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$A \geqslant B \geqslant 0$ ENSURES $B^rA^pB^r \geqslant (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)}$ FOR $1 \geqslant 2r \geqslant 0$, $p \geqslant s \geqslant 0$ WITH $p+2r \geqslant 2s$

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An operator means a bounded linear operator on a complex Hilbert space. The purpose of this paper is to investigate "operator inequalities which preserve some order" on operators A and B in case $A \geqslant B \geqslant 0$. Our central results are as follows. If $A \geqslant B \geqslant 0$, then for each r such that $1/2 \geqslant r \geqslant 0$

(i) $$B^rA^pB^r \geqslant (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)}$$

and

$$(A^rB^{p-s}A^r)^{(p+2r)/(p-s+2r)} \geqslant A^rB^pA^r$$

hold for each p and s such that $p \geqslant s \geqslant 0$ and $p+2r \geqslant 2s$. As an immediate consequence of these results, we shall show an elementary self-contained and alternative proof of the following. If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$

(i) $$(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$$

and

(ii) $$A^{(p+2r)/q} \geqslant (A^rB^pA^r)^{1/q}$$

hold for each p and q such that $p \geqslant 0$, $q \geqslant 1$ and $(1+2r)q \geqslant p+2r$.

The rest of this paper is devoted to show some applications of these operator inequalities and also to give related counterexamples.

1. STATEMENT OF MAIN RESULT

An operator A on a Hilbert space $\mathscr{H}$ is said to be positive if $(Ax, x) \geqslant 0$ for every vector x in $\mathscr{H}$. It is well known that $A \geqslant B \geqslant 0$ does not always ensure $A^2 \geqslant B^2$ in general. Although there is such counterexample, we shall show "operator inequalities which preserve order in some sense".

THEOREM 1. *If $A \geqslant B \geqslant 0$, then for each r such that $1/2 \geqslant r \geqslant 0$*

(i) $$B^rA^pB^r \geqslant (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)}$$

and

(ii) $$(A^rB^{p-s}A^r)^{(p+2r)/(p-s+2r)} \geqslant A^rB^pA^r$$

hold for each p and s such that $p \geqslant s \geqslant 0$ and $p + 2r \geqslant 2s$.

COROLLARY 1. *If $A \geqslant B \geqslant 0$, then for each r such that $1/2 \geqslant r \geqslant 0$*

(i) $$B^rA^pB^r \geqslant (B^rAB^r)^{(p+2r)/(1+2r)}$$

and

(ii) $$(A^rBA^r)^{(p+2r)/(1+2r)} \geqslant A^rB^pA^r$$

hold for each p with $2(1 + r) \geqslant p \geqslant 1$.

COROLLARY 2. *If $A \geqslant B \geqslant 0$, then for each r such that $1/2 \geqslant r \geqslant 0$*

(i) $$B^rA^pB^r \geqslant (B^rA^{p/2-r}B^r)^2$$

and

(ii) $$(A^rB^{p/2-r}A^r)^2 \geqslant A^rB^pA^r$$

hold for each $p \geqslant 2r \geqslant 0$.

2. PROOFS OF THE RESULTS

First of all, we cite the following result [5].

THEOREM A. *If $A \geqslant B \geqslant 0$, then $A^\alpha \geqslant B^\alpha$ for each $\alpha \in [0, 1]$.*

Proof of Theorem 1. Let $A^{(p-s)/2}B^r = UH$ be the polar decomposition of $A^{(p-s)/2}B^r$, that is, U is a partial isometry and H is a positive operator such that $H = [(A^{(p-s)/2}B^r)^*(A^{(p-s)/2}B^r)]^{1/2}$ with the kernel condition $N(U) = N(H)$ where $N(S)$ means the kernel of an operator S. Then we have

(1) $$B^rA^pB^r = B^rA^{(p-s)/2}A^sA^{(p-s)/2}B^r = = HU^*A^sUH$$

and

(2) $$H^2 = HU^*UH = B^rA^{(p-s)}B^r$$

because U^*U is the initial projection. By Theorem A, for any $r \in [0, 1/2]$ we have

$$A^{p-s+2r} = A^{(p-s)/2}A^{2r}A^{(p-s)/2} \geqslant$$

$$\geqslant A^{(p-s)/2}B^{2r}A^{(p-s)/2} = UH^2U^*. \tag{3}$$

The hypotheses $p + 2r \geqslant 2s$ and $p \geqslant s \geqslant 0$ ensure $s/(p - s + 2r) \in [0, 1]$, so that (3) implies the following (4) by Theorem A since U^*U is the initial projection

