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PERTURBATIONS OF NEST-SUBALGEBRAS OF VON NEUMANN ALGEBRAS

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In [6] E. C. Lance initiated the perturbation theory of nest algebras and proved that, roughly speaking, two nest algebras are close if and only if their invariant nests are close and in fact they are similar via an invertible operator close to the identity.

In [4] and [5] F. Gilfeather and D. R. Larson investigated a special class of reflexive algebras, the nest-subalgebras of von Neumann algebras.

In [2] K. R. Davidson suggested an analogue of Lance's results for nest-subalgebras of approximately finite von Neumann algebras.

More precisely, let H denote a Hilbert space, $B(H)$ the algebra of bounded operators on H and $M \subset B(H)$ a von Neumann algebra. Let $L \subset M$ be a nest (i.e. a totally ordered strongly closed family) of projections and define the algebras

$$\operatorname{Alg} L = \{x \in B(H)\,;\,(1-p)xp = 0\ (\forall)\, p \in L\}$$

(the nest algebra with respect to L) and $M \cap \operatorname{Alg} L$ the nest-subalgebra of M with respect to L.

The natural extension of Lance's result is that two nests L_1 and L_2 in M are close (i.e. there is a lattice isomorphism of L_1 onto L_2 close to the identity) if and only if the algebras $M \cap \operatorname{Alg} L_1$ and $M \cap \operatorname{Alg} L_2$ are close in the Hausdorff metric.

Unfortunately, this fails to be true if one does not take certain precautions on M.

A first necessary condition is that M must be a factor.

Indeed, suppose that the center of M is not trivial. We may consider then two nests $L_1 \subset L_2$ in the center, $L_1 \neq L_2$. It follows that

$$M \cap \operatorname{Alg} L_1 = M \cap \operatorname{Alg} L_2 = M$$

but however L_1 and L_2 are not close.

In this paper we show that this is the only obstruction and we obtain the desired perturbation results in arbitrary factors.

The main ingredient is that in this case nests have a reflexivity-type property (Lemma 3).

We also prove a von Neumann algebra analogue of W. Arveson's distance formula (Theorem 2), which removes the hyperfiniteness hypothesis for M, heavily used in [2] and [4].

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LEMMA 1. *Let M be a von Neumann algebra and $0 \neq p \leqslant q \leqslant r \neq 1$ be projections in M. If $S = (1-p)Mr$, then for every x in S*

$$\mathrm{dist}(x, qS(r-p)) = \max(\|xp\|, \|(1-q)x\|).$$

Proof. Consider the operators $a = qxp$, $b = (1-q)xp$, $c = (1-q)x(r-p)$. Since $x - (a+b+c)$ belongs to $qS(r-p)$, one has

$$\mathrm{dist}(x, qS(r-p)) = \mathrm{dist}(a+b+c, qS(r-p)).$$

Note that $xp = a+b$ and $(1-q)x = b+c$. We may clearly assume that $\|xp\| \leqslant 1$ and $\|(1-q)x\| \leqslant 1$, hence $a^*a + b^*b \leqslant p$ and $bb^* + cc^* \leqslant 1-q$. There exist contractions $u_0 \in pMp$, $v_0 \in (1-q)M(1-q)$, u and v in M such that

$$(a^*a)^{\frac{1}{2}} = u_0(p - b^*b)^{\frac{1}{2}} \quad \text{and} \quad (cc^*)^{\frac{1}{2}} = v_0((1-q) - bb^*)^{\frac{1}{2}},$$

so that $a = u(p - b^*b)^{\frac{1}{2}}$ and $c^* = v((1-q) - bb^*)^{\frac{1}{2}}$ ([3]). Moreover,

$$u = (q-p)up \quad \text{and} \quad v = (r-p)v(1-q).$$

Let $y = (p - b^*b)^{\frac{1}{2}} + b - b^* + ((1-q) - bb^*)^{\frac{1}{2}}$. Routine computations yield

$$(u + (1-q))y(p + v^*) = a + b + c - ubv^*,$$

$\|u + 1 - q\| \leqslant 1$, $\|p + v^*\| \leqslant 1$ and $yy^* = p + 1 - q$ hence $\|y\| = 1$. Finally, since $ubv^* \in qS(r-p)$, it follows that $\|a + b + c - ubv^*\| \leqslant 1$, hence

$$\mathrm{dist}(a + b + c, qS(r-p)) \leqslant 1.$$

On the other side it is easy to see that

$$\max(\|xp\|, \|(1-q)x\|) \leqslant \mathrm{dist}(x, qS(r-p))$$

and the conclusion follows.

