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# NILPOTENT OPERATORS AND SYSTEMS OF PROJECTIONS

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## INTRODUCTION

In [2], Corach, Porta and Recht studied the properties of $A_p = \{a \in A : p(a) = 0\}$, where $A$ is a complex Banach algebra and $p \in \mathbf{C}[x]$ has simple roots. This set is identified with the universal model $\left\{(P_1, \ldots, P_n) \in A^n : P_i^2 = P_i, P_iP_j = 0 \text{ when } i \neq j \text{ and } \sum_{i=1}^{n} P_i = 1\right\}$, the set of systems of projectors of $A$; an action of the group of invertible elements of $A$ on both sets is introduced: $u*a = uau^{-1}$, where $u$ is invertible and $a$ is either a root or a system of projectors. The main result establishes that $A_p$, with the action mentioned, is a union of homogeneous spaces.

This is no longer the case when the polynomial has multiple roots.

The analysis is reduced, essentially, to problems concerning nilpotent elements. Even so the problem remains difficult in the case of a well known example, the algebra $L(H)$ of all bounded linear operators acting on a complex, separable Hilbert space $H$, which is the case we will study here.

In [1], Fialkow and Herrero, proved that the map $\pi_T : G(H) \to S(T)$ ($\pi_T(U) = UTU^{-1}$) has local cross sections if and only if $T$ is similar to a nice Jordan nilpotent ([1], 16.1; see Section 3 below). Here $G(H)$ denotes the group of invertible elements of $L(H)$ and $S(T) = \{T' : T' \text{ is similar to } T, \text{ i.e., } T' = UTU^{-1}$ for some $U \in G(H)\}$ is the similarity orbit of $T$.

In this paper we consider the application $\varphi$, which assigns to a nilpotent operator $T$ of order $n$, the system of selfadjoint projectors wich triangularizes $T$, through the canonical decomposition of $T$, that is

$$\varphi(T) = (P_{\ker T}, P_{\ker T^2 \ominus \ker T}, \ldots, P_{\ker T^n \ominus \ker T^{n-1}}),$$

where $P_M$ denotes the orthogonal projection onto the subspace $M$.

We will define an action of $G(H)$ on these $n$-tuples in terms of a map (the Gram — Schmidt map; Section 1), that "orthonormalizes" a given system of idempotents, and is compatible with the maps $\pi_T$ and $\varphi$ (see Section 2).

The main result of this paper is the characterization of the continuity points of $\varphi$, considered as an application from the set $N_n(H)$ of nilpotent operators of order $n$ into the set of selfadjoint systems of projectors: $\varphi$ is continuous at $T$ if and only if $T$ is similar to $q_j \oplus q_n^{(\infty)}$, $0 \leqslant j \leqslant n-1$ (where $q_i$ is the Jordan nilpotent cell in $C^{i \times i}$ (Theorem 3.9)). Therefore the points of continuity of $\varphi$ form an open dense subset of $N_n(H)$. Moreover, $\varphi \mid S(T)$ is continuous if $\pi_T$ has local cross sections (Proposition 3.2).

In Section 4 we study the analogous problem in the Calkin algebra. A map $\tilde{\varphi}$ is defined from the orbit of similarity of $\pi(q_n^{(\infty)})$ (the class of $q_n^{(\infty)}$, which is open and dense in $N_n(A(H))$), into the systems of projectors of the Calkin algebra. This is done in a natural way, with the coordinates of $\tilde{\varphi}$ in the $C^*$-algebra generated by $\pi(q_n^{(\infty)})$ and $\pi(I)$. Unfortunately, there is no natural way, in general, to define $\tilde{\varphi}$ outside $S(\pi(q_n^{(\infty)}))$.

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NOTATIONS

In what follows, $K(H)$ will denote the ideal of compact operators and $A(H) = L(H)/K(H)$ is the Calkin algebra.

$$N_n(H) = \{T \in L(H) : T^n = 0 \text{ and } T^{n-1} \neq 0\}$$

$$I_n(H) = \left\{(P_1, \ldots, P_n) \in L(H)^n : P_i^2 = P_i,\ P_iP_j = 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{n} P_i = I\right\}$$

and

$$P_n(H) = \{(P_1, \ldots, P_n) \in I_n(H) : P_i^* = P_i\}.$$

$I_n(H)$ and $P_n(H)$ are closed subsets of $L(H)^n$ with its usual topology. $N_n(A(H))$, $I_n(A(H))$ and $P_n(A(H))$ are analogously defined. By subspace we mean a closed linear manifold of a Hilbert space. Given a subspace $S \subset H$ we denote by $P_S$ the projector (i.e. selfadjoint idempotent) onto $S$.

$q_k \in L(\mathbf{C}^k)$ is the operator defined by

$$q_k(e_i) = \begin{cases} 0 & \text{if } i = 1 \\ e_{i-1} & \text{if } i \neq 1 \end{cases},$$

where $\{e_i\}_{1 \leq i \leq k}$ is the canonical orthonormal basis of $\mathbf{C}^k$.

If $A \in L(H)$, then $\operatorname{rank}(A) = \dim R(A)$ and $\operatorname{nul}(A) = \dim \ker(A)$.

## 1. ORTHONORMALIZATION OF IDEMPOTENTS

Let $I(H) = \{P \in L(H) : P^2 = P\}$ be the set of all idempotents of $L(H)$, and let $P(H) = \{P \in I(H) : P^* = P\}$ be the selfadjoint ones.

