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LIFTINGS AND EXTENSIONS OF MAPS ON C^*-ALGEBRAS

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1. INTRODUCTION

Let B be a C^*-algebra with a two sided closed ideal J and let $\pi : B \to B/J$ be the quotient homomorphism. If E is either an operator system or a C^*-algebra then a map $\varphi : E \to B/J$ is said to have a lifting, or be liftable, if there exists a map $\psi : E \to B$ such that $\varphi = \pi\psi$. If E is contained in B/J and φ is the identity then the map is suppressed and ψ is said to be a lifting of E. Usually conditions are imposed on φ, and lifting maps of the same type are sought. If E is a finite dimensional operator system then unital positive maps have unital positive liftings [2], contractive maps have contractive liftings [11], but completely positive maps need not have completely positive liftings [3]. In light of these results we formulate and prove the strongest possible general lifting theorem. This asserts that every finite dimensional operator system in B/J has an n-positive unital lifting (Proposition 2.4), for each integer $n \geqslant 1$. The lifting theorems of Andersen [2] and Choi-Effros [11] are both special cases. The lifting of finite dimensional operator systems is a prelude to the lifting of C^*-algebras, as in [2] for C^*-algebras with the positive approximation property, or as in [10] for nuclear C^*-algebras. Here, we obtain n-positive liftings for separable C^*-algebras which have the n-positive approximation property.

In the third section this theory is applied to the reduced C^*-algebra of F_2, the free group on two generators, in order to answer a question which arises from recent work of Størmer [24]. In that paper the problem of extending a positive map $\varphi : A \to B(H)$ to a larger C^*-algebra B was considered, and this was shown to be possible if A is nuclear. An example was given to the contrary in the non-nuclear case in [18] and we strengthen this to n-positive maps here. Examples of non-extendible maps on operator systems have been known for some time [4], [24] and we incorporate these in a general theory. The fourth section is concerned with a short application to matrix ranges. Background material is to be found in [20], [21].

It is assumed throughout that all C^*-algebras and operator systems are unital, with 1 representing the identity. The C^*-algebra of $n \times n$ matrices is denoted M_n, and if $\varphi : A \to B$ is a linear map then $\varphi \otimes \mathrm{id}_n : A \otimes M_n \to A \otimes M_n$ is defined by $\varphi \otimes \mathrm{id}_n(a_{ij}) = (\varphi(a_{ij}))$. φ is n-positive if $\varphi \otimes \mathrm{id}_n$ is positive, and is completely positive if every $\varphi \otimes \mathrm{id}_n$ is positive.

2. n-POSITIVE LIFTINGS

In [2] Andersen proved, as a corollary of a difficult selection theorem for $A(K)$ spaces, that any finite dimensional operator system E in a quotient C^*-algebra B/J has a positive unital lifting $\varphi : E \to B$. A consequence of this result was the existence of positive unital liftings $\varphi : B/J \to B$ whenever B/J is separable and has the positive approximation property [2, Theorem 7]. We wish to obtain a generalization of this to n-positive liftings, but first take the opportunity to provide a simple proof of Andersen's theorem.

THEOREM 2.1 (Andersen). *Let E be a finite dimensional operator system in a quotient C^*-algebra B/J. Then there exists a unital positive lifting $\varphi : E \to B$.*

Proof. Working with a basis for E consisting of self-adjoint elements and the identity, it is easy to construct a self-adjoint unital lifting $\psi : E \to B$, which is automatically bounded since E is finite dimensional. Let

$$K = \{\psi(a)_- : a \in E,\ \|a\| = 1,\ a \geqslant 0\}$$

where x_- denotes the negative part of a self-adjoint C^*-algebra element x. Then K is a compact subset of J and is thus contained in $\overline{\mathrm{conv}}\{x_n\}$ for some sequence $\{x_n\}_{n=1}^{\infty} \subseteq J$ satisfying $\lim_{n\to\infty} \|x_n\| = 0$ [14]. By reordering $\{x_n\}$ if necessary, there exist integers $n_1 < n_2 < n_3 \ldots$ such that

$$2^{-(k+1)} \leqslant \|x_n\| < 2^{-k} \quad \text{for } n_k \leqslant n < n_{k+1}.$$

Since the positive part of the open unit ball of a C^*-algebra is upward filtering [15] there exists a sequence $\{j_k\}_{k=1}^{\infty} \subseteq J$ such that

$$\|j_k\| < 2^{-k} \quad \text{and} \quad j_k \geqslant x_n \quad \text{for} \quad n_k \leqslant n < n_{k+1}.$$

Define $j = \sum_{r=1}^{n_1-1} x_r + \sum_{k=1}^{\infty} j_k$ and observe that $j \geqslant x$ for all $x \in K$.

