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ON THE EXTENSION PROBLEM FOR A CLASS OF TRANSLATION INVARIANT POSITIVE FORMS

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To Mischa Cotlar, teacher and friend in his 75^{th} birthday

1. INTRODUCTION

Classical Toeplitz forms can be defined as the sesquilinear forms, in the vector space A of the finitely supported functions on a group Γ, that are invariant under translations, i.e., under the natural action of Γ on A.

However, many problems in Analysis lead to the consideration of forms of that kind but such that they are defined and invariant only in a subspace of A. Moreover, in some of these instances the form can be extended to an invariant form in the whole space A, so that the integral representation and unitary dilation properties of the Toeplitz forms can be used to solve the problem.

The present paper is concerned with the extension problem for a class of such generalized invariant forms which we call Toeplitz-Krein-Cotlar (T-K-C) forms. This notion includes two basic examples: first, the functions of positive type on a symmetric interval of the real line, for which M. G. Krein proved an extension theorem [25] that started a large flow of papers; second, a class of modified Toeplitz forms introduced by M. Cotlar in order to study the Hilbert transform [18] and later included in the notion of "Generalized Toeplitz Kernel" [6], which allows a unified approach to several topics, including the realization of linear systems ([13], [14], [19]).

Here we prove, by means of T-K-C forms, some extensions to the discrete plane of the above mentioned Krein's theorem, of the Helson-Szegö theorem [23], and of the Sz.-Nagy – Foiaş lifting of the commutant [27] as well as related properties.

In Section 2 the needed properties of Toeplitz forms are reviewed and then T-K-C forms are defined. In several instances the data of a given problem can be expressed by means of such a form and the following usually happens: the problem has a solution if and only if the form can be extended to a positive Toeplitz form; something similar happens with the uniqueness of the solution and with the description of all solutions when there is more than one. These considerations may be exemplified by the fundamental papers of Adamjan, Arov and Krein on moment theory ([1], [2], [3]), which are related to our subject as it is seen in [6],

[10] and [11]. In Section 3 we show that such an extension of a T-K-C form exists if and only if an associated family of isometries extends to a group of unitary operators, so the following result proves to be useful:

1) THEOREM A. *Let U, V be isometries with domains D_U, D_V and ranges R_U, R_V, respectively, which are closed subspaces of the same Hilbert space E, and such that $U^n D_V \subset D_U$ and $U^n R_V \subset D_U$ holds for $n = 0, 1 \ldots$. Then*

$$\langle U^n Vf, Vf' \rangle = \langle U^n f, f' \rangle, \quad \forall f, f' \in D_V, \; n = 1, 2, \ldots$$

is a necessary and sufficient condition for the existence of a Hilbert space F that contains E and of two commuting unitary operators in F, U' and V', that extend U and V, respectively.

In Section 4 the results of Section 3 are used to prove the extension properties of positive T-K-C forms from which the following Theorems B, C and D will be derived.

In Section 5 we give a discrete operator valued version of a result due to M. S. Livsic (see [16]) which extends to two variables the already mentioned theorem of Krein:

2) THEOREM B. *Let $\mathbf{Z}$ be the integer group, a and b two fixed natural numbers and $k\{(m, n) \in \mathbf{Z}^2 : |m| \leqslant a, \; |n| \leqslant b\} \to L(E)$ a function of positive type which takes its values in the set $L(E)$ of bounded operators in a Hilbert space E. Assume that one of the restrictions of k given by $k_1 = k(\cdot, 0)$ and $k_2 = k(0, \cdot)$ has only one extension of positive type to the whole set $\mathbf{Z}$, $k' : \mathbf{Z} \to L(E)$. Then there exists a function $K : \mathbf{Z}^2 \to L(E)$ which extends k and is of positive type. Thus, there exists a spectral measure θ in $\mathbf{T}^2$, where $\mathbf{T}$ is the circle group, such that*

$$\langle K(m, n)v, w \rangle = \int e^{i(ms+nt)} d\langle \theta(s, t)v, w \rangle$$

holds whenever $v, w \in E$, $(m, n) \in \mathbf{Z}^2$ and $|m| \leqslant a$, $|n| \leqslant b$.

Devinatz [20] proved Theorem B under the assumption that both k_1 and k_2 have only one extension of positive type.

