^0T(A) \to R^0T(B) \to R^0T(C) \xrightarrow{\delta^0_E} R^1T(A) \to \dots \to R^nT(C) \xrightarrow{\delta^n_E} R^{n+1}T(A) \to \dots$$

is exact.

If $\mathscr{S}$ is an abelian category, then $\mathrm{Hom}_{\mathscr{S}}(A, \cdot): \mathscr{S} \to \mathscr{A}b$ defines a left exact functor for any object A in $\mathscr{S}$; if $\mathscr{S}$ has enough injectives denote the right derived functors by $\mathrm{Ext}^n_{\mathscr{S}}(A, \cdot)$. $\mathrm{Ext}^*_{\mathscr{S}}$ may, of course, be viewed as a bifunctor (in which case a contravariant analog of (iii) holds in the first variable). It is useful to view elements of $\mathrm{Ext}^n_{\mathscr{S}}(A, B)$ as equivalence classes of n-fold exact sequences starting at B and ending at A, under the equivalence relation generated by morphisms between them:

$$\begin{array}{ccccccccccccc} 0 \to & B & \to & C_0 & \to & C_1 & \to \dots \to & C_{n-1} & \to & A & \to 0 \\ & \| & & \downarrow & & \downarrow & & \downarrow & & \| & \\ 0 \to & B & \to & C'_0 & \to & C'_1 & \to \dots \to & C'_{n-1} & \to & A & \to 0. \end{array}$$

Concatenation of such sequences gives the Yoneda pairing:

$$\mathrm{Ext}^n_{\mathscr{S}}(A, B) \times \mathrm{Ext}^m_{\mathscr{S}}(C, A) \to \mathrm{Ext}^{n+m}_{\mathscr{S}}(C, B).$$

On $\mathscr{S}(\Gamma)$ denote these bifunctors by Ext^*_Γ. Recall that the invariant section functor S_Γ is naturally isomorphic to $\mathrm{Hom}_\Gamma(\underset{\sim}{Z}, \cdot)$. Set $H^n(\Gamma, A) = \mathrm{Ext}^n_\Gamma(\underset{\sim}{Z}, A)$—this is called the nth cohomology group of Γ with coefficients in A. Note that $H^*(\Gamma, \underset{\sim}{Z}) = \mathrm{Ext}^*_\Gamma(\underset{\sim}{Z}, \underset{\sim}{Z})$ is a graded ring under the Yoneda pairing.

We consider now the dependence of Ext^*_Γ on Γ. Let $\varphi: \Lambda \to \Gamma$ be a groupoid homomorphism. Recall that $\varphi^*: \mathscr{S}(\Gamma) \to \mathscr{S}(\Lambda)$ is an exact functor; hence, if $0 \to B \to C_0 \to \dots \to C_{n-1} \to A \to 0$ is an n-fold exact sequence in $\mathscr{S}(\Gamma)$ then $0 \to \varphi^*(B) \to \varphi^*(C_0) \to \dots \to \varphi^*(C_{n-1}) \to \varphi^*(A) \to 0$ is an n-fold exact sequence in $\mathscr{S}(\Lambda)$. Since φ^* preserves equivalence classes of n-fold exact sequences, one has the following proposition.

8. PROPOSITION. *With $\varphi: \Lambda \to \Gamma$ as above, the map $\varphi^*: \mathrm{Ext}^n_\Gamma(A, B) \to \mathrm{Ext}^n_\Lambda(\varphi^*(A), \varphi^*(B))$ defines a natural transformation of bifunctors for each n. Note that since*

$\varphi^*(\underset{\sim}{Z}) \approx \underset{\sim}{Z}$, one has well-defined maps on cohomology $\varphi^* : H^*(\Gamma, A) \to H^*(\Lambda, \varphi^*(A))$. Furthermore, if $\pi_\psi : \Gamma^\psi \to \Gamma$ is the map discussed in § 0.5 iii), then $\pi_\psi^* : H^*(\Gamma, A) \to$ $\to H^*(\Gamma^\psi, \pi_\psi^*(A))$ is an isomorphism.

2. TWISTS

A twist is defined to be an extension of a sheaf groupoid Γ by a Γ-sheaf which is compatible with the action of Γ (by analogy with group extensions). Isomorphism classes of twists by a given Γ-sheaf form an abelian group: this assignment gives rise to a covariant half-exact functor which vanishes on injectives.

1. DEFINITION. Let Γ be a sheaf groupoid and A a Γ-sheaf. A *twist* by A over Γ is given by a sheaf groupoid Σ (with $\Sigma^0 = \Gamma^0$) together with two groupoid homomorphisms (which respect the identification of unit spaces):

$$A \xrightarrow[j]{} \Sigma \xrightarrow[\pi]{} \Gamma$$

such that j is injective, π is surjective, $\pi^{-1}(\Gamma^0) = j(A)$ and for all $\sigma \in \Sigma$ and $a \in A_{s(\sigma)}$ one has $\sigma j(a)\sigma^* = j(\pi(\sigma)a)$. Note that π is a local homeomorphism (write $\pi(\sigma) = \dot{\sigma}$).

By the usual abuse of language we refer to Σ as a twist when explicit reference to the maps j and π does not appreciably enhance clarity. The semi-direct product $A \times \Gamma = A * \Gamma = \{(a, \gamma) : a \in A_{r(\gamma)}\}$ is the trivial twist defined by the following formulas:

i) $s(a, \gamma) = s(\gamma)$,

ii) $(a, \gamma)^* = (-\gamma^* a, \gamma^*)$,

iii) if $(\gamma, \gamma') \in \Gamma^2$ put $(a, \gamma)(a', \gamma') = (a + \gamma a', \gamma\gamma')$.

2. PROPOSITION. *A twist Σ is isomorphic to $A \times \Gamma$ iff there is a groupoid homomorphism $\tau : \Gamma \to \Sigma$ with $\pi\tau = \mathrm{id}_\Gamma$.*

Proof. The isomorphism $\tilde{\tau} : A \times \Gamma \to \Sigma$ is given by $\tilde{\tau}(a, \gamma) = j(a)\tau(\gamma)$. ∎

Given two twists Σ_0 and Σ_1 by A_0 and A_1, the fiber product $\Sigma_0 \underset{\Gamma}{*} \Sigma_1 = \{(\sigma_0, \sigma_1) : \dot{\sigma}_0 = \dot{\sigma}_1\}$ is easily seen to be a twist by $A_0 \oplus A_1$.

3. DEFINITION. A morphism of twists is given by a commutative diagram:

$$(*) \qquad \begin{array}{ccccc} A & \longrightarrow & \Sigma & \longrightarrow & \Gamma \\ \downarrow f & & \downarrow \tilde{f} & & \| \\ B & \longrightarrow & \Lambda & \longrightarrow & \Gamma \end{array}$$

where f is a Γ-morphism and $\tilde{f}$ is a groupoid homomorphism. Clearly f is injective (resp. surjective) iff $\tilde{f}$ is injective (resp. surjective). Note that projection onto either factor $\Sigma_0 \underset{\Gamma}{*} \Sigma_1 \to \Sigma_i$ $(i = 0, 1)$ yields a morphism of twists compatible with projection onto a direct summand $A_0 \oplus A_1 \to A_i$.

4. LEMMA. *Given a morphism of twists as above* $(*)$ *there is a surjective twist morphism:*

$$\begin{array}{ccccc} B \oplus A & \longrightarrow & (B \times \Gamma) * \Sigma & \longrightarrow & \Gamma \\ {\scriptstyle g}\downarrow & & {\scriptstyle \bar{g}}\downarrow & & \| \\ B & \longrightarrow & \Lambda & \longrightarrow & \Gamma \end{array}$$

such that $g(b \oplus a) = b + f(a)$ *and* $\bar{g}((0, \dot{\sigma}), \sigma) = \bar{f}(\sigma)$ *for all* $\sigma \in \Sigma$.

Proof. Let the embedding $B \to \Lambda$ be denoted by j and set $\bar{g}((b, \dot{\sigma}), \sigma) = j(b)\bar{f}(\sigma)$. Evidently, $\bar{g}$ is compatible with g and it suffices to check that this defines a groupoid homomorphism. Given $(\sigma, \sigma') \in \Sigma^2$ and $b \in B_{r(\sigma)}$, $b' \in B_{r(\sigma')}$, one has $((b, \dot{\sigma}), \sigma)((b', \dot{\sigma}'), \sigma) = ((b + \dot{\sigma}b', \dot{\sigma}\dot{\sigma}'), \sigma\sigma')$;

$$\bar{g}((b, \dot{\sigma}), \sigma)\bar{g}((b', \dot{\sigma}'), \sigma') = j(b)\bar{f}(\sigma)j(b')\bar{f}(\sigma') = j(b + \dot{\sigma}b')\bar{f}(\sigma)\bar{f}(\sigma') =$$

$$= \bar{g}((b + \dot{\sigma}b', \dot{\sigma}\dot{\sigma}'), \sigma\sigma').$$

It is routine to verify that $\bar{g}$ preserves involution. ▨

Note that Λ is isomorphic to the quotient of $(B \times \Gamma) \underset{\Gamma}{*} \Sigma$ by the image of $\ker g$.

5. DEFINITION. Two twists by A, Σ and Σ', are said to be properly isomorphic if there is a twist morphism between them which preserves the inclusion of A:

$$\begin{array}{ccccc} A & \longrightarrow & \Sigma & \longrightarrow & \Gamma \\ \| & & \downarrow & & \| \\ A & \longrightarrow & \Sigma' & \longrightarrow & \Gamma. \end{array}$$

The collection of proper isomorphism classes of twists by A is denoted $T_\Gamma(A)$ (write $[\Sigma] \in T_\Gamma(A)$).

6. PROPOSITION. *If* Σ *is a twist by* A *and* $f \in \mathrm{Hom}_\Gamma(A, B)$, *there exists a twist by* B *denoted* $f_*\Sigma$, *which is unique up to proper isomorphism, and a twist morphism* $\bar{f}: \Sigma \to f_*\Sigma$ *(which is compatible with* f*).*

Proof. Let $j: A \to \Sigma$ denote the twist embedding and define $i: A \to (B \times \Gamma) \underset{\Gamma}{*} \Sigma$ by $i(a) = ((-f(a), x), j(a))$ for $a \in A_x$. Note that $i(A)$ is a normal subgroupoid (of § 0.5 v)); set $f_*\Sigma = (B \times \Sigma) \underset{\Gamma}{*} \Sigma / i(A)$. It is clear that $f_*\Sigma$ is a twist by B. The map

$\bar{f} : \Sigma \to f_* \Sigma$ is given as the composite

$$\Sigma \to (B \times \Gamma) \underset{\Gamma}{*} \Sigma \to f_* \Sigma.$$

Uniqueness follows from the preceding lemma. ☐

Given $f \in \mathrm{Hom}_\Gamma(A, B)$ define $f_* : T_\Gamma(A) \to T_\Gamma(B)$ by $f_*[\Sigma] = [f_* \Sigma]$. It follows that this assignment is functorial. Note that $f_*[A \times\!| \Gamma] = [B \times\!| \Gamma]$; moreover, $0_*[\Sigma] = [B \times\!| \Gamma]$ for any $[\Sigma] \in T_\Gamma(A)$. We introduce a binary operation on $T_\Gamma(A)$ so that it becomes an abelian group with neutral element $[A \times \Gamma]$. Let $\nabla^A : A \oplus A \to A$ be given by $\nabla^A(a, a') = a + a'$. Set $[\Sigma] + [\Sigma'] = \nabla^A_*[\Sigma \underset{\Gamma}{*} \Sigma']$ where $[\Sigma'], [\Sigma] \in T_\Gamma(A)$. It is immediate that $[\Sigma] + [\Sigma'] = [\Sigma'] + [\Sigma]$ (since $a + a' = a' + a$).