By compactness there exists a positive linear functional $\theta \in E^*$ such that $\theta(a) \geqslant \|a\|$ for all $a \in E^+$. Define $v: E \to B$ by $v(a) = \psi(a) + \theta(a)j$. Then if $a \in E^+$,

$\|a\| = 1$,

$$\nu(a) = \psi(a)_+ + (\theta(a)j - \psi(a)_-) \geqslant$$

$$\geqslant \psi(a)_+ + (j - \psi(a)_-) \geqslant 0$$

by the construction of j. Thus ν is a positive lifting of E, and it only remains to make it unital. Following [6], write $\nu(1) = 1 + k_1 - k_2$ where $k_1, k_2 \in J^+$ and choose a state ω on E. Define $\varphi : E \to B$ by

$$\varphi(a) = (1 + k_1)^{-\frac{1}{2}}(\nu(a) + \omega(a)k_2)(1 + k_1)^{-\frac{1}{2}}, \quad (a \in E)$$

and observe that φ is a unital positive lifting of E.

In order to generalize this theorem, the following technical lemma will be needed.

LEMMA 2.2. *Let E be an operator system and let B be a C^*-algebra. If $\psi : E \otimes M_n \to B \otimes M_n$ satisfies*

$$\psi(U^*XU) = U^*\psi(X)U$$

for all $X \in E \otimes M_n$ and all unitary matrices $U \in M_n$ then there exist $\varphi, \lambda : E \to B$ such that

$$\psi(X) = \varphi \otimes \mathrm{id}_n(X) + \lambda(\mathrm{trace}\, X) \otimes I_n, \quad X \in E \otimes M_n.$$

Proof. To avoid technical complications only the case $n = 2$ will be discussed. The calculations for $n \geqslant 3$ are in the same spirit, depending on a number of matrix identities.

(1) For $a \in E$, the identity

$$\begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{-it} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

leads to

$$\begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}\psi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{-it} & 0 \\ 0 & 1 \end{pmatrix} = \psi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and so $\psi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix. Thus there exist maps $\lambda, \mu : E \to B$ such that

$$\psi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mu(a) & 0 \\ 0 & \lambda(a) \end{pmatrix}.$$

(2) The identity

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}$$

gives

$$\psi\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}=\begin{pmatrix}\lambda(a) & 0\\ 0 & \mu(a)\end{pmatrix}.$$

(3) For any $t \in \mathbf{R}$

$$\begin{pmatrix}e^{it} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}e^{-it} & 0\\ 0 & 1\end{pmatrix}=e^{it}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}$$

and so

$$\begin{pmatrix}e^{it} & 0\\ 0 & 1\end{pmatrix}\psi\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}e^{-it} & 0\\ 0 & 1\end{pmatrix}=e^{it}\psi\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}.$$

Thus there exists a linear map $\varphi : E \to B$ such that

$$\psi\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \varphi(a)\\ 0 & 0\end{pmatrix}.$$

(4) Since $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ a & 0\end{pmatrix}$ it follows from (3) that

$$\psi\begin{pmatrix}0 & 0\\ a & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ \varphi(a) & 0\end{pmatrix}.$$

(5) The identity

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0 & a\\ a & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}$$

leads to

$$\begin{pmatrix}\varphi(a) & 0\\ 0 & -\varphi(a)\end{pmatrix}=\begin{pmatrix}\mu(a) & 0\\ 0 & \lambda(a)\end{pmatrix}-\begin{pmatrix}\lambda(a) & 0\\ 0 & \mu(a)\end{pmatrix}$$

and so $\varphi = \mu - \lambda$.

(6) Finally

$$\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu(a) & 0 \\ 0 & \lambda(a) \end{pmatrix} + \begin{pmatrix} 0 & \varphi(b) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varphi(c) & 0 \end{pmatrix} + \begin{pmatrix} \lambda(d) & 0 \\ 0 & \mu(d) \end{pmatrix} =$$

$$= \begin{pmatrix} \varphi(a) + \lambda(a+d) & \varphi(b) \\ \varphi(c) & \varphi(d) + \lambda(a+d) \end{pmatrix} =$$

$$= \varphi \otimes \mathrm{id}_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \lambda\left(\mathrm{trace}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \otimes I_2,$$

and the proof is complete.

If E is a finite dimensional operator system then a simple compactness argument gives a strictly positive state $\theta_0 \in E^*$ and a constant $k \in (0, 1)$ such that

$$\theta_0(a) \geqslant k\|a\| \quad (a \in E^+).$$

Many different choices of θ_0 and k are possible, but we fix one now for the remainder of the section.

Given any state α on E, the linear functional $\beta = (1-k)^{-1}(\theta_0 - k\alpha)$ is also a state, and so $\theta_0 = k\alpha + (1-k)\beta$. Write

$$E_0 = \{a \in E : \theta_0(a) = 0\}.$$

LEMMA 2.3. *If* $A \in E_0 \otimes M_n$ *satisfies*

$$I_n + A \geqslant 0$$

then $\|\mathrm{trace}\, A\| \leqslant n/k$.