In Section 6 we consider an angular problem in prediction theory and we prove the following:

3) THEOREM C. *Let Γ be a locally compact abelian group, $\Gamma^{\wedge}$ its dual and $X = \{X_s : s \in \Gamma \times \mathbf{Z}\}$ a wide sense stationary stochastic process with spectral measure μ. The angle between the subspaces generated by $\{X_{(t,n)} : t \in \Gamma, \; n \in \mathbf{Z}, \; n < 0\}$ and $\{X_{(t,n)} : t \in \Gamma, \; n \in \mathbf{Z}, \; n \geqslant 0\}$ is positive if and only if $d\mu = e^u \, d|\theta|$ holds, with u a real function $|\theta|$-essentially bounded in $\Gamma^{\wedge} \times \mathbf{T}$, $|\theta|$ the total variation of*

a complex Borel measure θ in $\Gamma^\wedge \times \mathbf{T}$ such that its Fourier transform $\theta^\wedge$ is zero in $\{(t, m) : t \in \Gamma,\ m \in \mathbf{Z},\ m < 0\}$ and

$$\sup\{|\arg \theta(\beta) : \beta \in \Gamma^\wedge \times \mathbf{T},\ \beta \text{ a Borel set}\} < \pi/2.$$

In Section 7 we consider the problems of the lifting of the commutant of two contraction semigroups. To those problems a T-K-C form is associated and it is shown that their solutions are related to the existence of positive Toeplitz extensions of such a form.

That approach enables us to prove in Section 8 the result we now state with definitions as in ([27], Section 1.6), while P_E^M always denotes the orthogonal projection of the Hilbert space M onto the closed subspace E, and $\bigvee \{E_a : a \in A\}$ is the minimal closed subspace of M that contains the set $E_a \subset M$ for every $a \in A$.

4) THEOREM D. *For $j = 1, 2$ let $\{T_j', T_j''\}$ be a commuting pair of contractions on a Hilbert space E_j, $\{W_j', W_j''\} \subset L(M_j)$ a minimal isometric dilation of $\{T_j', T_j''\}$ and $\{U_j', U_j''\} \subset L(F_j)$ a minimal unitary extension of $\{W_j', W_j''\}$.*

a) *Assume that at least one of the following inclusions holds:*

$$E_2 \subset \bigvee \{U_2'^m U_2''^n E_2 : m < 0,\ n \leqslant 0\}, \quad E_2 \subset \bigvee \{U_2'^m U_2''^n E_2 : m \leqslant 0,\ n < 0\},$$

$$E_1 \subset \bigvee \{U_1'^m U_1''^n E_1 : m > 0,\ n \geqslant 0\}, \quad E_1 \subset \bigvee \{U_1'^m U_1''^n E_1 : m \geqslant 0,\ n > 0\}.$$

Then for any $X \in L(E_1, E_2)$ such that $XT_1' = T_2'X$ and $XT_1'' = T_2''X$ there exists $\tau \in L(F_1, F_2)$ that verifies:

$$(\#) \qquad \tau U_1' = U_2'\tau,\ \tau U_1'' = U_2''\tau,\ P_{E_2}^{F_2}\tau|M_1 = XP_{E_1}^{M_1},\ \|\tau\| = \|X\|.$$

b) *Assume that*

$$M_2 = [\bigvee \{W_2'^m W_2''^n E_2 : m > 0,\ n \geqslant 0\}] \vee E_2 =$$

$$= [\bigvee \{W_2'^m W_2''^n E_2 : m \geqslant 0,\ n > 0\}] \vee E_2.$$

If $X \in (E_1, E_2)$ is such that $XT_1' = T_2'X$, $XT_1'' = T_2''X$ and there exists $\tau \in L(F_1, F_2)$ that verifies (#), then there exists $\gamma \in L(M_1, M_2)$ such that:

$$\gamma W_1' = W_2'\gamma,\ \gamma W_1'' = W_2''\gamma,\ P_{E_2}^{M_2}\gamma = XP_{E_1}^{M_1},\ \|\gamma\| = \|X\|.$$

In the Appendix we relate our approach with a theorem of Cooper on the extension of isometries.