Let now $L \subset M$ be a nest and $A = M \cap \operatorname{Alg} L$ be the corresponding nest-subalgebra.

THEOREM 2. *For every x in M*

$$\operatorname{dist}(x, A) = \sup_{p \in L} \|(1-p)xp\|.$$

Proof. By a slight variation of [1, Lemma 1] we may assume that

$$L = \{0 \neq p_1 \leqslant \ldots \leqslant p_n \neq 1\}, \quad n \geqslant 1.$$

For any x in M we note that

$$(1-p_1)Ap_n = (p_2 - p_1)A(p_n - p_1) \oplus (1-p_2)A(p_n - p_1) =$$

$$= (p_2 - p_1)M(p_n - p_1) \oplus (1-p_2)A(p_n - p_1),$$

hence

$$\operatorname{dist}(x, A) = \operatorname{dist}((1-p_1)xp_n, (1-p_1)Ap_n) \leqslant$$

$$\leqslant \operatorname{dist}((1-p_1)xp_n - (1-p_2)a(p_n - p_1), (p_2 - p_1)M(p_n - p_1))$$

for any a in A.

We apply Lemma 1 and obtain

$$\operatorname{dist}(x, A) \leqslant \max\{\|(1-p_1)xp_1\|, \|(1-p_2)xp_n - (1-p_2)a(p_n - p_1)\|\}$$

for any a in A, hence

$$\operatorname{dist}(x, A) \leqslant \text{may}\{\|(1-p_1)xp_1\|, \operatorname{dist}((1-p_2)xp_n, (1-p_2)Ap_n)\}$$

since

$$(1-p_2)A(p_n - p_1) = (1-p_2)Ap_n.$$

Suppose now that for some $k < n$

$$\operatorname{dist}(x, A) \leqslant$$

$$\leqslant \max\{\max_{i=1,k-1} \|(1-p_i)xp_i\|, \operatorname{dist}((1-p_k)xp_n, (1-p_k)Ap_n)\}.$$

Note that

$$(1-p_k)Ap_n = (p_{k+1} - p_k)M(p_n - p_k) \oplus (1-p_{k+1})A(p_n - p_k).$$

By taking into account Lemma 1, it follows again that

$$\operatorname{dist}((1-p_k)xp_n,\ (1-p_k)Ap_n) \leqslant$$

$$\leqslant \operatorname{dist}((1-p_k)xp_n - (1-p_{k+1})a(p_n-p_k), (p_{k+1}-p_k)M(p_n-p_k)) \leqslant$$

$$\leqslant \max\{\|(1-p_k)xp_k\|,\ \operatorname{dist}((1-p_{k+1})xp_n,\ (1-p_{k+1})Ap_n)\}.$$

At the last step one simply has

$$\operatorname{dist}((1-p_n)xp_n,\ (1-p_n)Ap_n) \leqslant \|(1-p_n)xp_n\|,$$

hence

$$\operatorname{dist}(x, A) \leqslant \max_{p\in L}\|(1-p)xp\|$$

by induction. Since the opposite inequality is immediate, it follows that $\operatorname{dist}(x, A) = \max_{p\in L}\|(1-p)xp\|$, which concludes the proof.

COROLLARY. *For every x in M*

$$\operatorname{dist}(x, \operatorname{Alg} L) = \operatorname{dist}(x, A).$$

(A similar but different result is Lemma 4.8 in [4].)

If $M = B(H)$, Theorem 2 is W. Arveson's distance formula [1]. We note that, excepting slight variations, the outline of the above proof is due, in the case $M = B(H)$, to S. C. Power [7].