7. THEOREM. *When endowed with the above operation $T_\Gamma(A)$ is an abelian group with neutral element $[A \times\!| \Gamma]$. Furthermore, $T_\Gamma : \mathscr{S}(\Gamma) \to \mathscr{A}b$ is a covariant half-exact functor.*

Proof. Give a twist Σ, the composite morphism

$$\Sigma \xrightarrow[\iota]{} (A \times\!| \Gamma) \underset{\Gamma}{*} \Sigma \to \nabla^A_*((A \times\!| \Gamma) \underset{\Gamma}{*} \Sigma)$$

(where $\iota(\sigma) = ((0, \dot{\sigma}), \sigma))$ is an isomorphism; hence, $[A \times\!| \Gamma] + [\Sigma] = [\Sigma]$ for all $[\Sigma] \in T_\Gamma(A)$. Define $\theta^A \in \mathrm{Hom}_\Gamma(A, A)$ by $\theta^A(a) = -a$; since $\nabla^A(\theta^a(a), a) = 0$ (i.e. $a - a = 0$) $[\Sigma] + \theta^A_*[\Sigma] = [\nabla^*_A(\Sigma \underset{\Gamma}{*} \theta^A_* \Sigma)] = [A \times \Gamma]$. Hence $T_\Gamma(A)$ is an abelian group with $-[\Sigma] = \theta^A_*[\Sigma]$. If $f \in \mathrm{Hom}_\Gamma(A, B)$ then $f_*([\Sigma] + [\Sigma']) = f_*[\Sigma] + f_*[\Sigma']$, by commutativity of the diagram:

$$\begin{array}{ccc} A \oplus A & \xrightarrow{(f, f)} & B \oplus B \\ \downarrow{\scriptstyle \nabla A} & & \downarrow{\scriptstyle \nabla B} \\ A & \xrightarrow{f} & B \end{array}.$$

It remains to show that T_Γ is half-exact; suppose that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of Γ-sheaves and that $[\Sigma] \in \ker g_*$, that is, $[g_* \Sigma] = [C \times \Gamma]$. One has the following surjective twist morphism:

$$\begin{array}{ccccc} B & \longrightarrow & \Sigma & \longrightarrow & \Gamma \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar{g}} & & \| \\ C & \longrightarrow & C \times \Gamma & \longrightarrow & \Gamma. \end{array}$$

Set $\Sigma' = \{\sigma \in \Sigma : \bar{g}(\sigma) = (0, \dot{\sigma})\}$; then Σ' is clearly a subgroupoid. In fact, Σ' is a twist by A ($\cong \ker g$) and hence, by the above proposition, $[\Sigma] = f_*[\Sigma']$. Hence $\ker g_* = \mathrm{Im}\, f_*$. ▨

In order to verify that T_Γ vanishes on injectives, it will be useful to identify $H_c^2(\Gamma, A)$ with a certain subgroup of $T_\Gamma(A)$, where $H_c^*(\Gamma, \cdot)$ denotes the continuous cocycle cohomology functors (see Addendum 1). Let $[\Sigma] \in T_\Gamma(A)$; Σ is said to be continuously split if there is a continuous section $h: \Gamma \to \Sigma$ of the twist projecion π (i.e. $\pi(h(\gamma)) = \gamma$ for all $\gamma \in \Gamma$). One defines a continuous two-cocyle $c_h: \Gamma^2 \to A$, which measures the extent to which h differs from a groupoid homomorphism, by the formula:

$$c_h(\gamma, \gamma') = h(\gamma)h(\gamma')h(\gamma\gamma')^* \quad \text{for all } (\gamma, \gamma') \in \Gamma^2.$$

It is not hard to see that Σ may be reconstructed from this cocycle (see [35]) and that cohomologous cocycles yield isomorphic twists. The reader is referred to Addendum 1 for details on the continuous cocyle cohomology. Of relevance here is its characterization as a relative cohomology theory; in particular, one has $H_c^n(\Gamma, J) = 0$ for all $n > 0$ if J is a relative injective.

A sheaf P over X is said to be of product type if there are abelian groups P^x for $x \in X$ such that $S(U, P) = \prod_{x \in U} P^x$ for each open subset $U \subset X$. Such sheave are known to be acyclic (i.e. $H^n(X, P) = 0$ for all $n > 0$).

8. PROPOSITION. *$T_\Gamma(Q) = 0$ if Q is an injective Γ-sheaf.*

Proof. It suffices to show that $T_\Gamma(J(P)) = 0$ if P is of product type. Now, $J(P)$ is itself of product type, since $S(U, J(P)) \cong \prod_{x \in U} (\prod_{\gamma \in s^{-1}(x)} P^{r(\gamma)})$; this will be used to show that any twist by $J(P)$ is continuously split. From this it will follow that $T_\Gamma(J(P)) \cong H_c^2(\Gamma, J(P)) = 0$ (since $J(P)$ is a relative injective). Let Σ be a twist by $J(P)$ with twist projection $\pi: \Sigma \to \Gamma$. Since π is a local homeomorphism, there is for each $\gamma \in \Gamma$ an open neighbourhood U_γ of γ and a continuous map $h_\gamma: U_\gamma \to \Sigma$ so that $\pi(h_\gamma(\gamma')) = \gamma'$ for all $\gamma' \in U_\gamma$. Hence, we may choose an open covering $\mathscr{U} = \{U_i : i \in I\}$ of Γ together with local sections $h_i: U_i \to \Sigma$; identifying $J(P)$ with its image in Σ, the formula $\lambda_{ij}(\gamma) = h_i(\gamma)h_j(\gamma)^*$ for $\gamma \in U_i \cap U_j$ yields a continuous $r^*(J(P))$ valued one-cocycle relative to $\mathscr{U}$. Since $r^*(J(P))$ is of product type, as well, the class of λ is trivial. Hence, Σ is continuously split. ▨

We conclude this section with a brief discussion concerning the dependence of T_Γ on Γ. Let $\varphi: \Lambda \to \Gamma$ be a groupoid homomorphism and Σ be a twist by A (over Γ). The fiber-product $\Lambda \underset{\Gamma}{*} \Sigma$ may be viewed as a twist over Λ by $\varphi^*(A)$:

$$\varphi^*(A) = \Lambda^0 * A \to \Lambda \underset{\Gamma}{*} \Sigma \to \Lambda.$$

We write $\varphi^*(\Sigma) = \Lambda \underset{\Gamma}{*} \Sigma$; the map $T_\Gamma(A) \to T_\Lambda(\varphi^*(A))$ given by $[\Sigma] \to [\varphi^*(\Sigma)]$ yields a natural transformation of functors $T_\Gamma \to T_\Lambda \circ \varphi^*$.

9. REMARK. The notion of twist considered in [21] is that of a topological extension of the groupoid Γ by the constant group bundle $\mathbf{T}\times\Gamma^0$ with trivial action. We use the term topological twist to distinguish this notion from that considered above. A topological groupoid Λ is said to be a topological twist over Γ if there is given a sequence of groupoids with common unit space (Γ^0)

$$\mathbf{T}\times\Gamma^0 \xrightarrow[i]{} \Lambda \xrightarrow[\pi]{} \Gamma$$

such that

i) π is a submersion (continuous open surjection admitting local sections),
ii) i is a homeomorphism onto $\pi^{-1}(\Gamma^0)$,
iii) $\lambda(t, s(\lambda))\lambda^* = (t, r(\lambda))$ for all $\lambda\in\Lambda$, $t\in\mathbf{T}$.

Note that Λ may be viewed as a principal $\mathbf{T}$-bundle over Γ via the map $(t, \lambda)\to\to(t, r(\lambda))\lambda$. The group of isomorphism classes of topological twists over Γ is denoted $\mathrm{Tw}(\Gamma)$ (in [21], Γ was required to be a relation, that is, a sheaf groupoid without isotropy). We show that $\mathrm{Tw}(\Gamma)$ may be identified with $T_\Gamma(\underset{\sim}{\mathbf{T}})$.

Given a topological twist Λ, let $\underset{\sim}{\Lambda}$ denote the set sheaf over Γ of germs of continuous local sections of $\pi:\Lambda\to\Gamma$. Given $\gamma\in\Gamma$, an element of $\underset{\sim}{\Lambda}_\gamma$ is given by a continuous section defined on an open neighbourhood of γ and two such are identified if they agree on some neighbourhood of γ. We equip $\underset{\sim}{\Lambda}=\coprod_\gamma \underset{\sim}{\Lambda}_\gamma$ with the usual sheaf topology. With groupoid structure inherited from Λ, it is imediate that $\underset{\sim}{\Lambda}$ is a twist by $\underset{\sim}{\mathbf{T}}$. Further, the map $\mathrm{Tw}(\Gamma)\to T_\Gamma(\underset{\sim}{\mathbf{T}})$ given by $[\Lambda]\to[\underset{\sim}{\Lambda}]$ is a monomorphism. To see that this map is surjective, let Σ be a twist by $\underset{\sim}{\mathbf{T}}$. Choose a family of local sections for the twist projection relative to an open covering of Γ, $\{h_i : U_i\to\to\Sigma\}$. Define a continuous $\mathbf{T}$-valued one-cocycle, $\lambda_{ij}:U_{ij}\to\mathbf{T}$, by the formula

$$\lambda_{ij}(\gamma) = h_i(\gamma)h_j(\gamma)^* .$$

Denote the associated circle-bundle by $\check{\Sigma}$ ($=\coprod_i \mathbf{T}\times U_i/\sim$ where $(t, \gamma)\in\mathbf{T}\times U_i\sim\sim(t', \gamma')\in\mathbf{T}\times U_j$ if $\gamma=\gamma'$ and $t'=t\lambda_{ij}(\gamma)$). There is a continuous surjection $\Sigma\to\check{\Sigma}$ (which sends the germ of h_i at $\gamma\in U_i$ to the class of $(1, \gamma)\in\mathbf{T}\times U_i$); $\check{\Sigma}$ is endowed with the unique structure of a topological groupoid which makes this map a homomorphism. Evidently, $\check{\Sigma}$ is a topological twist and $\Sigma\cong\underset{\sim}{\check{\Sigma}}$. Thus $\mathrm{Tw}(\Gamma)\cong T_\Gamma(\underset{\sim}{\mathbf{T}})$.

3. THE LONG EXACT SEQUENCE

This secion begins with the study of the left-exact functor $Z_\Gamma:\mathcal{S}(\Gamma)\to\mathcal{Ab}$ which assigns the group of continuous A-valued one-cocyles to a Γ-sheaf A. Its first derived functor is shown to be naturally isomorphic to the twist functor. Moreover, we show there to be a long exact sequence relating the cohomology of Γ and Γ^0 to the derived functors of Z_Γ. Fix a sheaf groupoid Γ.

1. DEFINITION. Let A be a Γ-sheaf and $f: \Gamma \to r^*(A)$ be a continuous section (viewed as a continuous map $f: \Gamma \to A$ with $f(\gamma) \in A_{r(\gamma)}$). Then f is said to be a continuous (A-valued) one-cocycle if

$$(*) \qquad f(\gamma\gamma') = f(\gamma) + \gamma f(\gamma') \quad \text{for all } (\gamma, \gamma') \in \Gamma^2 .$$

The group of all such is noted $Z_\Gamma(A)$.