Proof. Clearly A is self-adjoint and so if A is written in matrix form as (a_{ij}) with $a_{ij} \in E_0$, then each diagonal entry a_{ii} is self-adjoint, and $1 + a_{ii} \geqslant 0$. If α is any state on E, let β be the state satisfying

$$\theta_0 = k\alpha + (1-k)\beta.$$

Then

$$1 + \alpha(a_{ii}) \geqslant 0, \quad 1 + \beta(a_{ii}) \geqslant 0$$

from which it follows that

$$\alpha(a_{ii}), \beta(a_{ii}) \geqslant -1.$$

On the other hand

$$0 = \theta_0(a_{ii}) = k\alpha(a_{ii}) + (1-k)\beta(a_{ii}),$$

and so

$$\alpha(a_{ii}) = -\left(\frac{1}{k} - 1\right)\beta(a_{ii}) \leqslant \left(\frac{1}{k} - 1\right) \leqslant \frac{1}{k}$$

since $\beta(a_{ii}) \geqslant -1$. Thus

$$-1 \leqslant \alpha(a_{ii}) \leqslant 1/k$$

and since α was an arbitrary state, $\|a_{ii}\| \leqslant 1/k$. Consequently

$$\|\text{trace}\, A\| = \left\| \sum_{i=1}^{n} a_{ii} \right\| \leqslant \sum_{i=1}^{n} \|a_{ii}\| \leqslant n/k.$$

PROPOSITION 2.4. *Let E be a finite dimensional operator system in B/J. Then for each integer $n \geqslant 1$, E has an n-positive unital lifting.*

Proof. By Theorem 2.1 there exists a positive lifting $\varphi : E \otimes M_n \to B \otimes M_n$. Let G be the compact unitary group of M_n with normalized Haar measure μ, and define a new positive lifting ψ of $E \otimes M_n$ by

$$\psi(X) = \int_G U\varphi(U^* X U)U^* \,\mathrm{d}\mu(U)$$

for $X \in E \otimes M_n$. Then if $V \in G$

$$\psi(V^* X V) = \int_G V^*(VU)\varphi((VU)^* X(VU))(VU)^* V \,\mathrm{d}\mu(U) =$$

$$= V^* \psi(X) V$$

by invariance of μ. By Lemma 2.2 there exist maps τ, $\lambda : E \to B$ such that

$$\psi(X) = \tau \otimes \mathrm{id}_n(X) + \lambda(\text{trace}\, X) \otimes I_n, \quad (X \in E \otimes M_n).$$

If $a \in E^+$ then $a \otimes P_1 \geqslant 0$, where

$$P_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

and so $\psi(a \otimes P_1) \geqslant 0$. From this it follows that $\lambda(a) \geqslant 0$ by examination of the

(2, 2) entry, and thus $\lambda \geqslant 0$. In addition $\lambda(a) \in J$ and so $\lambda : E \to J$. Let $j_0 = \lambda(1) \in J^+$ and define $\sigma : E \to B$ by

$$\rho(a) = \tau(a) + \frac{2n}{k}\theta_0(a)j_0.$$

We now check that σ is an n-positive map.

If $X \in (E \otimes M_n)^+$ then X may be expressed as

$$X = M + Y, \quad M \in M_n, \quad Y \in E_0 \otimes M_n.$$

States are completely positive [22], and so $\theta_0 \otimes \mathrm{id}_n(X) \geqslant 0$, leading to $M \geqslant 0$ since Y is annihilated. If $\sigma \otimes \mathrm{id}_n(X + \varepsilon I_n) \geqslant 0$ for all $\varepsilon > 0$ then $\sigma \otimes \mathrm{id}_n(X) \geqslant 0$, and so without loss of generality we assume that M is invertible. Then there exists an invertible matrix T such that $T^*MT = I_n$, and so

$$T^*XT = I_n + T^*YT \geqslant 0.$$

Clearly

$$\sigma \otimes id_n(X) = T^{-1*}\sigma \otimes \mathrm{id}_n(T^*XT)T^{-1}$$

and so it suffices to check positivity of $\sigma \otimes \mathrm{id}_n$ on positive matrices of the form $I_n + A$ where $A \in E_0 \otimes M_n$.

By Lemma 2.3 $\|\mathrm{trace}\, A\| \leqslant n/k$ and thus $\|\mathrm{trace}(I_n + A)\| \leqslant n + n/k \leqslant 2n/k$ since $k < 1$. Then

$$\sigma \otimes \mathrm{id}_n(I_n + A) = \tau \otimes \mathrm{id}_n(I_n + A) + \frac{2n}{k} j_0 \otimes I_n =$$

$$= \psi(I_n + A) + \left(\frac{2n}{k} j_0 - \lambda(\mathrm{trace}(I_n + A)) \otimes I_n\right) \geqslant$$

$$\geqslant \psi(I_n + A) + \left(\frac{2n}{k} j_0 - \frac{2n}{k} j_0\right) \otimes I_n$$

since $\lambda \geqslant 0$ and $\mathrm{trace}(I_n + A) \leqslant (2n/k)1$. Thus

$$\sigma \otimes \mathrm{id}_n(I_n + A) \geqslant \psi(I_n + A) \geqslant 0$$

and σ is an n-positive lifting of E. The final argument, modifying σ to a unital n-positive lifting, has been given at the end of Theorem 2.1.