Given an idempotent $P \in I(H)$, there is a formula for the projector onto the range of $P$, $P_{R(P)} \in P(H)$, given by

$$P_{R(P)} = PP^*(I - (P - P^*)^2)^{-1}. \tag{1.1}$$

Indeed if

$$P = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} \begin{matrix} R(P) \\ R(P)^\perp \end{matrix},$$

then

$$P^* = \begin{pmatrix} I & 0 \\ A^* & 0 \end{pmatrix}, \quad PP^* = \begin{pmatrix} I + AA^* & 0 \\ 0 & 0 \end{pmatrix},$$

$$P - P^* = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \quad \text{and} \quad (P - P^*)^2 = \begin{pmatrix} -AA^* & 0 \\ 0 & -A^*A \end{pmatrix}.$$

Hence $I - (P - P^*)^2 = \begin{pmatrix} I + AA^* & 0 \\ 0 & I + A^*A \end{pmatrix}$ which is clearly invertible, and so (1.1) follows immediately. Thus, we can define a surjective map

$$\mathrm{GS}_2 : I(H) \to P(H) \quad \text{by } \mathrm{GS}_2(P) = P_{R(P)}, \quad P \in I(H).$$

More generally, we can define the map

$$\mathrm{GS}_n : I_n(H) \to P_n(H)$$

(which we call the Gram—Schmidt map) as follows: if $P = (P_1, \ldots, P_n) \in I_n(H)$, then

$$(1.2) \quad \begin{aligned} &1)\ \mathrm{GS}_n(P)_1 = \mathrm{GS}_2(P_1) \\ &2)\ \mathrm{GS}_n(P)_j = \mathrm{GS}_2\left(\sum_{i=1}^{j} P_i\right) - \mathrm{GS}_2\left(\sum_{i=1}^{j-1} P_i\right), \quad 1 < j \leqslant n. \end{aligned}$$

REMARK 1.3. $R(P_1 + \ldots + P_j) = R(\mathrm{GS}_n(P)_1 + \ldots + \mathrm{GS}_n(P)_j)$, $1 \leqslant j \leqslant n$. It is completely apparent that $\mathrm{GS}_n(P) \in P_n(H)$.

It is clear that the map $\mathrm{GS}_n(P)$ is continuous, and that through (1.1) it can be described in terms of $P_i$, $P_i^*$ and $I$ in such a way that $\mathrm{GS}_n(P)$ lies in the $C^*$-algebra generated by $\{P_i,\ 1 \leqslant i \leqslant n\}$.

Next we will see that $P \in I_n(H)$ and $\mathrm{GS}_n(P)$ are "similar", that is:

PROPOSITION 1.4. *If* $P = (P_1, \ldots, P_n) \in I_n(H)$ *and* $\mathrm{GS}_n(P) = (Q_1, \ldots, Q_n)$ *then* $U = \sum_{i=1}^{n} Q_i P_i \in G(H)$ *and* $Q_i = U P_i U^{-1}$, $1 \leqslant i \leqslant n$.

*Proof.* It is easy to see that $Q_i U = U P_i$, $1 \leqslant i \leqslant n$, so it suffices to prove that $U$ is invertible. First we will show that

$$P_j Q_i = 0 = Q_j P_i \quad \text{if } i < j.$$

If $i = 1$, $P_j Q_1 = P_j P_1 Q_1 = 0$. By induction, we deduce that if $j > i > 1$, then

$$P_j Q_i = P_j Q_i + P_j \sum_{l=1}^{i-1} Q_l = P_j \sum_{l=1}^{i} Q_l =$$

$$= P_j \left( \sum_{k=1}^{i} P_k \right) \left( \sum_{l=1}^{i} Q_l \right) = 0.$$

Similarly $Q_j P_i = 0$. So if $i < j$, $(Q_j P_i)^2 = 0$ and

$$I = \left( \sum_{i=1}^{n} Q_i \right) \left( \sum_{j=1}^{n} P_j \right) = \sum_{l=1}^{n} Q_l P_l + \sum_{i \neq j} Q_i P_j;$$

then

$$U = \sum_{l=1}^{n} Q_l P_l = I - \sum_{i \neq j} Q_i P_j;$$

But

$$(I - \sum_{i<j} Q_i P_j)(I + Q_1 P_2) = I - \sum_{\substack{i<j \\ (i,j) \neq (1,2)}} Q_i P_j$$

By induction, we obtain $UR_1 = I - \sum_{2 \leq i < j} Q_i P_j$, where $R_1 = (I + Q_1 P_2)(I + Q_1 P_3) \ldots (I + Q_1 P_n)$.

By another inductive argument, we see that

$$UR_1 R_2 \ldots R_{n-1} = I,$$

where

$$R_i = (I + Q_i P_{i+1})(I + Q_i P_{i+2}) \ldots (I + Q_i P_n) \quad (1 \leq i \leq n-1).$$

Thus $U$ is right invertible.

A similar argument shows that

$$R_1 R_2 \ldots R_{n-1} U = I.$$

Hence, $U$ is invertible ($U^{-1} = R_1 R_2 \ldots R_{n-1}$). ▨

REMARK 1.5. 1) The map $\mathrm{GS}_n : I_n(H) \to P_n(H)$ is a homotopic equivalence. Indeed, let

$$P = (P_1, \ldots, P_n) \in I_n(H);$$

by an argument analogous to the proof of Proposition 1.4,

$$\psi_t(P) = \sum_{i=1}^{n} [tP_i + (1-t)\mathrm{GS}_n(P)_i] P_i \in G(H), \quad t \in [0, 1].$$

Then $F(t, P) = (\psi_t(P) P_1 \psi_t(P)^{-1}, \ldots, \psi_t(P) P_n \psi_t(P)^{-1})$ is the required homotopy.

