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MOST QUASITRIANGULAR OPERATORS ARE TRIANGULAR, MOST BIQUASITRIANGULAR OPERATORS ARE BITRIANGULAR

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1. INTRODUCTION

Recently, Che-Kao Fong [3] proved that the set of all diagonal operators, acting on a complex separable infinite dimensional Hilbert space $\mathscr{H}$, includes a G_δ-dense subset of the class $\mathrm{Nor}(\mathscr{H})$ of all normal operators acting on $\mathscr{H}$. (Here, and in what follows, *operator* denotes a bounded linear transformation of the space into itself.) More precisely, if $\sigma(A)$ denotes the spectrum of $A \in \mathscr{L}(\mathscr{H})$ (the algebra of all operators), and $\sigma_0(A)$ is the set of all *normal eigenvalues* of A (that is, those isolated points λ of $\sigma(A)$ such that the Riesz idempotent corresponding to the clopen subset $\{\lambda\}$ has finite rank), then Fong's result can be described as follows:

$$\mathrm{Nor}(\mathscr{H})^0 = \{N \in \mathrm{Nor}(\mathscr{H}) : \mathscr{H} = \bigvee \{\ker(N - \lambda) : \lambda \in \sigma_0(N)\}\}$$

is a G_δ-dense subset of $\mathrm{Nor}(\mathscr{H})$. (Here $\bigvee$ denotes, as usual, "the closed linear span of".) It is convenient to remark that $\mathrm{Nor}(\mathscr{H})$ coincides with the norm-closure, $[\mathrm{Nor}(\mathscr{H})^0]^-$, of $\mathrm{Nor}(\mathscr{H})^0$; however, $\mathrm{Nor}(\mathscr{H})$ is not included in $\mathrm{Nor}(\mathscr{H})^0 + \mathscr{K}(\mathscr{H}) = \{N + K : N \in \mathrm{Nor}(\mathscr{H})^0,\ K \in \mathscr{K}(\mathscr{H})\}$, where $\mathscr{K}(\mathscr{H})$ denotes the ideal of all compact operators. To see this, observe that the essential spectrum, $\sigma_e(N) = \sigma(N + \mathscr{K}(\mathscr{H}))$, of every N in $\mathrm{Nor}(\mathscr{H})^0$ has empty interior.

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is called *triangular* if it admits an upper triangular matrix with respect to some orthonormal basis of $\mathscr{H}$. The operator T is called *quasitriangular* (*quasidiagonal*) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ in $\mathrm{PF}(\mathscr{H})$ (the family of all finite rank orthogonal projections, with the usual partial order induced by range inclusion) such that $P_n \to 1$ strongly and

$$\|(1 - P_n)TP_n\| \to 0 \quad (\|TP_n - P_nT\| \to 0, \text{ resp.}) \quad \text{as } n \to \infty.$$

It is well-known that if (QT) and $(\triangle)$ denote the class of all quasitriangular and, respectively, all triangular operators, then

$$(\mathrm{QT}) = (\triangle)^- = (\triangle) + \mathcal{K}(\mathcal{H}) =$$

$$= \{T \in \mathcal{L}(\mathcal{H}): \text{ Given } \varepsilon > 0 \text{ there exists } K_\varepsilon \in \mathcal{K}(\mathcal{H}), \text{ with } \|K_\varepsilon\| < \varepsilon, \text{ such that } T - K_\varepsilon \in (\triangle)\}$$

[5], [6]. (In particular, (QT) is a closed subset of $\mathcal{L}(\mathcal{H})$.)

Let $T \in (\mathrm{QT})$. In [8], [9], the author has completely characterized the sequences $\{\lambda_n\}_{n=1}^\infty$ of complex numbers such that there exist K in $\mathcal{K}(\mathcal{H})$, and an orthonormal basis (ONB) $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ satisfying

$$T - K = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & * & \\ & & \lambda_3 & & \\ & & & \cdot & \\ & & & & \cdot \\ & 0 & & & & \cdot \\ & & & & & & \cdot \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$$

(and the analogous result for the case when K is required to have norm smaller than a given $\varepsilon > 0$).

By using these results, and Fong's argument, it is possible to prove the following two analogues to the result of [3]. Theorems 1 and 2 below were conjectured by C.-K. Fong (personal communication).

THEOREM 1. *The set*

$$(\triangle)^0 = \{A \in (\triangle): (1) \textit{ The diagonal entries of } A \textit{ (with respect to some ONB) belong to } \sigma_0(A); \ (2) \dim\ker(A - \lambda) = 1 \textit{ for all } \lambda \in \sigma_0(A); \textit{ and } (3)\ \mathcal{H} = \bigvee\{\ker(A - \lambda) : \lambda \in \sigma_0(A)\}\}$$

is a G_δ-dense subset of (QT)*; moreover,*

$$(\mathrm{QT}) = [(\triangle)^0]^- = (\triangle)^0 + \mathcal{K}(\mathcal{H}) =$$

$$= \{T \in \mathcal{L}(\mathcal{H}): \textit{Given } \varepsilon > 0 \textit{ there exists } K_\varepsilon \in \mathcal{K}(\mathcal{H}), \textit{ with } \|K_\varepsilon\| < \varepsilon, \textit{ such that } T - K_\varepsilon \in (\triangle)^0\}.$$

Recall that $T \in \mathscr{L}(\mathscr{H})$ is *biquasitriangular* (class (BQT)) if both T and its adjoint T^* are quasitriangular; A is called *bitriangular* (class $(\mathrm{B}\triangle)$) if both A and A^* are triangular operators (not necessarily with respect to the same ONB).