COROLLARY 2.5. *If E is a finite dimensional operator system in B/J then for each integer $n \geqslant 1$ there exists an n-isometric unital lifting of E.*

Proof. Let $\varphi: E \to B$ be a $2n$-positive unital lifting of E, by Proposition 2.4. If $X \in E \otimes M_n$ then $\|X\| \leqslant 1$ if and only if $\begin{pmatrix} I_n & X \\ X^* & I_n \end{pmatrix} \geqslant 0$ in $E \otimes M_{2n}$. Then $\begin{pmatrix} I_n & \varphi \otimes \mathrm{id}_n(X) \\ \varphi \otimes \mathrm{id}_n(X)^* & I_n \end{pmatrix} \geqslant 0$, and so $\|\varphi \otimes \mathrm{id}_n(X)\| \leqslant 1$. Since any lifting is norm non-decreasing, it follows that φ is n-isometric.

We say that a C^*-algebra A has the n-positive (respectively n-contractive) approximation property if there exists a net $\{T_\lambda : A \to A\}_{\lambda \in \Lambda}$ of finite rank n-positive (respectively n-contractive) operators converging in the point norm topology to I. If A is separable then the net may be replaced by a sequence of operators $\{T_r\}_{r=1}^\infty$.

THEOREM 2.6. *Let A be a separable C^*-algebra with the n-positive (respectively n-contractive) approximation property, and let $\varphi: A \to B/J$ be an n-positive (respectively n-contractive) map. Then there exists an n-positive (respectively n-contractive) lifting $\psi: A \to B$ of φ, which may be chosen to be unital if φ is unital.*

Proof. Let $\{T_r\}_{r=1}^\infty$ be the sequence of approximating maps and let E_r be the operator system in B/J spanned by $\varphi T_r(A)$ and 1. By Proposition 2.4 or Corollary 2.5 there exists an n-positive (respectively n-contractive) lifting $\psi_r : E_r \to B$ and so $\varphi T_r : A \to B/J$ has an n-positive (respectively n-contractive) lifting $\psi_r \varphi T_r : A \to B$. The argument given by Arveson in [6] is valid here, and so the set of n-positive (respectively n-contractive) liftable maps of A into B/J is closed in the point norm topology. Since we have shown that each φT_r is liftable, we conclude that φ has an n-positive (respectively n-contractive) lifting ψ.

In the n-positive case the argument at the end of Theorem 2.1 will modify ψ to be unital and n-positive. If φ is unital and n-contractive then φ is n-positive and so has an n-positive lifting ψ. Since A is a C^*-algebra ψ is also n-contractive and so the proof is complete.

REMARK. The n-contractive part of this theorem generalizes the C^*-algebra version of a result of Choi-Effros [11, Theorem 2.6].

3. EXTENSIONS OF POSITIVE MAPS

The starting point for this section is a recent result due to Stormer [24] which asserts that if $A \subseteq B$ are unital C^*-algebras with A nuclear, then every unital positive map $\varphi: A \to B(H)$ has a positive extension $\psi: B \to B(H)$. Note that the completely positive version (with the nuclearity requirement dropped) is Arverson's Hahn-Banach theorem [4]. We begin by strengthening this result to n-positive maps.

PROPOSITION 3.1. *Let $A \subseteq B$ be unital C^*-algebras with A nuclear. Then every n-positive unital map $\varphi: A \to B(H)$ has an n-positive extension $\psi: B \to B(H)$.*

Proof. $A \otimes M_n$ is nuclear and $\varphi \otimes \mathrm{id}_n : A \otimes M_n \to B(H \otimes \mathbf{C}^n)$ is positive, so by [24, Theorem 3.14] there exists a positive unital extension $\lambda: B \otimes M_n \to$ $\to B(H \otimes \mathbf{C}^n)$. Since $B \otimes M_n$ is a C^*-algebra, λ is contractive. As in Proposition 2.4, let G be the unitary group of M_n with normalized Haar measure μ, which is both left and right invariant since G is compact. Define $\eta : B \otimes M_n \to B(H \otimes \mathbf{C}^n)$ by

$$\eta(X) = \iint_{G \times G} U\lambda(U^*XV)V^* \mathrm{d}\mu(U)\mathrm{d}\mu(V), \quad (X \in B \otimes M_n).$$

The invariance of μ implies that

$$\eta(CXD) = C\eta(X)D$$

for $X \in B \otimes M_n$, $C, D \in G$, and by linearity this relation holds for any matrices $C, D \in M_n$. By considering suitable choices of C and D as matrix units in M_n, it is easy to check that η has the form $\psi \otimes \mathrm{id}_n$ for some map $\psi: B \to B(H)$.

Now λ is contractive, and so η is contractive, from the definition. Moreover, if $X \in A \otimes M_n$ then, for $U, V \in G$,

$$\lambda(U^*XV) = \varphi \otimes \mathrm{id}_n(U^*XV) = U^*\varphi \otimes \mathrm{id}_n(X)V$$

and so

$$\varphi \otimes \mathrm{id}_n(X) = \iint_{G \times G} U\lambda(U^*XV)V^* \mathrm{d}\mu(U)\mathrm{d}\mu(V) = \eta(X) = \psi \otimes \mathrm{id}_n(X).$$

Thus ψ is an n-contractive extension of the unital map φ, and so is an n-positive extension.

