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A DERIVATION RANGE WHOSE CLOSURE INCLUDES THE THIN OPERATORS

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1. INTRODUCTION

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded, linear operators on a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$. The *inner derivation* induced by $A \in \mathscr{L}(\mathscr{H})$ is the map defined by $\delta_A(X) = [A, X] = AX - XA$ $(X \in \mathscr{L}(\mathscr{H}))$.

The celebrated result of A. Brown and C. M. Pearcy on the structure of commutators says that "$T \in \operatorname{ran} \delta_A$ for some A in $\mathscr{L}(\mathscr{H})$ if and only if T is not of the form $\lambda I + K$ for some non-zero λ in $\mathbf{C}$ and some $K \in \mathscr{K}(\mathscr{H})$", where $\operatorname{ran} \delta_A$ denotes the range of δ_A and $\mathscr{K}(\mathscr{H})$ is the ideal of all compact operators [4]. The subset $\{\lambda I + K : \lambda \in \mathbf{C},\ K \in \mathscr{K}(\mathscr{H})\}$ is the class of the *thin operators*.

At the Fifth International Conference on Operator Theory (Timişoara-Herculane, Romania, June, 1980), J. P. Williams raised the following question [14, Problem 4] (see also [10]):

$$\text{Must } \bigcup\{(\operatorname{ran} \delta_A)^- : A \in \mathscr{L}(\mathscr{H})\} = \mathscr{L}(\mathscr{H})?$$

The purpose of this article is to give an affirmative answer to Williams's problem:

THEOREM 1.1. *There exists $A \in \mathscr{L}(\mathscr{H})$ such that $(\operatorname{ran} \delta_A)^-$ includes the class of all thin operators.*

The article also includes some new results about the class of operators B such that $I \in (\operatorname{ran} \delta_B)^-$.

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2. JOEL ANDERSON OPERATORS

The identity is not a commutator; that is, $I \notin \operatorname{ran} \delta_A$ for any A in $\mathscr{L}(\mathscr{H})$. Nevertheless, J. H. Anderson proved the remarkable result that $I \in (\operatorname{ran} \delta_B)^-$ for a large class of operators [1]. The family

$$\mathrm{JA}(\mathscr{H}) = \{B \in \mathscr{L}(\mathscr{H}) : I \in (\operatorname{ran}\delta_B)^-\}$$

was further studied in [5] and [7]. It is known (and easy to prove) that $\mathrm{JA}(\mathscr{H})$ is invariant under similarity and that, if

$$\mathscr{A} = \{T \otimes I : T \in \mathscr{L}(\mathscr{H})\} \quad (\subset \mathscr{L}(\mathscr{H} \otimes \mathscr{H})),$$

then $\mathscr{A} \cap \mathrm{JA}(\mathscr{H} \otimes \mathscr{H})$ is a G_δ-dense subset of $\mathscr{A}$ [1], [7]; moreover, there exists R in $\mathrm{JA}(\mathscr{H})$ such that

$$\sigma(R) = \sigma_{\mathrm{lre}}(R) = \mathbf{D}^-,$$

where $\sigma(\cdot)$ and $\sigma_{\mathrm{lre}}(\cdot)$ denote the spectrum and, respectively, the intersection of the left essential and the right essential spectra, and $\mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ is the open unit disk [7, Lemma 3.1].

It follows from [3], [8, Chapter 5] that

$$\|R - N_k\| \to 0 \quad (k \to \infty)$$

for a suitable sequence $\{N_k\}_{k\geq 1}$ of nilpotent operators.

Indeed, beginning with an operator T in $\mathrm{JA}(\mathscr{H})$, there are many different ways to construct new members of the family. The key ingredients for the proof of Theorem 1.1 are Lemmas 2.1 and 2.2 (below), which belong to this circle of ideas.

LEMMA 2.1. *If $T \in \mathrm{JA}(\mathscr{H})$ and $W \in \mathscr{L}(\mathscr{H})$ is invertible, then $T \otimes W \in \in \mathrm{JA}(\mathscr{H} \otimes \mathscr{H})$.*

Proof. By hypothesis, there is a sequence $\{X_n\}_{n\geq 1}$ such that $\|[T, X_n] - I\| \to 0$ $(n \to \infty)$. It is straightforward to check that

$$\|[T \otimes W,\ X_n \otimes W^{-1}] - I \otimes I\| = \|[T, X_n] \otimes I - I \otimes I\| =$$

$$= \|([T, X_n] - I) \otimes I\| \to 0$$

$(n \to \infty)$.