THEOREM 2. *The set*

$$(\mathrm{B}\triangle)^0 = \{A \in (\mathrm{B}\triangle) : (1)\ \sigma_0(A) = \sigma_0(A^*)^* \text{ and the diagonal entries of } A\ (A^*) \text{ belong to } \sigma_0(A)\ (\sigma_0(A^*) \text{ resp.}); (2)\ \dim\ker(A-\lambda) = \dim\ker(A-\lambda)^* = 1 \text{ for all } \lambda \in \sigma_0(A); \text{ and } (3)\ \mathscr{H} = \bigvee\{\ker(A-\lambda) : \lambda \in \sigma_0(A)\} = \bigvee\{\ker(A-\lambda)^* : \lambda \in \sigma_0(A)\}\}$$

is a G_δ-dense subset of (BQT); *moreover,*

$$(\mathrm{BQT}) = [(\mathrm{B}\triangle)^0]^- = (\mathrm{B}\triangle)^0 + \mathscr{K}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : \text{Given } \varepsilon > 0 \text{ there exists } K_\varepsilon \in \mathscr{K}(\mathscr{H}), \text{ with } \|K_\varepsilon\| < \varepsilon, \text{ such that } T - K_\varepsilon \in (\mathrm{B}\triangle)^0\}.$$

Indeed, the proof of Theorem 2 requires a "symmetric" version for the class (BQT) of the "non-symmetric" results of [8], [9] for the class (QT) (see Section 3 below).

On the other hand, a well-known result of D. Voiculescu [15] shows that the sets

$$\mathrm{SNor}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : T \text{ is similar to a normal operator}\},$$

$$\mathrm{SFNor}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : T \text{ is similar to a normal operator with finite spectrum}\},$$

$$\mathrm{Alg}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : T \text{ is algebraic, i.e., } T \text{ satisfies a polynomial equation}\},$$

and

$$\mathrm{Alg}(\mathscr{H}) + \mathscr{K}(\mathscr{H})$$

are dense in (BQT).

THEOREM 3. SNor($\mathscr{H}$), SFNor($\mathscr{H}$), Alg($\mathscr{H}$) *and* Alg($\mathscr{H}$) + $\mathscr{K}(\mathscr{H})$ *are dense first category F_σ subsets of* (BQT).

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2. TRIANGULAR VERSUS QUASITRIANGULAR OPERATORS

Let $(\triangle)^0$ be defined as in Theorem 1. It follows from [8, Section 4], [9, Theorem 2.3] that, if $T \in (\mathrm{QT})$, then given $\varepsilon > 0$ there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $A = T - K_\varepsilon \in (\triangle)^0$. Since (QT) is invariant under compact perturbations [5], [7], we deduce that

$$(\mathrm{QT}) = [(\triangle)^0]^- = (\triangle)^0 + \mathscr{K}(\mathscr{H}).$$

It only remains to show that $(\triangle)^0$ is a G_δ in $\mathscr{L}(\mathscr{H})$. To this end, we proceed as in Fong's proof. Let $\{P_n\}_{n=1}^\infty \subset \mathrm{PF}(\mathscr{H})$ be an increasing sequence such that $P_n \uparrow 1$ (strongly, as $n \to \infty$), and let $(\mathrm{QT})^n$ be the set of all those $T \in (\mathrm{QT})$ such that there exists E in $\mathrm{PF}(\mathscr{H})$ satisfying

(i) $(1 - E)TE = 0$,

(ii) $\sigma(T \mid \operatorname{ran} E)$ consists of $\operatorname{rank} E$ distinct normal eigenvalues of T, and

(iii) $\|P_n E P_n - P_n\| < 1/n$.

If $A \in (\triangle)^0$ has a triangular matrix with diagonal entries $\{\lambda_n\}_{n=1}^\infty \subset \sigma_0(A)$ with respect to an ONB $\{e_n\}_{n=1}^\infty$, and E_m denotes the orthogonal projection onto $\bigvee \{e_n\}_{n=1}^m$, then $E_m \uparrow 1$, and therefore

$$\|P_n E_m P_n - P_n\| \to 0 \quad (m \to \infty)$$

for each $n = 1, 2, \ldots$. It readily follows from this (and our previous observations) that $(\mathrm{QT})^n$ is a dense open subset of (QT) including $(\triangle)^0$. Thus, $(\triangle)^0$ is included in the set

$$\bigcap_{n=1}^{\infty} (\mathrm{QT})^n,$$

which is a G_δ in $\mathscr{L}(\mathscr{H})$. (Recall that (QT) is a closed subset of $\mathscr{L}(\mathscr{H})$.)

On the other hand, if $B \in \bigcap_{n=1}^{\infty} (\mathrm{QT})^n$, then $\sigma_0(B)$ is necessarily a (denumerable) infinite set. Let $\sigma_0(B) = \{\mu_n\}_{n=1}^\infty$, and let F_m be the orthogonal projection onto $\ker \prod_{n=1}^{m} (B - \mu_n)$; then conditions (ii) and (iii) imply that $F_m \uparrow 1$ $(m \to \infty)$, and

$\dim\ker(B-\mu_n)=1$ for all $n=1,2,\ldots$. Clearly, B admits an upper triangular matrix representation with respect to the Gram-Schmidt orthonormalization of the sequence $\{f_n\}_{n=1}^{\infty}$, where f_n is a unit vector in $\ker(B-\mu_n)$, $n=1,2,\ldots$; moreover, the sequence of diagonal entries of this matrix representation of B coincides with the sequence $\{\mu_n\}_{n=1}^{\infty}$. Hence, $B\in(\triangle)^0$; that is,

$$(\triangle)^0=\bigcap_{n=1}^{\infty}(\mathrm{QT})^n$$

is a G_δ-dense subset of (QT).

The proof of Theorem 1 is now complete. ▨

3. BITRIANGULAR VERSUS BIQUASITRIANGULAR OPERATORS

Recall that $T\in\mathcal{L}(\mathcal{H})$ is semi-Fredholm if $\operatorname{ran}T$ is closed, and either $\ker T$ or $\ker T^*$ is finite dimensional. In this case, we define the index of T by

$$\operatorname{ind}T=\dim\ker T-\dim\ker T^*.$$

The reader is referred to [13] for properties of these operators. The semi-Fredholm domain of T is the open set $\rho_{s\text{-}F}(T)=\{\lambda\in\mathbf{C}:\lambda-T \text{ is semi-Fredholm}\}$. A celebrated result of C. Apostol, C. Foiaş and D. Voiculescu says that T is quasitriangular if and only if $\rho_{s\text{-}F}^{-}(T)=\{\lambda\in\rho_{s\text{-}F}(T):\operatorname{ind}(\lambda-T)<0\}$ is empty, and therefore $T\in(\mathrm{BQT})$ if and only if both $\rho_{s\text{-}F}^{-}(T)$ and $\rho_{s\text{-}F}^{+}(T)=\{\lambda\in\mathbf{C}:\operatorname{ind}(\lambda-T)>0\}$ are empty sets (see [1], [7, Chapter 6]).