$$A^s = (A^{p-s+2r})^{s/(p-s+2r)} \geqslant$$

$$\geqslant (UH^2U^*)^{s/(p-s+2r)} = UH^{2s/(p-s+2r)}U^*. \tag{4}$$

By (1), (4) and (2), we have the following (5) since U^*U is the initial projection

$$B^rA^pB^r \geqslant HU^*UH^{2s/(p-s+2r)}U^*UH = H^{2+2s/(p-s+2r)} =$$

$$= (H^2)^{(p+2r)/(p-s+2r)} = (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)} \tag{5}$$

so the proof of (i) is complete. We may assume the existence of B^{-1} without loss of generality. By hypothesis, $B^{-1} \geqslant A^{-1} \geqslant 0$. (i) ensures

$$A^{-r}B^{-p}A^{-r} \geqslant (A^{-r}B^{-(p-s)}A^{-r})^{(p+2r)/(p-s+2r)}$$

holds for r, p and s with the same conditions as (i) in Theorem 1. Taking inverses gives (ii), whence the proof of Theorem 1 is complete.

Proof of Corollary 1. Put $s = p - 1$ in Theorem 1, then the hypothesis $2(1 + r) \geqslant p \geqslant 1$ ensures the hypotheses in Theorem 1, so we have Corollary 1.

Proof of Corollary 2. Put $2s = p + 2r$ in Theorem 1, so $p \geqslant s \geqslant 0$ since $2p \geqslant p + 2r = 2s$ whenever $p \geqslant 2r \geqslant 0$, so we have Corollary 2.

3. APPLICATIONS OF THEOREM 1

Although the following results are shown in [3] using Hansen result [4], here we show an alternative self-contained proof by using Theorem 1.

THEOREM 2. *If $A \geqslant B \geqslant 0$, then for each r such that $1/2 \geqslant r \geqslant 0$*

$$(B^rA^pB^r)^{(1+2r)/(p+2r)} \geqslant B^{1+2r} \tag{i}$$

and

$$A^{1+2r} \geqslant (A^rB^pA^r)^{(1+2r)/(p+2r)} \tag{ii}$$

hold for each $p \geqslant 1$.

THEOREM B. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$*

(i) $$(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$$

and

(ii) $$A^{(p+2r)/q} \geqslant (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geqslant 0$, $q \geqslant 1$ and $(1+2r)q \geqslant p+2r$.

Proof of Theorem 2. For any given $p \geqslant 1$, there exists a nonnegative integer n such that $2^{n+1} \geqslant p \geqslant 2^n$. Then we have

(*) $$1 \geqslant p/2^{n+1} \geqslant p/2^n - 1 \geqslant 0.$$

As we can put $2s = p$ in Theorem 1 because this s satisfies $p \geqslant s \geqslant 0$ and $p + 2r \geqslant 2s$, we have

$$B^r A^p B^r \geqslant (B^r A^{p/2} B^r)^{(p+2r)/(p/2+2r)}.$$

Applying the same technique again in Theorem 1 for $p/2$ instead of p, we have

$$B^r A^{p/2} B^r \geqslant (B^r A^{p/2^2} B^r)^{(p/2+2r)/(p/2^2+2r)}.$$

Repeating this technique

(6) $$B^r A^{p/2^{k-1}} B^r \geqslant (B^r A^{p/2^k} B^r)^{q_k}$$

where $q_k = (p/2^{k-1} + 2r)/(p/2^k + 2r) \geqslant 1$ for $k = 1, 2, \ldots, n$.

Finally at $(n+1)$-th time we have the following (7) by Theorem 1

(7) $$B^r A^{p/2^n} B^r \geqslant (B^r A^{p/2^n - s} B^r)^{q_{n+1}}$$

where $q_{n+1} = (p/2^n + 2r)/(p/2^n - s + 2r) \geqslant 1$. We can choose s in Theorem 1 such that $s = p/2^n - 1$ in (*) because this s satisfies the conditions in Theorem 1 for $p/2^n$ instead of p, that is, $p/2^n \geqslant s \geqslant 0$ and $p/2^n + 2r \geqslant p/2^n \geqslant 2s$ by (*) Then we have

$$q_1 q_2 \ldots . q_n q_{n+1} = (p+2r)/(1+2r).$$

Combining (6) ($k = 1, 2, \ldots n$) and (7) for $s = p/2^n - 1$, Theorem A implies

$$(B^r A^p B^r)^{1/(q_1 q_2 \cdots q_n q_{n+1})} = (B^r A^p B^r)^{(1+2r)/(p+2r)} \geqslant B^r A B^r \geqslant B^{1+2r}$$

because $q_k \geqslant 1$ for $k = 1, 2, \ldots, n, n+1$. Whence we have (i). (ii) is also shown by the same way as the proof from (i) to (ii) in Theorem 1.