2. BASIC DEFINITIONS AND PROPERTIES

In what follows Γ will be a group with neutral element e, E a vector space and $A = A(\Gamma, E)$ the space of all functions $h : \Gamma \to E$ such that $\operatorname{supp} h = \{t \in \Gamma : h(t) \neq 0\}$ is a finite set. E may be identified in a natural way with a subspace of A: for each $v \in E$ let $h_v \in A$ be such that $\operatorname{supp} h_v = \{e\}$ and $h_v(e) = v$; so we may write $E \subset A$. The group Γ acts in A by means of the "translations" $\{S_t : t \in \Gamma\}$ given by $(S_t h)(s) = h(t^{-1}s)$, $h \in A$, $t, s \in \Gamma$. Let $\mathbf{C}$ be the set of complex numbers; a sesquilinear form $B : A \times A \to \mathbf{C}$ is said to be a Toeplitz form in A if

$$(1) \qquad B(S_t h, S_t h') = B(h, h')$$

for every $h, h' \in A$ and every $t \in \Gamma$.

A sesquilinear form B in a vector space is called positive if $B(h, h) \geqslant 0$ for every h belonging to that space. Positive Toeplitz forms are known to have special integral representation and unitary dilation properties which we now recall.

Every positive form B in A defines a possibly degenerate scalar product in A giving rise to a Hilbert space H such that there is a natural operator Π from A onto a dense subspace of H; let $\lambda : E \to H$ be the restriction to E of that operator. If B is also a Toeplitz form, the group of translations $\{S_t : t \in \Gamma\}$ generates a unitary representation $U = \{U_t : t \in \Gamma\}$ of Γ on $L(H)$ such that

$$U_t \Pi = \Pi S_t \quad \text{and} \quad \langle U_t \Pi h, U_s \Pi h' \rangle = B(S_t h, S_s h')$$

holds for every $t, s \in \Gamma$, $h, h' \in A$. Let E_0 be the closure in H of $\lambda(E)$; then

$$(1a) \qquad B(S_t v, S_s w) = \langle U_t \lambda v, U_s \lambda w \rangle_H = \langle P^H_{E_0} U_{s^{-1}t} \lambda v, \lambda w \rangle_{E_0}$$

for every $t, s \in \Gamma$, $v, w \in E$. Moreover, the so-called minimality condition holds:

$$(1b) \qquad H = \bigvee \{U_t E_0 : t \in \Gamma\}.$$

Assume that $H', U' = \{U'_t : t \in \Gamma\}$ and $\lambda' : E \to H'$ satisfy the same properties as H, U and λ; setting $L(U_t \lambda v) = U'_t \lambda' v$ we define a unitary operator $L : H \to H'$ such that $L\lambda = \lambda'$ and $LU_t = U'_t L$ for every $t \in \Gamma$; thus we may say that H, U and λ are essentially unique.

If there exists a locally convex topology on E such that the norm given by $\|v\| = |B(v, v)|^{1/2}$ is continuous, let E' be the dual space of E with that topology and $L^{-}(E, E')$ the space of all continuous antilinear operators from E to E'. Setting $k(t) = \lambda^* U_t \lambda$ we define a function $k : \Gamma \to L^{-}(E, E')$ that satisfies $B(S_t v, S_s w) = [k(s^{-1}t)v](w)$ and consequently $B(h, h') = \sum \{[k(s^{-1}t)h(t)]h'(s) : s, t \in \Gamma\}$,

$\forall h, h' \in A$. Conversely, if we start with a locally convex topological vector space E and a function $k : \Gamma \to L^-(E, E')$, the last equality defines a Toeplitz form B; k is said to be of positive type if and only if B is positive.

In particular, let E be a Hilbert space, so $L^-(E, E')$ may be identified with $L(E)$. If $\Gamma \to L(E)$ is a function of positive type and $k(e)$ equals 1, the identity operator, then $\langle v, v\rangle_E = B(v, v) = \langle \lambda v, \lambda v\rangle_H$ for every $v \in E$, so E can be identified with the closed subspace E_0 of H. From (1a) and (1b) it follows that

$$k(t) = P_E^H U_{t|E}, \ \forall\, t \in \Gamma, \quad \text{and} \quad H = \bigvee \{U_t E : t \in \Gamma\},$$

as is established by Naimark's famous dilation theorem. Thus, we shall say that, if (1a) and (1b) hold, $U = \{U_t : t \in \Gamma\}$ is the minimal unitary dilation of the positive Toeplitz form B.