Let $\sigma_l(T)$ ($\sigma_r(T)$) denote the left (right, resp.) spectrum of T. The following proposition is a mild improvement of the author's results about the structure of a triangular operator (see [7, Corollary 3.40], [10]).

PROPOSITION 4. *If $A\in\mathcal{L}(\mathcal{H})$ is a triangular operator with an upper triangular matrix $(a_{ij})_{i,j=1}^{\infty}$ (with respect to an ONB $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}$), and diagonal sequence* $\mathrm{d}(A)=\{a_{jj}\}_{j=1}^{\infty}$, *then*

(i) $\mathrm{d}(A)\subset\sigma(A)=\sigma_l(A)=\sigma_{lre}(A)\cup\rho_{s\text{-}F}^{+}(A)\cup\sigma_0(A)$, *so that* $\operatorname{ind}(\lambda-A)\geqslant 0$ *for all* $\lambda\in\rho_{s\text{-}F}(A)$. *(Here* $\sigma_{lre}(A)=\mathbf{C}\setminus\rho_{s\text{-}F}(A)$.*)*

(ii) *Every nonempty clopen subset of $\sigma(A)$ intersects* $\mathrm{d}(A)$, *and every component of $\sigma(A)$ intersects* $\mathrm{d}(A)^-$.

(iii) *Furthermore, if σ is a clopen subset of $\sigma(A)$, and $\mathcal{H}(A;\sigma)$ is the corresponding Riesz spectral subspace (so that $\mathcal{H}$ is the direct sum of $\mathcal{H}(A;\sigma)$ and $\mathcal{H}(A;\sigma(A)\setminus\sigma)$, $\sigma(A\mid\mathcal{H}(A;\sigma))=\sigma$ and $\sigma(A\mid\mathcal{H}(A;\sigma(A)\setminus\sigma))=\sigma(A)\setminus\sigma$ [13]), then*

$$\operatorname{card}\{j;\ a_{jj}\in\sigma\}=\dim\mathcal{H}(A;\sigma).$$

(In particular, every isolated point of $\sigma(A)$ belongs to $\mathrm{d}(A)$.*)*

(iv) *If* $\ker(\lambda - A)^* \neq \{0\}$, *then* $\lambda \in d(A)$, *so that the point spectrum*, $\sigma_p(A^*)$, *of* A^* *is an at most denumerable subset of* $d(A)^* = \{\bar{\lambda} : \lambda \in d(A)\}$; *furthermore*,

$$\dim\ker[(a_{hh} - A)^*]^k \leqslant \min[\operatorname{card}\{j : a_{jj} = a_{hh}\},\ \dim\ker(a_{hh} - A)^k]$$

for all $h, k = 1, 2, \ldots$.

(v) $\mathscr{H} = \bigvee \{\ker(a_{jj} - A)^k : j, k = 1, 2, \ldots\}$.

(vi) *If* π *is a bijection of the set* $\mathbf{N}$ *of all natural numbers onto itself, then* A *admits an upper triangular matrix representation with* $d(A) = \{a_{\pi(j),\pi(j)}\}_{j=1}^{\infty}$ *(with respect to some ONB of* $\mathscr{H}$*).*

Proof. Everything is proved in the above mentioned references, except (vi) which was proved in [9]), (v) (which is trivial: observe that e_h belongs to the linear span of $\{\ker(a_{jj} - A)^h : 1 \leqslant j \leqslant h\}$ $(h = 1, 2, \ldots)$, and the second part of (iv). To see this, observe that, if $x = \sum_{j=1}^{\infty} c_j e_j \in \ker(a_{hh} - A)^*$, then

$$0 = (A - a_{hh})^* x = \begin{pmatrix} \overline{(a_{11} - a_{hh})} c_1 \\ \bar{a}_{12} c_1 + \overline{(a_{22} - a_{hh})} c_2 \\ \bar{a}_{13} c_1 + \bar{a}_{23} c_2 + \overline{(a_{33} - a_{hh})} c_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

It readily follows that, if $a_{jj} \neq a_{hh}$, then c_j is a linear function of $c_1, c_2, \ldots, c_{j-1}$ $(j = 2, 3, \ldots)$.

We infer that $\dim\ker(a_{hh} - A)^*$ cannot exceed $\operatorname{card}\{j : a_{jj} = a_{hh}\}$. Moreover if P_m denotes the orthogonal projection of $\mathscr{H}$ onto the span of the first m coordinates, then an elementary analysis of the homogeneous linear system

$$P_m(A - a_{hh})^* x_m = 0 \quad (x_m \in \operatorname{ran} P_m = \bigvee \{e_j\}_{j=1}^m)$$

indicates that

$$\dim\{x_m \in \operatorname{ran} P_m : P_m(A - a_{hh})^* x_m = 0\} =$$

$$= \dim\{x_m \in \operatorname{ran} P_m : (A - a_{hh}) x_m = 0\} \leqslant \dim\ker(A - a_{hh}).$$

Since this holds for all $m = 1, 2, \ldots$, we deduce that

$$\dim\ker(A - a_{hh})^* \leqslant \dim\ker(A - a_{hh}).$$

By applying the same arguments to the triangular operator $(A - a_{hh})^k$ and its adjoint, we conclude that

$$\dim\ker[(A - a_{hh})^*]^k \leqslant \dim\ker(A - a_{hh})^k$$

for all $h, k = 1, 2, \dots$. ▨

COROLLARY 5. *If $A \in \mathscr{L}(\mathscr{H})$ is bitriangular, and A and A^* admit upper triangular matrices $(a_{ij})_{i,j=1}^{\infty}$ and, respectively, $(b_{ij})_{i,j=1}^{\infty}$ (with respect to two, not necessarily identical,* ONB's*), then*

(i) $\mathrm{d}(A) \subset \sigma(A) = \sigma_{\mathrm{lre}}(A) \cup \sigma_0(A)$, *so that* $\mathrm{ind}(\lambda - A) = 0$ *for all* $\lambda \in \rho_{\mathrm{s\text{-}F}}(A)$.

(ii) *Every nonempty clopen subset of* $\sigma(A)$ *intersects* $\mathrm{d}(A)$, *and every component of* $\sigma(A)$ *intersects* $\mathrm{d}(A)^-$.