Proof of Theorem B. In case $1 \geqslant p \geqslant 0$, if $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ by Theorem A, whence the results follows by easy calculation. We have only to show the following (8) for each $r \geqslant 0$ and $p \geqslant 1$;

$$(B^rA^pB^r)^{(1+2r)/(p+2r)} \geqslant B^{1+2r} \tag{8}$$

since (i) of Theorem B for values q larger than $q = (p + 2r)/(1 + 2r) \geqslant 1$ follows by Theorem A. Put $C = (B^rA^pB^r)^{1/q}$ and $D = B^{1+2r}$.

Then $C \geqslant D$ by Theorem 2 for $1/2 \geqslant r \geqslant 0$, this inequality means that (8) holds for $1/2 \geqslant r \geqslant 0$. Repeating (8) again for $1/2 \geqslant r_1 \geqslant 0$ and $p_1 \geqslant 1$

$$(D^{r_1}C^{p_1}D^{r_1})^{1/q_1} \geqslant D^{1+2r_1}$$

for $q_1 = (p_1 + 2r_1)/(1 + 2r_1)$; namely,

$$\{B^{(1+2r)r_1}(B^rA^pB^r)^{p_1/q}B^{(1+2r)r_1}\}^{1/q_1} \geqslant B^{(1+2r)(1+2r_1)}.$$

Put $p_1 = q \geqslant 1$. Then we have

$$\{B^{(1+2r)r_1+r}A^pB^{r+(1+2r)r_1}\}^{1/q_1} \geqslant B^{(1+2r)(1+2r_1)}. \tag{9}$$

Put $r_2 = (1 + 2r)r_1 + r$, then $q_1 = (p_1 + 2r_1)/(1 + 2r_1) = (p + 2r_2)/(1 + 2r_2)$ since $p_1 = q$ and $(1 + 2r)(1 + 2r_1) = 1 + 2r_2$. Consequently (9) means that (8) holds for $r_2 \in [0, 3/2]$ since r, $r_1 \in [0, 1/2]$ and repeating this method, (8) holds for each $r \geqslant 0$, so (i) is shown. (ii) is also shown by the same way as the proof from (i) to (ii) in Theorem 1.

4. REMARKS

REMARK 1. In Theorem 1, the condition $1/2 \geqslant r \geqslant 0$ is essential. We cite a counterexample to Theorem 1 without this condition. Put $r = 0.51$, $p = 2$ and $s = 1.51$. Then $p \geqslant s \geqslant 0$, $p + 2r = 2s$ but $1/2 \ngeqslant r \geqslant 0$. Take B and C as

$$B = \begin{pmatrix} -1 & 3 \\ 3 & 5 \end{pmatrix}^2 \geqslant 0, \quad C = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}^2 \geqslant 0.$$

Then $A = B + C \geqslant B \geqslant 0$, but the computer shows

$$D = B^rA^pB^r - (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)} = \begin{pmatrix} 132855.9099\ldots & 219528.9898\ldots \\ 219528.9898\ldots & 362730.6429\ldots \end{pmatrix}.$$

The eigenvalues of D are $-4.1722\ldots$ and $495590.7251\ldots$. Whence $B^rA^pB^r \ngeqslant$ $\ngeqslant (B^rA^{p-s}B^r)^{(p+2r)/(p-s+2r)}$.

REMARK 2. In Theorem 1, the condition $p + 2r \geqslant 2s$ is essential. We cite a counterexample to Theorem 1 without this condition. Put $r = 0.5$, $p = 2$ and $s = 1.51$. Then $1/2 = r \geqslant 0$, $p \geqslant s \geqslant 0$ but $p + 2r \ngeqslant 2s$. Take B and C as

$$B = \begin{pmatrix} 5 & 1/2 \\ 1/2 & 3 \end{pmatrix}^2 \geqslant 0, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^2 \geqslant 0.$$

Then $A = B + C \geqslant B \geqslant 0$, but the computer shows

$$D = B^r A^p B^r - (B^r A^{p-s} B^r)^{(p+2r)/(p-s+2r)} = \begin{pmatrix} 10587.9860\ldots & 9112.6930\ldots \\ 9112.6930\ldots & 7835.1473\ldots \end{pmatrix}.$$

The eigenvalues of D are $-4.4902\ldots$. and $18427.6236\ldots$. Whence $B^r A^p B^r \ngeqslant (B^r A^{p-s} B^r)^{(p+2r)/(p-s+2r)}$.