Hence, $T \otimes W \in \mathrm{JA}(\mathscr{H} \otimes \mathscr{H})$. ▨

LEMMA 2.2. *$\mathrm{JA}(\mathscr{H} \otimes \mathscr{H})$ contains a quasinilpotent operator Q.*

Proof. Cleary, if $\mathscr{A}_\infty := \{T \otimes I \in \mathscr{A} : \sigma(T) = \{0\}\}$ and $\mathscr{A}_m := \{T \otimes I \in \in \mathscr{A} : \sigma(T) \subset (1/m)\mathbf{D}\}$, then $\mathscr{A}_\infty = \bigcap_{m\geq 1} \mathscr{A}_m$, which is a G_δ in $\mathscr{A}$ $(\sigma(T \otimes I) = \sigma(T))$. Since $\mathscr{A}$ is a complete metric space, it follows from Alexandrov's Theorem and its proof (see, e.g., [9, Theorem 2—76, p. 85]), that $(\mathscr{A}_\infty, d)$ is a complete metric space, where d is the metric defined by

$$d(A, B) = \|A - B\| + \sum_{m\geq 1} 2^{-m}\varphi_m(A, B), \quad \text{where } f_m(A) = (\text{dist}[A, \mathscr{A}\setminus\mathscr{A}_m])^{-1}$$

and

$$\varphi_m(A, B) = |f_m(A) - f_m(B)|/[1 + |f_m(A) - f_m(B)|] \quad (m = 1, 2, \ldots).$$

Borrowing an argument from the proof of Corollary 4 of [1], we define

$$\mathscr{B}_n = \{T \otimes I \in \mathscr{A}_\infty : \text{dist}[I \otimes I, \operatorname{ran} \delta_{T\otimes I}] \geq 1/n\},$$

which is obviously a closed subset of $(\mathscr{A}_\infty, d)$ because d induces the same topology as the norm on $\mathscr{A}_\infty$.

CLAIM. *$\mathscr{B}_n$ is nowhere dense in $(\mathscr{A}_\infty, d)$ $(n = 1,2, \ldots)$.*

Let $T \otimes I \in \mathscr{A}_\infty$ and let $\varepsilon > 0$ be given. Since $\sigma(T \otimes I) = \{0\}$, it follows from the upper semicontinuity of the spectral radius, sp$(\cdot)$, that if $\|T \otimes I - A_h \otimes I\| \to 0$ $(h \to \infty)$, then $\lim_{h\to\infty} \text{sp}(A_h) = 0$ (see, e.g., [8, Chapter 1]). On the other hand, by using standard results of approximation, T can be *uniformly* approximated by operators whose spectra are closed disks centered at the origin, with arbitrarily small radii; furthermore, by using the Similarity Orbit Theorem [2, Theorem 9.2], we can find a sequence $\{\eta_h\}_{h=1}^{\infty}$ of positive reals decreasing to 0, and invertible operators $\{W_h\}_{h\geq 1}$ such that

$$\|T - W_h(\eta_h R)W_h^{-1}\| \to 0 \quad (h \to \infty).$$

Let $n_0(\varepsilon)$ be the first integer such that $\sum_{m>n_0(\varepsilon)} 2^{-m} < \varepsilon/3$. It is not difficult to verify that

$$(W_h(\eta_h R)W_h^{-1}) \otimes I \in \mathscr{A}_m \quad \text{and} \quad \varphi_m[T \otimes I, (W_h(\eta_h R)W_h^{-1}) \otimes I] < \varepsilon/3$$

for $1 \leq m \leq n_0(\varepsilon)$, provided h is large enough.