Assume moreover that Γ is a locally compact abelian (LCA) topological group, and that $B(S_1 v, w)$ depends continuously on t for every $v, w \in E$. Then U is strongly continuous and there exists a spectral measure θ on the dual group $\Gamma^\wedge$ such that $U_t = \int \chi(t) d\theta(\chi)$, and consequently

$$B(S_t v, S_s w) = \int \chi(t-s) d\langle B(\chi)v, w\rangle, \tag{1c}$$

for every $t, s \in \Gamma$, $v, w \in E$. In the case $E = \mathbf{C}^n$, B can be obtained as the Fourier transform of a positive measure matrix, in the following sense. If μ is a complex Borel measure on $\Gamma^\wedge$, its Fourier transform $\mu^\wedge : \Gamma \to \mathbf{C}$ is given by $\mu^\wedge(s) = \int \chi(s) d\mu(\chi)$. Let $\{v_1, \ldots, v_n\}$ be the canonic basis of E and let the measure μ_{jk} be given by $\mu_{jk}(\Delta) = \langle \theta(\Delta) v_j, v_k\rangle$ for every Borel set $\Delta \subset \Gamma^\wedge$, $j, k = 1, \ldots, n$. Then

$$\sum \{\mu_{jk}(\Delta)\lambda_j \bar{\lambda}_k : 1 \leqslant j,\ k \leqslant n\} =$$

$$= \langle \theta(\Delta) [\textstyle\sum \{\lambda_j v_j : 1 \leqslant j \leqslant n\}],\ [\textstyle\sum \{\lambda_j v_j : 1 \leqslant j \leqslant n\}]\rangle \geqslant 0$$

holds for every $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, so it is said that $\{\mu_{jk}\}_{1 \leqslant j,k \leqslant n}$ is a positive measure matrix. Moreover, B is given by

$$\begin{aligned} &B(S_t[\textstyle\sum \{\lambda_j v_j : 1 \leqslant j \leqslant n\}],\ S_s[\textstyle\sum \{\lambda'_j v_j : 1 \leqslant j \leqslant n\}]) = \\ &= \textstyle\sum \{\lambda_j \bar{\lambda}'_k \hat{\mu}_{jk}(t-s) : 1 \leqslant j,\ k \leqslant n\} \end{aligned} \tag{1d}$$

for every $\lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n \in \mathbf{C}$, $s, t \in \Gamma$.

Let us now give the precise definition of the generalization of Toeplitz forms that will be studied in this paper.

Let Γ, E, A and S_t be as above. We assume that $E = E_1 \times \ldots \times E_n$ is the cartesian product of n vector spaces, so that each element of A is a n-tuple $h = (h_1, \ldots, h_n)$ of finitely supported functions $h_j : \Gamma \to E_j$, $j = 1, \ldots, n$. To each family $\Omega = \{\omega_1, \ldots, \omega_n\}$ of n subsets of Γ we associate the subspace $A_\Omega \equiv A_\Omega(\Gamma, E)$ of $A \equiv A(\Gamma, E)$ defined by $A_\Omega = \{h = (h_1, \ldots, h_n) \in A : \operatorname{supp} h_j \subset \omega_j, j = 1, \ldots, n\}$. Let $B: A_\Omega \times A_\Omega \to \mathbf{C}$ be a sesquilinear form in A_Ω that is invariant under the translations S_t whenever that makes sense, i.e., such that (1) holds whenever h and h' belong to the set D'_t defined by $D'_t = \{g \in A_\Omega : S_t g \in A_\Omega\}$. Then we say that B or, more precisely, that $\{B; \Gamma, \Omega; E\}$ is a *Toeplitz-Krein-Cotlar form.* When $\Omega = \{\Gamma\}$ we have a Toeplitz form, which we denote by $\{B; \Gamma; E\}$.

When B is a positive T-K-C form, B and the vector space A_Ω generate a Hilbert space H_Ω and a natural operator π_Ω from A_Ω onto a dense subspace of H_Ω, while the family $\{S_{t|D'_t} : t \in \Gamma\}$ of linear operators in A_Ω generate a family of isometries $V = \{V_t : t \in \Gamma\}$ associated to the form B. In several cases it will be necessary to extend such a form B to a usual positive Toeplitz form $\{B'; \Gamma; E\}$. We shall now see that such an extension exists if and only if V can be extented to a unitary representation of Γ.

