J. OPERATOR THEORY
21(1989), 41–67

AN L^1-TYPE FUNCTIONAL CALCULUS FOR THE LAPLACE OPERATOR IN $L^p(\mathbf{R})$

WERNER J. RICKER

1. INTRODUCTION

The Laplace operator $L = -\mathrm{d}^2/\mathrm{d}x^2$ in $L^p(\mathbf{R})$, $1 < p < \infty$, with domain

$$\mathscr{D}(L) = \{f \in L^p(\mathbf{R});\ f' \in AC,\ f'' \in L^p(\mathbf{R})\}$$

is a closed, densely defined operator with spectrum $\sigma(L) = [0,\infty)$; here AC is the space of functions on $\mathbf{R}$ which are absolutely continuous on bounded intervals. For $0 < \omega < \pi$, let S_ω denote the open cone $\{z \in \mathbf{C}\setminus\{0\};\ |\arg(z)| < \omega\}$. Then it is known that L, considered in $L^p(\mathbf{R})$ for $1 < p < \infty$, has an $\mathscr{H}^\infty(S_\omega)$-functional calculus for every $0 < \omega < \pi$, [9]. Operators admitting an $\mathscr{H}^\infty(S_\omega)$-functional calculus for some $0 < \omega < \pi$ have recently been investigated in the Hilbert space setting by A. McIntosh [8] and A. Yagi [11]. The situation in Banach spaces is less clear and more complex; some positive results due to M. Cowling can be found in [2].

Getting back to the Laplace operator, it is known for $p = 2$ that L is self-adjoint and so L admits a far more extensive functional calculus than that based on $\mathscr{H}^\infty(S_\omega)$ for $0<\omega<\pi$. Indeed, it is possible to form a continuous linear operator $\psi(L)$ for every bounded Borel function ψ on $\mathbf{R}^+ = [0,\infty)$ via the spectral integral

$$\psi(L) = \int_{\sigma(L)} \psi(\lambda)\mathrm{d}E(\lambda),$$

where E is the resolution of the identity for L. The question arises of whether this is still the case for $p \neq 2$, that is, whether L is a scalar-type spectral operator (in the sense of N. Dunford [3])? Then L would again admit a functional calculus with respect to the bounded Borel functions on $\mathbf{R}^+$. Unfortunalely, this is not so [9]. Nevertheless, the aim of this note is to show that something positive can still be said when $p \neq 2$.

Fix $1 < p < \infty$ and let $\mathcal{M}^{(p)}$ denote the Banach algebra of all (equivalence classes of) functions $m: \mathbf{R} \to \mathbf{C}$ which are p-multipliers, equipped with the usual multiplier norm $\|\cdot\|_p$. That is, $\|m\|_p = \|m(D)\|$ where the operator norm of $m(D)$ is taken in the space $L^p(\mathbf{R})$. Here $D = -\mathrm{i}d/dx$ is the closed, densely defined operator in $L^p(\mathbf{R})$ with domain

$$\mathscr{D}(D) = \{f \in L^p(\mathbf{R}); f \in AC, f' \in L^p(\mathbf{R})\}$$

and $m(D)$ is the bounded operator in $L^p(\mathbf{R})$ specified by the p-multiplier m, that is,

$$(m(D)f)\hat{} = m\hat{f}, \quad f \in L^2 \cap L^p(\mathbf{R}),$$

where $\hat{}$ denotes the Fourier transform. Let $\gamma(x) = x^2$ for each $x \in \mathbf{R}$. Denote by $\mathscr{A}^{(p)}(\mathbf{R}^+)$ the Boolean algebra consisting of those measurable sets $E \subseteq \mathbf{R}^+$ for which $\chi_E \circ \gamma$ belongs to $\mathcal{M}^{(p)}$. So, if $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, then the bounded operator in $L^p(\mathbf{R})$ given by

$$P(E) = (\chi_E \circ \gamma)(D)$$

is a projection which commutes with L (see Section 4).

Let $\mathscr{L}(L^p(\mathbf{R}))$ denote the space of all continuous linear operators of $L^p(\mathbf{R})$ into itself. Then $P: \mathscr{A}^{(p)}(\mathbf{R}^+) \to \mathscr{L}(L^p(\mathbf{R}))$ is a multiplicative, finitely additive set function. For $p = 2$ the Boolean algebra of sets $\mathscr{A}^{(2)}(\mathbf{R}^+)$ is actually the σ-algebra of all measurable subsets of $\mathbf{R}^+$ and $P(\cdot)$, which is σ-additive with respect to the strong operator topology in $\mathscr{L}(L^2(\mathbf{R}))$, is the resolution of the identity for the self-adjoint operator L. However, if $p \neq 2$, then the range $\{P(E); E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ of $P(\cdot)$ is not uniformly bounded in $\mathscr{L}(L^p(\mathbf{R}))$, [9; Lemma 2]. Accordingly, the domain $\mathscr{A}^{(p)}(\mathbf{R}^+)$ cannot be enlarged so that $P(\cdot)$ has an extension (with values in $\mathscr{L}(L^p(\mathbf{R}))$) to a σ-additive spectral measure on the measurable subsets of $\mathbf{R}^+$, [3; XVII Lemma 3.3 and Corollary 3.10]. Despite this difficulty the operator L can still be considered, if suitably interpreted, as an unbounded scalar-type spectral operator in some well--defined sense; see (1) and (2) below. But, more importantly, it turns out that most of the well known results for scalar-type spectral operators remain valid for L. The reason is that it is still possible, due to some recent work of I. Kluvánek [6], to associate with P some sort of an L^1-type space of functions and an "integration process" such that the integration mapping $f \mapsto \int_0^\infty f\,dP$ is a continuous algebra homomorphism. This L^1-type functional calculus for L, as distinct from other functional calculi such as those based on $\mathrm{BV}(\mathbf{R}^+)$ or some $\mathscr{H}^\infty(S_\omega)$, say, is more in the spirit of the classical theory of scalar-type spectral operators. Namely, the operators in the range of the L^1-type functional calculus are approximable by linear combinations of disjoint projectors which commute with L.

It turns out, for each $\lambda \in \rho(L) = \mathbf{C}\setminus\sigma(L)$, that the function $R_\lambda : x \mapsto (x-\lambda)^{-1}$, $x \in \mathbf{R}^+$, belongs to this L^1-space and $\int_0^\infty R_\lambda \mathrm{d}P = (L-\lambda)^{-1}$. In particular, the operator $\int_0^\infty (f/g)\mathrm{d}P$ agrees with the usual definition of $(f/g)(L) = f(L)g(L)^{-1}$, specified via the partial fraction decomposition of f/g, whenever f and g are polynomials such that $\deg(f) \leqslant \deg(g)$ and g has no zeros in $\sigma(L)$. Actually more is true; the L^1-functions with respect to P include the elements of $\mathscr{H}^\infty(S_\omega)$ in the sense that the restriction to $\mathbf{R}^+$ of any element of $\mathscr{H}^\infty(S_\omega)$, $0 < \omega < \pi$, belongs to the L^1-space of P. Furthermore, for fixed $0 < \omega < \pi$, the imbedding which sends $f \in \mathscr{H}^\infty(S_\omega)$ to its restriction on $\mathbf{R}^+$ is continuous from $\mathscr{H}^\infty(S_\omega)$ into $L^1(P)$.

For each $n = 1, 2, \ldots$, the function $\lambda^{(n)} : x \mapsto x\chi_{[0,n]}(x)$, $x \in \mathbf{R}^+$, is P-integrable and

$$(1) \qquad \mathscr{D}(L) = \left\{ f \in L^p(\mathbf{R})\,;\, \lim_{n\to\infty} \left(\int_0^\infty \lambda^{(n)}\mathrm{d}P \right) f \text{ exists in } L^p(\mathbf{R}) \right\}$$

with

$$(2) \qquad Lf = \lim_{n\to\infty} \left(\int_0^\infty \lambda^{(n)}\mathrm{d}P \right) f, \quad f \in \mathscr{D}(L).$$

In addition, the Boolean algebra of projections $\{P(E)\,;\, E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ satisfies all the essential properties of a classical resolution of the identity except the boundedness requirement. For example, if $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, then the spectrum of the restriction of L to the range of $P(E)$ is contained in the closure $\bar{E}$ of E. Furthermore, an operator $T \in \mathscr{L}(L^p(\mathbf{R}))$ commutes with L, in the sense that $T(\mathscr{D}(L)) \subseteq \mathscr{D}(L)$ and $TLf = LTf$ for every $f \in \mathscr{D}(L)$, if and only if $TP(E) = P(E)T$ for every $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$. It is also the case that the bicommutant of L is precisely the weak-operator closed algebra generated by $\{P(E)\,;\, E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$. This is, perhaps, somewhat surprising since the Boolean algebra $\{P(E)\,;\, E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ is not bounded.

The paper is organised as follows. In Section 2 we give a brief outline of the salient features of Kluvánek's theory [6] and establish a few related results which will be needed in the sequel. Section 3 is devoted to a detailed study of the set function $P(\cdot)$ and its associated L^1-space. In the final section we establish the identities (1) and (2) along with the other results indicated above.

2. CLOSABLE SPECTRAL SET FUNCTIONS

In this section we wish to outline briefly a self-contained summary of those aspects of the integration theory developed in [6] which are needed in the sequel.

Let Γ be an algebra of subsets of some set Ω. The vector space of all Γ-simple functions is denoted by $\mathrm{sim}(\Gamma)$. If X is a Banach space, then an additive and multiplicative map $P: \Gamma \to \mathscr{L}(X)$ such that $P(\Omega) = I$, where $\mathscr{L}(X)$ is the space of all bounded linear operators of X into itself, is called a spectral set function. In this case P has an unique additive and multiplicative extension to $\mathrm{sim}(\Gamma)$ defined in an obvious way; its value at an element $f \in \mathrm{sim}(\Gamma)$ is denoted by $\int_\Omega f\,\mathrm{d}P$ or simply by $P(f)$. In particular, $P(\chi_U) = P(U)$ for every $U \in \Gamma$. By a P-null set is meant any subset $U \subseteq \Omega$ for which there exist sets $U_j \in \Gamma$ such that $P(U_j) = 0$, for every $j = 1, 2, \ldots$, and $U \subseteq \bigcup_{j=1}^{\infty} U_j$. For a function f on Ω, let

$$\|f\|_\infty = \inf\{\sup\{|f(\omega)|;\ \omega \in \Omega \setminus U\};\ U \in \mathscr{N}\},$$

where $\mathscr{N}$ is the family of all P-null sets. If $\|f\|_\infty = 0$, then f is said to be P-null. Let $L^\infty(P)$ be the algebra of all functions f on Ω such that for every $\varepsilon > 0$ there exists a function $g \in \mathrm{sim}(\Gamma)$ for which $\|f - g\|_\infty < \varepsilon$.