Fix this h. Since JA$(\mathscr{H})$ is invariant under similarity and multiplication by non-zero scalars, $(W_h(\eta_h R)W_h^{-1}) \otimes I \in \text{JA}(\mathscr{H} \otimes \mathscr{H})$. Therefore, there exists $X \in \mathscr{L}(\mathscr{H} \otimes \mathscr{H})$ such that

$$\|[(W_h(\eta R)W_h^{-1}) \otimes I, X] - I \otimes I\| < 1/4n.$$

On the other hand,

$$\|W_h(\eta_h R)W_h^{-1} - W_h(\eta_h N_k)W_h^{-1}\| \leq \eta_h\|W_h\| \cdot \|W_h^{-1}\| \cdot$$

$$\cdot \|R - N_k\| \to 0 \quad (k \to \infty).$$

Thus, if $N = W_h(\eta_h N_k)W_h^{-1}$ for some k large enough, then $N \otimes I$ is nilpotent, $\|[N, X] - I\| < 1/2n$ and

$$d(T \otimes I, N \otimes I) = \|T - N\| + \sum_{m \geq 1} 2^{-m}\varphi_m(T \otimes I, N \otimes I) < \|T - N\| +$$

$$+ \sum_{m \geq 1}^{n_0(\varepsilon)} 2^{-m}\varphi_m(T \otimes I, N \otimes I) + \sum_{m > n_0(\varepsilon)} 2^{-m} < 2(\varepsilon/3) + \varepsilon/3 = \varepsilon,$$

so that $N \otimes 1 \in \mathscr{A}_\infty \setminus \mathscr{B}_n$, and $N \otimes I$ can be chosen arbitrarily close to $T \otimes I$.

Hence, $\mathscr{B}_n$ is nowhere dense in $(\mathscr{A}_\infty, d)$.

By Baire's category theorem, $\mathscr{A}_\infty \setminus \bigcup_{n \geq 1} \mathscr{B}_n$ is a (d-dense and uniformly dense!) subset of second category in $\mathscr{A}_\infty$.

If $Q \in \mathscr{A}_\infty \setminus \bigcup_{n \geq 1} \mathscr{B}_n$, then Q is quasinilpotent and $I \otimes I \in (\operatorname{ran} \delta_Q)^-$. ▨

REMARKS 2.3. (i) It follows from [11, p. 522] that a quasinilpotent operator $Q \in \mathrm{JA}(\mathscr{H})$ cannot be essentially nilpotent, that is, $Q^k \notin \mathscr{K}(\mathscr{H})$ for any $k = 1, 2, \ldots$, and therefore Q is a *universal quasinilpotent* in the sense of [6], [8, Chapter 8]: every nilpotent operator in $\mathscr{L}(\mathscr{H})$ is the norm-limit of similarities of Q. From [6] and Lemma 2.2 (and its proof), we conclude that almost every quasinilpotent in $\mathscr{L}(\mathscr{H})$ is a universal quasinilpotent and belongs to $\mathrm{JA}(\mathscr{H})$.

(ii) Let Q be as above, and let Γ be a nonempty compact subset of $\mathbf{C}$. If $\{\lambda_n\}$ is a (finite or denumerable) dense subset of Γ, then $T = \sum_n^{\oplus} (\lambda_n + Q) \in \mathrm{JA}(\mathscr{H} \oplus \oplus \mathscr{H} \oplus \ldots)$, and $\sigma(T) = \sigma_{\mathrm{lre}}(T) = \Gamma$. This examples strongly enlarge the collection given in [7, Section 3].

(iii) By combining J. G. Stampfli's result [11] with the observations of [5] and [7], it is easy to see that if $\|A - W_k B W_k^{-1}\| \to 0$ $(k \to \infty)$ for some sequence $\{W_k\}_{k \geq 1}$ of invertible operators such that $\|W_k\| \cdot \|W_k^{-1}\| \leq C$ (for some $C \geq 1$, independent of k), $A = A_1 \oplus A_2$ and $A_1 \notin \mathrm{JA}$, then $B \notin \mathrm{JA}$. (For instance, if A_1 is polynomially compact, or a G_1-operator, etc.)

(iv) In a similar vein, if $A, X_n \in \mathscr{L}(\mathscr{H})$, $\|[A, X_n] - I\| \to 0$ $(n \to \infty)$ and f is an analytic function defined on some neighborhood of $\sigma(A)$ such that f' does not vanish on $\sigma(A)$, then $f(A) \in \mathrm{JA}(\mathscr{H})$. Indeed, a simple contour integral calculation shows that $[f(A), X_n] \to f'(A)$ (in the norm, as $n \to \infty$). Since $f'(A)$ is an invertible element of $\mathscr{A}'(f(A))$, it follows from [11, p. 521] that $I \in (\operatorname{ran} \delta_{f(A)})^-$.