For any algebra $A \subset B(H)$ define

$$\operatorname{Lat} A = \{p = p^2 = p^* \in B(H)\ ;\ (1-p)xp = 0\ (\forall) x \in A\}.$$

LEMMA 3. *Let M be a factor and $L \subset M$ be a nest of projections. Then*

$$M \cap \operatorname{Lat}(M \cap \operatorname{Alg} L) = L.$$

Proof. If p belongs to $M \cap \operatorname{Lat}(M \cap \operatorname{Alg} L)$ then p commutes with every projection in L. Suppose that there are projections $q \leqslant r$, $q \neq r$ in L such that

$$q - pq = p_0 \neq 0 \quad \text{and} \quad p(r-q) = q_0 \neq 0.$$

Since M is a factor, there is a partial isometry $x \in M$, $x \neq 0$ such that $xq_0 = x = p_0x$.

Clearly

$$(1-p)xp \neq 0 \quad \text{but} \quad (1-q)xq = 0 \quad (\forall)\, q \in L.$$

The contradiction shows that for every $q \leqslant r$ in L, $q \neq r$ one has either

$$q \leqslant p \quad \text{or} \quad p(r-q) = 0. \tag{1}$$

Suppose now that $q \leqslant p \leqslant r$ and $q \neq r$ are consecutive projections in L (i.e. $r - q$ is an atom in L).

If $p \neq q$ and $r \neq p$, choose $x \neq 0$ in M such that $x(p-q) = x = (r-p)x$. Again it follows that

$$(1-p)xp \neq 0 \quad \text{but} \quad (1-q)xq = 0 \quad (\forall)\, q \in L.$$

Consequently, either

$$p = q \quad \text{or} \quad p = r. \tag{2}$$

(1) and (2) show that, roughly speaking, p 'has no holes' in its decomposition with respect to L and that p is trivial on every atom of L. It follows that p belongs to L, which concludes the proof.

Recall that for two subalgebras A and B in $B(H)$, the Hausdorff distance between them is (slightly different but equivalent to that) given by

$$\operatorname{dist}(A, B) = \max\{\sup_{\substack{x\in A\\ \|x\|\leqslant 1}} \inf_{y\in B} \|x-y\|\,;\ \sup_{\substack{y\in B\\ \|y\|\leqslant 1}} \inf_{x\in A} \|x-y\|\}.$$

We can state now the main result.

THEOREM 4. *Let M be a factor and L_1, L_2 be nests of projections in M. The following statements are equivalent.*

i) *There is a lattice isomorphism of L_1 onto L_2 close to the identity.*

ii) *The algebras $M \cap \operatorname{Alg} L_1$ and $M \cap \operatorname{Alg} L_2$ are close in the Hausdorff metric.*

Proof. (i) $\Rightarrow$ (ii) Let x belong to $M \cap \operatorname{Alg} L_1$, $\|x\| \leqslant 1$. For every projection $p \in L_2$, choose $q \in L_1$ such that $\|p-q\| \leqslant \varepsilon$, hence $\|(1-p)xp\| \leqslant 2\varepsilon$. Theorem 2 implies now that $\operatorname{dist}(x, M \cap \operatorname{Alg} L_2) \leqslant 2\varepsilon$. If we reverse the roles of L_1 and L_2 we obtain (ii).

(ii) $\Rightarrow$ (i) is essentially [2, Theorem 4.4, ii $\Rightarrow$ i], so we shall only sketch the proof.

Let A be a subalgebra of M such that

$$\operatorname{dist}(M \cap \operatorname{Alg} L_1, A) < \varepsilon \leqslant 10^{-2}.$$

For every p in L_1, one can find a unique projection $\alpha(p)$ in $M \cap \mathrm{Lat}\, A$ such that $L_3 = \{\alpha(p)\,;\, p \in L_1\}$ is a nest and

$$\|p - \alpha(p)\| \leqslant 40\varepsilon \quad (\forall)\, p \in L_1 .$$

(See [2] for details.)

Now if $A = M \cap \mathrm{Alg}\, L_2$, Lemma 3 implies that actually L_3 is a subnest of L_2, close to L_1. By the previous implication, $M \cap \mathrm{Alg}\, L_1$ and $M \cap \mathrm{Alg}\, L_3$ are close, hence $M \cap \mathrm{Alg}\, L_2$ and $M \cap \mathrm{Alg}\, L_3$ are close. Since $M \cap \mathrm{Alg}\, L_2 \subset M \cap \mathrm{Alg}\, L_3$, it follows that in fact they are equal and one uses again Lemma 3 to obtain $L_2 = L_3$, which concludes the proof.

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