We come now to the central notion in [6]. A spectral set function $P: \Gamma \to \mathscr{L}(X)$ is said to be *closable* if

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} \int_\Omega f_j\,\mathrm{d}P \right\| = 0$$

whenever $f_j \in \mathrm{sim}(\Gamma)$, $j = 1, 2, \ldots$, are functions satisfying

$$\sum_{j=1}^{\infty} \left\| \int_\Omega f_j\,\mathrm{d}P \right\| < \infty \tag{3}$$

and $\sum_{j=1}^{\infty} f_j(\omega) = 0$ for every $\omega \in \Omega$ such that

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty. \tag{4}$$

If this is the case, then a function f on Ω is said to be *P-integrable* if, and only if, there exist functions $f_j \in \text{sim}(\Gamma)$, $j = 1, 2, \ldots$, satisfying (3), such that

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega) \tag{5}$$

holds for every $\omega \in \Omega$ for which the inequality (4) holds. The closability of P guarantees that the operator $\sum_{j=1}^{\infty} \int_\Omega f_j \, dP$, denoted by $P(f)$ or $\int_\Omega f \, dP$, is a well-defined element of $\mathscr{L}(X)$. That is, it is independent of the particular sequence of functions $\{f_j\}_{j=1}^{\infty}$ in $\text{sim}(\Gamma)$ satisfying the above stated properties. In this case, $\left\{\omega \in \Omega;\ \sum_{j=1}^{\infty} |f_j(\omega)| = \infty\right\}$ is a P-null set [6; Proposition 2]. The space of all P-integrable functions is denoted by $L(P)$. It turns out that $L(P) \subseteq L^\infty(P)$ and $\|f\|_\infty \leqslant \|P(f)\|$, for every $f \in L(P)$. In addition, if $f, g \in L(P)$, then also $fg \in L(P)$ and $P(fg) = P(f)P(g)$. That is, $L(P)$ is an algebra of functions. If $f \in L(P)$, then

$$\|P(f)\| = \inf \sum_{j=1}^{\infty} \|P(f_j)\|, \tag{6}$$

where the infimum is taken over all choices of functions $f_j \in \text{sim}(\Gamma)$, $j = 1, 2, \ldots$, satisfying condition (3), such that the equality (5) holds for every $\omega \in \Omega$ for which the inequality (4) does [6; Lemma 3]. We note that the Beppo-Levi theorem is valid. That is, if $\{f_j\}_{j=1}^{\infty} \subseteq L(P)$ are functions satisfying (3) and f is a function on Ω such that the equality (5) holds for every $\omega \in \Omega$ for which the inequality (4) holds, then $f \in L(P)$ and $\int_\Omega f \, dP = \sum_{j=1}^{\infty} \int_\Omega f_j \, dP$, [6; Proposition 4]. However, the Dominated Convergence theorem may not be valid in this setting; see Remark 1(iii) below.

Concerning the spectrum, it is the case that

$$\sigma\left(\int_\Omega f \, dP\right) = \bigcap_{U \in \mathcal{N}} \overline{\{f(\omega);\ \omega \in \Omega \setminus U\}}, \quad f \in L(P). \tag{7}$$

Since the functional $f \mapsto \left\|\int_\Omega f \, dP\right\|$ is a seminorm on $L(P)$ it is possible to form the associated normed space, denoted by $L^1(P)$, by identifying any two elements of $L(P)$ for which this seminorm vanishes on their difference. It turns out that $L^1(P)$ is complete and the integration mapping

$$f \mapsto \int_\Omega dP, \quad f \in L(P),$$

induces an isomorphism of the Banach algebra $L^1(P)$ onto the uniformly closed algebra in $\mathscr{L}(X)$ generated by $\{P(U);\ U \in \Gamma\}$. This algebra is denoted by $\langle P \rangle$. Using the fact that $\|f\|_\infty = 0$ holds for a function $f \in L(P)$ if, and only if, $P(f) = 0$, it follows that $L^1(P)$ can be realized as equivalence classes (modulo P-a.e.) of P-integrable functions on Ω. If $f \in L(P)$, we will also denote $\|P(f)\|$ by $\|f\|_1$ or $\|f\|_{L^1(P)}$.

Fix $1 < p < \infty$ and let $X = L^p(\mathbf{R})$. Let $\mathscr{A}^{(p)}$ be the family of those measurable sets $E \subseteq \mathbf{R}$ for which $\chi_E \in \mathscr{M}^{(p)}$. Then $\mathscr{A}^{(p)}$ is a Boolean algebra of sets and $Q\colon \mathscr{A}^{(p)} \to \mathscr{L}(L^p(\mathbf{R}))$ defined by

$$Q(E) = \chi_E(D), \quad E \in \mathscr{A}^{(p)},$$

is a closable spectral set function with the property that a set $U \subseteq \mathbf{R}$ is Q-null if, and only if, it is null with respect to Lebesgue measure on $\mathbf{R}$, [6; Proposition 12]. Since there is no possibility of confusion, once $p \in (1, \infty)$ is fixed, we do not indicate the dependence on p in the notation for Q.

A function f of bounded variation on $\mathbf{R}$ has a decomposition of the form $f_1 + f_2 + f_3$ with $f_1 \in AC$, f_2 continuous and singular (i.e. its derivative is zero a.e.) and f_3 a jump function. If f vanishes at a point (or at $-\infty$), then there is an unique decomposition of this type with all three components f_1, f_2 and f_3 vanishing at that point. If f_2 is identically zero, then f is called non-singular.

LEMMA 1. ([6; Proposition 16]). *Let $p \in (1, \infty)$. If f is a non-singular function of bounded variation on $\mathbf{R}$ such that $f(-\infty) = 0$, then f is Q-integrable.*

Let $f\colon \mathbf{R} \to \mathbf{C}$ be a function. Then $\bar{f}$ and $\tilde{f}$ denote the functions $x \mapsto \overline{f(x)}$, $x \in \mathbf{R}$, (the bar denotes complex conjugation) and $x \mapsto f(-x)$, $x \in \mathbf{R}$, respectively. If $f \in \mathscr{M}^{(p)}$, then $\bar{f}$ and $\tilde{f}$ are also p-multipliers and $\|\bar{f}\|_p = \|f\|_p = \|\tilde{f}\|_p$. In particular, if $E \in \mathscr{A}^{(p)}$, then the set $-E = \{-t;\ t \in E\}$ also belongs to $\mathscr{A}^{(p)}$ since $\tilde{\chi}_E = \chi_{-E}$.

LEMMA 2. *Let $p \in (1, \infty)$.*

(i) *If $f \in L(Q)$, then also $\bar{f} \in L(Q)$ and $\|\bar{f}\|_{L^1(Q)} = \|f\|_{L^1(Q)}$. In particular, both* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are Q-integrable.*

(ii) *If $f \in L(Q)$, then also $\tilde{f} \in L(Q)$ and $\|\tilde{f}\|_{L^1(Q)} = \|f\|_{L^1(Q)}$.*

(iii) *If $f \in L(Q)$, then $f \in \mathscr{M}^p$ and the operator $\int_{\mathbf{R}} f\,dQ$ coincides with the multiplier operator $f(D)$ as defined in Section* 1.

Proof. (i) Let $f = \sum_{k=1}^{n} \alpha_k \chi_{E(k)}$ where the $E(k)$ are pairwise disjoint elements of $\mathscr{A}^{(p)}$. Then $\bar{f} = \sum_{k=1}^{n} \bar{\alpha}_k \chi_{E(k)}$ belongs to $\mathrm{sim}(\mathscr{A}^{(p)}) \subseteq L(Q)$ and $\int_{\mathbf{R}} \bar{f}\,dQ =$

$= \sum_{k=1}^{n} \bar{\alpha}_k Q(E(k))$. But, it is readily checked that $\sum_{k=1}^{n} \bar{\alpha}_k Q(E(k)) = \bar{f}(D)$ where $\bar{f}(D)$ is the multiplier operator corresponding to $\bar{f}$ (see Section 1). It follows from the comments prior to the lemma and the definition of $|||\cdot|||_p$ that

$$\text{(8)} \qquad \|\bar{f}\|_{L^1(Q)} = \|\bar{f}(D)\| = |||\bar{f}|||_p = |||f|||_p = \|f(D)\| = \|f\|_{L^1(Q)}.$$

If now $f \in L(Q)$, then there exist functions $f_j \in \operatorname{sim}(\mathscr{A}^{(p)})$, $j = 1, 2, \ldots$, satisfying (3) — with Q in place of P and $\Omega = \mathbf{R}$ — such that (5) holds for every $\omega \in \Omega$ for which (4) holds. Then $\{\bar{f}_j\}_{j=1}^{\infty} \subseteq \operatorname{sim}(\mathscr{A}^{(p)})$ and $\overline{f(\omega)} = \sum_{j=1}^{\infty} \bar{f}_j(\omega)$ whenever $\sum_{j=1}^{\infty} |\overline{f_j(\omega)}| < \infty$. It follows from (8) that $\sum_{j=1}^{\infty} \left\| \int_{\mathbf{R}} \bar{f}_j \mathrm{d}Q \right\| < \infty$ and hence, $\bar{f} \in L(Q)$. The equality of norms $\|\bar{f}\|_{L^1(Q)} = \|f\|_{L^1(Q)}$ follows from (6) and (8).

The final statement of part (i) is clear from the linearity of integral and the identities $\operatorname{Re}(f) = (1/2)(f + \bar{f})$ and $\operatorname{Im}(f) = (1/2)\mathrm{i}(\bar{f} - f)$.

(ii) follows from a similar argument as for (i) using the identity $Q(-E) = \tilde{\chi}_E(D)$, $E \in \mathscr{A}^{(p)}$.

(iii) There exist functions $f_j \in \operatorname{sim}(\mathscr{A}^{(p)}) \subseteq \mathscr{M}^{(p)}$ such that $\sum_{j=1}^{\infty} \left\| \int_{\mathbf{R}} f_j \,\mathrm{d}Q \right\| < \infty$ and $f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$ for every $\omega \in \mathbf{R}$ for which $\sum_{j=1}^{\infty} |f_j(\omega)| < \infty$. It was noted earlier that $\left\{ \omega \in \mathbf{R} \,;\, \sum_{j=1}^{\infty} |f_j(\omega)| = \infty \right\}$ is Q-null and hence, also null for Lebesgue measure. So, $\psi_n = \sum_{j=1}^{n} f_j$, $n = 1, 2, \ldots$, converges to f a.e. on $\mathbf{R}$. Since each $\psi_n \in \mathscr{M}^{(p)}$ and

$$\sup_n |||\psi_n|||_p \leqslant \sum_{j=1}^{\infty} |||f_j|||_p = \sum_{j=1}^{\infty} \left\| \int_{\mathbf{R}} f_j \,\mathrm{d}Q \right\| < \infty$$

it follows from a standard result about multipliers that the limit function $f \in \mathscr{M}^{(p)}$ and $\psi_n(D) \to f(D)$ in the weak operator topology. But, $\psi_n(D) = \sum_{j=1}^{n} f_j(D) = \sum_{j=1}^{n} \int_{\mathbf{R}} f_j \,\mathrm{d}Q$. Since $\int_{\mathbf{R}} f \,\mathrm{d}Q = \sum_{j=1}^{\infty} \int_{\mathbf{R}} f_j \mathrm{d}Q$ (by definition) and the series converges in the uniform operator topology, then certainly $\sum_{j=1}^{n} \int_{\mathbf{R}} f_j \,\mathrm{d}Q$, $n = 1, 2, \ldots,$ converges to the operator $\int_{\mathbf{R}} f\,dQ$ in the weak operator topology. It follows that $\int_{\mathbf{R}} f\,dQ = f(D)$.

REMARK 1. (i) It follows from (7) and Lemma 2(i) that the Banach algebra $\langle Q\rangle \subseteq \mathscr{L}(L^p(\mathbf{R}))$ has the property that each of its elements T has a Cartesian decomposition $T = A + iB$ where A and B both belong to $\langle Q\rangle$ and have real spectrum. In particular, $AB = BA$.

(ii) Lemma 2(iii) implies that $\|f\|_{L^1(Q)} = \|f\|_p$ for every $f \in L(Q)$.

(iii) If $p \neq 2$, then the space $L^1(Q)$ does not have the lattice property of a classical L^1-space. Indeed, whenever $f \geqslant 0$ is a Q-integrable function which is not zero almost everywhere there exists a measurable function $g\colon \mathbf{R} \to \mathbf{C}$ such that $|g| \leqslant f$ (a.e.) but g is not Q-integrable. This follows from the inclusion $L(Q) \subseteq \mathscr{M}^{(p)}$ and [5; Theorem 1.12].