REMARK. The restriction that φ be unital is not essential. If φ is n-positive and $\varphi(1) = T \in B(H)$ then, modifying an argument of Choi-Effros [12, Lemma 2.2], there exists a unital n-positive map $\xi: A \to B(H)$ such that $\varphi(a) = T^{1/2}\xi(a)T^{1/2}$ for $a \in A$. Proposition 3.1 can then be applied to ξ.

In [24] Stormer conjectured that his extension theorem was no longer true if A were not assumed to be nuclear. An example to this effect was given in [18], but we present here a related and stronger result. Let F_2 denote the free group on two generators. When F_2 acts on $K = l_2(F_2)$ by left translation the resulting C^*-subalgebra of $B(K)$ is denoted $C^*_\lambda(F_2)$ and is a quotient of the group C^*-algebra $C^*(F_2)$ by an ideal J. Let $C^*(F_2)$ be faithfully represented on some separable Hilbert space H.

THEOREM 3.2. *For each integer $n \geqslant 1$ there exists an n-positive unital map $\varphi_n: C_\lambda^*(F_2) \to B(H)$ which has no positive extension to $B(K)$.*

Proof. $C_\lambda^*(F_2)$ is separable and has the n-positive approximation property for each positive integer n [13]. By Theorem 2.6, applied to the identity map of $C_\lambda^*(F_2)$ onto $C^*(F_2)/J$, there exists for each $n \geqslant 1$ an n-positive unital lifting $\varphi_n: C_\lambda^*(F_2) \to \to C^*(F_2)$. Since $C^*(F_2) \subseteq B(H)$ we may regard the larger algebra as the range of φ_n. The verification that no φ_n has a positive extension to $B(K)$ now follows by combining the argument of [18] with that of [11, Theorem 4.5] to show that the quotient map $C^*(F_2) \to C_\lambda^*(F_2)$ does not lift to an extendible positive map.

COROLLARY 3.3. *For each integer $n \geqslant 1$ there exists an integer $k(n)$ and an n-positive unital map $\psi_n: C_\lambda^*(F_2) \to M_{k(n)}$ which has no positive extension to $B(K)$.*

Proof. Suppose that this were false for some integer n. Then all unital n-positive maps of $C_\lambda^*(F_2)$ into matrix algebras would have positive extensions to $B(K)$. Since $B(H)$ is the w*-closure of an increasing union of matrix algebras, a simple limit argument would then imply that $\varphi_n: C_\lambda^*(F_2) \to B(H)$, constructed in Theorem 3.2, has a positive extension to $B(K)$. A contradiction has been reached.

REMARK. With some simple exceptions, $B(H)$ could be replaced in Theorem 3.2 by any unital C^*-algebra. Let A be a unital C^*-algebra which has irreducible representations on Hilbert spaces of arbitrarily large dimension (including ∞). Fix $n \geqslant 1$ and recall from [19] that there exist completely positive maps $\tau_r: M_{k(n)} \to A$, $\sigma_r: A \to M_{k(n)}$ such that $\|\sigma_r\tau_r - \mathrm{id}\| < 1/r$ for $r \geqslant 1$. If $\psi_n: C_\lambda^*(F_2) \to M_{k(n)}$ is the non-extendible n-positive map of Corollary 3.3 then $\tau_r\psi_n: C_\lambda^*(F_2) \to A$ is n-positive. At least one of these must be non-extendible otherwise, after composition with σ_r, a limit argument would construct a positive extension of ψ_n.

We now construct non-extendible positive maps on operator systems by methods which allow us to obtain the explicit examples of [4, A2] and [24] as special cases. These positive maps are actually order isomorphisms. We need a general result, which is an abstraction of the proof of [24, Proposition 3.15].

LEMMA 3.4. *Let E, E', F, F' be operator systems with $E \subset E'$ and $F \subset F'$. Suppose that there is a positive unital projection π from F' onto F and a positive linear map φ from E onto F which has a positive right inverse ψ^{-1}.*

Then φ extends to a positive linear map $\bar{\varphi}$ from E' into F' if and only if there is a positive unital projection from E' onto E.

Proof. If such an extension $\bar{\varphi}$ exists, then the map $\varphi^{-1}\bar{\varphi}$ is a positive unital projection from E' onto E. Conversely if there is a positive projection ρ from E' onto E then $\bar{\varphi} = \varphi\rho$ is a positive extension of φ.

As a first application, we assume that $F' = B(H)$ for some Hilbert space H, F is the range of a positive unital projection on $B(H)$ and F is not order isomorphic to an abelian C^*-algebra. (For example F could be an injective non-abelian C^*-algebra.) Let X be the w*-closure of the pure states of F and define the canonical map $\psi: F \to C(X)$ by $\psi(x)(p) = p(x)$, $x \in F$, $p \in X$. Then ψ is an order isomorphism from F onto an operator system $E \subseteq C(X)$. Let $\varphi = \psi^{-1}$.

PROPOSITION 3.5. *φ does not extend to a positive map $\bar{\varphi}$: $C(X) \to B(H)$.*

Proof. If such an extension did exist then E would be the range of a positive projection on $C(X)$ and hence would be order-isomorphic to an abelian C^*-algebra [17, Theorem 4] contrary to assumption.