3. THIN OPERATORS IN THE CLOSURE OF A DERIVATION RANGE

It follows from the above lemmas that if $U \in \mathscr{L}(\mathscr{H})$ is the bilateral shift and $Q \in \mathscr{L}(\mathscr{R})$ $(\mathscr{R} = \mathscr{H} \otimes \mathscr{H})$ is one of the operators given by Lemma 2.2, then

$$A = (I + Q) \otimes U \in \mathrm{JA}(\mathscr{R} \otimes \mathscr{H}).$$

Write $\mathscr{R} \otimes \mathscr{H} = \sum_{n \in \mathbf{Z}}^{\oplus} \mathscr{R}_n$ ($I \otimes U$ maps $\mathscr{R}_n$ isometrically onto $\mathscr{R}_{n+1}$, for each n in $\mathbf{Z}$). Let $(A_{ij})_{i,j \in \mathbf{Z}}$ denote the matrix of A with respect to this orthogonal direct sum decomposition; then

$$A_{ij} = \begin{cases} I + Q, & \text{if } i = j + 1,\ j \in \mathbf{Z} \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 3.1. *Let $C = (C_{ij})_{i,j \in \mathbf{Z}}$ be the matrix of an operator $C \in \{A\}'$ (= the commutant of A) with respect to the above decomposition; then*

$$C_{ij} = C_{0,i-j} \quad \textit{for all } i, j \in \mathbf{Z},$$

where $C_{0n} \in \{Q\}'$ for all $n \in \mathbf{Z}$.

In particular, $\{A\}' \cap \mathscr{K}(\mathscr{R} \otimes \mathscr{H}) = \{0\}$.

Proof. Let $C_n := C_{0n}$ ($n \in \mathbf{Z}$). Since A is invertible ($A^{-1} = (I + Q)^{-1} \otimes U^{-1}$) and commutes with C, $C = ACA^{-1}$, whence we obtain

$$0 = (C - ACA^{-1})_{ij} = C_{ij} - (I + Q)C_{i-1,j-1}(I + Q)^{-1}.$$

By induction (separate the cases $i \geqslant 0$ and $i < 0$), for all $i, j \in \mathbf{Z}$, we have

$$C_{ij} = (I + Q)^i C_{j-i}(I + Q)^{-i} \quad (i \in \mathbf{Z}).$$

Therefore,

$$\sup_i \|(I + Q)^i C_n (I + Q)^{-i}\| \leqslant \|C\| < \infty \quad \text{for each } n \in \mathbf{Z}.$$

Since Q is quasinilpotent, it follows from a result of J. P. Williams [13, Theorem 2] that $C_n \in \{Q\}'$ for all n in $\mathbf{Z}$. (Observe that $I + Q = e^S$ for some quasinilpotent S such that $\{S\}' = \{Q\}'$.)

It is completely apparent that C cannot be compact, unless $C = 0$. Therefore $\{A\}' \cap \mathscr{K}(\mathscr{R} \otimes \mathscr{H}) = \{0\}$. ▨

Proof of Theorem 1.1. Let $A = (I + Q) \otimes U \in \mathrm{JA}(\mathscr{R} \otimes \mathscr{H})$ be the operator defined at the introduction of this section. By definition of the class JA, $(\operatorname{ran} \delta_A)^- \supset \supset \{\lambda I_{\mathscr{R} \otimes \mathscr{H}} : \lambda \in \mathbf{C}\}$.

On the other hand, by Lemma 3.1, A commutes with no non-zero compact operator. A fortiori, $\{A\}'$ does not contain any non-zero trace class operator. But, by another result of J. P. Williams [12, Corollary 1], this latter condition is equivalent to

$$(\operatorname{ran} \delta_A)^- \supset \mathscr{K}(\mathscr{R} \otimes \mathscr{H}).$$

Hence, $(\operatorname{ran} \delta_A)^-$ includes the class of all thin operators. ▨

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