2) The invertible $U$ of 1.4 belongs to the $C^*$-algebra generated by $P_i$, $1 \leq i \leq n$.

Given a fixed $P = (P_1, \ldots, P_n) \in I_n(H)$, we define the map

$$\mathscr{S}_p : G(H) \to I_n(H), \quad \mathscr{S}_p(V) = (VP_1 V^{-1}, \ldots, VP_n V^{-1}), \quad V \in G(H).$$

Clearly, $\mathscr{S}_P$ is continuous. It is known that $\mathscr{S}_P(G(H)) = S(P)$ is the component of $P$ in $I_n(H)$ (cf. [2]). Moreover, given $P' = (P'_1, \ldots, P'_n) \in \mathscr{S}_P(G(H))$ there is a neighborhood $\mathscr{U}$ of $P'$ in $I_n(H)$ such that for $Q = (Q_1, \ldots, Q_n)$ in $\mathscr{U}$, $\sum_{i=1}^{n} Q_i P'_i \in G(H)$ because $\sum_{i=1}^{n} Q_i P'_i$ is close to $I$.

PROPOSITION 1.6. *Let $P \in I_n(H)$. Then for every $P' \in \mathscr{S}_P(G(H))$ there exists a neighborhood $\mathscr{U}_{P'} \subset I_n(H)$ and a local cross section $s_{P'} : \mathscr{U}_{P'} \to G(H)$, that is, $\mathscr{S}_P \circ s_{P'} = \mathrm{id}\, \mathscr{U}_{P'}$.*

*Proof.* If $P' = P$, we define

$$s_P : \mathscr{U}_P \to G(H) \quad \text{by } s_P(Q) = \sum_{i=1}^{n} Q_i P_i .$$

If $P' \neq P$, let $U \in G(H)$ such that $P' = UPU^{-1}$. Here we take $\mathscr{U}_{P'} = U\mathscr{U}_P U^{-1}$ and $s_{P'}(Q) = Us_P(U^{-1}QU)$. It is straightforward to verify that this is, indeed, a local cross section. The continuity is trivial.

Finally observe that, from (1.2) and (1.6), for a fixed $n$-tuple $P \in P_n(H)$ we obtain a map $\overline{\mathscr{S}}_P : G(H) \to P_n(H)$, as follows:

By conjugation of $P$ we obtain elements of $I_n(H)$, not necessarily in $P_n(H)$. Define

(1.7) $$\overline{\mathscr{S}}_P(V) = \mathrm{GS}_n(\mathscr{S}_P(V)), \quad V \in G(H).$$

(1.8) Clearly, the local cross section $s_{P'}$ of (1.6) ($P' \in \mathscr{S}_P(G(H))$) restricted to $\mathscr{V}_{P'} = \mathscr{U}_{P'} \cap P_n(H)$, is a local cross section for the map $\mathscr{S}_P$.

## 2. NILPOTENT OPERATORS AND $P_n(H)$

Given a nilpotent operator $T \in N_n(H) = \{T \in L(H) : T^n = 0,\ T^{n-1} \neq 0\}$, there is a decomposition of $H$ into $n$ subspaces $H_1, \ldots, H_n$ such that the matrix of $T$ in that decomposition is strictly upper triangular

$$T = \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ & 0 & A_{23} & \cdots & \vdots \\ & & 0 & \ddots & \vdots \\ & & & 0 & A_{n-1,n} \\ & & & & 0 \end{bmatrix}$$

where $A_{ij} = P_{H_i} T P_{H_j}$, $H_1 = \ker T$ and $H_{i+1} = \ker T_{i+1} \ominus \ker T^i$, $1 \leqslant i \leqslant n-1$. This is known as the canonical decomposition of $T$.

So, to each $T \in N_n(H)$ we can assign in a natural way an element of $P_n(H)$. We define:

(2.1) $$\varphi : N_n(H) \to P_n(H) \text{ by}$$
$$\varphi(T) = (P_{\ker T}, P_{\ker T^2 \ominus \ker T}, \ldots, P_{\ker T^n \ominus \ker T^{n-1}}).$$

In this paper we will study some facts about this application $\varphi$. Clearly $P_{\ker T^{i+1} \ominus \ker T^i} = P_{\ker T^{i+1}} - P_{\ker T^i}$. So in many cases it will suffice to study the properties of $T \to P_{\ker T^i}$ as those of $\varphi_1(T) = P_{\ker T}$.

REMARKS 2.2. 1) It follows from spectral theory that $\varphi_1(T) = P_{\ker T} = \chi_{\{0\}}(T^*T)$ (cf. [5], 12.29).

2) If $P = (P_1, \ldots, P_n) \in P_n(H)$ and there exists $T \in N_n(H)$ with $\varphi(T) = P$, then

$$\operatorname{rank}(P_1) \geqslant \operatorname{rank}(P_2) \geqslant \ldots \geqslant \operatorname{rank}(P_n) \neq 0.$$

Indeed from the fact that $T(\ker T^i) \subset \ker T^{i-1}$, $2 \leqslant i \leqslant n$, and $\ker T^i = \ker T^{i-1} \oplus R(P_i)$, we have $R(P_i) \cong \ker T^i \ominus \ker T^{i-1}$. Hence the application $T : \ker T^i \to \ker T^{i-1}$ induces $\bar{T} : \ker T^i \ominus \ker T^{i-1} \to \ker T^{i-1} \ominus \ker T^{i-2}$ which is a monomorphism of vector spaces. So $\operatorname{rank}(P_i) \leqslant \operatorname{rank}(P_{i-1})$, $2 \leqslant i \leqslant n$.