A simple concrete example is provided by the case where $F = V + iV$, with V a finite dimensional spin factor [1]. Thus $F = \text{span}\{1, s_1, \ldots, s_n\}$, where $s_1, \ldots, s_n$ are symmetries in $B(H)$ satisyfing $s_j s_k + s_k s_j = 2\delta_{jk}1$. According to [16] there exists a positive projection from $B(H)$ onto F. The pure state space X of F may be identified with the unit sphere in $\mathbf{R}^n$ [1]. It is readily verified that in this case E is the linear span of 1 and the coordinate functions $x_1, \ldots, x_n$. The order isomorphism φ is given by

$$\varphi(\alpha_0 1 + \alpha_1 x_1 + \ldots + \alpha_n x_n) = \alpha_0 1 + \alpha_1 s_1 + \ldots + \alpha_n s_n.$$

When $n = 2$, $s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $X = T$, the unit circle in $\mathbf{R}^2$, we recover Arveson's example [4, A2], together with a new proof that φ is not extendible.

The operator system E constructed by the above method is of necessity infinite dimensional. The following example, which was shown to us by M.-D. Choi, shows that we can choose E to be finite dimensional if we drop the requirement that φ be an order isomorphism.

Let $E = \{(a, b, c, d) \in \ell_4^\infty : a - b = c - d\}$, and define $\varphi: E \to M_2$ by $\varphi(a, b, c, d) = \frac{1}{2}\begin{pmatrix} a + b & a - b \\ c - d & c + d \end{pmatrix}$. Then φ is positive and unital but $\|\varphi(1 + i, 1 - i, -1 + i, -1 - i)\| = 2$, so that $\|\varphi\| > 1$. Hence φ does not extend to a positive map on ℓ_4^∞.

PROPOSITION 3.6. *Let A be an injective C^*-algebra and $\varphi: E \to A \subseteq B(H)$ be as in Proposition 3.5. The following are equivalent:*

(1) *φ is extendible (to a positive map $\bar{\varphi}$: $C(X) \to B(H)$),*

(2) $\|\varphi\| = 1$,

(3) *A is abelian.*

Proof. (1) $\Rightarrow$ (2). $\bar{\varphi}$ is unital, so $\|\varphi\| = \|\bar{\varphi}\| = 1$.

(2) $\Rightarrow$ (3). If A is nonabelian then there exists $a \in A$, $\|a\| = 1$ such that $a^2 = 0$. Then $|f(a)| \leqslant 1/2$ for each state of A. Thus $\|\psi(a)\| \leqslant 1/2$ (where $\psi: A \to C(X)$ is the canonical order isomorphism defined by $\psi(a)(f) = f(a)$). Now $\varphi = \psi^{-1}$ so $\|\varphi(\psi(a))\| = 1 = \|a\| \geqslant 2\|\psi(a)\|$. Therefore $\|\psi\| \geqslant 2$.

(3) $\Rightarrow$ (1). If A is abelian then φ is completely positive [22] and so φ is extendible [4].

We now study non-commutative operator systems and non-extendible order-isomorphisms on them from a rather different viewpoint from that adopted in [24, Proposition 3.15]. The following result is obtained by modifying the arguments in [5, page 286] to deal with positive, instead of completely positive maps. $K(H)$ denotes the algebra of compact operators on a Hilbert space H.

THEOREM 3.7. *Let $\varphi: B(H) \to B(H)$ be a positive unital map and let $F = \{x \in B(H) : \varphi(x) = x\}$ be irreducible. Suppose that there exists $x_0 \in F$, such that $d(x_0, K(H)) < \|x_0\|$. Then F_{sa} is a* JC*-algebra.*

Proof. This closely follows the argument in [5]. First note that there is a central projection $e \in B(H)^{**}$ such that $B(H)$ is $*$-isomorphic to $B(H)^{**}e$ and $K(H)(1-e)=0$. Note also that e is a minimal projection, since $B(H)^{**}e$ is a factor. By [5, p. 286], there exists a normal positive projection ψ on $B(H)^{**}$ such that $\psi\varphi = \psi$ on $B(H)$ and $F \subseteq \psi(B)$. Let p be the support projection of ψ. Then $px = xp$ for all $x \in F$ [16, Lemma 1.2(2)]. Hence $pex = pxe = xpe$ for all $x \in F$, so that pe is a central projection in $B(H)^{**}e$. Thus $pe = 0$ or $pe = 1$. We prove that the first possibility does not occur.

Suppose that $pe = 0$. Then $x_0 = \psi(x_0) = \psi(px_0p) = \psi(p(1-e)x_0(1-e)p) = \psi((1-e)x_0)$. Now there exists $y \in K(H)$ such that $\|x_0 - y\| < \|x_0\|$ and $(1-e)y = 0$. Therefore $\|x_0\| = \|\psi((1-e)(x_0 - y))\| \leqslant \|x_0 - y\| < \|x_0\|$. This contradiction shows that $pe \neq 0$. Hence $pe = e$.