REMARK 3. In Corollary 1, the condition $2(1 + r) \geqslant p \geqslant 1$ is essential. We cite also a counterexample to Corollary 1 without this condition. Put $r = 0.5$, $p = 3.01$. Then $1/2 = r \geqslant 0$, $p \geqslant 1$ but $2(1 + r) \ngeqslant p$. Take B and C as

$$B = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}^2 \geqslant 0, \quad C = \begin{pmatrix} 3 & -1 \\ -1 & 1/4 \end{pmatrix}^2 \geqslant 0.$$

Then $A = B + C \geqslant B \geqslant 0$, but the computer shows

$$D = B^r A^p B^r - (B^r A B^r)^{(p+2r)/(1+2r)} = \begin{pmatrix} 244030.0141\ldots & 176015.0777\ldots \\ 176015.0777\ldots & 126937.4248\ldots \end{pmatrix}.$$

The eigenvalues of D are $-12.8470\ldots$ and $370980.2861\ldots$. Whence $B^r A^p B^r \ngeqslant (B^r A B^r)^{(p+2r)/(1+2r)}$.

REMARK 4. When we give an alternative self-contained proof of Theorem 2, we iterate the following result.

PROPOSITION 1. *If* $A \geqslant B \geqslant 0$, *then*

$$(B^r A^p B^r)^{(p-s+2r)/(p+2r)} \geqslant B^r A^{p-s} B^r \tag{10}$$

and

$$(A^r B^{p-s} A^r)^{(p-s+2r)/(p+2r)} \geqslant A^r B^{p-s} A^r \tag{11}$$

hold under the same conditions as in Theorem 1.

As stated in the proof of Theorem 2, Proposition 1 is merely a simple corollary of Theorem 1 by Theorem A since $(p - s + 2r)/(p + 2r) \leqslant 1$. But Proposition 1 does not always imply Theorem 1. We would like to remark that an alternative proof of Proposition 1 can be given by Hansen result [4] by the same technique as in [3], but we may not give a direct proof of Theorem 1 by merely using Hansen result [4].

5. COUNTEREXAMPLES

In answer to the conjecture at the end of [1], we have given the following result [2].

THEOREM C. *Let A, B and C be nonnegative Hermitian matrices such that $C \geqslant A$ and $C \geqslant B$. There exist A, B and C such that*

$$\sqrt{2}C \geqslant (A^2 + B^2)^{1/2}$$

does not always hold.

The following variation on Theorem C sheds some further light on the question.

THEOREM D. *Let A, B and C be nonnegative Hermitian matrices such that $C \geqslant A$ and $C \geqslant B$. There exist A, B and C such that $kC \geqslant (A^p + B^p)^{1/p}$ will ail for any $p > 1$ and any positive constant $k > 0$.*

Proof of Theorem D. Take A, B and C on two dimensional Euclidean space as follows:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}, \quad C = A + B.$$

As easily seen, A and B are both projections and $C \geqslant A \geqslant 0$ and $C \geqslant B \geqslant 0$. Also we can verify that C has eigenvalues $2\cos^2(\theta/2)$ and $2\sin^2(\theta/2)$ and $(A^2 + B^2)^{1/2} = (A + B)^{1/2} = C^{1/2}$. Thus the following inequality

$$(**) \qquad \sqrt{2}C \geqslant (A^2 + B^2)^{1/2}$$

amounts to

$$\sqrt{2}C \geqslant C^{1/2}$$

that is,

$$\sqrt{2}\left(2\cos^2\frac{\theta}{2}\right) \geqslant \left(2\cos^2\frac{\theta}{2}\right)^{1/2} \quad \text{and} \quad \sqrt{2}\left(2\sin^2\frac{\theta}{2}\right) \geqslant \left(2\sin^2\frac{\theta}{2}\right)^{1/2},$$

equivalently

$$\cos\frac{\theta}{2} \geqslant \frac{1}{2} \quad \text{and} \quad \sin\frac{\theta}{2} \geqslant \frac{1}{2}.$$

The second inequality implies $\theta \geqslant \pi/3$, so for small θ the inequality (**) fails. By the same way, in fact, any inequality

$$kC \geqslant (A^p + B^p)^{1/p}$$

for any numbers $p > 1$ and $c > 0$ will fail for θ small enough.