We conclude this section with a technical lemma which is needed later. First however, we need some further notation. Let $I(n) = (2^n, 2^{n+1}]$, for every integer $n \in \mathbf{Z}$, in which case $\Delta = \{I(n)\,;\ n \in \mathbf{Z}\} \cup \{-I(n);\ n \in \mathbf{Z}\}$ is a dyadic decomposition of $\mathbf{R}\setminus\{0\}$. If $\varphi\colon \mathbf{R} \to \mathbf{C}$ is a function and J is an interval in $\mathbf{R}$, then $V(\varphi|J)$ denotes the total variation of the restriction $\varphi|J$ of φ to J.

LEMMA 3. *Let $1 < p < \infty$. Let $\psi\colon \mathbf{R}\setminus\{0\} \to \mathbf{R}$ be a symmetric function (i.e. $\psi(-x) = \psi(x)$, $x \in \mathbf{R}\setminus\{0\}$) which is bounded and of class C^∞. Suppose that $\sup\{V(\psi|I(n));\ n \in \mathbf{Z}\} < \infty$ and $\{x \in \overline{I(n)}\,;|\psi(x)| = \alpha\}$ is a finite set for each $n \in \mathbf{Z}$ and $\alpha > 0$. Then ψ is Q-integrable.*

Proof. Since ψ is symmetric it follows that

$$M = \sup\{V(\psi|J);\ J \in \Delta\} < \infty \tag{9}$$

and hence $\psi \in \mathscr{M}^{(p)}$ by the Marcinkiewicz multiplier theorem. We may assume that $p \neq 2$ as the conclusion is obvious in that case. Accordingly, choose $r \in (1, \infty)$ such that p lies between 2 and r, in which case there exists $\theta \in (0, 1)$ such that $p^{-1} = (1/2)\,\theta + (1-\theta)r^{-1}$. Let $\varepsilon > 0$ be given. Since ψ is bounded there exists an integer $k > 0$ such that $\{\psi(x)\,; x \neq 0\} \subseteq [-k\varepsilon, k\varepsilon]$. For every $n \in \mathbf{Z}$, each set $\{x \in \overline{I(n)}\,; |\psi(x)| = s\varepsilon\}$, $s = 1, 2, \ldots, k$, is finite and hence the sets

$$E(n, s)^+ = \{x \in \overline{I(n)}\,; \psi(x) \geqslant s\varepsilon\} \quad \text{and} \quad E(n, s)^- = \{x \in \overline{I(n)}\,; \psi(x) \leqslant -s\varepsilon\}$$

are either empty or the finite union of closed, disjoint intervals contained in $\overline{I(n)}$. Accordingly, the characteristic function $\chi^+_{n,s}$ (resp. $\chi^-_{n,s}$) of the set $(-E(n, s)^+) \cup E(n, s)^+$ intersected with $J(n) = (-I(n)) \cup I(n)$ (resp. $(-E(n,s)^-) \cup E(n,s)^-$

intersected with $J(n)$) is a symmetric r-multiplier for every $n \in \mathbf{Z}$ and $1 \leqslant s \leqslant k$. Define $\varphi_s^{(n)} = \varepsilon\chi_{n,s}^+ - \varepsilon\chi_{n,s}^-$, for each $n \in \mathbf{Z}$ and $1 \leqslant s \leqslant k$, and let $\varphi^{(n)} = \sum_{s=1}^{k} \varphi_s^{(n)}$, $n \in \mathbf{Z}$. Then, for each $n \in \mathbf{Z}$, it is the case that

(i) $\varphi^{(n)}$ and $\varphi_s^{(n)}$, $1 \leqslant s \leqslant k$, are symmetric and supported in $J(n)$,

(ii) $|\varphi^{(n)}(x)| \leqslant |\psi(x)|$ for each $x \in J(n)$,

(iii) $\sup\{|\psi(x) - \varphi^{(n)}(x)|;\ x \in J(n)\} \leqslant \varepsilon$, and

(iv) $V((\varphi^{(n)} - \psi)|\pm I(n)) \leqslant V(\psi|I(n))$ and $V((s\varphi_s^{(n)} - \psi)|\pm I(n)) \leqslant V(\psi|I(n))$, for each $s = 1, \ldots, k$.

It follows from (iv) and subadditivity of total variation that

(v) $V(\varphi^{(n)}) \leqslant 4V(\psi|I(n))$ and $V(s\varphi_s^{(n)}) \leqslant 4V(\psi|I(n))$, for each $n \in \mathbf{Z}$ and $s = 1, \ldots, k$.

Let φ_s be the function on $\mathbf{R}\setminus\{0\}$ specified to be $\varphi_s^{(n)}$ on $J(n)$, $n \in \mathbf{Z}$. The inequalities (v) show that

$$\sup\{V(s\varphi_s^{(n)} \mid J);\ J \in \Delta\} < \infty, \quad 1 \leqslant s \leqslant k,$$

and hence the Marcinkiewicz multiplier theorem (together with (9)) asserts that $s\varphi_s$ is an r-multiplier and there exists a constant α_r, depending only on r, such that

$$\|s\varphi_s\|_r \leqslant \alpha_r\{\|s\varphi_s\|_\infty + \sup_{J\in\Delta} V(s\varphi_s^{(n)}|J)\} \leqslant \alpha_r\{\|\psi\|_\infty + 4M\}.$$

Accordingly, the function φ_s, $1 \leqslant s \leqslant k$, which takes its values in $\{0, \pm\varepsilon\}$, is also an r-multiplier. We remark that if $F(s)^+ = \varphi_s^{-1}(\{\varepsilon\})$ and $F(s)^- = \varphi_s^{-1}(\{-\varepsilon\})$, then

$$\varphi_s = \varepsilon \sum_{n\in\mathbf{Z}} \chi_{n,s}^+ - \varepsilon \sum_{n\in\mathbf{Z}} \chi_{n,s}^- = \varepsilon\chi_{F(s)^+} - \varepsilon\chi_{F(s)^-} = \varphi_s^+ - \varphi_s^-$$

where $\varphi_s^+ = \max\{0, \varphi_s\}$ (resp. $\varphi_s^- = \max\{0, -\varphi_s\}$) denotes the positive (resp. negative) part of φ_s, $1 \leqslant s \leqslant k$. It is clear from the definition of $\varphi_s^{(n)}$ that $(\varphi_s^{(n)})^+ = \varepsilon\chi_{n,s}^+$ and $(\varphi_s^{(n)})^- = \varepsilon\chi_{n,s}^-$. Since $V((\varphi_s^{(n)})^+)$ and $V((\varphi_s^{(n)})^-)$ do not exceed $V(\varphi_s^{(n)})$, for every $n \in \mathbf{Z}$ and $1 \leqslant s \leqslant k$, it follows from (v) again and the Marcinkiewicz multiplier theorem that φ_s^+ and φ_s^- are also (symmetric) r-multipliers. Accordingly, each φ_s, $1 \leqslant s \leqslant k$, and hence also $\varphi = \sum_{s=1}^{k} \varphi_s$, is a linear combination of symmetric r-multipliers of the form χ_E, $E \in \mathscr{A}^{(p)}$.

It follows from (iii) and the definition of φ that $\|\psi - \varphi\|_\infty \leqslant \varepsilon$ and from (iv) and (9) that

$$\sup\{V((\varphi - \psi)|J);\ J \in \Delta\} \leqslant M.$$

So, the Marcinkiewicz multiplier theorem implies that

$$|||\psi - \varphi|||_r \leqslant \alpha_r(\varepsilon + M). \tag{10}$$

Since $|||\cdot|||_2 = \|\cdot\|_\infty$ we have also

$$|||\psi - \varphi|||_2 \leqslant \varepsilon. \tag{11}$$

Then the logarithmic convexity of the norm in L^q-spaces together with (10) and (11) implies that

$$|||\varphi - \psi|||_p \leqslant \varepsilon^\theta[\alpha_r(\varepsilon + M)]^{1-\theta}.$$

Let ε^* be the solution of $(\varepsilon^*)^\theta[\alpha_r(\varepsilon^* + M)]^{1-\theta} = \varepsilon$, that is $(\varepsilon^*)^{\theta/(1-\theta)}(\varepsilon^* + M) = \alpha_r^{-1}\varepsilon^{1/(1-\theta)}$. The solution exists because the function $f(x)=x^\beta(x + M)$, $x \geqslant 0$, with $\beta = [\theta/(1 - \theta)] > 0$ is strictly increasing, $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Going through the same argument with ε^* in place of ε gives a p-multiplier φ which is a linear combination of symmetric p-multipliers of the form χ_E, $E \in \mathscr{A}^{(p)}$ (i.e. $\varphi \in \operatorname{sim}(\mathscr{A}^{(p)})$) and satisfies (cf. Remark 2(ii))

$$\|\varphi - \psi\|_{L^1(Q)} = |||\varphi - \psi|||_p < \varepsilon.$$

The completeness of $L^1(Q)$ and Remark 1(ii) imply that ψ is Q-integrable. ▨

3. THE SPACE $L^1(P)$

It was noted earlier that L is not a scalar-type spectral operator in $L^p(\mathbf{R})$, $p \neq 2$, in the classical sense. Nevertheless, there is available a large family of projection operators $P(J)$, $J \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, as defined in Section 1, which commute with L (cf. Section 4). We begin by showing that P is a closable spectral set function for which the following preliminary result is needed. If $f \in L(P)$, then the operator $\int_{\mathbf{R}^+} f\mathrm{d}P$ will also be denoted by $\int_0^\infty f\,\mathrm{d}P$.

LEMMA 4. *Let* $1 < p < \infty$. *If* $f \in \operatorname{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$, *then* $f \circ \gamma \in \operatorname{sim}(\mathscr{A}^{(p)})$ *and*

$$\int_0^\infty f\,\mathrm{d}P = \int_{\mathbf{R}} (f \circ \gamma)\mathrm{d}Q.$$

Proof. Let $f = \sum_{k=1}^{n} \alpha_k \chi_{E(k)}$ where the $E(k)$ are pairwise disjoint elements of $\mathscr{A}^{(p)}(\mathbf{R}^+)$ and let $E = \bigcup_{k=1}^{n} E(k)$. Then

$$f \circ \gamma = \sum_{k=1}^{n} \alpha_k \chi_{E(k)^{1/2}} + \sum_{k=1}^{n} \alpha_k \chi_{-E(k)^{1/2}} - f(0)\chi_E(0)\chi_{\{0\}} \tag{12}$$

and hence, $f \circ \gamma \in \operatorname{sim}(\mathscr{A}^{(p)})$. We have used the fact that if J belongs to $\mathscr{A}^{(p)}(\mathbf{R}^+)$ then $J^{1/2}$ and $-J^{1/2}$ belong to $\mathscr{A}^{(p)}$ since $\chi_{J^{1/2}} = (\chi_J \circ \gamma)\chi_{[0,\infty)}$ and $\chi_{-J^{1/2}} = (\chi_J \circ \gamma)\chi_{(-\infty,0]}$. By definition of P it follows that

$$\int_0^\infty f \, \mathrm{d}P = \sum_{k=1}^{n} \alpha_k P(E(k)) = \sum_{k=1}^{n} \alpha_k [Q(E(k)^{1/2}) + Q(-E(k)^{1/2})].$$

But, the right-hand-side is precisely $\int_{\mathbf{R}} (f \circ \gamma) \mathrm{d}Q$; see (12). ▨

THEOREM 1. *Let $1 < p < \infty$. The spectral set function $P: \mathscr{A}^{(p)}(\mathbf{R}^+) \to \mathscr{L}(L^p(\mathbf{R}))$ is closable.*

Proof. That $\mathscr{A}^{(p)}(\mathbf{R}^+)$ is a Boolen algebra of subsets of $\mathbf{R}^+$ follows from the observation that the collection of functions $f: \mathbf{R}^+ \to \mathbf{C}$ for which $f \circ \gamma \in \mathscr{M}^{(p)}$ forms an algebra under pointwise operations. It follows from the formula

$$P(J) = (\chi_J \circ \gamma)(D) = Q(J^{1/2}) + Q(-J^{1/2}), \quad J \in \mathscr{A}^{(p)}(\mathbf{R}^+), \tag{13}$$

and properties of Q that P is a spectral set function.