Now we define $P_n^+(H) = \{(P_1, \ldots, P_n) \in P_n(H) : \operatorname{rank}(P_n) \neq 0, \operatorname{rank}(P_i) \leqslant \operatorname{rank}(P_{i-1}),\ 2 \leqslant i \leqslant n\}$. It is not difficult to check that:

PROPOSITION 2.3. *The image of $\varphi$ coincides with $P_n^+(H)$.*

Given a fixed $T \in N_n(H)$, we have a natural map

(2.4) $$\pi_T : G(H) \to N_n(H) \ \text{ by } \ \pi_T(V) = VTV^{-1}, \quad V \in G(H).$$

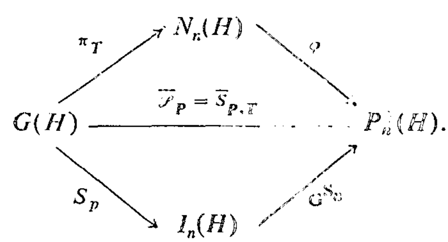
Combining this map with $\varphi$ we obtain, for a fixed $P = (P_1, \ldots, P_n) \in P_n^+(H)$ and a $T \in N_n(H)$ such that $\varphi(T) = P$, an application

(2.5) $$\bar{S}_{P,T} : G(H) \to P_n^+(H) \ \text{ by } \ \bar{S}_{P,T}(V) = \varphi(\pi_T(V)).$$

At first sight this application depends on $T$ (the mention of $P$ in the notation is redundant, for $P$ depends on $T$).

PROPOSITION 2.6. *The application $\bar{S}_{P,T}$ coincides with $\bar{\mathscr{S}}_P$ (of (1.7)) and so it is continuous and does not depend on $T$ but on $\varphi(T) = P$.*

*Moreover, we can construct the following commutative diagram:*



*Proof.* First we fix $U \in G(H)$.

$$UPU^{-1} = (UP_1U^{-1}, \ldots, UP_nU^{-1}) \in I_n(H),$$

and so

$$\begin{aligned} &R(UP_1U^{-1} + \ldots + UP_iU^{-1}) = \\ (2.7) \qquad &= R(UP_1U^{-1}) \oplus \ldots \oplus R(UP_iU^{-1}), \quad 1 \leqslant i \leqslant n \end{aligned}$$

(the direct sum is not necessarily orthogonal).

If $i = 1$, $\ker(UTU^{-1}) = U\ker T = UR(P_1) = R(UP_1U^{-1})$ so $\varphi_1(UTU^{-1}) = P_{R(UP_1U^{-1})} = \mathrm{GS}_n(\mathcal{S}_P(U))_1$ as desired.

If $i > 1$, let $M_i = \ker(UT^iU^{-1}) \ominus \ker(UT^{i-1}U^{-1})$. So $\varphi_i(UTU^{-1}) = P_{M_i}$. But for $1 \leqslant j \leqslant n$ we have

$$\ker(UT^jU^{-1}) = U(R(P_1 + \ldots + P_j)) = R(UP_1U^{-1} + \ldots + UP_jU^{-1}) =$$
$$= R(UP_1U^{-1}) \oplus \ldots \oplus R(UP_jU^{-1})$$

by (2.7). Then

$$M_i = \ker(UT^iU^{-1}) \ominus \ker(UT^{i-1}U^{-1}) = [R(UP_1U^{-1}) \oplus \ldots \oplus R(UP_iU^{-1})] \ominus$$
$$\ominus [R(UP_1U^{-1}) \oplus \ldots \oplus R(UP_{i-1}U^{-1})] = R(\mathrm{GS}_n(\mathcal{S}_P(U))_i)$$

which completes the proof. ▨

COROLLARY 2.8. *Given $P \in P_n^+(H)$ and $T \in N_n(H)$ with $\varphi(T) = P$, there exist a neighborhood $V_P \subset P_n^+(H)$ of $P$ and a local cross section $t_P : V_P \to N_n(H)$, of $\varphi$, i.e. $\varphi \circ t_P = \mathrm{id}_{V_P}$, such that $t_P(P) = T$ and $t_P(V_P) \subset S(T)$.*

*Proof.* We define $t_P = \pi_T \circ s_P$, where $s_P : V_P \to G(H)$ is the local cross section given in (1.8). Clearly, $t_P$ is continuous and $\varphi \circ \pi_T \circ s_P = \check{S}_{P,T} \circ s_P = \check{S}_P \circ s_P = \mathrm{id}_{V_P}$ by (2.7). It is immediate that $t_P(P) = T$ and $t_P(V_P) \subset S(T)$. ▨

## 3. POINTS OF CONTINUITY OF THE CANONICAL DECOMPOSITION OF A NILPOTENT OPERATOR

In this section we will see that $\varphi : N_n(H) \to P_n^+$ is not continuous in general. We will characterize the points of continuity of $\varphi$, which form an open dense subset of $N_n(H)$.

It is not difficult to find examples where the application $\varphi_1$, $\varphi_1(T) = P_{\ker(T)}$ is not continuous (and therefore neither is $\varphi$).