2. REMARKS. Let $f: \Gamma \to r^*(A)$ be a continuous section: the map $\sigma_f : \Gamma \to A \times\!\!| \Gamma$ given by $\sigma_f(\gamma) = (f(\gamma), \gamma)$ is a groupoid homomorphism iff f satisfies the cocycle identity $(*)$. Thus, $Z_\Gamma(A)$ may be identified with the collection of trivializing sections of the semi-direct product $A \times\!\!| \Gamma$. Note that if $\varphi : \Lambda \to \Gamma$ is a homomorphism, there is the usual natural transformation of functors $\varphi^* : Z_\Gamma(A) \to Z_\Lambda(\varphi^* A))$ given by $(\varphi^* f)(\lambda) = f(\varphi(\lambda))$.

It is immediate that Z_Γ is a left-exact functor. The obstruction to right-exactness appears in the form of a twist. Indeed, given a short exact sequence of Γ-sheaves:

$$0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0$$

and $f \in Z_\Gamma(C)$, we form a twist by A as a subgroupoid of $B \times\!\!| \Gamma$. Put $\Sigma_f = \{(b, \gamma) \in B \times\!\!| \Gamma : f(\gamma) = h(b)\}$ and note that this gives a twist by A (when A is identified with its image in B). It is routine to verify that the mapping $\mu : Z_\Gamma(C) \to T_\Gamma(A)$ given by $f \to [\Sigma_f]$ is a group homomorphism.

3. PROPOSITION. *With the situation as above, the sequence*

$$0 \to Z_\Gamma(A) \xrightarrow[g_*]{} Z_\Gamma(B) \xrightarrow[h_*]{} Z_\Gamma(C) \xrightarrow[\mu]{} T_\Gamma(A) \xrightarrow[g_*]{} T_\Gamma(B) \xrightarrow[h_*]{} T_\Gamma(C)$$

is exact. Further, μ is natural with respect to short exact sequences in $\mathscr{S}(\Gamma)$.

Proof. We need only verify exactness at $Z_\Gamma(C)$ and $T_\Gamma(A)$. Given $f \in Z_\Gamma(C)$ we show that $\Sigma_f \cong A \times\!\!| \Gamma$ iff there is $k \in Z_\Gamma(B)$ such that $f = h_* k$. Suppose there is such a k, then $\sigma_k : \Gamma \to B \times\!\!| \Gamma$ defined by $\sigma_k(\gamma) = (k(\gamma), \gamma)$ trivializes Σ_f (note $f(\gamma) = h(k(\gamma))$. Conversely, any such trivialization arises from a lifting of f. Hence $\ker \mu = \operatorname{Im} h_*$. Next, we check $\operatorname{Im} \mu = \ker g_*$. Suppose Σ is a twist by A such that $g_* \Sigma \cong B \times\!\!| \Gamma$; there ist hen a twist embedding $\Sigma \to B \times\!\!| \Gamma$ which when composed with the map $B \times\!\!| \Gamma \to C \times\!\!| \Gamma$ gives a twist morphism $l : \Sigma \to C \times\!\!| \Gamma$ which vanishes on A. Hence l factors through Γ ; let $f \in Z_\Gamma(C)$ correspond to the resulting trivialization (i.e. $l(\sigma) = (f(\dot{\sigma}), \dot{\sigma})$). Clearly $[\Sigma] = [\Sigma_f]$ and $\ker g_* \subset \operatorname{Im}\mu$. The converse is obvious. Given a morphism of short exact sequences in $\mathscr{S}(\Gamma)$,

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

one verifies that the diagram

$$\begin{array}{ccc} Z_\Gamma(C) & \xrightarrow[\mu]{} & T_\Gamma(A) \\ \downarrow & & \downarrow \\ Z_\Gamma(C') & \xrightarrow[\mu]{} & T_\Gamma(A') \end{array}$$

is commutative. ▨

4. COROLLARY. *T_Γ is naturally isomorphic to the first right derived functor of* Z_Γ.

Proof. $T_\Gamma(Q) = 0$ if Q is injective. ▨

Given a Γ-sheaf A, we define a map $d : S(A) \to Z_\Gamma(A)$ by the formula:

$$(df)(\gamma) = f(r(\gamma)) - \gamma f(s(\gamma)) \quad \text{for all } \gamma \in \Gamma.$$

We check that df satisfies the cocycle formula (*); for all $(\gamma, \gamma') \in \Gamma^2$ one has

$$(df)(\gamma\gamma') = f(r(\gamma\gamma')) - \gamma\gamma' f(s(\gamma\gamma')) =$$

$$= f(r(\gamma)) - \gamma f(s(\gamma)) + \gamma f(r(\gamma')) - \gamma\gamma' f(s(\gamma')) = (df)(\gamma) + \gamma (df)(\gamma').$$

Since $df = 0$ iff $f \in S_\Gamma(A)$, the sequence, $0 \to S(A) \to S(A) \to Z_\Gamma(A)$ is exact. We define a map $\eta : Z_\Gamma(A) \to \operatorname{Ext}^1_\Gamma(\underset{\sim}{\mathbf{Z}}, A)$; given $h \in Z_\Gamma(A)$ we endow $A \oplus \underset{\sim}{\mathbf{Z}}$ with an action of Γ compatible with the given action on A:

$$\gamma(a, k) = (\gamma a + kh(\gamma), k) \quad \text{for all } a \in A_{s(\gamma)} \text{ and } k \in \mathbf{Z}.$$

Denote the resultant Γ-sheaf by $E(h)$ and note that one obtains a short exact sequence of Γ-sheaves:

$$0 \to A \to E(h) \to \underset{\sim}{\mathbf{Z}} \to 0.$$

Let $\eta(h) \in \operatorname{Ext}^1_\Gamma(\underset{\sim}{\mathbf{Z}}, A)$ be the class of this sequence. One verifies that η is a natural transformation of functors $Z_\Gamma \to \operatorname{Ext}^1_\Gamma(\underset{\sim}{\mathbf{Z}}, \cdot)$.

5. LEMMA. $\operatorname{Im} d = \ker \eta$.

Proof. We show $\operatorname{Im} d \subset \ker\eta$. Given $f \in S(A)$, we must show that $E(df) \cong$ $\cong A \oplus \underset{\sim}{\mathbf{Z}}$ (as Γ-sheaves). It suffices to find an equivariant splitting $g: \underset{\sim}{\mathbf{Z}} \to E(df)$.

Set $g(k, x) = (kf(x), k)$; for all $\gamma \in \Gamma$ one has:

$$\gamma g(k, s(\gamma)) = (\gamma kf(s(\gamma)) + kdf(\gamma), k) =$$

$$= (\gamma kfs(\gamma)) + kf(r(\gamma)) - k\gamma f(s(\gamma)), k) = (kf(r(\gamma)), k) = g(k, r(\gamma)).$$

Hence $E(df)$ splits as desired. Conversely, suppose $\eta(h) = 0$; there is then an equivariant splitting $g : \underset{\sim}{Z} \to E(h)$ of the form $g(k, x) = (kf(x), k)$ for some $f \in S(A)$. It follows that $h = df$. ▨

6. Remark. Given a short exact sequence of Γ-sheaves

$$0 \to A \xrightarrow{i} E \underset{g}{\overset{f}{\rightleftarrows}} \underset{\sim}{Z} \to 0$$

with sheaf splitting $g \in \mathrm{Hom}(\underset{\sim}{Z}, E^0)$ (so that $f^0 g = \mathrm{id}_{\underset{\sim}{Z}}$), we associate an element $h \in Z_\Gamma(A)$ as a measure of the extent g fails to be equivariant. Let $h \in S(r^*(A))$ be given by the formula:

$$i(h(\gamma)) = \gamma g(1, s(\gamma)) - g(1, r(\gamma)) \quad \text{for all } \gamma \in \Gamma.$$

It follows that $h \in Z_\Gamma(A)$ and $E \cong E(h)$.

Suppose $\varphi : \Lambda \to \Gamma$ is a groupoid homomorphism, then applying φ^* to the above short exact sequence (with sheaf splitting) yields the commutativity of the square:

$$\begin{array}{ccc} Z_\Gamma(A) & \xrightarrow{\eta} & \mathrm{Ext}^1_\Gamma(\underset{\sim}{Z}, A) \\ \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle\varphi^*} \\ Z_\Lambda(\varphi^*(A)) & \xrightarrow{\eta} & \mathrm{Ext}^1_\Lambda(\underset{\sim}{Z}, \varphi^*(A)). \end{array}$$

Let Z^n_Γ denote the nth right derived functor of Z_Γ (so $Z^0_\Gamma \cong Z_\Gamma$ and $Z^1_\Gamma \cong T_\Gamma$). Note that any Γ-sheaf A may be embedded in an injective Q which is acyclic as a sheaf (i.e. $H^n(\Gamma^0, Q^0) = 0$ for $n > 0$).

7. Theorem. *Given an injective resolution of a Γ-sheaf A*

$$A \to Q_0 \to Q_1 \to Q_2 \to \dots,$$

the sequence of complexes

$$0 \to S_\Gamma(Q_*) \to S(Q_*) \xrightarrow{d} Z_\Gamma(Q_*) \to 0$$

is exact. The resulting long exact sequence

$$0 \to H^0(\Gamma, A) \to H^0(\Gamma^0, A^0) \xrightarrow{d^0} Z^0_\Gamma(A) \xrightarrow{\partial^0} H^1(\Gamma, A) \to \dots$$

$$\dots \xrightarrow{\partial^{n-1}} H^n(\Gamma, A) \to H^n(\Gamma^0, A^0) \xrightarrow{d^n} Z^n_\Gamma(A) \xrightarrow{\partial^n} H^{n+1}(\Gamma, A) \to \dots$$

is independent of the injective resolution and functorial in A.

Proof. If Q is injective, $d\colon S(Q) \to Z_\Gamma(Q)$ is surjective by the above lemma ($\mathrm{Ext}^1_\Gamma(\underset{\sim}{Z}, Q) = 0$). ☐

8. Remarks. i) Identifying $H^1(\Gamma, A)$ with $\mathrm{Ext}^1_\Gamma(\underset{\sim}{Z}, A)$, one checks that $\partial^0 = -\eta$.

ii) If $\varphi : \Lambda \to \Gamma$ is a groupoid homomorphism, then $\varphi^* : \mathscr{S}(\Gamma) \to \mathscr{S}(\Lambda)$ induces a (natural) morphism of long exact sequences:

$$\begin{array}{ccccccccc}
\dots \to & H^n(\Gamma, A) & \longrightarrow & H^n(\Gamma^0, A^0) & \longrightarrow & Z^n_\Gamma(A) & \xrightarrow{\partial^n} & H^{n+1}(\Gamma, A) & \to \dots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \to & H^n(\Lambda, \varphi^*A) & \to & H^n(\Lambda^0, \varphi^*A^0) & \to & Z^n_\Lambda(\varphi^*A) & \xrightarrow{\partial^n} & H^{n+1}(\Lambda, \varphi^*A) & \to \dots .
\end{array}$$

The map $\partial^1 : T_\Gamma(A) \to H^2(\Gamma, A)$ will play an important role in the next section — it will be useful to know when two twists give rise to the same second cohomology class.