Suppose that $f_j \in \operatorname{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$, $j = 1, 2, \ldots,$ are functions satisfying (3) and $\sum_{j=1}^{\infty} f_j(\omega) = 0$ for every $\omega \in \mathbf{R}^+$ such that (4) holds. By (3) and Lemma 4 it follows that $\sum_{j=1}^{\infty} \left\| \int_{\mathbf{R}} (f_j \circ \gamma) \mathrm{d}Q \right\| < \infty$. Furthermore, if $u \in \mathbf{R}$ and $\sum_{j=1}^{\infty} |(f_j \circ \gamma)(u)|$ is finite, then $\sum_{j=1}^{\infty} |f_j(u^2)| < \infty$ and so $\sum_{j=1}^{\infty} f_j(u^2) = 0$ (since $u^2 \in \mathbf{R}^+$), that is, $\sum_{j=1}^{\infty} (f_j \circ \gamma)(u) = 0$. Then the closability of Q (cf. Section 2) and Lemma 4 imply that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} \int_0^\infty f_j \, \mathrm{d}P \right\| = \lim_{n\to\infty} \left\| \sum_{j=1}^{n} \int_{\mathbf{R}} (f_j \circ \gamma) \mathrm{d}Q \right\| = 0.$$ ▨

Theorem 1 ensures the existence of the space $L^1(P)$. To make a more detailed study of this space in relation to the operator L we require the following result.

THEOREM 2. *Let $1 < p < \infty$. Let $f: \mathbf{R}^+ \to \mathbf{C}$ be a measurable function. Then $f \in L(P)$ if, and only if, $f \circ \gamma \in L(Q)$. In this case,*

$$\int_0^\infty f\,\mathrm{d}P = \int_{\mathbf{R}} (f \circ \gamma)\mathrm{d}Q. \tag{14}$$

Proof. Suppose $f \circ \gamma \in L(Q)$. Then there exist functions $f_j \in \mathrm{sim}(\mathscr{A}^{(p)})$, $j = 1, 2, \ldots$, such that $\sum_{j=1}^{\infty} \left\| \int_{\mathbf{R}} f_j\,\mathrm{d}Q \right\| < \infty$ and $\sum_{j=1}^{\infty} f_j(u) = (f \circ \gamma)(u) = f(u^2)$ for every $u \in \mathbf{R}$ such that $\sum_{j=1}^{\infty} |f_j(u)| < \infty$. Define $g_j = f_j \chi_{[0,\infty)}$, $j = 1, 2, \ldots$. Then $g_j \in \mathscr{A}^{(p)}$ and $\sum_{j=1}^{\infty} g_j(\omega) = f(\omega^2)$ for all $\omega \geqslant 0$ such that $\sum_{j=1}^{\infty} |f_j(\omega)| < \infty$. So, for $s \geqslant 0$ we have

$$f(s) = \sum_{j=1}^{\infty} g_j(s^{1/2}) \quad \text{whenever} \quad \sum_{j=1}^{\infty} |g_j(s^{1/2})| = \sum_{j=1}^{\infty} |f_j(s^{1/2})| < \infty. \tag{15}$$

Fix j. Suppose $f_j = \sum_{k=1}^{n(j)} \alpha_k^{(j)} \chi_{E(k,j)}$ with the sets $E(k, j)$, $1 \leqslant k \leqslant n(j)$, pairwise disjoint in $\mathscr{A}^{(p)}$. Then $g_j = \sum_{k=1}^{n(j)} \alpha_k^{(j)} \chi_{E(k,j)^+}$ where $E(k, j)^+ = E(k, j) \cap \mathbf{R}^+$. So, if $s \geqslant 0$, then

$$g_j(s^{1/2}) = \sum_{k=1}^{n(j)} \alpha_k^{(j)} \chi_{E(k,j)^+}(s^{1/2}) = \sum_{k=1}^{n(j)} \alpha_k^{(j)} \chi_{E^2(k,j)^+}(s), \tag{16}$$

where $E^2(k, j)^+ = \{t^2 ; t \in E(k, j)^+\}$. Since

$$(\chi_{E^2(k,j)^+}) \circ \gamma = \chi_{E(k,j)}\chi_{[0,\infty)} + (\chi_{-E(k,j)^+}) - \chi_{E(k,j)}(0)\chi_{[0,\infty)}(0)\chi_{\{0\}}, \tag{17}$$

for each $1 \leqslant k \leqslant n(j)$, it is clear that $s \mapsto g_j(s^{1/2})$, $s \geqslant 0$, belongs to $\mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$ for each $j = 1, 2, \ldots$. It follows from (13), (16) and (17) that

$$\int_0^\infty g_j(s^{1/2})\mathrm{d}P(s) = \sum_{k=1}^{n(j)} \alpha_k^{(j)} P(E^2(k, j)^+) =$$

$$= \sum_{k=1}^{n(j)} \alpha_k^{(j)} [Q(E(k, j)^+) + Q(-E(k, j)^+)] = \int_{\mathbf{R}} (\chi_{[0,\infty)} f_j)(s)\mathrm{d}Q(s) + \tag{18}$$

$$+ \int_{\mathbf{R}} (\chi_{[0,\infty)} f_j)^{\sim}(s)\,\mathrm{d}Q(s) = Q(\mathbf{R}^+) \int_{\mathbf{R}} f_j\mathrm{d}Q + \int_{\mathbf{R}} (\chi_{[0,\infty)} f_j)^{\sim}\mathrm{d}Q.$$

By Lemma 2(ii) and the multiplicativity of Q we have

$$\left\|\int_{\mathbf{R}}(\chi_{[0,\infty)}f_j)^{\sim}\mathrm{d}Q\right\| = \left\|\int_{\mathbf{R}}(\chi_{[0,\infty)}f_j)\mathrm{d}Q\right\| \leqslant \|Q(\mathbf{R}^+)\|\cdot\left\|\int_{\mathbf{R}}f_j\,\mathrm{d}Q\right\|$$

and hence (18) implies that

$$\sum_{j=1}^{\infty}\left\|\int_0^{\infty}g_j(s^{1/2})\mathrm{d}P(s)\right\| \leqslant 2\,\|Q(\mathbf{R}^+)\|\cdot\sum_{j=1}^{\infty}\left\|\int_{\mathbf{R}}f_j\mathrm{d}Q\right\| < \infty.$$

Combining this with (15) we have, by definition of P-integrability, that $f\in L(P)$ and

(19) $$\int_0^{\infty}f\,\mathrm{d}P = \sum_{j=1}^{\infty}\int_0^{\infty}g_j(s^{1/2})\mathrm{d}P(s) = \sum_{j=1}^{\infty}\int_{\mathbf{R}}\{\chi_{[0,\infty)}f_j + (\chi_{[0,\infty)}f_j)^{\sim}\}\mathrm{d}Q;$$

see (18). For each $j = 1, 2, \ldots$, let $E(j) = \bigcup_{k=1}^{n(j)}E(k,j)^+$ and define

$$h_j = \chi_{[0,\infty)}f_j + (\chi_{[0,\infty)}f_j)^{\sim} - \chi_{E(j)}(0)f_j(0)\chi_{\{0\}}.$$

Then $h_j\in\mathrm{sim}(\mathscr{A}^{(p)})$ is symmetric, $h_j(\omega) = f_j(|\omega|)$, $\omega\in\mathbf{R}$, and

$$\int_{\mathbf{R}}\{\chi_{[0,\infty)}f_j + (\chi_{[0,\infty)}f_j)^{\sim}\}\mathrm{d}Q = \int_{\mathbf{R}}h_j\mathrm{d}Q,\quad j = 1, 2, \ldots .$$

In particular, $\sum_{j=1}^{\infty}|h_j(\omega)| < \infty$ if, and only if, $\sum_{j=1}^{\infty}|f_j(|\omega|)| < \infty$. It follows from (19) that

(20) $$\sum_{j=1}^{\infty}f\,\mathrm{d}P = \sum_{j=1}^{\infty}\int_{\mathbf{R}}h_j\,\mathrm{d}Q.$$

Furthermore, if $\omega\in\mathbf{R}$ satisfies $\sum_{j=1}^{\infty}|h_j(\omega)| < \infty$, then $\sum_{j=1}^{\infty}|f_j(|\omega|)| < \infty$ and so

$$(f\circ\gamma)(\omega) = f(\omega^2) = \sum_{j=1}^{\infty}f_j(|\omega|) = \sum_{j=1}^{\infty}h_j(\omega).$$

Since $f\circ\gamma\in L^1(Q)$ it follows from the definition of integral that $\int_{\mathbf{R}}(f\circ\gamma)\mathrm{d}Q =$

$= \sum_{j=1}^{\infty}\int_{\mathbf{R}}h_j\,\mathrm{d}Q$. Combining this with (20) gives (14).

The converse statement follows from the definition of integral and Lemma 4. ▨

The following result is an immediate consequence of Lemma 2(i), Theorem 2 and the identities

$$\mathrm{Re}(f \circ \gamma) = \mathrm{Re}(f) \circ \gamma \quad \text{and} \quad \mathrm{Im}(f \circ \gamma) = \mathrm{Im}(f) \circ \gamma,$$

valid for any function $f: \mathbf{R}^+ \to \mathbf{C}$.

COROLLARY 2.1. *Let* $1 < p < \infty$. *If* $f \in L(P)$, *then also the functions* $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ *and* $\bar{f}$ *are P-integrable.*

REMARK 2. (i) The obvious analogue of Remark 1(i) applies to the Banach algebra $\langle P \rangle$.

(ii) Theorem 2 and Remark 1(ii) imply that

$$\|f\|_{L^1(P)} = \|f \circ \gamma\|_p = \|f \circ \gamma\|_{L^1(Q)}, \quad f \in L(P).$$

(iii) As for Q, if $p \neq 2$, then the space $L^1(P)$ does not satisfy the lattice property of a classical L^1-space. Indeed, let $f \geqslant 0$ be a P-integrable function which is not zero almost everywhere. Remark 1(iii) guarantees the existence of a measurable function $g: \mathbf{R} \to \mathbf{C}$ which is not Q-integrable and satisfies $|g| \leqslant f \circ \gamma$ (a.e.). So at least one of the functions $g\chi_{[0,\infty)}$ or $g\chi_{(-\infty,0)}$ is not Q-integrable, say $g\chi_{[0,\infty)}$. Accordingly, the function $(g\chi_{[0,\infty)})^\sim$ is not Q-integrable; see Lemma 2(ii). It follows that $h = g\chi_{[0,\infty)} + (g\chi_{[0,\infty)})^\sim$ also fails to be Q-integrable (the summands are disjointly supported). If we define $\psi(x) = g(x^{1/2})$, $x \in \mathbf{R}^+$, then ψ is measurable and satisfies $|\psi| \leqslant f$ but ψ is not P-integrable as $h = \psi \circ \gamma$ is not Q-integrable (cf. Theorem 2).