Let $(\alpha_n)_{n\in\mathbf{N}}$ be a sequence verifying

$$\begin{cases}\alpha_n \to 0 & (n\to\infty)\\ \alpha_n = 0 & \text{if } n \text{ is odd.}\end{cases}$$

$D = D_{(\alpha_n)}$ is the diagonal operator with weights $\alpha_n$ on $\ell^2(N)$ for a given orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$, that is, $De_k = \alpha_k e_k$. Let $S$ be the unilateral shift and $T = SD$. It is clear that $T^2 = 0$. Now let $T_n = P_n T$, where $P_n$ is the projector onto the first $n$ coordinates. Then $T_n^2 = 0$, and

$$\|T - T_n\| \leqslant \sup_{k\geqslant n} |\alpha_k| \xrightarrow[(n\to\infty)]{} 0.$$

But $I - P_{\ker(T)}$ has infinite rank, and $I - P_{\ker(T_n)}$ $(n\in\mathbf{N})$ have finite rank, so that $P_{\ker(T_n)} \nrightarrow P_{\ker(T)}$.

PROPOSITION 3.1. *Let $T\in N_n(H)$ be a nilpotent such that $\varphi$ is continuous at $T$. If $S$ is similar to $T$ then $\varphi$ is continuous at $S$.*

*Proof.* Let $(S_k)_{k\geqslant 1}$ be a sequence such that $S_k\in N_n(H)$, $S_k \xrightarrow[(k\to\infty)]{} S$. If $S = UTU^{-1}$ with $U\in G(H)$, then it is clear that $U^{-1}S_kU \xrightarrow[(k\to\infty)]{} T$. Hence $\varphi(U^{-1}S_kU) \xrightarrow[(k\to\infty)]{} \varphi(T)$.

On the other hand, using that

$$\varphi(S_k) = \mathrm{GS}_n(U\varphi(U^{-1}S_kU)U^{-1})$$

and

$$\varphi(S) = \mathrm{GS}_n(U\varphi(T)U^{-1})$$

and the fact that $\mathrm{GS}_n$ is continuous. we conclude that $\varphi(S_k) \xrightarrow[(k\to\infty)]{} \varphi(S)$. ▨

Now we will state a result relating the continuity of $\varphi$ restricted to $S(T)$ and the existence of local cross sections of the map $\pi_T : G(H)\to S(T)$.

PROPOSITION 3.2. *If $\pi_T : G(H)\to S(T)$ has local cross sections, then $\varphi \mid S(T)$ is continuous.*

*Proof.* By Proposition 3.1, it suffices to prove that $\varphi\mid S(T)$ is continuous at $T$. Let $W_kTW_k^{-1} \xrightarrow[(k\to\infty)]{} T$ and let $\mathcal{U}_T$ be a neighborhood of $T$ in $S(T)$ with $\mathrm{sc}_T : \mathcal{U}_T \to G(H)$ a local cross section. Then for $k$ large, $W_kTW_k^{-1}\in\mathcal{U}_T$. So by (2.6) we have

$$\varphi(W_kTW_k^{-1}) = \varphi\circ\pi_T(\mathrm{sc}_T(W_kTW_k^{-1})) = \bar{S}_P(\mathrm{sc}_T(W_kTW_k^{-1}))$$

where both $\bar{S}_P$ and $\mathrm{sc}_T$ are continuous. ▨

COROLLARY 3.3. *If $S(T)$ is open in $N_n(H)$ and has local cross sections, then $\varphi$ is continuous at $T$.*

L. A. Fialkow and D. A. Herrero ([1], Theorem 16.1) have characterized the operators such that $S(T)$ has local cross sections. If $T$ is nilpotent, then $S(T)$ has local cross sections if and only if $T$ is similar to $\bigoplus_{j=1}^{n} q_j^{(\alpha_j)}$, where $0 \leqslant \alpha_j < \infty$ for every $j$ but one index $j_0$, and $\alpha_{j_0} = +\infty$.

This leads us to study the problem of the continuity of $\varphi$ for certain classes of nilpotents, esspecially Jordan and nice Jordan nilpotents, whose similarity orbits have convenient properties.

We define

$$J_n(H) = \left\{T \in N_n(H) : T \sim \bigoplus_{j=1}^{n} q_j^{(\alpha_j)},\ 0 \leqslant \alpha_j \leqslant \infty\right\}$$

Jordan nilpotents of order $n$,

$$NJ_n(H) = \left\{T \in N_n(H) : T \sim q_n^{(\infty)} \oplus \bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)},\ \sum_{j=1}^{n-1} \alpha_j < \infty\right\}$$

nice Jordan nilpotents of order $n$,

$$Q_j = q_j \oplus q_n^{(\infty)}, \quad 1 \leqslant j \leqslant n-1, \quad Q_0 = q_n^{(\infty)}$$

and

$$N_n^+(H) = \{T \in L(H) : T \in K(H) \text{ and } T^{n-1} + T^* \text{ is Fredholm}\}.$$

The following is well known ([3], Chapters 7, 8).

THEOREM 3.4. i) *$J_n(H)$ is dense in $N_n(H)$,* ([3], Theorem 7.15).

ii) $q_1^{(\infty)} \oplus q_n^{(\infty)} \in S(Q_j)^-$, $0 \leqslant j \leqslant n-1$.

iii) $S(q_1^{(\infty)} \oplus q_n^{(\infty)})^- = N_n(H) - N_n^+(H)$, ([3], Corollary 8.2).

iv) *$S(Q_i)^- \cap N_n^+$ consists of operators similar to $q_n^{(\infty)} \oplus \bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)}$ such that $\sum_{j=1}^{n-1} j\alpha_j < \infty$, $\sum_{j=1}^{n-1} j\alpha_j = nk + i$ for some $k \geqslant 0$, and* $\operatorname{rank}\left[\left(\bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)}\right)^m\right] \leqslant \operatorname{rank}[(q_i \oplus q_n^{(k)})^m]$, $1 \leqslant m \leqslant n$.

v) *$S(Q_j)$ is open in $S(Q_j)^-$,* ([3], Proposition 8.19).