(iii) *Furthermore, if σ is a clopen subset of $\sigma(A)$, and $\mathscr{H}(A; \sigma)$ is the corresponding Riesz spectral subspace, then*

$$\mathrm{card}\{j : a_{jj} \in \sigma\} = \dim \mathscr{H}(A; \sigma).$$

(iv) $\sigma_{\mathrm{p}}(A) = \mathrm{d}(A)$, $\sigma_{\mathrm{p}}(A^*) = \mathrm{d}(A)^*$ *and there exists a bijective mapping π from* $\mathbf{N}$ *onto itself such that*

$$b_{hh} = \bar{a}_{\pi(h),\pi(h)} \quad (h = 1, 2, \dots)$$

and

$$\dim\ker[(b_{hh} - A)^*]^k = \dim\ker(a_{\pi(h),\pi(h)} - A)^k \leqslant$$

$$\leqslant \mathrm{card}\{j : b_{jj} = b_{hh}\}$$

for all $h, k = 1, 2, \dots$.

(v) $\mathscr{H} = \bigvee \{\ker(a_{jj} - A)^k : j, k = 1, 2, \dots\} = \bigvee \{\ker[(a_{jj} - A)^*]^k : j, k = 1, 2, \dots\}$.

(vi) *If π is a bijection of* $\mathbf{N}$ *onto itself, then A (A^*) admits an upper triangular matrix representation with* $\mathrm{d}(A) = \{a_{\pi(j),\pi(j)}\}_{j=1}^{\infty}$ ($\mathrm{d}(A^*) = \{b_{\pi(j),\pi(j)}\}_{j=1}^{\infty}$, *resp.*). *In particular, A^* admits a representation of that type with* $\mathrm{d}(A^*) = \mathrm{d}(A)^*$ *(in the sense of sequences).*

Proof. (i), (ii), (iii), (v), and the first part of (vi) follow immediately from the corresponding parts of Proposition 4.

Clearly, $\mathrm{d}(A) \subset \sigma_{\mathrm{p}}(A)$ and $\mathrm{d}(A^*) \subset \sigma_{\mathrm{p}}(A^*)$. On the other hand, by Proposition 4(iv), $\sigma_{\mathrm{p}}(A^*) \subset \mathrm{d}(A)^*$ and $\sigma_{\mathrm{p}}(A) \subset \mathrm{d}(A^*)^*$. Thus,

$$\sigma_{\mathrm{p}}(A) \subset \mathrm{d}(A^*)^* \subset \sigma_{\mathrm{p}}(A^*)^* \subset \mathrm{d}(A) \subset \sigma_{\mathrm{p}}(A) \subset \mathrm{d}(A^*)^* \subset \sigma_{\mathrm{p}}(A^*)^*,$$

whence it follows that $\sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{p}}(A^*)^* = \mathrm{d}(A) = \mathrm{d}(A^*)^*$ (as subsets of the complex plane).

It readily follows that the set $\mathrm{d}(A)$ is independent of the particular representation of A as an upper triangular matrix. Furthermore, by using this fact and Proposition 4(iv), we see that

$$\dim\ker(A-\lambda)^k = \dim\ker[(A-\lambda)^*]^k$$

for all $\lambda \in \mathbf{C}$ and all $k = 1, 2, \ldots$; moreover, $\dim\ker(A-\lambda)^k$ cannot exceed $\operatorname{card}\{j : a_{jj} = \lambda\}$.

If $\operatorname{card}\{j : a_{jj} = \lambda\}$ is not finite, then the proof of Proposition 4(iv) shows that $\ker(A-\lambda)$ is infinite dimensional. A fortiori so are $\ker(A-\lambda)^k$ and $\ker[(A-\lambda)^*])^k$ for all $k = 1, 2, \ldots$.

If $\operatorname{card}\{j: a_{jj} = \lambda\}$ is finite, then the same proof shows that

$$\operatorname{card}\{j : a_{jj} = \lambda\} = \dim\ker(A-\lambda)^{p(\lambda)} = \dim \bigvee \{\ker(A-\lambda)^k\}_{k=1}^{\infty} =$$

$$= \dim\ker[(A-\lambda)^*]^{p(\lambda)} = \dim \bigvee \{\ker[(A-\lambda)^*]^k\}_{k=1}^{\infty}$$

for some $p(\lambda)$ large enough.

The existence of a bijection π of $\mathbf{N}$ onto itself such that $b_{hh} = a_{\pi(h),\pi(h)}$ $(h = 1, 2, \ldots)$ and $\dim\ker[(b_{hh} - A)^*]^k = \dim\ker(a_{\pi(h),\pi(h)} - A)^k$ for all $h, k = 1, 2, \ldots$ follows immediately from the first part of the proof.

The second part of (vi) is a consequence of (iv). ∎

Now we are in a position to prove symmetric versions of the main results of [9] for the class (BQT).

THEOREM 6. *Let T be a biquasitriangular operator such that $\sigma(T) = \sigma_e(T)$, and let $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that*

(i) $\lambda_j \in \sigma(T)$ *for all* $j = 1, 2, \ldots$

and

(ii) $\operatorname{card}\{j : \lambda_j \in \sigma\}$ *is an infinite set for each clopen subset σ of $\sigma(T)$.*

Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $A = T - K_\varepsilon$ is bitriangular, $\sigma(A) = \sigma_e(A)$, and the diagonal sequence $\mathrm{d}(A)$ coincides with Γ.

Proof. Let $\{P_n\}_{n=1}^{\infty} \subset \mathrm{PF}(\mathscr{H})$ be an increasing sequence such that $P_n \uparrow 1$ $(n \to \infty)$. By using [8, Section 4], [9, Theorem 2.3], we can find $K_1 \in \mathscr{K}(\mathscr{H})$, with $\|K_1\| < \varepsilon/2$, and $E_1 \in \mathrm{PF}(\mathscr{H})$ such that $(1 - E_1)(T - K_1)E_1 = 0$, $E_1P_1E_1 = P_1$, and

$$A_1 = T - K_1 \mid \operatorname{ran} E_1 = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & * & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \lambda_{p_1} \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_{p_1} \end{matrix},$$

(where $\{e_j\}_{j=1}^{p_1}$ is an ONB of $\operatorname{ran} E_1$) for some $p_1 \geqslant 1$; moreover,

$$T - K_1 = \begin{pmatrix} A_1 & B_1' \\ 0 & T_1 \end{pmatrix} \begin{matrix} \operatorname{ran} E_1 \\ \ker E_1 \end{matrix},$$

where T_1 is a biquasitriangular operator on $\ker E_1$ satisfying $\sigma(T_1) = \sigma_e(T_1)$.