Theorem 2 shows that the richness of the space of P-integrable functions depends on that of the space $L(Q)$ and, of course, on the Boolean algebra of sets $\mathscr{A}^{(p)}(\mathbf{R}^+)$. For example, Lemma 1 and Theorem 2 imply that every non-singular function of bounded variation on $[0, \infty)$ which vanishes at infinity is P-integrable. Or, if $0 < \omega < \pi$, then the restriction to $\mathbf{R}^+$ of any element from $\mathscr{H}^\infty(S_\omega)$ belongs to $L(P)$; see Theorem 4 below. It is worth noting that there are elements in $L(P)$ which are not of bounded variation. For example, if $s \in \mathbf{R}$, then the function $x \mapsto e^{isx}$, $x \in \mathbf{R}$, is Q-integrable [6; Corollary 19] and so Theorem 2 implies that the function $\psi_s: x \mapsto e^{isx^{1/2}}$, $x \in \mathbf{R}^+$, is P-integrable. We have used here the identity

$$(\psi_s \circ \gamma)(x) = e^{is|x|} = e^{-isx}\chi_{(-\infty,0)}(x) + e^{isx}\chi_{[0,\infty)}(x), \quad x \in \mathbf{R},$$

which also shows that

$$\int_0^\infty e^{isx^{1/2}} \mathrm{d}P(x) = T_{-s}Q((-\infty, 0)) + T_sQ(\mathbf{R}^+)$$

where T_u is the operator, in $L^p(\mathbf{R})$, of translation by $u \in \mathbf{R}$. However, if $s \neq 0$, then ψ_s is not of bounded variation. Similarly, there exist elements of $L(P)$ which are not the restriction to $\mathbf{R}^+$ of any element of $\mathscr{H}^\infty(S_\omega)$ for any $0 < \omega < \pi$. Indeed, if ψ_s were the restriction to $\mathbf{R}^+$ of a holomorphic function in S_ω, then this would have to be the function $z \mapsto e^{isz^{1/2}}$, $z \in S_\omega$, which is not bounded when $s < 0$.

Concerning the richness of $\mathscr{A}^{(p)}(\mathbf{R}^+)$ it clearly contains all intervals in $\mathbf{R}^+$ since $\mathscr{A}^{(p)}$ is known to contain all finite unions of intervals in $\mathbf{R}$. To be sure there are sets in $\mathscr{A}^{(p)}(\mathbf{R}^+)$ which are more complex than intervals or finite disjoint unions of intervals. Indeed, if F is any measurable subset of $\mathbf{R}$ which is symmetric about zero (i.e. $\chi_F = \tilde{\chi}_F$) and such that $\chi_F \in \mathscr{M}^{(p)}$, then the set $F^2 = \{t^2 ; t \in F\}$ belongs to $\mathscr{A}^{(p)}(\mathbf{R}^+)$. A large class of such sets can be constructed as follows. Let $\{\lambda_k\}_{k \in \mathbf{Z}}$ be a Hadamard sequence of non-negative numbers, that is,

$$\inf\{\lambda_{k+1}/\lambda_k ; k \in \mathbf{Z}\} \geqslant r > 1.$$

Then $\mathbf{R}\setminus\{0\}$ can be decomposed into the union of disjoint intervals of the form $(\alpha, \beta]$ say, whose vertices belong to $\{\pm\lambda_k ; k \in \mathbf{Z}\}$. The Marcinkiewicz multiplier theorem implies that any bounded function on $\mathbf{R}$ which is constant on each such interval belongs to $\mathscr{M}^{(p)}$. Let $\{J(k) ; k \in \mathbf{Z}\}$ be the decomposition of $(0, \infty)$ consisting of intervals of the type indicated with vertices from $\{\lambda_k ; k \in \mathbf{Z}\}$. So, if $\{\alpha_k ; k \in \mathbf{Z}\}$ is any 0-1 valued sequence, then the function $m : \mathbf{R}^+ \to \mathbf{C}$ defined by $m = \sum_{k \in \mathbf{Z}} \alpha_k \chi_{J(k)}$ determines an element of $\mathscr{A}^{(p)}(\mathbf{R}^+)$, namely the set $m^{-1}(\{1\})$. This follows from the observation that $\{\lambda_k^{1/2}; k \in \mathbf{Z}\}$ is also a Hadamard sequence and

$$m \circ \gamma = \sum_{k \in \mathbf{Z}} \alpha_k (\chi_{J(k)^{1/2}} + \chi_{-J(k)^{1/2}}).$$

There are still other types of sets in $\mathscr{A}^{(p)}(\mathbf{R}^+)$. It is known that for each connected subset K of the circle group $\mathbf{T} = \{z \in \mathbf{C} ; |z| = 1\}$ and for each $s \in \mathbf{R}$ the function $t \mapsto \chi_K(e^{ist})$, $t \in \mathbf{R}$, belongs to $\mathscr{M}^{(p)}$; see [4; Lemma 6], for example. Accordingly, if $s \in \mathbf{R}$ and $K \subseteq \mathbf{T}$ are such that $F = \{t \in \mathbf{R} ; e^{ist} \in K\}$ is symmetric about zero, then $F^2 \in \mathscr{A}^{(p)}(\mathbf{R}^+)$.

The following result shows that $\langle P \rangle$ contains the resolvent operators of L.

THEOREM 3. *Let* $1 < p < \infty$. *If* $\lambda \in \rho(L)$, *then the function* $R_\lambda \colon \omega \mapsto (\omega - \lambda)^{-1}$, $\omega \in \mathbf{R}^+$, *is P-integrable and* $\int_0^\infty R_\lambda \mathrm{d}P = (L - \lambda)^{-1}$. *In particular,* $(L - \lambda)^{-1} \in \langle P \rangle$ *for every* $\lambda \notin \sigma(L)$.

Proof. If u is a square root of λ, in which case $\{-u, u\} \subseteq \mathbf{C}\setminus\mathbf{R}$, then $R_\lambda \circ \gamma = (\psi_1 - \psi_2)/2u$ where $\psi_1(\omega) = (\omega - u)^{-1}$ and $\psi_2(\omega) = (\omega + u)^{-1}$, for each $\omega \in \mathbf{R}$. Since both ψ_1 and ψ_2 are absolutely continuous on $\mathbf{R}$ (as their derivatives belong to $L^1(\mathbf{R})$) and vanish at $-\infty$, it follows from Lemma 1 that $\psi_1, \psi_2 \in L(Q)$ and hence also $R_\lambda \circ \gamma \in L(Q)$. Then Theorem 2 implies that $R_\lambda \in L(P)$ and

$$(21) \qquad \int_0^\infty R_\lambda \mathrm{d}P = \int_{\mathbf{R}} (R_\lambda \circ \gamma)\,\mathrm{d}Q = (2u)^{-1}\left\{\int_{\mathbf{R}} \psi_1 \mathrm{d}Q - \int_{\mathbf{R}} \psi_2 \mathrm{d}Q\right\}.$$

Now ψ_1 and ψ_2 are both p-multipliers (since $L(Q) \subseteq \mathcal{M}^{(p)}$, for example) and it is not difficult to check that $\psi_1(D) = (D-u)^{-1}$ and $\psi_2(D) = (D+u)^{-1}$. It follows from (21) and Lemma 2(iii) that

$$\int_0^\infty R_\lambda \mathrm{d}P = (2u)^{-1}\{(D-u)^{-1} - (D+u)^{-1}\}.$$

But, the right-hand-side is precisely $(D-u)^{-1}(D+u)^{-1} = (L-\lambda)^{-1}$ by the resolvent identities for D. ▨

We conclude this section with the following result.

THEOREM 4. *Let* $1 < p < \infty$. *If* $0 < \omega < \pi$, *then* $\mathcal{H}^\infty(S_\omega) \subseteq L(P)$ *in the sense that the restriction to* $\mathbf{R}^+$ *of any element* $\psi \in \mathcal{H}^\infty(S_\omega)$ *is P-integrable. Furthermore, if this restriction is denoted by* $\psi_{\mathbf{R}^+}$, *then*

$$\|\psi_{\mathbf{R}^+}\|_{L^1(P)} \leqslant \alpha_p \|\psi\|_\infty \Big/ \sin\left(\frac{1}{2}\cdot\omega\right), \qquad \psi \in \mathcal{H}^\infty(S_\omega)$$

where α_p *is a constant depending only on* p.

Proof. If $\psi \in \mathcal{H}^\infty(S_\omega)$, then it is clear that $\psi \circ \gamma \in \mathcal{H}^\infty(-S_{(1/2)\omega} \cup S_{(1/2)\omega})$, where the same symbol γ denotes the extension to $\mathbf{C}$ of $\gamma : x \mapsto x^2$, $x \in \mathbf{R}$, as defined earlier. It is established in [9; Section 2] that

$$(22) \qquad |(\psi \circ \gamma)'(x)| \leqslant \|\psi \circ \gamma\|_\infty \Big/ |x| \sin\left(\frac{1}{2}\,\omega\right), \qquad x \in \mathbf{R}\setminus\{0\},$$

from which it follows that $\psi \circ \gamma|\mathbf{R}$ is a p-multiplier and

$$\|\psi \circ \gamma|\mathbf{R}\|_p \leqslant \alpha_p \|\psi \circ \gamma\|_\infty \Big/ \sin\left(\frac{1}{2}\cdot\omega\right) = \alpha_p \|\psi\|_\infty \Big/ \sin\left(\frac{1}{2}\cdot\omega\right)$$

for some constant α_p depending only on p; see [9]. Granted for the moment that $\psi \circ \gamma|\mathbf{R} \in L(Q)$, it follows from Theorem 2 that $\psi_{\mathbf{R}_+} \in L(P)$. Then Remark 1(ii) and

Lemma 2(iii) imply that

$$\|\psi_{\mathbf{R}^+}\|_{L^1(P)} = \|\psi \circ \gamma|\mathbf{R}\|_{L^1(Q)} = \|\psi \circ \gamma|\mathbf{R}\|_{p} \leqslant \alpha_p \|\psi\|_\infty \Big/ \sin\left(\frac{1}{2}\omega\right).$$

So, it remains to establish that $\psi \circ \gamma|\mathbf{R} \in L(Q)$ whenever $\psi \in \mathscr{H}^\infty(S_\omega)$. Since $f \in L(Q)$ if, and only if, both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Q-integrable (cf. Lemma 2(i)) and $\psi \circ \gamma|\mathbf{R}$ is symmetric, it suffices to show that any $\mathbf{R}$-valued, symmetric function φ on $\mathbf{R}\setminus\{0\}$ which is the real (or imaginary) part of the restriction to $\mathbf{R}\setminus\{0\}$ of an element from $\mathscr{H}^\infty(-S_{(1/2)\omega} \cup S_{(1/2)\omega})$, say $\varphi_\sim$, is Q-integrable. But, this follows from Lemma 3. Indeed, in the notation of Lemma 3 it follows from (22), which is actually valid for any element of $\mathscr{H}^\infty(-S_{(1/2)\omega} \cup S_{(1/2)\omega})$ and not just those functions of the form $\psi \circ \gamma$ (see the proof in [9]), that

$$\sup\{V(\varphi \mid I(n))\,;\, n \in \mathbf{Z}\} \leqslant \|\varphi_\sim\|_\infty \ln(2) \Big/ \sin\left(\frac{1}{2}\omega\right) < \infty.$$

That $\{x \in \overline{I(n)}\,;\, |\varphi(x)| = \alpha\}$ is a finite set for each $n \in \mathbf{Z}$ and $\alpha > 0$ is known; see [7; Theorem 8.2], for example. ▨

4. THE CONNECTION BETWEEN L AND P AND SOME APPLICATIONS

The aim of this section is to establish the identities (1) and (2) and to deduce some consequences from them. First we need some further notation and preliminary lemmas.