9. Proposition. *Let Σ and Σ' be two twists by A (over Γ); then $\partial^1[\Sigma] = \partial^1[\Sigma']$ iff there is a local homeomorphism $\psi\colon X \to \Gamma^0$ with $[\pi^*_\psi(\Sigma)] = [\pi^*_\psi(\Sigma')]$ (regarded as elements of $T_{\Gamma^\psi}(\pi^*_\psi A)$).*

Proof. If $\partial^1[\Sigma] = \partial^1[\Sigma']$ there is $g \in H^1(\Gamma^0, A^0)$ such that $[\Sigma] - [\Sigma'] = d^1 g$. There is a local homeomorphism $\psi\colon X \to \Gamma^0$ such that $\psi^* g = 0 \in H^1(X, \psi^* A^0)$. One has the commutative diagram

$$\begin{array}{ccccc}
H^1(\Gamma^0, A^0) & \xrightarrow{d^1} & T_\Gamma(A) & \xrightarrow{\partial^1} & H^2(\Gamma, A) \\
\downarrow \pi^*_\psi & & \downarrow \pi^*_\psi & & \vdots \\
H^1((\Gamma^\psi)^0, \pi^*_\psi A^0) & \xrightarrow{d^1} & T_{\Gamma^\psi}(\pi^*_\psi A) & \xrightarrow{\partial^1} & H^2(\Gamma^\psi, \pi^*_\psi A)
\end{array}$$

with $(\Gamma^\psi)^0 = X$ and $\pi^*_\psi A^0 = \psi^* A^0$. Hence, $[\pi^*_\psi \Sigma] - [\pi^*_\psi \Sigma'] = \pi^*_\psi(d^1 g) = 0$. The converse is obvious.

4. EQUIVARIANT C^*-BUNDLES

The cohomological apparatus developed above is applied to the study of actions of a sheaf groupoid on elementary C^*-bundles. Such actions will be classified up to a suitable notion of Morita equivalence by an invariant taking values in $H^2(\Gamma, \underset{\sim}{T})$.

Fell's notion of Banach bundle [11] seems better suited to the setting of groupoid actions than the equivalent formalism of continuous fields. The reader is urged to consult Addendum 2 for the appropriate definitions and a preliminary treatment of elementary C^*-bundles.

The groupoids considered in this section are assumed to be locally compact, Hausdorff, and second countable; Banach bundles are assumed to be strongly separable.

1. DEFINITION. An action of a groupoid Γ on a Banach bundle E fibred over Γ^0 is given by a continuous map $\tau : \Gamma * E \to E$ (where $\Gamma * E = \{(\gamma, e) : e \in E_{s(\gamma)}\} \subset \subset \Gamma \times E$) satisfying the conditions:

i) $\tau_\gamma : E_{s(\gamma)} \to E_{r(\gamma)}$ is an isometric isomorphism for each $\gamma \in \Gamma$,

ii) $\tau_{\gamma\gamma'} = \tau_\gamma \circ \tau_{\gamma'}$ for all $(\gamma, \gamma') \in \Gamma^2$.

The pair (E, τ) is said to be a *Banach Γ-bundle*. If each fiber is a Hilbert space the pair is called a *Hilbert Γ-bundle*. If E is a C^*-bundle and each τ_γ is a $*$-isomorphism, then (E, τ) is called a *C^*-Γ-bundle* and an elementary C^*-Γ-bundle if E is elementary.

2. REMARK. To check continuity it suffices to show that $\gamma \to \tau_\gamma(f(s(\gamma)))$ is a continuous section of $r^*(E)$, for every f in a family of sections of E large enough to determine the topology of E.

Let Σ be a topological twist over a sheaf groupoid Γ and define an action of $\mathbf{T}$ on Σ by $t\sigma = (t, r(\sigma))\sigma$ for $t \in \mathbf{T}$, $\sigma \in \Sigma$ (this action makes Σ a principal $\mathbf{T}$-bundle over Γ).

3. DEFINITION. A Hilbert Σ-bundle (V, ρ) is called a *twist representation* of Σ if

$$\rho_{t\sigma}(v) = t\rho_\sigma(v) \quad \text{for all } t \in \mathbf{T},\ \sigma \in \Sigma, \text{ and } v \in V_{s(\sigma)}.$$

4. EXAMPLE. Given a topological twist Σ over Γ we construct a twist representation of Σ. The Hilbert bundle V is realized as that associated to a Hilbert $C_0(\Gamma^0)$--module. Let $D = \{f \in C_c(\Sigma) : f(t\sigma) = tf(\sigma) \text{ for } t \in \mathbf{T} \text{ and } \sigma \in \Sigma\}$ and note that if $g, f \in D$ then $\overline{g(t\sigma)}f(t\sigma) = \overline{g(\sigma)}f(\sigma)$. This defines an element of $C_c(\Gamma)$ which we denote $[\bar{g}f]$. Let $\langle \cdot , \cdot \rangle$ be the $C_0(\Gamma^0)$-valued sesquilinear form given by

$$\langle g, f\rangle(x) = \sum_{s(\gamma)=x} [\bar{g}f](\gamma) \quad \text{for } x \in \Gamma^0 \text{ and } f, g \in D.$$

The completion of D with respect to the norm $\|f\| = \|\langle f, f\rangle\|_\infty^{1/2}$ is a (right) Hilbert $C_0(\Gamma^0)$-module (see [21], § 2). Let V be the associated Hilbert bundle (i.e. $\bar{D} = C_0(V)$) and define the action $\rho : \Sigma * V \to V$ by the formula

$$(\rho_\sigma(f))(\sigma') = f(\sigma'\sigma) \quad \text{for } \sigma \in \Sigma,\ f \in V_{s(\sigma)},\ s(\sigma') = r(\sigma).$$

Since $\rho_{t\sigma}(f) = t\rho_\sigma(f)$, (V, ρ) is a twist representation of Σ.

5. PROPOSITION. *If (V, ρ) is a twist representation of Σ, then the formula*

$$\operatorname{Ad}\rho_\sigma(a) = \rho_\sigma a \rho_\sigma^* \quad \textit{for } \sigma \in \Sigma,\ a \in \mathscr{K}(V)_{s(\sigma)}$$

defines an action of Γ on the elementary C^*-bundle $\mathscr{K}(V)$.

Proof. It suffices to check that the formula is unambiguous — if $\dot{\sigma} = \dot{\sigma}'$ there is $t \in \mathbf{T}$ such that $\rho_{\sigma'} = t\rho_\sigma$; hence $\operatorname{Ad}\rho_\sigma = \operatorname{Ad}\rho_{\sigma'}$. ∎

The elementary C^*-Γ-bundle $(\mathscr{K}(V), \operatorname{Ad}\rho)$ is said to be associated to the twist representation (V, ρ).

6. DEFINITION. Suppose (E, τ) is an elementary C^*-Γ-bundle; a multiplier $m \in M(E)$ is said to be *invariant* if $\tau_\gamma(me) = m\tau_\gamma(e)$ and $\tau_\gamma(em) = \tau_\gamma(e)m$ for all $\gamma \in \Gamma$ and $e \in E_{s(\gamma)}$. The collection of invariant multipliers is denoted $M(E)^\tau$.

7. REMARK. If $p \in M(E)^\tau$ is a projection such that $p_x \neq 0$ for all $x \in \Gamma^0$, set $E^p = \{e \in E : e = pep\}$, and let τ^p denote the restriction of τ to E^p. It is clear that (E^p, τ^p) is an elementary C^*-Γ-bundle.

8. DEFINITION. Two elementary C^*-Γ-bundles (E, τ) and (E', τ') are to be *Morita equivalent*, write $(E, \tau) \underset{M}{\sim} (E', \tau')$, if there is an elementary C^*-Γ-bundle (F, α) and projections $p, p' \in M(F)^\alpha$, with $pp' = 0$ and $p_x \neq 0$ and $p'_x \neq 0$ for all $x \in \Gamma^0$, such that $(E, \tau) \cong (F^p, \alpha^p)$ and $(E', \tau') \cong (F^{p'}, \alpha^{p'})$ (cf. Theorem 1.1 in [1]).

9. PROPOSITION. *If (V, ρ) and (V', ρ') are twist representations of Σ then $(\mathscr{K}(V), \operatorname{Ad}\rho) \underset{M}{\sim} (\mathscr{K}(V'), \operatorname{Ad}\rho')$.*

Proof. Let $(V \oplus V', \rho \oplus \rho')$ denote the direct sum of the twist representations (V, ρ) and (V', ρ'), that is, set

$$(\rho \oplus \rho')_\sigma(v + v') = \rho_\sigma(v) + \rho'_\sigma(v') \quad \text{for } \sigma \in \Sigma,\ v \in V_{s(\sigma)},\ v' \in V'_{s(\sigma)}.$$

Let $p, p' \in M(\mathscr{K}(V \oplus V'))^{\operatorname{Ad}(\rho\oplus\rho')}$ be the projections onto the summands V, V'. It follows that $(\mathscr{K}(V), \operatorname{Ad}\rho) \cong (\mathscr{K}(V \oplus V')^p, \operatorname{Ad}(\rho \oplus \rho')^p)$ and that $(\mathscr{K}(V'), \operatorname{Ad}\rho') \cong$ $\cong (\mathscr{K}(V \oplus V')^{p'}, \operatorname{Ad}(\rho \oplus \rho')^{p'})$. ∎

One forms the tensor product of two elementary C^*-Γ-bundles, (E, τ) and (E', τ'), in a straightforward manner. Let $\tau \otimes \tau'$ be the action of Γ on $E \otimes E'$ given by:

$$(\tau \otimes \tau')_\gamma(e \otimes e') = \tau_\gamma(e) \otimes \tau_\gamma(e') \quad \text{for } \gamma \in \Gamma,\ e \in E_{s(\gamma)},\ e' \in E'_{s(\gamma)}.$$

One checks that $(E \otimes E', \tau \otimes \tau')$ is an elementary C^*-Γ-bundle.

10. PROPOSITION. *If $(E, \tau), (E', \tau'), (F, \alpha)$, and (F', α') are elementary C^*-Γ-bundles such that $(E, \tau) \underset{M}{\sim} (F, \alpha)$ and $(E', \tau') \underset{M}{\sim} (F', \alpha')$, then $(E \otimes E', \tau \otimes \tau') \underset{M}{\sim} (F \otimes F', \alpha \otimes \alpha')$.*

Proof. It suffices to check that if (D, β) and (D', β') are elementary C^*-Γ-bundles and $p \in M(D)^\beta$ and $p' \in M(D')^{\beta'}$ are projections ($p_x \neq 0$, $p'_x \neq 0$ for all $x \in \Gamma^0$) then $(D^p \otimes (D')^{p'}, \beta^p \otimes (\beta')^{p'} \cong ((D \otimes D')^{p\otimes p'}, (\beta \otimes \beta')^{p\otimes p'})$; but this is immediate. ▨

If Σ and Σ' are two topological twists over Γ, let $\Sigma \underset{\mathbf{T}}{*} \Sigma'$ be the topological twist corresponding to $[\underset{\sim}{\Sigma}] + [\underset{\sim}{\Sigma'}]$ in $T_\Gamma(\underset{\sim}{\mathbf{T}})$. Indeed, $\Sigma \underset{\mathbf{T}}{*} \Sigma'$ is given as the quotient of the fiber-product $\Sigma \underset{\mathbf{T}}{*} \Sigma' = \{(\sigma, \sigma') \in \Sigma \times \Sigma' : \dot{\sigma} = \dot{\sigma}'\}$ by the equivalence relation $(t\sigma, \sigma') \sim (\sigma, t\sigma')$ for $t \in \mathbf{T}$ (see [21], § 4.2).