Now we can apply the same argument to T_1^* and $P_2' = (P_2 \vee E_1) - E_1$, in order to find $K_2 \in \mathscr{K}(\mathscr{H})$, with $\|K_2\| < \varepsilon/4$, and $E_2 \in \mathrm{PF}(\mathscr{H})$ such that $E_1K_2 = K_2E_1 = 0$, $\operatorname{ran} E_2 \supset P_2\mathscr{H} \vee E_1\mathscr{H} \vee (T-K_1)^*E_1\mathscr{H}$, $(1-E_2)(T-K_1-K_2)^*E_2 = 0$, $E_2P_2E_2 = P_2$, $E_2P_2'E_2 = P_2'$,

$$(T - K_1 - K_2)^* \mid \operatorname{ran} E_2 = \begin{pmatrix} \bar\lambda_1 & & & & & & & & \\ & \bar\lambda_2 & & & * & & & & \\ & & \ddots & & & & & & \\ & & & \bar\lambda_{p_1} & & & & & \\ & & & & \bar\lambda_{p_1+1} & & & & \\ & 0 & & & & \ddots & & & \\ & & & & & & & & \bar\lambda_{p_2} \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_{p_1} \\ f_{p_1+1} \\ \vdots \\ f_{p_2} \end{matrix}$$

($\{f_j\}_{j=1}^{p_2}$ is an ONB of $\operatorname{ran} E_2$, $p_2 > p_1$),

$$T - K_1 - K_2 = \begin{pmatrix} A_1 & B_1 & 0 \\ 0 & C_1 & 0 \\ 0 & D_1' & T_2 \end{pmatrix} \begin{matrix} \operatorname{ran} E_1 \\ \operatorname{ran}(E_2 - E_1) \\ \ker E_2 \end{matrix},$$

and T_2 is a biquasitriangular operator on $\ker E_2$ such that $\sigma(T_2) = \sigma_e(T_2)$.

By an obvious inductive argument (by applying the results of [8], [9] alternatively to T_{2n} and to T_{2n+1}^*), we can find compact operators $K_1, K_2, \ldots, K_n, \ldots$ and an increasing sequence $\{E_n\}_{n=0}^\infty \subset \mathrm{PF}(\mathscr{H})$ ($E_0 = 0$) such that the following conditions are satisfied

$$\|K_n\| < \varepsilon/2^n,$$

$$E_nK_{n+1} = K_{n+1}E_n = 0,$$

$$\operatorname{ran} E_{2n-1} \supset P_{2n-1}\mathscr{H} \vee E_{2n-2}\mathscr{H} \vee \left(T - \sum_{j=1}^{2n-2} K_j\right)^* E_{2n-2}\mathscr{H},$$

$$(1 - E_{2n-1})\left(T - \sum_{j=1}^{2n-1} K_j\right) E_{2n-1} = 0,$$

$$T - \sum_{j=1}^{2n-1} K_j \,|\, \operatorname{ran} E_{2n-1} = \begin{pmatrix} \lambda_1 & & & & * \\ & \lambda_2 & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \lambda_{p_{2n-1}} \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_{p_{2n-1}} \end{matrix}$$

($\{e_j\}_{j=1}^{p_{2n-1}}$ is an ONB of $\operatorname{ran} E_{2n-1}$ ($p_{2n-1} > p_{2n-2}$) which includes the vectors in the basis constructed for $\operatorname{ran} E_{2n-2}$),

$$\operatorname{ran} E_{2n} \supset P_{2n}\mathscr{H} \vee E_{2n-1}\mathscr{H} \vee \left(T - \sum_{j=1}^{2n-1} K_j\right) E_{2n-1}\mathscr{H},$$

$$(1 - E_{2n})\left(T - \sum_{j=1}^{2n} K_j\right)^* E_{2n} = 0,$$

and

$$\left(T - \sum_{j=1}^{2n} K_j\right)^* \Big|\, \operatorname{ran} E_{2n} = \begin{pmatrix} \bar{\lambda}_1 & & & & * \\ & \bar{\lambda}_2 & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \bar{\lambda}_{p_{2n}} \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_{p_{2n}} \end{matrix}$$

($\{f_j\}_{j=1}^{p_{2n}}$ is an ONB of $\operatorname{ran} E_{2n}$ ($p_{2n} > p_{2n-1}$) which includes the vectors in the basis constructed for $\operatorname{ran} E_{2n-2}$) for all $n = 1, 2, \ldots$.

Then

$$K_\varepsilon = \sum_{j=1}^{\infty} K_j \in \mathscr{K}(\mathscr{H}), \quad \|K_\varepsilon\| < \varepsilon,$$

(the series converges in the norm topology),

$$\bigvee \{\operatorname{ran} E_n\}_{n=0}^{\infty} \supset \bigvee \{\operatorname{ran} P_n\}_{n=1}^{\infty} = \mathscr{H},$$

and it is not difficult to check that $A = T - K_\varepsilon$ admits a staircase representation of the form (compare with [2]!)

$$A = \left(\begin{array}{ccccccccc} A_1 & B_1 & & & & & & & \\ & C_1 & & & & & 0 & & \\ & D_1 & A_2 & B_2 & & & & & \\ & & & C_2 & & & & & \\ & & & D_2 & A_3 & B_3 & & & \\ & & & & & C_3 & \cdot & & \\ & & & & & D_3 & \cdot & \cdot & \\ & 0 & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot \\ & & & & & & & & \cdot \end{array}\right) \begin{array}{l} \operatorname{ran} E_1 \\ \operatorname{ran}(E_2 - E_1) \\ \operatorname{ran}(E_3 - E_2) \\ \operatorname{ran}(E_4 - E_3) \\ \operatorname{ran}(E_5 - E_4) \\ \operatorname{ran}(E_6 - E_5) \\ \operatorname{ran}(E_7 - E_6) \\ \cdot \\ \cdot \\ \cdot \end{array},$$