LEMMA 5. *Let $1 < p < \infty$. Let $\{\alpha(n)\}_{n=1}^\infty$ be a sequence of non-negative numbers which increases to infinity. If $\chi_{\alpha(n)}$ denotes the characteristic function of $[-\alpha(n), \alpha(n)]$, for each $n = 1, 2, \ldots$, then $\lim_{n\to\infty} \chi_{\alpha(n)}(D) = \lim_{n\to\infty} Q([-\alpha(n), \alpha(n)]) = I$ with respect to the strong operator topology in $\mathscr{L}(L^p(\mathbf{R}))$.*

Proof. Since $\{\chi_{\alpha(n)}\}_{n=1}^\infty$ converges pointwise on $\mathbf{R}$ to the constant function 1 and

$$\sup\{\|\chi_{\alpha(n)}\|_p\,;\; n = 1,\ 2, \ldots\} < \infty,$$

it follows from a standard convergence result from multiplier theory that $\chi_{n(\alpha)}(D) \to I$ in the weak operator topology. In addition, $\chi_{\alpha(n)}(D)\chi_{\alpha(m)}(D) = \chi_{\alpha(k)}(D)$ where $\alpha(k) = \min\{\alpha(n), \alpha(m)\}$ and hence, it follows that $\chi_{\alpha(n)}(D) \to I$ in the strong operator topology, [1; Theorem 1]. ▨

Let T be a closed, densely defined operator in a Banach space X. Then its dual operator T^* is the operator with domain $\mathscr{D}(T^*)$ consisting of all

$x^* \in X^*$ (the continuous dual space of X) such that there exists $y^* \in X^*$ with the property

$$\langle Tx, x^* \rangle = \langle x, y^* \rangle, \quad x \in \mathscr{D}(T).$$

The element $y^* \in X^*$ is uniquely determined by x^*. Therefore a linear operator T^* with domain $\mathscr{D}(T^*)$ can be defined by setting $T^*x^* = y^*$. In particular,

$$\langle Tx, x^* \rangle = \langle x, T^*x^* \rangle, \quad x \in \mathscr{D}(T), \quad x^* \in \mathscr{D}(T^*). \tag{23}$$

The operator T^* is always closed. If X is reflexive, then T^* is also densely defined and $T^{**} = T$.

Of relevance to this paper is the set up when $p \in (1, \infty)$, $X = L^p(\mathbf{R})$ and T is the Laplace operator L as defined in Section 1. In this case we will denote L by $L_{(p)}$ if it is necessary to indicate in which space L is being considered. Then $L^*_{(p)} = L_{(q)}$ where $p^{-1} + q^{-1} = 1$; this identity includes the statement that $\mathscr{D}(L^*_{(p)})$, as defined via duality, is precisely $\mathscr{D}(L_{(q)})$ as defined in Section 1. We remark that if $m \in \mathscr{M}^{(p)}$, then also $m \in \mathscr{M}^{(q)}$ and the dual operator of $m(D_{(p)})$ is precisely $m(D_{(q)})$; here $D_{(p)}$ denotes the operator D of Section 1 considered in the space $L^p(\mathbf{R})$. In particular, $\mathscr{A}^{(p)}(\mathbf{R}^+) = \mathscr{A}^{(q)}(\mathbf{R}^+)$. For the case $1 < p \leqslant 2$ it is known that the operators D and L and their respective domains can be described in terms of the Fourier transform. Namely, an element f of $L^p(\mathbf{R})$ belongs to $\mathscr{D}(D)$ (resp. $\mathscr{D}(L)$) if, and only if, there exists $g \in L^p(\mathbf{R})$, necessarily unique, such that $\hat{g}(\xi) = \xi\hat{f}(\xi)$, $\xi \in \mathbf{R}$ (resp. $\hat{g}(\xi) = \xi^2\hat{f}(\xi)$), in which case $Df = g$ (resp. $Lf = g$).

LEMMA 6. *Let $1 < p < \infty$ and $\lambda^{(n)}$, $n = 1, 2, \ldots$, denote the function $\xi \mapsto \xi\chi_{[0,n]}(\xi)$, $\xi \in \mathbf{R}^+$. Then the identities*

$$\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f = Q([-n^{1/2}, n^{1/2}])Lf, \quad f \in \mathscr{D}(L), \tag{24}$$

are valid (in $L^p(\mathbf{R})$), for every $n = 1, 2, \ldots$. In particular, $\lambda^{(n)} \in L(P)$.

Proof. Since $(\lambda^{(n)} \circ \gamma)(\xi) = \xi^2\chi_{[-n^{1/2}, n^{1/2}]}(\xi)$, $\xi \in \mathbf{R}$, it follows from Lemma 1 and Theorem 2 that $\lambda^{(n)} \in L(P)$, $n = 1, 2, \ldots$.

Fix n. Suppose that $1 < p \leqslant 2$. If $f \in \mathscr{D}(L)$, then

$$\left(\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f\right)^\wedge(\xi) = ((\lambda^{(n)} \circ \gamma)(D)f)^\wedge(\xi) = \xi^2\chi_{[-n^{1/2}, n^{1/2}]}(\xi)\hat{f}(\xi). \tag{25}$$

On the other hand, also

$$(Q([-n^{1/2}, n^{1/2}])Lf)^\wedge(\xi) = \chi_{[-n^{1/2}, n^{1/2}]}(\xi)(Lf)^\wedge(\xi) = \xi^2\chi_{[-n^{1/2}, n^{1/2}]}(\xi)\hat{f}(\xi). \tag{26}$$

Of course, we have used here Lemma 2(iii). Then (24) follows from (25), (26) and the uniqueness of Fourier transforms.

Suppose now that $2 \leqslant p < \infty$. Let $P_{(p)}$ (resp. $Q_{(p)}$) denote the spectral set function P (resp. Q) considered in $L^p(\mathbf{R})$ and use a similar notation when considering $L^q(\mathbf{R})$ where $p^{-1} + q^{-1} = 1$. Then $P_{(p)}(E)^* = P_{(q)}(E)$, for every $E \in \mathscr{A}^{(p)}(\mathbf{R}^+) = = \mathscr{A}^{(q)}(\mathbf{R}^+)$, and $\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right)^* = \int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}$. Fix $f \in C_c^\infty(\mathbf{R})$. Then

$$\langle Q([-n^{1/2}, n^{1/2}])_{(p)} L_{(p)} f, g\rangle = \langle L_{(p)} f, Q([-n^{1/2}, n^{1/2}])_{(q)} g\rangle =$$

$$= \langle (L_{(p)} f)\hat{\ }, (Q([-n^{1/2}, n^{1/2}])_{(q)} g)\hat{\ }\rangle, \quad g \in C_c^\infty(\mathbf{R}),$$

where the element $Q([-n^{1/2}, n^{1/2}])_{(q)} g$ of $L^q(\mathbf{R})$ is identified with a tempered distribution in the usual way. Since $(L_{(p)} f)\hat{\ }(\xi) = \xi^2 \hat{f}(\xi)$ is a Schwartz function and the Fourier transform of the tempered distribution $Q([-n^{1/2}, n^{1/2}])_{(q)} g$ is the tem pered distribtion associated with the function $\chi_{[-n^{1/2}, n^{1/2}]}\hat{g}$ (using the fact that $\chi_{[-n^{1/2}, n^{1/2}]} \in \mathscr{M}^{(q)}$) it follows that

$$\langle (L_{(p)} f)\hat{\ }, (Q([-n^{1/2}, n^{1/2}])_{(q)} g)\hat{\ }\rangle = \langle \xi^2 \hat{f}(\xi), \chi_{[-n^{1/2}, n^{1/2}]}\hat{g}\rangle.$$

But, by the definition of multiplication of tempered distributions by polynomials it follows that

$$\langle \xi^2 \hat{f}(\xi), \chi_{[-n^{1/2}, n^{1/2}]}\hat{g}\rangle = \langle \hat{f}, \xi^2 \chi_{[-n^{1/2}, n^{1/2}]}\hat{g}\rangle = \langle \hat{f}, (\lambda^{(n)} \circ \gamma)\hat{g}\rangle.$$

Since $\lambda^{(n)} \circ \gamma \in \mathscr{M}^{(q)}$ the tempered distribution $(\lambda^{(n)} \circ \gamma)\hat{g}$ can be identified with the Fourier transform of the tempered distribution associated with the element $(\lambda^{(n)} \circ \gamma)(D_{(q)}) g = \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) g$ of $L^q(\mathbf{R})$. Accordingly,

$$\langle Q([-n^{1/2}, n^{1/2}]_{(p)} L_{(p)} f, g\rangle = \left\langle \hat{f}, \left(\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) g\right)\hat{\ }\right\rangle =$$

$$= \left\langle f, \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) g\right\rangle = \left\langle \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right) f, g\right\rangle,$$

for every $g \in C_c^\infty(\mathbf{R})$. Since $C_c^\infty(\mathbf{R})$ separates points of $L^p(\mathbf{R})$ we can conclude that

$$(27) \qquad Q([-n^{1/2}, n^{1/2}])_{(p)} L_{(p)} f = \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right) f.$$

This is valid for every $f \in C_c^\infty(\mathbf{R})$. To establish (27) for arbitrary $f \in \mathscr{D}(L_{(p)})$ we note that both $Q([-n^{1/2}, n^{1/2}])_{(p)}$ and $\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}$ are elements of $\mathscr{L}(L^p(\mathbf{R}))$ and that $C_c^\infty(\mathbf{R})$ is dense in the Sobolev space $W^{2,p}(\mathbf{R})$ which, as a linear space, coincides with $\mathscr{D}(L_{(p)})$. Thus, (24) is established. ∎

THEOREM 5. *Let* $1 < p < \infty$ *and* $\lambda^{(n)}$, $n = 1, 2, \ldots$, *denote the function* $\xi \mapsto \xi\chi_{[0,n]}(\xi)$, $\xi \in \mathbf{R}^+$. *Then* $\lambda^{(n)} \in L(P)$, $n = 1, 2, \ldots$. *In addition,*

$$(1) \qquad \mathscr{D}(L) = \left\{f \in L^p(\mathbf{R})\,;\, \lim_{n\to\infty}\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f \text{ exists in } L^p(\mathbf{R})\right\}$$

and

$$(2) \qquad Lf = \lim_{n\to\infty}\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f, \quad f \in \mathscr{D}(L).$$

Proof. Let $f \in \mathscr{D}(L)$. Then it follows from (24) and Lemma 5 that $\lim_{n\to\infty}\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f$ exists in $L^p(\mathbf{R})$ and equals Lf. So, f belongs to the right-hand-side of (1) and (2) is satisfied.

Conversely, suppose that $f \in L^p(\mathbf{R})$ and $g = \lim_{n\to\infty}\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f$ exists in $L^p(\mathbf{R})$. If $1 < p \leqslant 2$, then the Fourier transform is a continuous mapping of $L^p(\mathbf{R})$ into $L^q(\mathbf{R})$, where $p^{-1} + q^{-1} = 1$, and so

$$(28) \qquad \hat{g} = \lim_{n\to\infty}\left(\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f\right)^{\wedge}$$

in $L^q(\mathbf{R})$. Since $\chi_{[-n^{1/2}, n^{1/2}]} \to 1$ pointwise on $\mathbf{R}$, it follows from (25), (26) and (28) that

$$(29) \qquad \xi^2 \hat{f}(\xi) = \lim_{n\to\infty}\left(\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f\right)^{\wedge}(\xi),$$

for a.e. $\xi \in \mathbf{R}$. Then (28) and (29) show that $\hat{g}(\xi) = \xi^2\hat{f}(\xi)$ and so the definitions of L and $\mathscr{D}(L)$ in terms of the Fourier transform imply that $f \in \mathscr{D}(L)$ and $Lf = g$. So, (1) and (2) are established if $p \in (1, 2]$.