11. PROPOSITION. *If (V, ρ) and (V', ρ') are twist representations of Σ and Σ', then $(V \otimes V', \rho \otimes \rho')$ define a twist representation of $\Sigma \underset{\mathbf{T}}{*} \Sigma'$. Furthermore, one has* $(\mathscr{K}(V \otimes V'), \mathrm{Ad}(\rho \otimes \rho')) \cong (\mathscr{K}(V) \otimes \mathscr{K}(V'), \mathrm{Ad}\,\rho \otimes \mathrm{Ad}\,\rho')$.

Proof. For $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ with $\dot{\sigma} = \dot{\sigma}' = \gamma$, let

$$(\rho \otimes \rho')_{(\sigma,\sigma')} : (V \otimes V')_{s(\gamma)} \to (V \otimes V')_{r(\gamma)}$$

be given by

$$(\rho \otimes \rho')_{(\sigma,\sigma')}(v \otimes v') = \rho_\sigma(v) \otimes \rho_\sigma(v') \quad \text{for } v \in V_{s(\gamma)},\ v' \in V'_{s(\gamma)}.$$

The verification of the second assertion is routine. ▨

Given an elementary C^*-Γ-bundle (E, τ) with $\delta(E) = 0$, we show below that it is isomorphic to one of the form $(\mathscr{K}(V), \mathrm{Ad}\,\rho)$ for some twist representation (V, ρ). By stabilization it will be sufficient to consider actions of Γ on the constant bundle $\mathscr{K} \times \Gamma^0$. Let P_Γ denote the collection of strongly continuous homomorphisms $\pi : \Gamma \to \mathrm{Aut}(\mathscr{K})$ (so $\pi_{\gamma\gamma'} = \pi_\gamma \pi_{\gamma'}$ for $(\gamma, \gamma') \in \Gamma^2$ and $\gamma \to \pi_\gamma(a)$ is continuous for each $a \in \mathscr{K}$). For $\pi \in P_\Gamma$, let $\bar{\pi} : \Gamma^*(\mathscr{K} \times \Gamma^0) \to \mathscr{K} \times \Gamma^0$ denote the corresponding action (i.e. $\bar{\pi}_\gamma(a, s(\gamma)) = (\pi_\gamma(a), r(\gamma))$); clearly every action of Γ on $\mathscr{K} \times \Gamma^0$ is of this form. It is a standard fact that $\mathrm{Aut}(\mathscr{K}) \cong U(\mathscr{H})/\mathbf{T}$ where $U(\mathscr{H})$ is the group of unitary operators on a Hilbert space $\mathscr{H}$ equipped with the strong operator topology, (indeed, one has $\mathrm{Ad}: U(\mathscr{H}) \to \mathrm{Aut}(\mathscr{K})$ is onto). For $\pi \in P_\Gamma$ set

$$\Sigma(\pi) = \{(u, \gamma) \in U(\mathscr{H}) \times \Gamma : \pi_\gamma = \mathrm{Ad}\,u\},$$

and note that $\Sigma(\pi)$ is a topological twist over Γ. A twist representation of $\Sigma(\pi)$ on the constant Hilbert bundle $\mathscr{H} \times \Gamma^0$ is given by

$$\rho^\pi_{(u,\gamma)}(\xi, s(\gamma)) = (u\xi, r(\gamma)) \quad \text{for } (u, \gamma) \in \Sigma(\pi) \text{ and } \xi \in \mathscr{H}.$$

One checks that $(\mathscr{K}(\mathscr{H} \times \Gamma^0), \mathrm{Ad}\,\rho^\pi) = (\mathscr{K} \times \Gamma^0, \bar{\pi})$.

12. Proposition. *If (E, τ) is an elementary C^*-Γ-bundle with $\delta(E) = 0$, then there is $\pi \in P_\Gamma$ and a projection $p \in M(\mathscr{K} \times \Gamma^0)^{\bar{\pi}}$ such that $(E, \tau) \cong ((\mathscr{K} \times \Gamma^0)^p, \bar{\pi}^p)$. Furthermore, (E, τ) is isomorphic to an elementary C^*-Γ-bundle associated to a twist representation of $\Sigma(\pi)$.*

Proof. Let ι denote the trivial action of Γ on $\mathscr{K} \times \Gamma^0$ and consider the elementary C^*-Γ-bundle $((\mathscr{K} \times \Gamma^0) \otimes E, \ \iota \otimes \tau)$. Choose a rank-one projection $q \in \mathscr{K}$ and note that $q \otimes 1_E \in M((\mathscr{K} \times \Gamma^0) \otimes E)^{\iota \otimes \tau}$. Since $\delta(E) = 0$, $(\mathscr{K} \times \Gamma^0) \otimes E$ is isomorphic to $\mathscr{K} \times \Gamma^0$. Hence, there is $\pi \in P_\Gamma$ such that

$$((\mathscr{K} \times \Gamma^0) \otimes E, \ \iota \otimes \tau) \cong (\mathscr{K} \times \Gamma^0, \ \bar{\pi});$$

let $p \in M(\mathscr{K} \times \Gamma^0)^{\bar{\pi}}$ be the projection corresponding to $q \otimes 1_E$ under this identification. By construction one has $(E, \tau) \cong ((\mathscr{K} \times \Gamma^0)^p, \bar{\pi}^p)$. Let $V(p) \subset \mathscr{H} \times \Gamma^0$ be the sub-bundle $\{(\xi, x) \in \mathscr{H} \times \Gamma^0 : p_x\xi = \xi\}$. By the invariance of p under $\bar{\pi}$, one has

$$p_{r(\sigma)}\rho^{\pi}_{\sigma} = \rho^{\pi}_{\sigma} p_{s(\sigma)} \quad \text{for all } \sigma \in \Sigma(\pi).$$

Thus, one obtains a twist representation of $\Sigma(\pi)$ on $V(p)$ by restriction; denote this twist representation by $(V(p), \rho^{\pi} | V(p))$. It follows that $(E, \tau) \cong (\mathscr{K}(V(p)), \mathrm{Ad}\, \rho^{\pi} | V(p))$. ∎

13. Remark. Given $\pi, \pi' \in P_\Gamma$, it is not hard to see that $\Sigma(\pi) \cong \Sigma(\pi')$ iff π and π' are cocycle equivalent, (alternatively, exterior equivalent) that is, there is a strongly continuous map $u: \Gamma \to U(\mathscr{H})$ such that (cf. [25], Proposition 2.5):

i) $\pi'_\gamma = \mathrm{Ad}\, u_\gamma \pi_\gamma$ for $\gamma \in \Gamma$,

ii) $u_{\gamma\gamma'} = u_\gamma \pi_\gamma(u_{\gamma'})$ for $(\gamma, \gamma') \in \Gamma^2$.

Exterior equivalence follows by defining $\tau : \Gamma \to \mathrm{Aut}(M_2(\mathscr{K}))$ in the usual manner:

$$\tau_\gamma \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \pi'_\gamma(a_{11}) & u_\gamma \pi_\gamma(a_{12}) \\ \pi_\gamma(a_{21}) u^*_\gamma & \pi_\gamma(a_{22}) \end{pmatrix}.$$

Choosing an isomorphism $\mathscr{K} \otimes \mathscr{K} \cong \mathscr{K}$ induces an isomorphism $\mathrm{Aut}(\mathscr{K} \otimes \mathscr{K}) \cong$ $\cong \mathrm{Aut}(\mathscr{K})$ and hence a map $P_\Gamma \times P_\Gamma \to P_\Gamma$ denoted $(\pi, \pi') \to \pi \otimes \pi'$. One has that $\Sigma(\pi \otimes \pi') \cong \Sigma(\pi) \mathbin{\underset{\mathbf{T}}{*}} \Sigma(\pi')$ (see Proposition 11); and, hence, exterior equivalence classes (in P_Γ) form an abelian group (under tensor products) isomorphic to $T_\Gamma(\mathbf{T})$.

Two elements $\pi, \pi' \in P_\Gamma$ are said to be cohomologous if there is a strongly continuous map $h : \Gamma^0 \to \mathrm{Aut}(\mathscr{K})$ such that:

$$\pi_\gamma h(s(\gamma)) = h(r(\gamma))\pi'_\gamma \quad \text{for all } \gamma \in \Gamma$$

(cf. [25], Proposition 2.5); that is, π and π' are cohomologous iff $(\mathscr{K} \times \Gamma^0, \bar{\pi}) \cong$ $\cong (\mathscr{K} \times \Gamma^0, \bar{\pi}')$. The obstruction to lifting h to a continuous map $\tilde{h}: \Gamma^0 \to U(\mathscr{H})$ (so that $h(x) = \mathrm{Ad}\, \tilde{h}(x)$) is an element $w(h) \in H^1(\Gamma^0, \underline{\mathbf{T}})$ (cf. [30]; the associated principal T-bundle over Γ^0 is given by $W(h) = \{(u, x) : h(x) = \mathrm{Ad}\,u\} \subset \subset U(\mathscr{H}) \times \Gamma^0)$. Further, one has $[\underset{\sim}{\Sigma}(\pi)] = [\underset{\sim}{\Sigma}(\pi')] + d^1(w(h))$ (see § 3.7 and [21], § 4.5).

14. PROPOSITION. *Let* $\pi, \pi' \in P_\Gamma$; *then* $(\mathscr{H} \times \Gamma^0, \overline{\pi}) \underset{M}{\sim} (\mathscr{H} \times \Gamma^0, \overline{\pi}')$ *iff* $\partial^1[\underset{\sim}{\Sigma}(\pi)] = \partial^1[\Sigma(\pi')]$.

Proof. " $\Rightarrow$ " By Theorem 3.4 of [1] we may assume there to be a $\pi'' \in P_\Gamma$ such that π' and π'' are cohomologous and π and π'' are exterior equivalent (cf. [4], § 9); then $[\Sigma(\pi)] = [\Sigma(\pi'')] = [\Sigma(\pi')] + d^1(w)$ for some $w \in H^1(\Gamma^0, \underset{\sim}{\mathrm{T}})$ and $\partial^1[\underset{\sim}{\Sigma}(\pi)] = \partial^1[\Sigma'(\pi')]$ since $\ker \partial^1 = \mathrm{Im}\, d^1$.

„$\Leftarrow$" Choose π'' cohomologous to π' such that $\Sigma(\pi) \cong \Sigma(\pi'')$. ▨

15. DEFINITION. Let $\mathscr{E}_\Gamma$ denote the collection of isomorphism classes of elementary C^*-Γ-bundles; set $\mathscr{E}_\Gamma^0 = \{[E, \tau] \in \mathscr{E}_\Gamma : \delta(E) = 0\}$.

16. COROLLARY. *The map* $\delta_\Gamma : \mathscr{E}_\Gamma^0 \to H^2(\Gamma, \underset{\sim}{\mathrm{T}})$ *given by* $\delta_\Gamma(E, \tau) = \partial^1[\underset{\sim}{\Sigma}(\pi)]$, *where* $\pi \in P_\Gamma$ *is such that* $(\mathscr{H} \times \Gamma^0, \overline{\pi}) \cong ((\mathscr{H} \times \Gamma^0) \otimes E, \iota \otimes \tau)$, *is a complete invariant for Morita equivalence.*

Extending the invariant δ_Γ will require consideration of pull-back bundles. If $\varphi : \Lambda \to \Gamma$ is a homomorphism of sheaf groupoids and (E, τ) is an elementary C^*-Γ-bundle, set $\varphi^*(E) = E * \Lambda^0$ and define an action $\varphi^*(\tau)$ of Λ on $\varphi^*(E)$ by the formula:

$$\varphi^*(\tau)_\lambda(e, s(\lambda)) = (\tau_{\varphi(\lambda)}(e), r(\lambda)) \quad \text{for } \lambda \in \Lambda,\ e \in E_{s(\varphi(\lambda))}.$$

One checks that the pair $(\varphi^*(E), \varphi^*(\tau))$ is an elementary C^*-Λ-bundle. If $m \in M(E)^\tau$ then $\varphi^*(m) \in M(\varphi^*(E))^{\varphi^*(\tau)}$ (see Addendum 2).