$A \in (\mathrm{B}\triangle)$, and $\mathrm{d}(A) = \mathrm{d}(A^*) = \Gamma$. ▨

By comparing this result with Corollary 5, it is easily seen that Theorem 6 is the best possible result along these lines for the case when $T \in (\mathrm{BQT})$ and $\sigma(T) = \sigma_e(T)$. For the more general case, we have the following analog of [9, Corollary 2.4]:

COROLLARY 7. *Let $T \in (\mathrm{BQT})$, and let $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that*

(i) $\sigma_0(T) \subset \Gamma \subset \sigma_{lre}(T) \cup \sigma_0(T) \cup (\mathrm{interior}[\sigma(T) \setminus \sigma_e(T)])$;

(ii) *For each nonempty clopen subset σ of $\sigma(T)$,*

$$\operatorname{card}\{j : \lambda_j \in \sigma\} = \dim \mathscr{H}(T; \sigma);$$

(iii) $\Gamma_0 = \{\lambda_j \in \Gamma : \lambda_j \in \mathrm{interior}[\sigma(T) \setminus \sigma_e(T)]\}$ *is a finite (possible empty) or denumerable sequence whose limit points belong to the boundary $\partial\sigma_e(T)$ of $\sigma_e(T)$;*

(iv) *For each open set Ω such that $\Omega \cap \sigma_e(T) \neq \emptyset$, but $\partial\Omega \cap \sigma_e(T) = \emptyset$,*

$$\operatorname{card}\{j : \lambda_j \in \Omega\} = \aleph_0 .$$

Then, for each $\varepsilon > 0$ it is possible to find $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that

(1) $A = T - K_\varepsilon \in (\mathrm{B}\triangle)$;

(2) $\mathrm{d}(A) = \Gamma$;

(3) $\sigma(A) = [(T) \setminus \mathrm{interior}(\sigma(T) \setminus \sigma_e(T))] \cup \Gamma_0$, $\sigma_e(A) = \sigma_e(T)$;

(4) *if $\Delta_\varepsilon = \{\lambda \in \sigma_0(T) : \mathrm{dist}[\lambda, \sigma_e(T)] \geqslant \varepsilon\}$, then $\mathscr{H}(A; \Delta_\varepsilon) = \mathscr{H}(T; \Delta_\varepsilon)$, and $A \mid \mathscr{H}(A; \Delta_\varepsilon) = T \mid \mathscr{H}(T; \Delta_\varepsilon)$;*

(5) *if* $\lambda \in \sigma_0(T)$, *then* $\dim \mathscr{H}(A; \{\lambda\}) = \dim \mathscr{H}(T; \{\lambda\})$ *and* $A \mid \mathscr{H}(A; \{\lambda\})$ *is similar to* $T \mid \mathscr{H}(T; \{\lambda\})$.

Furthermore,

(6) *if each* $\lambda \in \Gamma_0$ *is associated to a Jordan nilpotent* $J(\lambda)$ *acting on a space of dimension* $\mathrm{d}(\lambda) = \mathrm{card}\{j : \lambda_j = \lambda\}$, *then* K_ε *can be chosen so that* $A - \lambda \mid \mathscr{H}(A; \{\lambda\})$ *is similar to* $J(\lambda)$.

If the size of the compact perturbation is irrelevant, then we have the following analog [9, Corollary 2.5]:

COROLLARY 8. *Let* $T \in (\mathrm{BQT})$, *and let* $\Gamma = \{\lambda_j\}_{j=1}^\infty$ *be a sequence of complex numbers such that all the limit points of* Γ *belong to* $\sigma_e(T)$ *and, moreover, for each open set* Ω *such that* $\Omega \cap \sigma_e(T) = \emptyset$, *but* $\partial\Omega \cap \sigma_e(T) = \emptyset$, $\mathrm{card}\{j;\ \lambda_j \in \Omega\} = \aleph_0$; *then there exists* $K \in \mathscr{K}(\mathscr{H})$ *such that* $A = T - K$ *is bitriangular with diagonal sequence* $\mathrm{d}(A) = \Gamma$.

Conversely, if $C \in \mathscr{K}(\mathscr{H})$ *and* $B = T - C$ *is bitriangular, then the diagonal sequence* $\Gamma = \mathrm{d}(B)$ *satisfies the two conditions mentioned above.*

The proofs of Corollaries 7 and 8 follow from Theorem 6 and ad hoc modifications of the proofs given in [9]. The details are left to the reader.

COROLLARY 9. $(\mathrm{B}\triangle)^0$ *is dense in* (BQT); *furthermore, given* T *in* (BQT) *and* $\varepsilon > 0$, *there exists* $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, *with* $\|K_\varepsilon\| < \varepsilon$, *such that* $A = T - K_\varepsilon \in (\mathrm{B}\triangle)^0$.

Proof. Let $T \in (\mathrm{BQT})$ and let $\varepsilon > 0$ be given. By using Corollary 7, we can find $K_1 \in \mathscr{K}(\mathscr{H})$, with $\|K_1\| < \varepsilon/2$, such that $A_1 = T - K_1 \in (\mathrm{B}\triangle)$ and

$$\sigma_0(T) \subset \mathrm{d}(A) \subset \sigma_0(T) \cup \partial\sigma_e(T).$$

Now it is a straightforward exercise to find $K_2 \in \mathscr{K}(\mathscr{H})$, with $\|K_2\| < \varepsilon/2$, such that $A = T - (K_1 + K_2) \in (\mathrm{B}\triangle)^0$. (Roughly speaking: push the points in the diagonal of A_1 to distinct nearby points in $\mathbf{C} \setminus \sigma_e(A)$.)

Clearly, $K_\varepsilon = K_1 + K_2$ satisfies our requirements. ∎

With Corollary 9, the proof of Theorem 2 follows by straightforward modifications of the proof of Theorem 1:

Instead of $(\mathrm{QT})^n$ defined by (i), (ii) and (iii) as above, consider the subsets $(\mathrm{BQT})^n$ of all those operators $T \in (\mathrm{BQT})$ such that there exist $E, F \in \mathrm{PF}(\mathscr{H})$ satisfying

(i′) $(1 - E)TE = (1 - F)T^*F = 0$,

(ii′) $\sigma(T \mid \mathrm{ran}\,E)$ consists of $\mathrm{rank}\,E$ distinct normal eigenvalues of T, and $\sigma(T^* \mid \mathrm{ran}\,F)$ consists of $\mathrm{rank}\,F$ distinct normal eigenvalues of T^*, and

(iii′) $\|P_nEP_n - P_n\| < 1/n$ and $\|P_nFP_n - P_n\| < 1/n$ $(n = 1, 2, \ldots)$.