Suppose then that $2 \leqslant p < \infty$. Let q satisfy $p^{-1} + q^{-1} = 1$. Then $1 < q \leqslant 2$ and so we know that $\left\{\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) h\right\}_{n=1}^\infty$ converges, in $L^q(\mathbf{R})$, to $L_{(q)}h$ whenever $h \in \mathscr{D}(L_{(q)})$. In particular,

$$(30) \qquad \langle f, L_{(q)}h \rangle = \lim_{n\to\infty} \left\langle f, \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)} h\right\rangle, \quad h \in \mathscr{D}(L_{(q)}).$$

It follows from (30) the identities $\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right)^* = \int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}$, $n = 1, 2, \ldots,$ and the fact that $\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right) f \to g$ in $L^p(\mathbf{R})$, that

$$\langle f, L_{(q)}h \rangle = \langle g, h \rangle, \quad h \in \mathscr{D}(L_{(q)}).$$

By definition of the dual operator this means that $f \in \mathscr{D}(L_{(q)}^*) = \mathscr{D}(L_{(p)})$) and $L_{(p)}f = L_{(q)}^* f = g$. So, (1) and (2) also hold for $2 \leqslant p < \infty$. ▨

REMARK 3 (i). It is clear from the proof of Theorem 5 (see also Lemma 5) that the functions $\lambda^{(n)}$, $n = 1, 2, \ldots$, in the statement of Theorem 5 can be replaced by functions $\xi \to \xi\chi_{[0,\beta(n)]}(\xi)$, $\xi \in \mathbf{R}^+$, for $n = 1, 2, \ldots$, where $\{\beta(n)\}_{n=1}^\infty$ is any non-negative sequence increasing to infinity.

(ii) The descriptions of $\mathscr{D}(L)$ and L as given by (1) and (2) are analogues of the "same" statements which are well known for (unbounded) scalar-type spectral operators; see [3; pp. 2251], for example.

We wish now to show that L exhibits still further similarities with scalar-type spectral operators. Indeed, the associated Boolean algebra of projections $\{P(E)\,;\, E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ satisfies all the required properties of being a classical resolution of the identity for L (see Definition 2.1 of Chapter XVIII in [3]) except the σ-additivity condition. Namely,

$$(31.1) \qquad \mathscr{D}(L) \supseteq \mathscr{R}(P(E)) = \{P(E)f\,;\, f \in L^p(\mathbf{R})\}$$

whenever $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$ is a bounded set,

$$(31.2) \qquad P(E)\mathscr{D}(L) \subseteq \mathscr{D}(L) \quad \text{and} \quad P(E)Lf = LP(E)f, \quad f \in \mathscr{D}(L),$$

for every $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, and

(31.3) if $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, then $\sigma(L_{P(E)}) \subseteq \bar{E}$, where $L_{P(E)}$ denotes the restriction of L to the closed subspace $\mathscr{R}(P(E))$.

To establish (31.2) we note that if $f \in \mathscr{D}(L)$, then Lemma 6 implies

$$\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) P(E) f = P(E)\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) f = P(E) Q([n^{1/2}, n^{1/2}]) Lf,$$

for each $n = 1, 2, \ldots$. Accordingly, Lemma 5 and the continuity of $P(E)$ imply that $\lim_{n\to\infty} \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P\right) P(E) f$ exists in $L^p(\mathbf{R})$ and equals $P(E)Lf$. It follows from Theorem 5 that $P(E) f \in \mathscr{D}(L)$ and $LP(E)f = P(E)Lf$.

To check (3.11) suppose firstly that $1 < p \leqslant 2$. Then

$$(P(E)f)^\wedge(\xi) = [\chi_{E^{1/2}}(\xi) + \chi_{-E^{1/2}}(\xi)]\hat{f}(\xi), \quad f \in L^p(\mathbf{R}).$$

So, to show that $P(E)f \in \mathscr{D}(L)$ it suffices, by the definition of L and $\mathscr{D}(L)$ in terms of the Fourier transform, to show that $\xi \mapsto \xi^2 \chi_F(\xi)$, $\xi \in \mathbf{R}$, is a p-multiplier whenever $F \in \mathscr{A}^{(p)}$ is a bounded set in $\mathbf{R}$. Since $F \subseteq [-\alpha, \alpha]$ for some $\alpha > 0$ and $\xi^2 \chi_F(\xi) = \xi^2 \chi_{[-\alpha,\alpha]}(\xi) \chi_F(\xi)$ it suffices to show that $\xi \to \xi^2 \chi_{[-\alpha,\alpha]}(\xi)$ is a p-multiplier. But, this is clear since the specified function is of bounded variation on $\mathbf{R}$. Suppose that $p \in (2, \infty)$. If q satisfies $p^{-1} + q^{-1} = 1$, in which case $1 < q \leqslant 2$, then we have just seen that $\mathscr{R}(P(E)_{(q)}) \subseteq \mathscr{D}(L_{(q)})$. So, Theorem 5 shows that

$$\sup \left\{ \left\| \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) P(E)_{(q)} g \right\|_q ; n = 1, 2, \ldots \right\} < \infty,$$

for every $g \in L^p(\mathbf{R})$ and hence, the Uniform Boundedness Principle, together with the identities $\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right)^* = \int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}$, $n = 1, 2, \ldots$, imply that

$$\sup_n \left\| \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right) P(E)_{(p)} \right\| = \sup_n \left\| \left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(q)}\right) P(E)_{(q)} \right\| < \infty.$$

Since the sequence of operators $\left\{\left(\int_0^\infty \lambda^{(n)} \mathrm{d}P_{(p)}\right) P(E)_{(p)}\right\}_{n=1}^\infty$ converges (cf. proof of

(31.2)) on the dense subspace $\mathscr{D}(L_{(p)})$ of $L^p(\mathbf{R})$ it actually converges at every $f \in L^p(\mathbf{R})$. Then Theorem 5 shows that $P(E)_{(p)}f \in \mathscr{D}(L_{(p)})$ for every $f \in L^p(\mathbf{R})$.

To establish (31.3) let $J = \bar{E}$. If $\lambda \notin J$, then the square roots of λ, denoted by $\pm u$ say, do not belong to the closed set $-J^{1/2} \cup J^{1/2}$. Accordingly, there is $\varepsilon > 0$ with the property that if $B(u, \varepsilon)$ and $B(-u, \varepsilon)$ are the open balls in $\mathbf{C}$ of radius ε centred at u and $-u$, respectively, then

$$[B(u, \varepsilon) \cup B(-u, \varepsilon)] \cap [-J^{1/2} \cup J^{1/2}] = \emptyset.$$

Let $h_{\lambda,\varepsilon}(\xi) = (\xi^2 - \lambda)^{-1}\chi_{\lambda,\varepsilon}(\xi)$, $\xi \in \mathbf{R}$, where $\chi_{\lambda,\varepsilon}$ is the characteristic function of $\mathbf{R}$ intersected with the complement (in $\mathbf{C}$) of $B(u, \varepsilon) \cup B(-u, \varepsilon)$. It can be assumed that ε is chosen so that $\mathbf{R} \cap [B(u, \varepsilon) \cup B(-u, \varepsilon)]$ is empty if $\lambda \notin \mathbf{R}^+$. The function $h_{\lambda,\varepsilon}$ is of bounded variation and so belongs to $\mathscr{M}^{(p)}$ for every $1 < p < \infty$. Let

$$h_\lambda(\xi) = (\chi_{E^{1/2}}(\xi) + \chi_{-E^{1/2}}(\xi))h_{\lambda,\varepsilon}(\xi), \quad \xi \in \mathbf{R}.$$

Then $h_\lambda \in \mathscr{M}^{(p)}$ and

$$h_\lambda(\xi) = \begin{cases} (\xi^2 - \lambda)^{-1} & \text{if } \xi \in E^{1/2} \cup (-E^{1/2}) \\ 0 & \text{otherwise.} \end{cases}$$

Since $P(E) = Q(E^{1/2}) + Q(-E^{1/2})$ is a multiplier operator it follows that $h_\lambda(D)$ commutes with $P(E)$ and so the range $\mathscr{R}(P(E))$ of $P(E)$ is invariant for $h_\lambda(D)$. Accordingly, the restriction, $h_\lambda(D)_{P(E)}$, of $h_\lambda(D)$ to $\mathscr{R}(P(E))$ is an element of $\mathscr{L}(\mathscr{R}(P(E)))$. If $1 < p \leqslant 2$, then using the formulation of L and $\mathscr{D}(L)$ in terms of the Fourier transform, it can be checked that $h_\lambda(D)_{P(E)}$ is the resolvent operator of $L_{P(E)}$ at the point λ (in the space $\mathscr{L}(\mathscr{R}(P(E)))$, of course). By duality, the restriction of $h_\lambda(D_{(p)})^* = h_\lambda(D_{(q)})$ to the range of $P(E)^*_{(p)} = P(E)_{(q)}$ is the resolvent operator of $L_{P(E)_{(q)}}$ at λ; here q satisfies $p^{-1} + q^{-1} = 1$. This shows, for all $p \in (1, \infty)$, that $\sigma(L_{P(E)}) \subseteq J = \bar{E}$ and establishes (31.3).

REMARK 4. It has been noted (for $p \neq 2$) that the only essential property of a resolution of the identity for L that the set function $P(\cdot)$ fails to satisfy in the classical sense is that $\mathscr{A}^{(p)}(\mathbf{R}^+)$ is not a σ-algebra and $P(\cdot)$ is not σ-additive. There is available, however, the following (weaker) substitute for this property, a special case of which is Lemma 5.

THEOREM 6. *Let* $\{E(n)\}_{n=1}^{\infty} \subseteq \mathscr{A}^{(p)}(\mathbf{R}^+)$ *be a sequence of sets such that*

$$E = \{\omega \in \mathbf{R}^+ ; \lim_{n\to\infty} \chi_{E(n)}(\omega) \text{ exists}\}$$

is a measurable set with $\mathbf{R}^+ \setminus E$ *null for Lebesgue measure and* $\sup\{\|P(E(n))\| ; n =$

$= 1, 2, \ldots\} < \infty$. *If*

$$E^{(1)} = \{\omega \in \mathbf{R}^+ ; \lim_{n\to\infty} \chi_{E(n)}(\omega) = 1\},$$

then $E, E^{(1)} \in \mathscr{A}^{(p)}(\mathbf{R}^+)$ *and* $\{P(E(n))\}_{n=1}^{\infty}$ *converges to* $P(E^{(1)})$ *in the strong operator topology.*

Proof. If $F = E^{1/2} \cup (-E^{1/2})$, then $F \subseteq \mathbf{R}$ is a measurable set such that $\mathbf{R}\setminus F$ is null for Lebesgue measure and hence $\chi_F \in \mathscr{M}^{(p)}$ with $\chi_F(D) = I$. Since $\chi_F = \chi_E \circ \gamma$ it follows that $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$ and $P(E) = \chi_F(D) = I$. Let $F(n) = E(n)^{1/2} \cup \cup (-E(n))^{1/2}$, $n = 1, 2, \ldots$, and $F^{(1)} = (E^{(1)})^{1/2} \cup (-E^{(1)})^{1/2}$. Using the identities

$$\chi_{E(n)}(\xi^2) = (\chi_{E(n)} \circ \gamma)(\xi) = \chi_{F(n)}(\xi), \quad \xi \in \mathbf{R},$$

valid for each $n = 1, 2, \ldots$,

$$\chi_{E^{(1)}}(\xi^2) = (\chi_{E^{(1)}} \circ \gamma)(\xi) = \chi_{F^{(1)}}(\xi), \quad \xi \in \mathbf{R},$$

and the definition of $E^{(1)}$ it follows that $\chi_{F(n)} \to \chi_{F^{(1)}}$ pointwise a.e. on $\mathbf{R}$. Since

$$\sup_n \|\chi_{F(n)}\|_p = \sup_n \|\chi_{F(n)}(D)\| = \sup_n \|P(E(n))\| < \infty,$$

it follows from a standard multiplier convergence theorem that the limit function (a.e.) $\chi_{F^{(1)}} \in \mathscr{M}^{(p)}$ and $\chi_{F(n)}(D) \to \chi_{F^{(1)}}(D)$ in the weak operator topology. That is, $E^{(1)} \in \mathscr{A}^{(p)}(\mathbf{R}^+)$ and $P(E(n)) \to P(E^{(1)})$ in the weak operator topology. Ordering the sets $\{E(n)\}$ by inclusion it follows that

$$P(E(n))P(E(m)) = P(E(n) \cap E(m)) = P(E(n)), \quad \text{whenever } E(n) \subseteq E(m).$$

Accordingly, the projections $\{P(E(n))\}$ are naturally ordered in the sense of [1] and so Theorem 1 of [1] implies that $P(E(n)) \to P(E^{(1)})$ in the strong operator topology of $\mathscr{L}(L^p(\mathbf{R}))$. ▨

One of the main results from the classical theory of scalar-type spectral operators (possibly unbounded) states that an operator commutes with a scalar-type spectral operator if, and only if, it commutes with its resolution of the identity; see [3; XV, Corollary 3.7 and XVIII, Corollary 2.4], for example. The proof of this fact relies heavily on the countable additivity of the resolution of the identity. So, it is perhaps somewhat surprising that this result still holds for L and $P(\cdot)$.