17. PROPOSITION. *Suppose* (E, τ) *and* (E', τ') *are elementary* C^*-Γ-*bundles and let* $\varphi : \Lambda \to \Gamma$ *be a homomorphism of sheaf groupoids. Then:*

i) *If* $(E, \tau) \underset{M}{\sim} (E', \tau')$ *then* $(\varphi^*(E), \varphi^*(\tau)) \underset{M}{\sim} (\varphi^*(E'), \varphi^*(\tau'))$,

ii) $(\varphi^*(E \otimes E'), \varphi^*(\tau \otimes \tau')) \cong (\varphi^*(E) \otimes \varphi^*(E'), \varphi^*(\tau) \otimes \varphi^*(\tau'))$.

If $\pi \in P_\Gamma$ *then* $\varphi^*(\pi) \in P_\Lambda$ *where* $\varphi^*(\pi)_\lambda = \pi_{\varphi(\lambda)}$; *moreover,* $\underset{\sim}{\Sigma}(\varphi^*(\pi)) \cong \varphi^*(\underset{\sim}{\Sigma}(\pi))$.

Proof. The verification of these assertion is straightforward. ▨

Let $\varphi^* : \mathscr{E}_\Gamma \to \mathscr{E}_\Lambda$ be a given by $\varphi^*[E, \tau] = [\varphi^*(E), \varphi^*(\tau)]$.

Let $\psi : X \to Z$ be a local homeomorphism and let E be an elementary C^*-bundle over Z. We define an action τ of $R(\psi)$ on $\psi^*(E) = E * X$ by:

$$\tau_{(x, x')}(e, x') = (e, x) \quad \text{for } (x, x') \in R(\psi),\ e \in E_{\psi(x')}.$$

Evidently, the pair $(\psi^*(E), \tau)$ is an elementary C^*-$R(\psi)$-bundle. Observe that $M(E) \cong M(\psi^*(E))^\tau$.

18. LEMMA. *Every elementary C^*-$R(\psi)$-bundle is of this form.*

Proof. Suppose (F, α) is an elementary C^*-$R(\psi)$-bundle. Consider the equivalence relation on F given by α, that is, $f \in F_x \sim f' \in F_{x'}$ if $\psi(x) = \psi(x')$ and $f = \alpha_{(x,x')}(f')$, and put $E = F/\sim$. We show that E is an elementary C^*-bundle over Z when endowed with the quotient topology. The bundle projection $p: E \to Z$ is given by:

$$p[f] = \psi(x) \quad \text{if } f \in F_x;$$

evidently, $F_{\psi(x)} \cong E_x$ so the fibers are elementary C^*-algebras. The continuous sections $C(E)$ may be identified with the α invariant sections of F viz. $C(E) \cong \{g \in C(F) : g(x) = \alpha_{(x,x')}g(x')\}$. To show that E satisfies Fell's condition, choose $z \in Z$ and $x \in X$ with $\psi(x) = z$; let U be an open neighbourhood of x such that $\psi|U$ is injective. Since F satisfies Fell's condition there is $p \in C_0(F)$ and a neighbourhood V of x with $V \subset U$ such that $p_x = 0$ for $x \notin U$ and p_x is a rank one-projection for all $x \in V$. Now let $q \in C(F)$ be defined by

$$q_x = \begin{cases} \alpha(x, x')p_{x'} & \text{if there is } x' \in U \text{ with } \psi(x) = \psi(x') \\ 0 & \text{otherwise.} \end{cases}$$

Under the above identification q provides a continuous section of E which is a rank-one projection over the neigbourhood $\psi(V)$ of z. One checks that

$$(F, \alpha) \cong (\psi^*(E), \tau).$$

Let Γ be a sheaf groupoid.

19. THEOREM. *There is a (surjective) map $\delta_\Gamma : \mathscr{E}_\Gamma \to H^2(\Gamma, \mathbb{T})$ satisfying the following properties:*

i) $\delta_\Gamma(E, \tau) = \delta_\Gamma(E', \tau')$ *iff* $(E, \tau) \underset{M}{\sim} (E', \tau')$;

ii) $\delta_\Gamma(E \otimes F, \tau \otimes \alpha) = \delta_\Gamma(E, \tau) + \delta_\Gamma(F, \alpha)$;

iii) *If* $\pi \in P_\Gamma$ *then* $\delta_\Gamma(\mathscr{K} \times \Gamma^0, \bar{\pi}) = \partial^1[\Sigma(\pi)]$;

iv) *If* $\varphi: \Lambda \to \Gamma$ *is a homomorphism,* $\delta_\Lambda \varphi^* = \varphi^* \delta_\Gamma$.

Proof. First observe that if $\psi: X \to \Gamma^0$ is a local homeomorphism, then $\pi_\psi^* : \mathscr{E}_\Gamma \to \mathscr{E}_{\Gamma^\psi}$ is a bijection which preserves tensor-products. Further, if (E, τ) and (E', τ') are elementary C^*-Γ-bundles, then $(E, \tau) \underset{M}{\sim} (E', \tau')$ iff $(\pi_\psi^*(E), \pi_\psi^*(\tau)) \underset{M}{\sim} (\pi_\psi^*(E'), \pi_\psi^*(\tau'))$. We extend δ_Γ from $\mathscr{E}_\Gamma^0$ to $\mathscr{E}_\Gamma$; let (E, τ) be an elementary C^*-Γ-bundle and choose a local homeomorphism $\psi: X \to \Gamma^0$ such that $\psi^*\delta(E) = 0$.

Set $\delta_\Gamma(E, \tau) = (\pi_\psi^*)^{-1}(\delta_{\Gamma^\psi}(\pi_\psi^*(E), \pi_\psi^*(\tau)))$ — (recall that one has the isomorphism $\pi_\psi^* : H^*(\Gamma, \mathbb{T}) \cong H^*(\Gamma^\psi, \mathbb{T})$) and note that δ_Γ is well-defined. It remains to check that the invariant is natural with respect to homomorphisms. Let $\varphi : \Lambda \to \Gamma$ be a homomorphism of sheaf groupoids; if $\pi \in P_\Gamma$ then $\varphi^*[\mathcal{K}\times\Gamma^0, \overline{\pi}] = [\mathcal{K}\times\Lambda^0, \overline{\pi\circ\varphi}]$ and $\varphi^*(\underset{\sim}{\Sigma}(\pi)) \cong \underset{\sim}{\Sigma}(\pi\circ\varphi)$. Hence,

$$\delta_\Lambda(\varphi^*[\mathcal{K}\times\Gamma^0, \overline{\pi}]) = \partial^1(\varphi^*[\underset{\sim}{\Sigma}(\pi)]) = \varphi^*\partial^1[\underset{\sim}{\Sigma}(\pi)] = \varphi^*(\delta_\Gamma(\mathcal{K}\times\Gamma^0, \overline{\pi})).$$

By tensoring with the constant bundle $\mathcal{K}\times\Gamma^0$ we may assume that (iv) holds for $\mathcal{E}^0_\Gamma$. For arbitrary $[E, \tau] \in \mathcal{E}_\Gamma$, choose a local homeomorphism $\psi: X \to \Gamma^0$ such that $\psi^*(\delta(E)) = 0$ and note that $\theta: X * \Lambda^0 \to \Lambda^0$ is a local homeomorphism ($X * \Lambda^0 = \{(x, y) \in X \times \Lambda^0 : \psi(x) = \psi(y)\}$) with $\theta^*(\delta(\varphi^*(E))) = 0$. Let $\tilde{\varphi}: \Lambda^0 \to \Gamma^\psi$ be given by $\tilde{\varphi}(x, \lambda, x') = (x, \varphi(\lambda), x')$ and note that one has a commutative diagram of sheaf groupoids:

$$\begin{array}{ccc} \Lambda^0 & \xrightarrow{\tilde{\varphi}} & \Gamma^\psi \\ \downarrow{\scriptstyle \pi_\theta} & & \downarrow{\scriptstyle \pi_\psi} \\ \Lambda & \xrightarrow[\varphi]{} & \Gamma. \end{array}$$

It follows that $\pi_\theta^* \circ \varphi^*[E, \tau] = \tilde{\varphi}^* \circ \pi_\psi^*[E, \tau]$. Since $\pi_\psi^*[E, \tau] \in \mathcal{E}^0_{\Gamma^\psi}$, $\delta_\Lambda \theta(\tilde{\varphi}^*(\pi_\psi^*[E, \tau])) = \tilde{\varphi}^*(\delta_{\Gamma^\psi}(\pi_\psi^*[E, \tau]))$. Naturality follows from the commutativity of the diagram:

$$\begin{array}{ccc} H^2(\Lambda^0, \underset{\sim}{\mathrm{T}}) & \xleftarrow{\tilde{\varphi}^*} & H^2(\Gamma^\psi, \underset{\sim}{\mathrm{T}}) \\ \uparrow{\scriptstyle \pi_\theta^*} & & \uparrow{\scriptstyle \pi_\psi^*} \\ H^2(\Lambda, \underset{\sim}{\mathrm{T}}) & \xleftarrow{\varphi^*} & H^2(\Gamma, \underset{\sim}{\mathrm{T}}) \end{array}$$

and the fact that π_θ^* and π_ψ^* are isomorphisms. ▨

If $\alpha: G \to \mathrm{Aut}(A)$ is an action of a discrete group on a separable continuous trace algebra, let $\Gamma_\alpha = G\times\hat{A}$ denote the sheaf groupoid associated to the action of G on the spectrum of A. Let $E(A)$ denote the elementary C^*-bundle associated to A (so $A \cong C_0(E(A))$) and $\tilde{\alpha}$ the action of Γ_α on $E(A)$.

20. Corollary. *If* $\alpha: G \to \mathrm{Aut}(A)$ *and* $\beta: G \to \mathrm{Aut}(B)$ *are actions of a discrete group G on the (separable) continuous trace algebras A and B, then* α *and* β *are Morita equivalent (see* [4], 3.1*) iff there is an isomorphism* $\varphi: \Gamma_\alpha \to \Gamma_\beta$ *such that* $\delta_{\Gamma_\alpha}(E(A), \tilde{\alpha}) = \varphi^*\delta_{\Gamma_\beta}(E(B), \tilde{\beta})$.

5. ADDENDA

Addendum 1. *Continuous cocycle cohomology*. For the convenience of the reader, we show that the complex of (non-normalized) continuous cochains may be realized as that resulting from the application of the invariant section functor to a relatively injective resolution (cf. [17], [22]). The cohomology of this complex is then given by the functors H_c^*, which from the viewpoint of relative homological algebra (cf. [24], Chapter IX) are the derived functors of the invariant section functor with respect to a smaller class of short exact sequences.

Let Γ be a sheaf groupoid.