To complete the proof it is enough to show that F_{sa} is a Jordan algebra: that is $x^2 \in F_{sa}$ whenever $x \in F_{sa}$. Let $z = \varphi(x^2) = x^2$. Then $z = \varphi(x^2) - \varphi(x)^2 \geqslant 0$ and $\psi(z) = 0$, since $\psi\varphi = \psi$ on $B(H)$. It follows that $pzp = 0$, and therefore $ez = 0$, since $pe = e$. Hence $z = 0$, since the map $x \to ex$ is an isomorphism. Thus $\varphi(x^2) = x^2$. This proves the result.

COROLLARY 3.8. *If π is a positive unital projection on $B(H)$ whose range is irreducible and contains a non-zero compact operator, then $\pi(B(H))_{sa}$ is a* JC*-algebra.*

COROLLARY 3.9. *Let E be an irreducible operator system which contains a non-zero compact operator. Let F be an operator system on a Hilbert space K which is the range of a positive unital projection on $B(K)$. Suppose that $\varphi: E \to F$ is an order-isomorphism from E onto F. If φ is extendible then E_{sa} is a* JC*-algebra.*

Proof. By Lemma 3.4 there exists a positive unital projection from $B(H)$ onto E. Therefore E_{sa} is a JC-algebra by Corollary 3.8.

REMARK. The fact that the positive map of [24, Example 3.16] is not extendible is an immediate consequence of this result. In that particular case F_{sa} was the JC-algebra of real symmetric 2×2 matrices. In general, it follows from [16] that F_{sa} will be order-isomorphic to a JC-algebra.

We now give an example of a 2-positive map on a finite dimensional JC-algebra which has no 2-positive extension. This contrasts with the case of finite dimensional C^*-algebras, which are of course nuclear.

EXAMPLE 3.10. Let $D = \{x \oplus x^t : x \in M_2\}$, where x^t denotes the usual transpose of a complex 2×2 matrix x. D is naturally embedded as a subspace of M_4. The self-adjoint part of D is a JC-algebra which is Jordan-isomorphic to $(M_2)_{sa}$.

According to [25] there exists a positive linear map $\varphi: M_2 \to M_4$ which is not decomposable in the sense of [23]. Thus, φ cannot be expressed as a sum of completely positive and completely copositive maps. Define a linear map $\psi: D \to M_4$ by $\psi(x \oplus x^t) = \varphi(x)$.

We first claim that ψ is 2-positive. For if not, there exist elements $a, b, c \in M_2$ such that

$$\begin{pmatrix} a \oplus a^t & b \oplus b^t \\ b^* \oplus b^{*t} & c \oplus c^t \end{pmatrix} \geqslant 0, \quad \text{but} \quad \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(b^*) & \varphi(c) \end{pmatrix} \ngeqslant 0.$$

Replacing c by $c + \varepsilon 1$ if necessary, where $\varepsilon > 0$, we may suppose that c is invertible. By [9, Lemma 2.1], $a \geqslant bc^{-1}b^*$ and $a^t \geqslant b^t c^{-1t} b^{*t}$, or equivalently, $a \geqslant b^* c^{-1} b$. However $\varphi(a) \ngeqslant \varphi(b)\varphi(c)^{-1}\varphi(b)^*$. Write $x = c^{-1/2} bc^{-1/2}$ and $y = c^{-1/2}ac^{-1/2}$. Then $y \geqslant x^*x$, $y \geqslant xx^*$, but $\varphi_0(y) \ngeqslant \varphi_0(x)\varphi_0(x)^*$ where $\varphi_0 : M_2 \to M_4$ is the positive unital map defined by $\varphi_0(z) = \varphi(c)^{-1/2}(c^{1/2}zc^{1/2})\varphi(c)^{-1/2}$. However, this contradicts 25, Theorem 5.2], where it is shown that such φ_0 must satisfy a "Strong Kadison Inequality". Therefore ψ is 2-positive.

Now suppose that ψ extends to a 2-positive map $\bar{\psi}: M_4 \to M_4$. Then define positive maps ψ_1 and ψ_2 on M_2 by $\psi_1(x) = \bar{\psi}(x \oplus 0)$ and $\psi_2(x) = \bar{\psi}(0 \oplus x^t)$. Clearly ψ_1 is 2-positive, hence completely positive and ψ_2 is 2-copositive, hence completely copositive. However $\varphi = \psi_1 + \psi_2$, which contradicts the fact that φ is not decomposable.

We have therefore shown that ψ is a 2-positive map on D which does not extend to a 2-positive map on M_4.

REMARKS. 1. The first part of the proof actually shows that every positive map from D into $B(H)$ is 2-positive.

2. The map ψ above does extend to a positive map on M_4, since there is a positive projection from M_4 onto D.

4. MATRIX RANGES

Recall from [7] that the numerical range $W(a)$ of a C^*-algebra element $a \in A$ is defined to be $\{\varphi(a) : \varphi$ is a state on $A\}$. If an ideal J is specified then the essential numerical range $W_e(a)$ is defined to be $W(\pi(a))$ where $\pi: A \to A/J$ is the quotient homomorphism. A state is a completely positive unital map $\varphi: A \to \mathbf{C}$, and so the higher order matrix ranges $W_n(a)$ are defined by replacing states by completely positive unital M_n-valued maps. As before the essential matrix ranges $W_{ne}(a)$ are defined to be $W_n(\pi(a))$. These were introduced in [4] and studied in [20], [21]. It is always true that $W_{ne}(a) \subseteq W_n(a)$ and a natural question is whether an ideal perturbation $a + j$ of a can be found for which $W_n(a + j) = W_{ne}(a)$. This was solved positively in [20], [21], but the methods of § 2 allow us to state a stronger result.