THEOREM 7. *Let* $1 < p < \infty$. *A bounded operator* T *in* $L^p(\mathbf{R})$ *commutes with* L, *in the sense that* $T(\mathscr{D}(L)) \subseteq \mathscr{D}(L)$ *and* $TLf = LTf$, $f \in \mathscr{D}(L)$, *if, and only if,* $TP(E) = P(E)T$ *for every* $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$.

Proof. Suppose that $TP(E) = P(E)T$, $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$. Then it is clear that $T\left(\int_0^\infty f\,\mathrm{d}P\right) = \left(\int_0^\infty f\,\mathrm{d}P\right)T$, for every $f \in \mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$ and hence, by definition of P-integrability also for every $f \in L(P)$. In particular, if $\lambda^{(n)}$ are the functions specified in Theorem 5, then

$$T\left(\int_0^\infty \lambda^{(n)}\mathrm{d}P\right) = \left(\int_0^\infty \lambda^{(n)}\mathrm{d}P\right)T, \quad n = 1, 2, \ldots .$$

It then follows easily from Theorem 5 and the continuity of T that $T(\mathscr{D}(L)) \subseteq \mathscr{D}(L)$ and $TLf = LTf$, $f \in \mathscr{D}(L)$.

The converse implication is more difficult to establish. We give only a brief sketch of the proof; details can be found in [10]. So, if T commutes with L, then it can be shown that $T(L - \lambda)^{-1} = (L - \lambda I)^{-1}T$, $\lambda \in \rho(L)$. From this it is possible to deduce that T commutes with $P(J)$ whenever J is an open, bounded interval in $\mathbf{R}^+ = \sigma(L)$. Now, it is possible to approximate $\psi(D)$ in the weak operator topology by linear combinations of projections $(\chi_J \circ \gamma)(D)$, $J \subseteq \mathbf{R}^+$, J an interval, whenever ψ is a symmetric element from $C_c^\infty(\mathbf{R}) \cap \mathscr{M}^{(p)}$ and hence, $T\psi(D) = \psi(D)T$. Since the operator $\psi(D)$, for arbitrary symmetric elements $\psi \in \mathscr{M}^{(p)}$, can be approximated in the weak operator topology by operators of the type $\varphi(D)$, where $\varphi \in C_c^\infty(\mathbf{R}) \cap \mathscr{M}^{(p)}$ is symmetric, and every p-multiplier $\chi_E \circ \gamma$, $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$, is symmetric, it follows that

$$TP(E) = T(\chi_E \circ \gamma)(D) = (\chi_E \circ \gamma)(D)T = P(E)T,$$

for every $E \in \mathscr{A}^{(p)}(\mathbf{R}^+)$. ▨

The final result is an analogue of the von Neumann bicommutant theorem. One version of von Neumann's theorem states that the weak-operator closed algebra generated by the resolution of the identity of a self-adjoint operator T in a separable Hilbert space coincides with the bicommutant of T; see [3; XVII, Theorem 3.22 and Corollary 3.17] for the case when T is bounded. If we interpret the spectral set function $P : \mathscr{A}^{(p)}(\mathbf{R}^+) \to \mathscr{L}(L^p(\mathbf{R}))$ as the resolution of the identity for L (which is justified by Theorem 5) and define the commutant $\{L\}^c$ of L as in the statement of Theorem 7, then the following result is indeed an analogue of the bicommutant theorem.

THEOREM 8. *Let $1 < p < \infty$. Then the bicommutant $\{L\}^{cc}$ of L coincides with the weak-operator closed algebra generated by $\{P(E); E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$.*

Proof. If we combine Theorem 7 above with [10; Theorem 3] it follows that

(32) $\{L\}^{cc} = \{P(E)\,;\, E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}^{cc} = \{\varphi(D)\,;\, \varphi \in \mathscr{M}^{(p)} \text{ and } \varphi \text{ symmetric}\}.$

Now, the bicommutant of a family of operators is always weak-operator closed. Accordingly, it suffices to show, by (32), that any operator of the form $\varphi(D)$, $\varphi \in \mathcal{M}^{(p)}$ and φ symmetric, can be approximated in the weak operator topology by elements from the linear span of $\{P(E)\,;\,E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$, that is, by operators of the form $(\psi \circ \gamma)(D) = \int_0^\infty \psi \, \mathrm{d}P$ where $\psi \in \mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$. That this is possible is established in the proof of Theorem 1 in [10]. ▨

REMARK 5. It is known that the weak-operator closed algebra generated by the resolution of the identity of a scalar-type spectral operator actually coincides with the commutant of the resolution of the identity whenever there exists a cyclic vector, [3; XVII, Theorem 3.20]. In anticipation of a similar result being valid for the Laplace operator L there arises the question of whether the spectral set function P admits a cyclic vector, that is, whether there exists an element $g \in L^p(\mathbf{R})$ such that the linear span of $\{P(E)g\,;\,E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ is dense in $L^p(\mathbf{R})$? That this is not the case (for $1 < p \leqslant 2$) can be argued as follows.

Let R be the bounded operator in $L^p(\mathbf{R})$ defined by

$$Rf : x \mapsto f(-x), \quad x \in \mathbf{R},$$

for each $f \in L^p(\mathbf{R})$. Suppose that P did admit a cyclic vector, say g. Noting that each element in the linear span of $\{P(E); E \in \mathscr{A}^{(p)}(\mathbf{R}^+)\}$ is of the form $(\varphi \circ \gamma)(D)$ for some $\varphi \in \mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$, it follows that there exists a sequence $\{\psi_n\} \subseteq \mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$ such that $\{(\psi_n \circ \gamma)(D)g\}$ converges to $g + Rg$ in $L^p(\mathbf{R})$. Since the Fourier transform is continuous from $L^p(\mathbf{R})$ into $L^q(\mathbf{R})$, where $p^{-1} + q^{-1} = 1$, it follows that $\{(\psi_n \circ \gamma)\hat{g}\}$ converges to $\hat{g} + (Rg)^\wedge$ in $L^q(\mathbf{R})$. By passing to a subsequence if necessary, it may be assumed that $\{(\psi_n \circ \gamma)\hat{g}\}$ converges a.e. on $\mathbf{R}$ to the function $\hat{g} + (Rg)^\wedge = \hat{g} + R\hat{g}$. Accordingly, the complement (in $\mathbf{R}$) of the set

$$F = \{\omega \in \mathbf{R}\,;\, \lim_{n\to\infty}(\psi_n \circ \gamma)(\omega)\hat{g}(\omega) = \hat{g}(\omega) + \hat{g}(-\omega)\}$$

is null for Lebesgue measure. Now, the set F can be decomposed into the disjoint union of three measurable sets, say U, V and W, where $W = F\setminus(U \cup V)$, $U = \{\omega \in F\,;\,\hat{g}(\omega) + \hat{g}(-\omega) \neq 0$ and $\hat{g}(\omega) = \hat{g}(-\omega)\}$ and $V = \{\omega \in F\,;\,\hat{g}(\omega) + \hat{g}(-\omega) \neq 0$ and $\hat{g}(\omega) \neq \hat{g}(-\omega)\}$. In addition, the sets U and W are symmetric about zero and $-V \subseteq (\mathbf{R}\setminus F)$. Accordingly, $-V$ and hence, also V, is a null set. So, U and W are disjoint, symmetric, measurable sets with $\mathbf{R}\setminus(U \cup W)$ a null set. It follows that

$$\hat{g} = \hat{g}\chi_U + \hat{g}\chi_W$$

where $\hat{g}\chi_U$ (resp. $\hat{g}\chi_W$) is an even (resp. odd) function a.e. on $\mathbf{R}$.

Let $h \in L^p(\mathbf{R})$. Then there exists a sequence $\{\varphi_n\}$ in $\mathrm{sim}(\mathscr{A}^{(p)}(\mathbf{R}^+))$ such that $\{(\varphi_n \circ \gamma)(D)g\}$ converges to h in $L^p(\mathbf{R})$ and hence, $\{(\varphi_n \circ \gamma)\hat{g}\}$ converges to $\hat{h}$ in $L^q(\mathbf{R})$. Passing to a subsequence if necessary, it follows, using the disjointness of U and W, that $\{(\varphi_n \circ \gamma)\hat{g}\chi_U\}$ converges to $\hat{h}\chi_U$ a.e. on $\mathbf{R}$ and $\{(\varphi_n \circ \gamma)\hat{g}\chi_W\}$ converges to $\hat{h}\chi_W$ a.e. on $\mathbf{R}$. Since $(\varphi_n \circ \gamma)\hat{g}\chi_U$ is an even function (a.e.) and $(\varphi_n \circ \gamma)\hat{g}\chi_W$ is an odd function (a.e.), for each $n = 1, 2, \ldots,$ it follows that $\hat{h}\chi_U$ is even (a.e.) and $\hat{h}\chi_W$ is odd (a.e). This is supposed to be true for arbitrary $h \in L^p(\mathbf{R})$ which it is clearly not. This gives the desired contradiction.

Acknowledgement. The support of the Centre for Mathematical Analysis (Australian National University) is gratefully acknowledged. The author wishes to thank Professor Michael Cowling for valuable discussions.

REFERENCES

1. BARRY, J. Y., On the convergence of ordered sets of projections, *Proc. Amer. Math. Soc.*, **5**(1954), 313–314.
2. COWLING, M., Square functions in Banach spaces, in *Miniconference on linear analysis and function spaces* (Canberra, 1984), *Proc. Centre Math. Anal.* (Australian National University), **9**(1985), 177–184.
3. DUNFORD, N.; SCHWARTZ, J. T., *Linear operators. III: Spectral operators*, Wiley-Interscience, New York, 1971.
4. GILLESPIE, T. A., A spectral theorem for L^p translations, *J. London Math. Soc. (2)*, **11**(1975), 499–508.
5. HÖRMANDER, L., Estimates for translation invariant operators in L^p spaces, *Acta Math.*, **104**(1960), 93–140.
6. KLUVÁNEK, I., Scalar operators and integration, *J. Austral. Math. Soc. (Ser. A)*, **45**(1988), 401–420.
7. KRABBE, G. L., Convolution operators that satisfy the spectral theorem, *Math. Z.*, **70**(1959), 446–462.
8. MCINTOSH, A., Operators which have an H^∞-functional calculus, in *Miniconference on operator theory and partial differential equations* (Maquarie University, 1986), *Proc. Centre Math. Anal.* (Australian National University), **14**(1986), 210–231.
9. RICKER, W., Spectral properties of the Laplace operator in $L^p(\mathbf{R})$, *Osaka J. Math.*, **25** (1988), 399–410.
10. RICKER, W., The commutant of the Laplace operator in $L^p(\mathbf{R})$, *Math. Nachr.*, to appear.
11. YAGI, A., Coincidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs, *C.R. Acad. Sci. Paris (Ser. I)*, **299**(1984), 173–176.

WERNER J. RICKER
Mathematisches Institut der Universität Tübingen,
Auf der Morgenstelle 10,
D-7400 Tübingen 1,
Federal Republic of Germany.

Permanent address:
School of Mathematics,
University of N. S. W.,
Kensington, N. S. W., 2033,
Australia.

Received September 10, 1987.