We would like to remark that a counterexample in Theorem D also gives a counterexample to Theorem E and Theorem F.

THEOREM E. *Let A, B and C be nonnegative Hermitian matrices such that $C \geqslant A$ and $C \geqslant B$. There exist A, B and C such that*

$$kC^{(p+2r)/q} \geqslant \{C^r(A^p + B^p)C^r\}^{1/q}$$

will fail for any $p > 1$, $r \geqslant 0$, $q \geqslant 1$ and any positive constant k.

THEOREM F. *Let A, B and C be nonnegative Hermitian matrices such that $C \geqslant A$ and $C \geqslant B$. There exist A, B and C such that*

$$kC^{(1+2r)/q} \geqslant \{C^r(A^p + B^p)^{1/p}C^r\}^{1/q}$$

will fail for any $p > 1$, $r \geqslant 0$, $q \geqslant 1$ and any positive constant k.

6. OPERATOR INEQUALITIES AS AN APPLICATION OF THEOREM B

Although there exist counterexamples Theorem C, Theorem D, Theorem E and Theorem F, using Theorem B we can show the following results for operators on an arbitrary dimensional Hilbert space related to these counterexamples.

THEOREM 3. *If $C \geqslant A \geqslant 0$ and $C \geqslant B \geqslant 0$, then for each $r \geqslant 0$*

$$2^{p/q}C^{(p+2r)/q} \geqslant \{C^r(A + B)^p C^r\}^{1/q}$$

holds for each p and q such that $p \geqslant 0$, $q \geqslant 1$ and $(1 + 2r)q \geqslant p + 2r$.

COROLLARY 3. *If $C \geqslant A \geqslant 0$ and $C \geqslant B \geqslant 0$, then for each $r \geqslant 0$*

$$2^{p(1+2r)/(p+2r)}C^{1+2r} \geqslant \{C^r(A + B)^p C^r\}^{(1+2r)/(p+2r)}$$

holds for each $p \geqslant 1$.

COROLLARY 4. *If* $C \geqslant A \geqslant 0$ *and* $C \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$

$$2C^{(p+2r)/p} \geqslant \{C^r(A+B)^p C^r\}^{1/p}$$

holds for each $p \geqslant 1$.

COROLLARY 5. *If* $C \geqslant A \geqslant 0$ *and* $C \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$

$$2^{p/(p+2r)}C \geqslant \{C^r(A+B)^p C^r\}^{1/(p+2r)}$$

holds for each $p \geqslant 1$.

COROLLARY 6. *If* $C \geqslant A \geqslant 0$ *and* $C \geqslant B \geqslant 0$, *then*

$$2C^2 \geqslant \{C(A+B)^2 C\}^{1/2}.$$

Proof of Theorem 3. $2C \geqslant A + B \geqslant 0$ by the hypotheses, so that the proof of Theorem 3 follows by (ii) of Theorem B.

Proof of Corollary 3. Put $q = (p+2r)/(1+2r)$ and $p \geqslant 1$ in Theorem 3.

Proof of Corollary 4. Put $p = q$ in Theorem 3.

Proof of Corollary 5. Put $q = p + 2r$ and $p \geqslant 1$ in Theorem 3.

Proof of Corollary 6. Put $p = 2$ and $r = 1$ in Corollary 4.

We would like to express our gratitude to Professor Roger Howe for kindly giving us the counterexample in Theorem D.

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Added in proof. We would like to cite several proofs to Theorem B as follows:

6. FUJII, M., Furuta's inequality and its mean theoretic approach, preprint.
7. FURUTA, T., A proof via operator means of an order preserving inequality, *Linear Algebra Appl.*, **113**(1989), 129—130.
8. FURUTA, T., An elementary proof of an order preserving inequality, preprint.
9. KAMEI, E., A satellite to Furuta's inequality, *Math. Japon.*, **33**(1988), 883—886.

Finally we would like to cite the following papers on some applications of Theorem B.

10. BACH, E.; FURUTA, T., Order preserving operator inequalities, *J. Operator Theory*, **19**(1988), 341—346.
11. FURUTA, T. The operator equation $T(H^{1/n}T)^n = k$, *Linear Algebra Appl.*, **109**(1988), 149—152.