THEOREM 4.1. *Let E be a finite dimensional operator system in a C^*-algebra A with an ideal J. Given an integer n, there exists a linear map $\tau: E \to J$ such that $W_n(a + \tau(a)) = W_{ne}(a)$ for all $a \in E$.*

Proof. $\pi(E)$ is a finite dimensional operator system in A/J and so has an n-positive unital lifting $\psi: \pi(E) \to A$. Define $\tau: E \to J$ by

$$\tau(a) = \psi\pi(a) - a \quad (a \in E).$$

It suffices to show that $W_n(\psi\pi(a)) \subseteq W_n(\pi(a))$. If $\varphi: A \to M_n$ is unital and completely positive then $\varphi\psi: \pi(E) \to M_n$ is n-positive, and so completely positive [8]. It thus has a completely positive unital extension $\theta: A/J \to M_n$. If $a \in E$ then

$$\theta(\pi(a)) = \varphi\psi\pi(a) = \varphi(a + \tau(a))$$

and the inclusion is proved.

REMARK. In the presence of the n-positive approximation property the same argument extends this result to infinite dimensional operator systems.

REFERENCES

1. ALFSEN, E. M.; SHULTZ, F. W., State spaces of Jordan algebras, *Acta Math.*, **140**(1978), 155–190.
2. ANDERSEN, T. B., Linear extensions, projections, and split faces, *J. Funct. Anal.*, **17**(1974), 161–173.
3. ANDERSON, J., A C^*-algebra for which Ext(A) is not a group, *Ann. of Math.*, **107**(1978), 455–458.

4. ARVESON, W. B., Subalgebras of C^*-algebras, *Acta Math.*, **123**(1969), 141–224.
5. ARVESON, W. B., Subalgebras of C^*-algebra. II, *Acta Math.*, **128**(1972), 271–308.
6. ARVESON, W. B., Notes on extensions of C^*-algebras, *Duke Math. J.*, **44**(1977), 329–355.
7. BONSALL, F.; DUNCAN, J., *Complete normed algebras*, Springer Verlag, Berlin, 1973.
8. CHOI, M. D., Positive linear maps on C^*-algebras, *Canad. J. Math.*, **24**(1972), 520–529.
9. CHOI, M. D., Some assorted inequalities for positive linear maps on C^*-algebras, *J. Operator Theory*, **4**(1980), 271–285.
10. CHOI, M. D.; EFFROS, E., The completely positive lifting problem for C^*-algebras, *Ann. of Math.*, **104**(1976), 585–609.
11. CHOI, M. D.; EFFROS, E., Lifting problems and the cohomology of C^*-algebras, *Canad. J. Math.*, **29**(1977), 1092–1111.
12. CHOI, M. D.; EFFROS, E., Injectivity and operator spaces, *J. Funct. Anal.*, **24**(1977), 156–209.
13. DE CANNIÈRE, J.; HAAGERUP, U., Multipliers of the Fourier algebras of some Lie groups and their discrete subgroup, *Amer. J. Math.*, **107**(1985), 455–500.
14. DIESTEL, J., *Sequences and series in Banach spaces*, Springer Verlag, Berlin, 1984.
15. DIXMIER, J., *C^*-algebras*, North Holland, Amsterdam, 1977.
16. EFFROS, E.; STØRMER, E., Positive projections and Jordan structure operator algebras, *Math. Scand.*, **45**(1979), 127–138.
17. FRIEDMAN, Y.; RUSSO, B., Contractive projections on $C_0(K)$, *Trans. Amer. Math. Soc.*, **273**(1982), 57–73.
18. ROBERTSON, A. G., A non-extendible positive map on the reduced C^*-algebra of a free group, *Bull. London Math. Soc.*, **18**(1986), 389–391.
19. SMITH, R. R., Completely bounded maps between C^*-algebras, *J. London Math. Soc.*, **27**(1983), 157–166.
20. SMITH, R. R.; WARD, J. D., Matrix ranges for Hilbert space operators, *Amer. J. Math.*, **102**(1980), 1031–1081.
21. SMITH, R. R.; WARD, J. D., Locally isometric liftings from quotient C^*-algebras, *Duke Math. J.*, **47**(1980), 621–631.
22. STINESPRING, W. F., Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.*, **6**(1955), 211–216.
23. STØRMER, E., Decomposition of positive projections on C^*-algebras, *Math. Ann.*, **247**(1980), 21–41.
24. STØRMER, E., Extension of positive maps into $B(H)$, *J. Funct. Anal.*, **66**(1986), 235–254.
25. WORONOWICZ, S. L., Positive linear maps on low-dimensional matrix algebras, *Rep. Math. Phys.*, **10**(1976), 165–183.

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