REMARK 3.5. 1) From (3.4 iv)) it follows that

$$S(Q_j)^- \cap N_n^+(H) \subset NJ_n(H).$$

2) If $T \sim q_n^{(\infty)} \oplus \bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)}$, $d_T = \sum_{j=1}^{n-1} j\alpha_j < \infty$ and $d_T = nk + r$, then $T \in S(Q_r)^-$, moreover $\bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)} \in S(q_r \oplus q_n^{(k)})^-$ in $L(\mathbf{C}^{d_T})$, ([3], Chapter 2).

COROLLARY 3.6. *$\bigcup_{j=0}^{n-1} S(Q_j)$ is dense in $N_n(H)$.*

*Proof.* Using 3.4(i) it suffices to prove that $J_n(H) \subset \left[\bigcup_{j=0}^{n-1} S(Q_j)\right]^-$, and using 3.4 (ii) and (iii) it is enough to see that

$$J_n(H) \cap N_n^+(H) = NJ_n(H) \subset \left[\bigcup_{j=0}^{n-1} S(Q_j)\right]^-,$$

which follows by (3.4). ▨

COROLLARY 3.7. *$S(Q_j)$ is open in $N_n(H)$, $0 \leqslant j \leqslant n-1$.*

*Proof.* This is a consequence of (3.4(v)), (3.6) and the fact that $Q_j \notin S(Q_r)^-$ if $r \neq j$, which holds by (3.4(iv)). ▨

COROLLARY 3.8.

$$N_n(H) \cap N_n^+(H) = NJ_n(H).$$

*Proof.* Let $T \in N_n(H) \cap N_n^+(H)$; by (3.6) there exists $j$, $0 \leqslant j \leqslant n-1$, such that $T \in S(Q_j)^- \cap N_n^+(H)$. Then, using (3.5(1)), $T \in NJ_n(H)$. The other inclusion is clear. ▨

Now we are able to characterize the points of continuity of $\varphi$.

THEOREM 3.9. *Let $T \in N_n(H)$. Then $\varphi$ is continuous at $T$ if and only if*

$$T \sim q_j \oplus q_n^{(\infty)} \quad \textit{for some } j,\ 0 \leqslant j \leqslant n-1.$$

*In particular, the points of continuity of $\varphi$ form an open dense subset of $N_n(H)$.*

*Proof.* By (3.3) and (3.7) it is clear that if $T \in \bigcup_{j=0}^{n-1} S(Q_j)$, then $\varphi$ is continuous at $T$.

Assume that $T \notin \bigcup_{j=0}^{n-1} S(Q_j)$, we will consider two cases.

*First:* $T \notin N_n^+(H)$. By (3.4(iii)), there exists a sequence $\{W_k\}_{k \in \mathbf{N}}$ in $G(H)$ such that

$$W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1} \xrightarrow[(k \to \infty)]{} T.$$

Let $H_1$ be the subspace of $H$ associated to $q_1^{(\infty)}$ and $H_2$ the one associated to $q_n^{(\infty)}$. We define, for a fixed $S \subset H_1$, $\dim S = 2$, the operators $R_m = (1/m)q_2 \oplus \oplus q_1^{(\infty)} \oplus q_n^{(\infty)}$ where $q_2$ acts on $S$, $q_1^{(\infty)}$ on $H_1 \ominus S$ and $q_n^{(\infty)}$ on $H_2$.

For a fixed $k \in \mathbf{N}$

$$W_k R_m W_k^{-1} \xrightarrow[(m\to\infty)]{} W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1}.$$

Let $B_k = W_k R_{m_k} W_k^{-1}$: if $m_k$ is large enough, then

$$\| B_k - W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1} \| < 1/k.$$

It is clear that $B_k \xrightarrow[(k\to\infty)]{} T$. But $\ker B_k \subset \ker(W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1})$ with codimension 1 for every $k \in \mathbf{N}$. And then

$$\| P_{\ker[W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1}]^{-1}} - P_{\ker B_k} \| = 1$$

for every $k \in \mathbf{N}$.

Now, because $W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1} \xrightarrow[(k\to\infty)]{} T$ if $P_{\ker[W_k(q_1^{(\infty)} \oplus q_n^{(\infty)})W_k^{-1}]} \not\xrightarrow[(k\to\infty)]{} P_{\ker T}$, then $\varphi$ is not continuous at $T$. Otherwise, $P_{\ker B_k} \not\xrightarrow[(k\to\infty)]{} P_{\ker T}$, so $\varphi$ is not continuous in this case.

*Second*: $T \in N_n(H) \cap N_n'(H)$. By (3.8) we know that $T \in NJ_n(H)$. Suppose that $T \sim q_n^{(\infty)} \oplus \bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)} = Q$ with $d_Q = \sum_{j=1}^{n-1} j\alpha_j < \infty$ and either $\alpha_j > 1$ for some index $j$ or there is more than one $\alpha_j$ with $\alpha_j > 0$ (i.e. $Q$ is not similar to any of the $Q_r$, $0 \leqslant r \leqslant n-1$).