DEFINITION. A Γ-morphism $f: A \to B$ is said to be *strict* if there is a sheaf morphism $h: B^0 \to A^0$ such that $f^0 = f^0 h f^0$ and $h = h f^0 h$ (that is, $(\ker f)^0$ is a direct summand of A^0 and $(\operatorname{Im} f)^0$ is a direct summand of B^0).

A sequence of Γ-morphisms $\ldots \xrightarrow{f} A \xrightarrow{g} \ldots$ is said to be *strict exact at* A if it is exact and both f and g are strict.

Recall that a Γ-sheaf J is a relative injective, if for any strict Γ-monomorphism $j: A \to B$ one has $j^*: \operatorname{Hom}_\Gamma(B, J) \to \operatorname{Hom}_\Gamma(A, J)$ is surjective. A relatively injective resolution of a Γ-sheaf A is given by a sequence

$$0 \to A \to J^0 \to J^1 \to \ldots$$

where each J^n is a relative injective and the sequence is strict exact at each point. Let $H_c^*(\Gamma, A)$ denote the cohomology of the complex $S_\Gamma(J^*)$ (note that this is independent, up to natural isomorphism, of the relatively injective resolution chosen). The covariant functors $H_c^n(\Gamma, \cdot)$ are characterized by the properties:

i) $H_c^0(\Gamma, A) \cong S_\Gamma(A)$;

ii) $H_c^n(\Gamma, J) = 0$ for $n > 0$ and J a relative injective;

iii) For each strict short exact sequence $0 \to A \to B \to C \to 0$ there are natural connecting maps δ_c^n such that the following is a long exact sequence:

$$0 \to H_c^0(\Gamma, A) \to H_c^0(\Gamma, B) \to H_c^0(\Gamma, C) \xrightarrow{\delta_c^0} H_c^1(\Gamma, A) \to \ldots$$

$$\ldots \to H_c^n(\Gamma, C) \xrightarrow{\delta_c^n} H^{n+1}(\Gamma, A) \to \ldots .$$

We construct a relatively injective resolution of a Γ-sheaf A.

Set $\Gamma^{n+1} = \{(\gamma_0, \ldots, \gamma_n) : \gamma_i \in \Gamma, s(\gamma_i) = r(\gamma_{i+1})\}$ (with the relative topology of the $(n+1)$-fold product) and define $\tilde{r}, \tilde{s}: \Gamma^{n+1} \to \Gamma^0$ by $\tilde{r}(\gamma_0, \ldots, \gamma_n) = r(\gamma_0)$, and $\tilde{s}(\gamma_0, \ldots, \gamma_n) = s(\gamma_n)$. $J^n = J^n(A)$ is defined by the presheaf $S(U, J^n) = S(\tilde{s}^{-1}(U), \tilde{r}^*(A))$. Thus, an element $f \in J_x^n$ is represented by a continuous function $f: \tilde{s}^{-1}(U) \to A$ with $f(\gamma_0, \ldots, \gamma_n) \in A_{r(\gamma_0)}$ and where U is some neighbourhood of x. The action of Γ on J^n is defined by "right translation", just as with $J(A)$ (cf. § 1); we write:

$$(\gamma f)(\gamma_0, \ldots, \gamma_n) = f(\gamma_0, \ldots, \gamma_n \gamma) \quad \text{where } f \in J^n_{s(\gamma)}, \gamma \in \Gamma.$$

Note that $J^0(A) = J(A^0)$ and $J^{n+1}(A) \cong J(J^n(A)^0)$; hence, J^n is a relative injective for each $n \geqslant 0$.

Define $e: A \to J^0$ by $e(a)(\gamma) = \gamma a$ (for $a \in A_{s(\gamma)}$) and $d_n: J^n \to J^{n+1}$ by

$$(d_n f)(\gamma_0, \ldots, \gamma_{n+1}) = \gamma_0 f(\gamma_1, \ldots, \gamma_{n+1}) + \sum_{k=1}^{n} (-1)^{k+1} f(\gamma_0, \ldots, \gamma_k \gamma_{k+1}, \ldots, \gamma_{n+1}).$$

Hence, one has a complex of Γ-sheaves:

$$0 \to A \xrightarrow{e} J^0 \xrightarrow{d_0} J^1 \xrightarrow{d_1} \dots .$$

To show that this yields a relatively injective resolution it suffices to show that the underlying complex of sheaves has a contracting homotopy. Let $t\colon (J^0)^0 \to A^0$ be given by $t(f) = f(x)$ for $f \in J_x^0$ and let $s_n : (J^{n+1})^0 \to (J^n)^0$ be given by $(s_n f)(\gamma_0, \dots, \gamma_n) = (-1)^{n+1} f(\gamma_0, \dots, \gamma_n, s(\gamma_n))$. Claim:

i) $te^0 = \mathrm{id}_A^0$,

ii) $e^0 t + s_0 d_0^0 = \mathrm{id}_{J^0}^0$,

iii) $d_{n-1}^0 s_{n-1} + s_n d_n^0 = \mathrm{id}_{J^n}^0$.

We verify (iii); given $f \in J^n$

$$(s_n d_n^0 f)(\gamma_0, \dots, \gamma_n) = (-1)^{n+1}(d_n^0 f)(\gamma_0, \dots, \gamma_n, s(\gamma_n)) =$$

$$= (-1)^{n+1}[\gamma_0 f(\gamma_1, \dots, \gamma_n, s(\gamma_n)) +$$

$$+ \sum_{k=0}^{n-1} (-1)^{k+1} f(\gamma_0, \dots, \gamma_k\gamma_{k+1}, \dots, \gamma_n, s(\gamma_n))] + f(\gamma_0, \dots, \gamma_n)\cdot$$

$$\cdot(d_{n-1}^0 s_{n-1} f)(\gamma_0, \dots, \gamma_n) = \gamma_0(s_{n-1} f)(\gamma_1, \dots, \gamma_n) +$$

$$+ \sum_{k=0}^{n-1} (-1)^{k+1}(s_{n-1} f)(\gamma_0, \dots, \gamma_k\gamma_{k+1}, \dots, \gamma_n) =$$

$$= (-1)^n[\gamma_0 f(\gamma_1, \dots, \gamma_n, s(\gamma_n)) + \sum_{k=0}^{n-1} (-1)^{k+1} f(\gamma_0, \dots, \gamma_k\gamma_{k+1}, \dots, \gamma_n, s(\gamma_n))].$$

Combining these two equations yields (iii). Verification of (i) and (ii) is left to the reader. The continuous cocycle cohomology $H_c^*(\Gamma, A)$ may thus be identified with the cohomology of the complex:

$$S_\Gamma(J^0) \to S_\Gamma(J^1) \to \dots .$$

Note that $S(J^n) = \{f\colon \Gamma^{n+1} \to A : f(\gamma_0, \dots, \gamma_n) \in A_{r(\gamma_0)}\}$ and $f \in S_\Gamma(J^n)$ iff $f(\gamma_0, \dots$ $\dots, \gamma_n\gamma) = f(\gamma_0, \dots, \gamma_n)$ whenever $s(\gamma_n) = r(\gamma)$. Hence $f \in S_\Gamma(J^n)$ iff it is independent of the last variable and one may identify $S_\Gamma(J^n)$ with $S(J^{n-1})$ (note $S_\Gamma(J^0)$ is identified with $S(A)$). Under these identifications the boundary map $(d_n)_{\otimes} : S_t(J^n) \to$ $\to S_t(J^{n+1})$ is given by

$$((d_n)_{\otimes} f)(\gamma_0, \ldots, \gamma_n) = \gamma_0 f(\gamma_1, \ldots, \gamma_n) + \sum_{k=0}^{n-1} (-1)^{k+1} f(\gamma_0, \ldots, \gamma_k\gamma_{k+1}, \ldots, \gamma_n) +$$

$$+ (-1)^{n+1} f(\gamma_0, \ldots, \gamma_{n-1}) \quad \text{for } n > 0$$

$$((d_0)_{\otimes} f)(\gamma) = \gamma f(s(\gamma)) - f(r(\gamma)).$$

This yields the standard complex of (non-normalized) continuous cochains (cf. [35]).

ADDENDUM 2. *Elementary C^*-bundles.* For the convenience of the reader, we assemble a collection of standard results mostly relating to continuous trace algebras but phrased in bundle-theoretic terms. In [11] Fell introduced the notion of Banach bundle as an alternative to the formalism of continuous fields of Banach spaces (see [7], Chapter 10). It the base space in question is either locally compact or paracompact, it follows from a result of Douady and dal Soglio-Hérault (appended to Fell's monograph [12]) that the two points of view are equivalent. We will tacitly assume that the base space of each Banach bundle considered is locally compact and that the bundle itself is strongly separable (i.e., the total space of the bundle is second countable; see [11], Proposition 1.8). This will ensure that the Banach space of continuous sections vanishing at infinity is separable (see [12], Proposition 10.10). We recall Fell's definitions (see [12]).

Let $\pi: E \to X$ be a continuous open surjection such that each fiber $E_x = \pi^{-1}(x)$ is a (non-trivial) Banach space. The pair (E, π) is said to be a Banach bundle with total space E and base space X if the following maps are continuous:

i) $E \to \mathbf{R}^+$ given by $e \to \|e\|$,

ii) $E*E \to E$ given by $(e, e') \to e + e'$,

iii) $\mathbf{C} \times E \to E$ given by $(\lambda, e) \to \lambda e$,

and if for every net $\{e_\alpha\} \subset E$ such that $\|e_\alpha\| \to 0$ and $\pi(e_\alpha) \to x$, one has $e_\alpha \to 0 \in E_x$. We shall also say that E is a Banach bundle over X. The continuous sections are denoted $C(E)$ (or $C(X, E)$ if the dependence on X is to be made explicit); Douady and dal Soglio-Hérault show that for every $e \in E$ there is $f \in C(E)$ such that $f(\pi(e)) =$ $= e$. Moreover, given a collection of Banach spaces $\{E_x : x \in X\}$ and a linear subspace $V \subset \prod_x E_x$ such that $\{v_x : v \in V\}$ is dense in E_x for each x and the map $x \to$ $\to \|v_x\|$ is continuous for each $v \in V$, there is a unique topology on $\bigsqcup E_x$ making it into a Banach bundle E such that $V \subset C(E)$ (cf. [11], Proposition 1.6). Hence there is a one-to-one correspondence between continuous fields of Banach spaces and Banach bundles. Let $C_0(E)$ denote the Banach space of continuous sections which vanish at infinitiy (note that $C_0(E)$ is the closure of $C_c(E)$, the compactly supported continuous sections, in the supremum norm); by our standing assumption that Banach bundles be strongly separable, $C_0(E)$ is separable.

If each fiber E_x in a Banach bundle (E, π) is a Hilbert space, (E, π) is called a Hilbert bundle; the inner-product $(\cdot\,, \cdot)$ is clearly continuous $E * E \to \mathbf{C}$. We require that $(\cdot\,, \cdot)$ be conjugate-linear in the first variable so that $C_0(E)$ becomes a right Hilbert $C_0(X)$-module with inner-product $\langle f, g\rangle(x) = (f(x), g(x))$ for $f, g \in C_0(E)$.