Now it is possible to show that (BQT)^n is an open dense subset of (BQT), and

$$(\mathrm{B}\triangle)^0 = \bigcap_{n=1}^{\infty} (\mathrm{BQT})^n$$

is a G_δ-dense subset of (BQT). ▨

The details are left to the reader.

REMARKS. (i) The definition of $(\triangle)^0$ is slightly redundant. Indeed, it follows from the last part of the proof of Theorem 1 that (1) follows from (2) and (3) (and therefore, this condition can be eliminated from the definition); more precisely, $(\triangle)^0$ can be re-defined as

$$(\triangle)^0 = \{A \in \mathscr{L}(\mathscr{H}) : (2)\ \dim\ker(A-\lambda) = 1 \text{ for all } \lambda \in \sigma_0(A); \text{ and}$$
$$(3)\ \mathscr{H} = \bigvee \{\ker(A-\lambda) : \lambda \in \sigma_0(A)\}\}.$$

However, the "temporary definition" given in Theorem 1 makes it easier to understand the structure of these operators.

(ii) The same argument (but a longer proof) as in Theorem 1 can be used to show that

$$(\triangle)^1 = \{A \in \mathscr{L}(\mathscr{H}) : (3')\ \mathscr{H} = \bigvee \{\ker(A-\lambda)^k : \lambda \in \sigma_0(A),\ k = 1, 2, \ldots\}\}$$
$$(= \{A \in \mathscr{L}(\mathscr{H}) : (3'')\, \mathscr{H} = \bigvee \{\mathscr{H}(A; \{\lambda\}) : \lambda \in \sigma_0(A)\}\})$$

is also a G_δ-dense subset of (QT) consisting exclusively of triangular operators.

(iii) Similarly, $(\mathrm{B}\triangle)^0$ can be re-defined as

$$(\mathrm{B}\triangle)^0 = \{A \in \mathscr{L}(\mathscr{H}) : (2')\ \dim\ker(A-\lambda) = 1 \text{ for all } \lambda \in \sigma_0(A); \text{ and}$$
$$\mathscr{H} = \bigvee \{\ker(A-\lambda) : \lambda \in \sigma_0(A)\} =$$
$$= \bigvee \{\ker(A-\lambda)^* : \lambda \in \sigma_0(A)\}\},$$

because (2′) and (3) actually imply (1), (2), (3), and the fact that this class is a subset of $(\mathrm{B}\triangle)$. (To see this, use Corollary 5.)

Furthermore, as in the case of (QT), the proof of Theorem 2 can be modified to show that

$$(\mathrm{B}\triangle)^1 = \{A \in \mathscr{L}(\mathscr{H}) : (3')\, \mathscr{H} = \bigvee \{\ker(A-\lambda)^k : \lambda \in \sigma_0(A), k = 1, 2, \ldots\} =$$
$$= \bigvee \{\ker[(A-\lambda)^*]^k : \lambda \in \sigma_0(A),\ k = 1, 2, \ldots\}\}$$
$$(= \{A \in \mathscr{L}(\mathscr{H}) : (3'')\ \mathscr{H} = \bigvee \{\mathscr{H}(A; \{\lambda\}) : \lambda \in \sigma_0(A)\} =$$
$$= \bigvee \{\mathscr{H}(A^*; \{\bar{\lambda}\}) : \lambda \in \sigma_0(A)\}\})$$

is a G_δ-dense subset of (BQT) included in $(\mathrm{B}\triangle)$.

(iv) If $A \in (\mathrm{B}\triangle)$, Γ is a subset of $d(A)$, and

$$\mathcal{M}(\Gamma) = \bigvee \{\ker(A-\lambda)^k : \lambda \in \Gamma,\ k = 1, 2, \ldots\},$$

then $\mathcal{M}(\Gamma) + \mathcal{M}(d(A) \setminus \Gamma)$ is dense in $\mathcal{H}$ (Corollary 5).

CONJECTURE. $\mathcal{M}(\Gamma) \cap \mathcal{M}(d(A) \setminus \Gamma) = \{0\}$ for all $\Gamma \subset d(A)$.

4. SOME DENSE FIRST CATEGORY F_σ SUBSETS OF (BQT)

First of all, observe that $\mathrm{Alg}(\mathcal{H}) = \bigcup_{n=1}^{\infty} \mathrm{Alg}_n(\mathcal{H})$, where

$$\mathrm{Alg}_n(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : p(T) = 0 \text{ for some monic polynomial } p \text{ of degree } n\}$$

($n = 1, 2, \ldots$), and $\mathrm{Alg}_n(\mathcal{H})$ is a *closed* subset of $\mathcal{L}(\mathcal{H})$. Indeed, if $\|T_k - T\| \to 0$ ($k \to \infty$) and $p_k(T_k) = 0$ for some monic polynomial p_k of degree n, it is easily seen that $p_k(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j^{(k)})$ can be chosen so that $\lambda_j^{(k)} \in \sigma(T_k)$ for all $j = 1, 2, \ldots, n$; by passing, if necessary, to a subsequence, we can directly assume that $\lambda_j^{(k)} \to \lambda_j$ ($k \to \infty$) for suitably chosen points λ_j ($j = 1, 2, \ldots, n$). Now it is straightforward to check that

$$p(T) = 0, \quad \text{where } p(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j):$$

that is, $T \in \mathrm{Alg}_n(\mathcal{H})$.

On the other hand, for every A in $\mathrm{Alg}_n(\mathcal{H})$, $\sigma(A)$ has at most n points, and there exist $\lambda \in \sigma(A)$ and an infinite dimensional subspace $\mathcal{M}$ such that

$$A = \begin{pmatrix} \lambda & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H} \ominus \mathcal{M} \end{matrix}.$$

If M is any normal operator acting on $\mathcal{M}$, then

$$A(M) = \begin{pmatrix} \lambda + M & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H} \ominus \mathcal{M} \end{matrix} \in (\mathrm{BQT}),$$

and $\|A - A(M)\| = \|M\|$ can be chosen arbitrarily small, but $A(M) \notin \mathrm{Alg}(\mathcal{H})$ for any M with an infinite spectrum.