Using (3.1) it suffices to see that $\varphi$ is not continuous at $Q$. Let

$$d_Q = \sum_{j=1}^{n-1} j\alpha_j = nk + r, \quad 0 \leqslant k, \ 0 \leqslant r < n$$

then, because of (3.5(iii)), $Q \in S(Q_r)^-$, $Q \notin S(Q_r)$ and $\bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)} \in S(q_r \oplus q_n^{(k)})^-$.

Suppose $k \neq 0$. Let $\{R_m\} \subset S(q_r \oplus q_n^{(k)})$ be a sequence such that $R_m \xrightarrow[(m\to\infty)]{} \bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)}$. We have that $\left(\bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)}\right)^{n-1} = 0$, but

$$\mathrm{nul}(R_m^{n-1}) = d_Q - k.$$

So $R_m \oplus q_n^{(\infty)} \xrightarrow[(m\to\infty)]{} Q$ and $\ker[(R_m \oplus q_n^{(\infty)})^{n-1}] \subset \ker Q^{n-1}$ with codimension $k > 0$ for every $m \in \mathbf{N}$.

Therefore

$$\|P_{\ker[(R_m \oplus q_n^{(\infty)})^{n-1}]} - P_{\ker Q^{n-1}}\| = 1$$

for every $m \in \mathbf{N}$. It is clear then that $\varphi$ cannot be continuous at $Q$.

If $k = 0$, $\bigoplus_{j=1}^{n-1} q_j^{(\alpha_j)} \in S(q_r)^-$, then $\alpha_j = 0$ for $j \geqslant r$. A formal repetition of the above argument shows that $\|P_{\ker[(R_m \oplus q_n^{(\infty)})^{r-1}]} - P_{\ker(Q^{r-1})}\| = 1$ for every $m \in \mathbf{N}$. Once again, we conclude that $\varphi$ cannot be continuous at $Q$. ▨

## 4. THE CANONICAL DECOMPOSITION IN THE CALKIN ALGEBRA

In this section our aim is to assign an $n$-tuple of selfadjoint "orthogonal" idempotents to a given nilpotent of order $n$ of the Calkin algebra $A(H)$.

Unfortunately, this cannot always be done in a natural way (see Remark 4.7, below).

However, there is a way to define the desired map for a class of nilpotents which is dense and open in $N_n(A(H))$. To do so we need some results about similarity orbits in the Calkin algebra.

LEMMA 4.1. (See [3], Proposition 7.22). *Let $T \in L(H)$, with $\pi(T) \in N_n(A(H))$. The following are equivalent:*

*(i) $T \sim J + K$, where $J \in NJ_n(H)$ and $K$ is compact.*

*(ii) $T = WJV + K$, where $J \sim q_n^{(\infty)}$, $W, V$ are Fredholm operators such that $\pi(W)\pi(V) = \pi(V)\pi(W) = \pi(1)$ and $K$ is compact.*

*(iii) $T^j + T^{*n-j}$ is Fredholm for every $1 \leqslant j \leqslant n-1$.*

*(iv) $T^j + T^{*n-j}$ is Fredholm for some $j$, $1 \leqslant j \leqslant n-1$.*

*In other words, we have that*

$$\pi(NJ_n(H)) = S(\pi(q_n^{(\infty)})) = \{t \in N_n(A(H)) : t^{n-1} + t^* \in G(A(H))\}.$$

PROPOSITION 4.2. *If $T \in NJ_n(H)$, then*

$$P_{R(T^{n-k})} = T^{n-k}T^{*n-k}[(T^{n-k} + T^{*k})(T^{*n-k} + T^k) + P]^{-1} \tag{4.3}$$

*where $P = P_{\ker T^k \ominus R(T^{n-k})}$ has finite rank.*

*Proof.* Let $A = (T^{n-k} + T^{*k})(T^{*n-k} + T^k)$. $A$ is a selfadjoint Fredholm operator and $A + P_{\ker(A)} \in G(H)$; moreover, $A$ is the sum of the operators $T^{n-k}T^{*n-k}$

and $T^{*k}T^k$, whose ranges are orthogonal. Thus, $\ker(A) = \ker(T^k) \cap \ker(T^{*n-k}) = \ker(T^k) \ominus R(T^{n-k})$, and

$$I = (A + P)(A + P)^{-1} =$$

$$= T^{n-k}T^{*n-k}(A + P)^{-1} + P(A + P)^{-1} + T^{*k}T^k(A + P)^{-1}.$$

It is easy to see that

$$T^{n-k}T^{*n-k}P = PT^{n-k}T^{*n-k} = 0$$

$$T^{*k}T^kP = PT^{*k}T^k = 0$$

and that $T^{n-k}T^{*n-k}$ and $T^{*k}T^k$ commute with $A$.

So we have

$$T^{n-k}T^{*n-k}(A + P)^{-1} + T^{*k}T^k(A + P)^{-1} + P = I$$

where the three summands are selfadjoint and the product of any two different is zero.

We conclude that $T^{n-k}T^{*n-k}(A + P)^{-1}$ is the projector onto $R(T^{n-k})$.

The fact that $\operatorname{rank}(P) < \infty$ is clear by using that $A$ is Fredholm. ∎

REMARK 4.4. The same formula holds if $T \in J_n(H)$.