If each fiber in a Banach bundle (E, π) is a C^*-algebra and both multiplication, $E*E \to E$ (write $(e, e') \to ee'$) and involution $E \to E$ (write $e \to e^*$) are continuous, then (E, π) is said to be a C^*-bundle. Note that $C_0(E)$ is a C^*-algebra with pointwise multiplication and involution. We say that (E, π) is an elementary C^*-bundle if each fiber is elementary (i.e. $E_x \cong \mathscr{K}(\mathscr{H})$ the algebra of compact operators on a Hilbert space $\mathscr{H}$) and if for every $x \in X$ there is an open neighbourhood U of x and $p \in C_0(E)$ such $p(x')$ is a rank-one projection for all $x' \in U$ (this is called Fell's condition). If (E, π) is an elementary C^*-bundle then $C_0(E)$ is a continuous trace algebra; conversely, given a continuous trace algebra A, there is a unique elementary C^*-bundle $\pi : E \to \hat{A}$ such that $A \cong C_0(E)$. We define the Dixmier-Douady invariant of an elementary C^*-bundle E to be that of the associated algebra of sections $C_0(E)$. Thus, if E is an elementary C^*-bundle over X, one obtains an element $\delta(E) \in H^2(X, \underset{\sim}{\mathbf{T}})$. In order to state the relevant properties of this invariant in this context, we must introduce a few more definitions.

If V is a Hilbert bundle over X, we define the associated elementary C^*-bundle $\mathscr{K}(V)$ in such a way that $C_0(\mathscr{K}(V)) = \mathscr{K}(C_0(V))$ (cf. [7], 10.7), where $\mathscr{K}(C_0(V))$ is the algebra of compact operators on the Hilbert module $C_0(V)$.

Set $\mathscr{K}(V)_x = \mathscr{K}(V_x)$ and let the topology of $\mathscr{K}(V)$ be defined by the finite rank operators. Given an elementary C^*-bundle E, one has $\delta(E) = 0$ iff $E \cong \mathscr{K}(V)$ for some Hilbert bundle V (see [7], Theorem 10.7.15).

Suppose E and F are elementary C^*-bundles over X. The tensor-product $E \otimes F$ is again an elementary C^*-bundle over X with fiber $(E \otimes F)_x = E_x \otimes F_x$ and with bundle topology determined by the family of sections $C_0(E) \underset{C_0(X)}{\bullet} C_0(F)$ (where $\bullet$ denotes algebraic tensor product). One has $\delta(E \otimes F) = \delta(E) + \delta(F)$ (see [6], Theorem 1).

We say that an elementary C^*-bundle E is stable if $E \cong (\mathscr{K} \times X) \otimes E$ (alternatively, $C_0(E) \cong \mathscr{K} \otimes C_0(E)$) where $\mathscr{K} \times X$ is the trivial bundle with constant fiber $\mathscr{K}$. Since such a bundle is evidently locally trivial we may invoke Theorem 10.8.4 of [7] (see [8], Theorem 11) which says in our context that there is a bijective correspondence between isomorphism classes of stable elementary C^*-bundles over X and $H^2(X, \underset{\sim}{\mathbf{T}})$ (given by $E \to \delta(E)$).

It follows that if E and F are any elementary C^*-bundles over X then $\delta(E) = \delta(F)$ iff $(\mathscr{K} \times X) \otimes E \cong (\mathscr{K} \times X) \otimes F$ (since $\delta(\mathscr{K} \times X) = 0$).

As stable isomorphism and strong Morita equivalence are one and the same for separable C^*-algebras [1], it is proper to think of the Dixmier-Douady invariant as a complete invariant for strong Morita equivalence of elementary C^*-bundles over a given space. Instead of introducing the notion of equivalence bimodule in the bundle context we resort to the linking algebra characterization (see [1], Theorem 1.1).

If E is an elementary C^*-bundle over X and $m \in M(C_0(E))$ is a multiplier, then m defines two fiber preserving bundle maps on E by right and left multiplication which we write $e \in E \to em$ and $e \in E \to me$; there is a family of multipliers $m_x \in M(E_x)$ such that if $e \in E_x$ then $em = em_x$ and $me = m_x e$ (and such a family of multipliers yields a multiplier for $C_0(E)$ precisely if the resulting maps on E are continuous). Write $M(C_0(E)) = M(E)$ when multipliers are regarded as bundle maps. If $p \in M(E)$ is a projection such that $p_x \neq 0$ for all $x \in X$, set $E^p = \{e \in E : e = pep\}$. Evidently, E^p is an elementary C^*-bundle and $\delta(E) = \delta(E^p)$ (one has $C_0(E^p) = pC_0(E)p$).

Let E and E' be elementary C^*-bundles over X; E and E' are said to be strongly Morita equivalent if there is an elementary C^*-bundle F and projections $p, p' \in M(F)$ with $pp' = 0$ and $p_x \neq 0$, $p'_x \neq 0$ for all x, such that $E \cong F^p$ and $E' \cong F^{p'}$. In this case, it is clear that $C_0(E)$ and $C_0(E')$ are strongly Morita equivalent in the usual sense (in a manner that respects the identification of spectra). One has $\delta(E) = \delta(F)$ iff E and F are strongly Morita equivalent.

Let (E, π) be a Banach bundle over X and $f: Z \to X$ a continuous map; set $f^*(E) = \{(e, z) : \pi(e) = f(z)\} = E * Z \subset E \times Z$ and observe that $f^*(E)$ is a Banach bundle over Z. If E is an elementary C^*-bundle then so is $f^*(E)$; furthermore, one has $\delta(f^*(E)) = f^*(\delta(E))$ (see [34], Proposition 1.4). If $m \in M(E)$ let $f^*(m) \in M(f^*(E))$ be given by $f^*(m)(e, z) = (me, z)$ and $(e, z)f^*(m) = (em, z)$. If F is another elementary C^*-bundle over X one has $f^*(E \otimes F) \cong f^*(E) \otimes f^*(F)$ (the isomorphism is given by $(e \otimes g, z) \to (e, z) \otimes (g, z)$).

ADDENDUM 3. *Ultraliminary groupoids.* The cohomology of a certain class of sheaf groupoids is "computed" in terms of the cohomology of an approximating sequence of subgroupoids. We shall make use of $\varprojlim^1$, the (right) derived functor of $\varprojlim$ for inverse sequences of abelian groups (see [38]). Briefly, given an inverse sequence of short exact sequences

$$0 \to A_k \to B_k \to C_k \to 0,$$

one obtains a six-term exact sequence:

$$(\#) \qquad 0 \to \varprojlim A_k \to \varprojlim B_k \to \varprojlim C_k \to \varprojlim{}^1 A_k \to \varprojlim{}^1 B_k \to \varprojlim{}^1 C_k \to 0,$$

since the higher derived functors vanish.

We associate a sheaf groupoid to a sequence of local homeomorphisms

$$X_0 \xrightarrow[\varphi_0]{} X_1 \xrightarrow[\varphi_1]{} X_2 \xrightarrow[\varphi_2]{} \dots ;$$

set $\Gamma_k = R(\varphi_{k-1} \circ \ldots \circ \varphi_0)$ for $k > 0$ (and $\Gamma_0 = X_0$), and note that $\Gamma_k \subset \Gamma_{k+1} \subset X_0 \times X_0$. Put $\Gamma = \bigcup_k \Gamma_k$ and note that Γ is a sheaf groupoid with unit space $\Gamma^0 = X_0$ ($= \Gamma_k^0$ for all k). Such a groupoid is called ultraliminary (cf. [20], § 4). An action of Γ on a sheaf A (over X_0) is then given by a family of compatible Γ_k-actions and one has $S_\Gamma(A) = \bigcap_k S_{\Gamma_k}(A)$. Recall that $\mathscr{S}(\Gamma_k)$ and $\mathscr{S}(X_k)$ are equivalent categories (see § 0.9) and let A^k denote the sheaf over X_k corresponding to A; one has $S_{\Gamma_k}(A) \cong S(A^k)$.

PROPOSITION. *With notation as above there is a short exact sequence*

$$(*) \qquad 0 \to \varprojlim{}^1 H^{n-1}(X_k, A^k) \to H^n(\Gamma, A) \xrightarrow[\zeta_n]{} \varprojlim H^n(X_k, A^k) \to 0,$$

for each $n > 0$, where ζ_n is the map given by functoriality.

Proof. We verify $(*)$ for $n = 1$, and then use a dimension shift argument for $n > 1$. Recall that we may regard $H^1(\Gamma, A)$ as isomorphism classes of short exact sequences of Γ-sheaves:

$$0 \to A \to B \to \underset{\sim}{Z} \to 0.$$

That ζ_1 is onto follows from the fact that a Γ-action is determined by its restrictions to Γ_k. We identify $\ker \zeta_1$ with a subquotient of $Z_\Gamma(A)$. Consider the following subgroup of $Z_\Gamma(A)$:

$$H = \{h \in Z_\Gamma(A) : \forall k \geqslant 0,\ \exists f \in S(A) \text{ such that } h(\gamma) =$$
$$= f(r(\gamma)) - \gamma f(s(\gamma))\ \forall \gamma \in \Gamma_k\}.$$

One checks that $\operatorname{Im} d \subset H$ and $\ker \zeta_1 \cong H/\operatorname{Im} d$. Moreover

$$H \cong \varprojlim S(A)/S_{\Gamma_k}(A);$$

by applying (#) to the inverse sequence of short exact sequences:

$$0 \to S_{\Gamma_k}(A) \to S(A) \to S(A)/S_{\Gamma_k}(A) \to 0$$

one obtains

$$\ker \zeta_1 \cong H/\operatorname{Im} d \cong \varprojlim{}^1 S_{\Gamma_k}(A) \cong \varprojlim{}^1 H^0(X_k, A^k)$$

(since $\varprojlim{}^1 S(A) = 0$). Thus, $(*)$ holds for $n = 1$.

We prove the general case by induction. Suppose (*) holds for n. Let $Q = Q(A)$ be the Γ-injective constructed in § 1. Since Q^k is of product type one has that $H^n(X_k, Q^k) = 0$. Consider the square:

$$\begin{array}{ccc} H^n(\Gamma, Q/A) & \xrightarrow[\zeta_n]{} & \varprojlim H^n(X_k, Q^k/A^k) \\ \downarrow & & \downarrow \\ H^{n+1}(\Gamma, A) & \xrightarrow[\zeta_{n+1}]{} & \varprojlim H^{n+1}(X_k, A^k). \end{array}$$

Since the vertical arrows are isomorphisms and the square commutes, ζ_{n+1} is onto.

To show that $\ker \zeta_{n+1} \cong \varprojlim^1 H^n(X_k, A^k)$ follows as above if $n > 1$.

In order to check that $\ker \zeta_2 \cong \varprojlim^1 H^1(X_k, A^k)$ we need to show

$$(**) \qquad \varprojlim{}^1 H^0(X_k, Q^k/A^k) \cong \varprojlim{}^1 H^1(X_k, A^k).$$

Note that $\varprojlim^1 H^0(X_k, Q^k) = 0$ (since $H^1(\Gamma, Q) = 0$); applying (#) to the inverse sequence:

$$0 \to H^0(X_k, A^k) \to H^0(X_k, Q^k) \to H^0(X_k, Q^k)/H^0(X_k, A^k) \to 0$$

shows that $\varprojlim^1 H^0(X_k, Q^k)/H^0(X_k, A^k) = 0$. Finally, applying (#) to the inverse sequence:

$$0 \to H^0(X_k, Q^k)/H^0(X_k, A^k) \to H^0(X_k, Q^k/A^k) \to H^1(X_k, A^k) \to 0$$

yields (**). ▨

Ultraliminary groupoids with X_0 compact were called hyperfinite relations in [21], § 6. The above proposition has its origin in the calculation of the twist group of a hyperfinite relation (cf. [21], §6.11). This result resembles Milnor's $\varprojlim^1$ sequence for computing the cohomology of a union of CW-complexes (see [26]).

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