It readily follows that $\mathrm{Alg}_n(\mathcal{H})$ has empty interior in (BQT), and therefore $\mathrm{Alg}(\mathcal{H})$ is a first category F_σ-dense subset of (BQT).

Similarly, we can write

$$\mathrm{Alg}(\mathscr{H}) + \mathscr{K}(\mathscr{H}) = \bigcup_{n=1}^{\infty} [\mathrm{Alg}_n(\mathscr{H}) + \mathscr{K}(\mathscr{H})],$$

and show that $[\mathrm{Alg}_n(\mathscr{H}) + \mathscr{K}(\mathscr{H})]$ is closed in $\mathscr{L}(\mathscr{H})$ and nowhere dense in (BQT), and that $\mathrm{Alg}(\mathscr{H}) + \mathscr{K}(\mathscr{H})$ is a first category F_σ-dense subset of (BQT). (Use the results of [14]:

$$\mathrm{Alg}_n(\mathscr{H}) + \mathscr{K}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : p(T) \in \mathscr{K}(\mathscr{H}) \text{ for some monic polynomial } p \text{ of degree } n\}.)$$

In [12], the author proved that, if

$$JA(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : \inf_{X \in \mathscr{L}(\mathscr{H})} \|(TX - XT) - 1\| = 0\},$$

then

$$[(\mathrm{BQT}) \cap JA(\mathscr{H})] + \mathscr{K}(\mathscr{H})$$

is a G_δ-dense subset of (BQT).

Since $\{[(\mathrm{BQT}) \cap JA(\mathscr{H})] + \mathscr{K}(\mathscr{H})\} \cap \mathrm{SNor}(\mathscr{H}) = \emptyset$ [12], it readily follows that $\mathrm{SNor}(\mathscr{H})$ is a first category dense subset of (BQT). A fortiori, the dense subset $\mathrm{SFNor}(\mathscr{H})$ is also first category in (BQT).

Thus, it only remains to show that $\mathrm{SNor}(\mathscr{H})$ and $\mathrm{SFNor}(\mathscr{H})$ are F_σ's in $\mathscr{L}(\mathscr{H})$.

(a) $\mathrm{SNor}(\mathscr{H}) = \bigcup_{n=1}^{\infty} \mathrm{SNor}_n(\mathscr{H})$, where

$$\mathrm{SNor}_n(\mathscr{H}) = \{T = WNW^{-1} : N \text{ is normal, } W \text{ is invertible, and } \|W\| \cdot \|W^{-1}\| \leqslant n\}.$$

Suppose $\|T_k - T\| \to 0$ $(k \to \infty)$ for some sequence $\{T_k = W_k N_k W_k^{-1}\}_{k=1}^{\infty}$ in $\mathrm{SNor}_n(\mathscr{H})$. Since

$$\|(\lambda - T_k)^{-1}\| \leqslant n\|(\lambda - N_k)^{-1}\| = n/(\mathrm{dist}[\lambda, \sigma(N_k)]) = n/(\mathrm{dist}[\lambda, \sigma(T_k)])$$

for all $\lambda \notin \sigma(T_k) = \sigma(N_k)$, it readily follows from [11, Lemma 1] that

$$\sigma(T_k) \to \sigma(T) \quad (k \to \infty)$$

in the Hausdorff metric; moreover, a closer analysis indicates that there exist normal operators M_k such that

$$\sigma(M_k) = \sigma(T) \quad (k = 1, 2, \ldots),$$

$$\dim\ker(\lambda - M_k) = \dim\ker(\lambda - T)$$

for each isolated point λ of $\sigma(T)$ (for all $k = 1, 2, \ldots$), and

$$\|T_k - W_k M_k W_k^{-1}\| \leqslant n\|N_k - M_k\| \to 0 \quad (k \to \infty).$$

Thus, $\|T_k' - T\| \to 0$ $(k \to \infty)$, where $T_k' = W_k M_k W_k^{-1}$.

Let $N = M_1$. It is not difficult to check (see, e.g., [7, Chapter 4]) that

$$M_k \in \{UNU^* : U \text{ is unitary}\}^- \quad (k = 1, 2, \ldots).$$

Therefore, we can find unitary operators U_k $(k = 1, 2, \ldots)$ such that the sequence

$$T_k'' = W_k(U_k N U_k^*)W_k^{-1} = (W_k U_k)N(W_k U_k)^{-1}$$

converges in the norm to T.

Since

$$\|W_k U_k\| \cdot \|(W_k U_k)^{-1}\| = \|W_k\| \cdot \|W_k^{-1}\| \leqslant n,$$

it readily follows from D. W. Hadwin's results [4, Theorem 3.5] that $T \in \mathrm{SNor}(\mathscr{H})$.

We conclude that

$$\mathrm{SNor}(\mathscr{H}) = \bigcup_{n=1}^{\infty} [\mathrm{SNor}_n(\mathscr{H})]^-$$

is an F_σ subset. (Indeed, a more careful analysis of Hadwin's proof, together with the result of [16], indicate that $\mathrm{SNor}_n(\mathscr{H})$ is actually a closed subset of $\mathscr{L}(\mathscr{H})$.)

(b) $\mathrm{SFNor}(\mathscr{H}) = \bigcup_{n=1}^{\infty} \mathrm{SFNor}_n(\mathscr{H})$, where

$$\mathrm{SFNor}(\mathscr{H}) = \{T \in \mathrm{SNor}_n(\mathscr{H}) : \sigma(T) \text{ has at most } n \text{ points}\} =$$

$$= \mathrm{SNor}_n(\mathscr{H}) \cap \mathrm{Alg}_n(\mathscr{H})$$

is a closed subset of $\mathscr{L}(\mathscr{H})$. Therefore, $\mathrm{SFNor}(\mathscr{H})$ is an F_σ subset.

The proof of Theorem 3 is now complete. ☐

After this article was submitted, the author has written two closely related papers in this area: "Most quasidiagonal operators are not block-diagonal", and

"The Jordan form of a bitriangular operator" (joint work with K. R. Davidson). In this latter paper, the authors show that a bitriangular operator is quasisimilar to a denumerable direct sum of translations of nilpotent Jordan cells.

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