We are now able to define the analogous of the mapping $\varphi$ for $t \in S(\pi(q_n^{(\infty)}))$. Let

$$\gamma_k(t) = t^{n-k}t^{*n-k}(t^{*n-k} + t^k)^{-1}(t^{n-k} + t^{*k})^{-1}$$

and

$$\tilde{\varphi}(t) = (\gamma_1(t), \gamma_2(t) - \gamma_1(t), \ldots, \gamma_n(t) - \gamma_{n-1}(t)). \tag{4.5}$$

THEOREM 4.6. *The map*

$$\tilde{\varphi} : S(\pi(q_n^{(\infty)})) \to P_n(A(H))$$

*is continuous. If $t \in S(\pi(q_n^{(\infty)}))$, $T \in N_n(H)$ and $\pi(T) = t$, then*

$$\tilde{\varphi}(t) = (\pi\varphi_1(T), \ldots, \pi\varphi_n(T)).$$

*Proof.* The continuity of $\tilde{\varphi}$ is obvious.

If $T \in N_n(H)$, $\pi(T) = t \in S(\pi(q_n^{(\infty)}))$, by using (3.8) we deduce that $T \in NJ_n(H)$ so that (4.3) holds.

Therefore

$$P_{\ker(T^k)} - P_{R(T^{n-k})} = P_{\ker T^k \ominus R(T^{n-k})}$$

has finite rank.

Then

$$\gamma_k(t) = \pi(P_{R(T^{n-k})}) = \pi(P_{\ker T^k})$$

so that $\tilde{\varphi}_k(t) = \pi\varphi_k(T)$. ▨

REMARKS 4.7. 1) The $n$-tuple of "orthogonal" idempotents $\tilde{\varphi}(t)$ of Theorem 4.6 triangularizes $t$.

2) If $T \in S(q_n^{(\infty)})$, then

$$P_{\ker T^k} = T^{n-k}T^{*n-k}(T^{*n-k} + T^k)^{-1}(T^{n-k} + T^{*k})^{-1}, \quad 1 \leqslant k \leqslant n-1,$$

because $R(T^{n-k}) = \ker T^k$ and $T^{n-k} + T^{*k}$ is invertible.

3) If $T \in J_n(H)$, then $\varphi_i(T)$ belongs to the $C^*$-algebra $C^*(T)$ generated by $T$ and $I$. Indeed, using ([3], 7.13), $R(T^k)$ is closed, so 0 is an isolated point of $\sigma(T^{*k}T^k)$ $1 \leqslant k \leqslant n-1$, then

$$P_{\ker T^k} = \chi_{\{0\}}(T^{*k}T^k) \in C^*(T).$$

4) Let $N_n(A(H)) = \{a \in A(H): a^n = 0, \ a^{n-1} \neq 0\}$. According to C. Olsen, [4], given $t \in N_n(A(H))$ there exists $T \in N_n(H)$ such that $\pi(T) = t$, where $\pi: L(H) \to$ $\to A(H)$ is the natural projection.

One would like to define

$$\tilde{\varphi}(t) = (\pi\varphi_1(T), \ldots, \pi\varphi_n(T)). \tag{4.8}$$

But this is not well-defined in general.

If 0 is an isolated point of the essential spectrum of $T^{*k}T^k$ for every $k$, $1 \leqslant k \leqslant$ $\leqslant n-1$, then it follows from [3], 7.17, that $T$ can be chosen so that 0 is an isolated point of the spectrum of $T^{*k}T^k$ (this is equivalent to saying that $T \in J_n(H)$). In this case by using the same argument as in 4.7.3, we can define

$$\tilde{\gamma}_k(t) = -\frac{1}{2\pi i}\int_{\partial B_k} (\lambda - t^{*k}t^k)^{-1}\,\mathrm{d}\lambda$$

where $B_k$ is an open neighborhood of 0 in $C$ such that $\overline{B}_k \cap \sigma(t^{*k}t^k) = 0$. Moreover

$$\tilde{\varphi}(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t) - \tilde{\gamma}_1(t), \ldots, \tilde{\gamma}_{n-1}(t) - \tilde{\gamma}_{n-2}(t), 1 - \tilde{\gamma}_{n-1}(t))$$

which verifies (4.8), for every $T \in J_n(H)$ such that $\pi(T) = t$. Also it is clear that $\tilde{\varphi}(t) \in C^*(t)$.

A formal repetition of the proof of Theorem 3.9 shows that $\tilde{\varphi}$ is continuous at $t$ if and only if $T \in NJ_n(H)$ (equivalently, $t \in S(\pi(q_n^{(\infty)}))$).

5) On the other hand, if 0 is not an isolated point of the essential spectrum of $T^{*k}T^k$ for some $k$, $1 \leqslant k \leqslant n-1$, then there is no way to define $T$, with $\pi(T) = t$, so that $P_{\ker T^h} \in C^*(T)$, and no natural way to define $\tilde{\varphi}(t)$. For instance, it is a classi-

cal exercise that, whatever definition we choose for $T$, there exist three pairwise orthogonal idempotents, $P_1$, $P_2$ and $P_3$ of infinite rank such that $P_1 + P_2 + P_3 = P_{\ker T^{n-k}}$, $P_j P_{\ker T^{n-k}} = P_{\ker T^{n-k}} P_j = P_j$ $(j = 1,2,3)$ and $P_j(T^k T^{*k})$ and $(T^k T^{*k})P_j$ are compact operators for $j = 1, 2$. Clearly, there is no natural definition for $\tilde{\varphi}(t)$, in this case.

(Consider, for instance, the order two nilpotent

$$T = \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} \begin{matrix} L^2([0,1],\, \mathrm{d}x) \\ L^2([0,1],\, \mathrm{d}x) \end{matrix},$$

where $H$ is the "multiplication by $x$" on $L^2([0,1], \mathrm{d}x)$.)

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