For each $t \in \Gamma$ let D_t be the domain of V_t; thus $D_e = H_\Omega$. We say that $U = \{U_t : t \in \Gamma\}$ is a minimal unitary extension of V if U is a unitary representation of Γ on a Hilbert space H that contains H_Ω, such that $U_{t|D_t} = V_t$ for every $t \in \Gamma$ and $H = \bigvee\{U_t H_\Omega : t \in \Gamma\}$. We identify two such extensions when they are equivalent under a unitary isomorphism which leaves invariant every element of H_Ω. A positive Toeplitz extension of $\{B; \Gamma, \Omega; E\}$ is any positive Toeplitz form $\{B'; \Gamma; E\}$ such that $B'(h, h') = B(h, h')$ for every $h, h' \in A_\Omega$. Then:

2) LEMMA. *Let B be a positive* T-K-C *form and V the associated family of isometries. If we associate to each positive Toeplitz form its minimal unitary dilation, we get a bijection between the set $\mathscr{B}$ of positive extensions of B and the set $\mathscr{U}$ of minimal unitary extensions of V.*

Proof. i) Let B' be a positive Toeplitz extension of B and $U = \{U_t : t \in \Gamma\} \subset L(H)$ its minimal unitary dilation. The restriction of Π to A_Ω may be identified with Π_Ω and H_Ω can be considered as a closed subspace of H. For every $t \in \Gamma$ and $h \in D'_t$ it is true that $U_t(\Pi h) = \Pi(S_t h) = \Pi_\Omega(S_t h) = V_t(\Pi_\Omega h)$, so $U_{t|D_t} = V_t$. From $H_\Omega \subset H$ it follows that $\bigvee\{U_t H_\Omega : t \in \Gamma\} \subset H$. In order to prove the converse inclusion, since $H = \bigvee\{U_t(\Pi E) : t \in \Gamma\}$, it is enough to show that $U_{t'}\Pi v'_j \in \bigvee\{U_t H_\Omega : t \in \Gamma\}$ holds for every $t' \in \Gamma$ and $v'_j = (v_1, \ldots, v_n)$ such that $v_i = 0$ if $i \neq j$, $j = 1, \ldots, n$; now, if $t_j \in \omega_j$ is such that $S_{t_j} v'_j \in A_\Omega$, then $U_{t'}\Pi v'_j = U_{t'} U_{t_j^{-1}} \Pi_\Omega S_{t_j} v'_j \in \bigvee\{U_t H_\Omega : t \in \Gamma\}$.

ii) Conversely, let $U \subset L(H)$ be a minimal unitary extension of V. For v_j' and t_j as before set $\Pi v_j' = U_{t_j^{-1}} \Pi_\Omega S_{t_j} v_j'$; then a unique linear extension $\Pi : A \to H$ of Π_Ω such that $U_t \Pi = \Pi S_t$, $\forall\, t \in \Gamma$, is well defined. A positive Toeplitz extension B' of B is defined by

$$B'(h, h') = \langle \Pi h, \Pi h' \rangle_H , \quad \forall h, h' \in A.$$

Let E' be the closure in H of $\Pi(E)$; since

$$B'(S_t v, S_s w) = \langle \Pi S_t v, \Pi S_s w \rangle_H = \langle P^H_{E'} U_{s^{-1}t} \Pi v, \Pi w \rangle_{E'},$$

U is unitary dilation of B'. The minimality follows from

$$U_{t' t_j} \Pi v' \in \bigvee \{ U_t(\Pi E) : t \in \Gamma \}, \quad \forall t' \in \Gamma.$$

Q.E.D.

The result we have just established is our basic lemma.

3. UNITARY EXTENSIONS OF ISOMETRIES

The following result will give unitary extensions of families of isometries associated to some T-K-C forms.

1) THEOREM. *Let Γ be an abelian group with neutral element e and $\Gamma_1 \subset \Gamma$ a sub-semigroup containing e and with the property that for any $s \in \Gamma$ there exists $t \in \Gamma_1$ such that $s + t \in \Gamma_1$. Let $U = \{U_t : t \in \Gamma_1\} \subset L(E)$ be an isometric representation of Γ_1 on a Hilbert space E and V an isometry with domain D_V and range R_V which are subspaces of E.*

Then U can be extended to a unitary representation $U' = \{U_t' : t \in \Gamma\}$ of Γ of a Hilbert space F containing E and V can be extended to a unitary operator $V' \in L(F)$ such that

$$U_t' V' = V' U_t', \quad \forall\, t \in \Gamma,$$

if and only if

a) $$\langle U_t V f, U_s V g \rangle = \langle U_t f, U_s g \rangle, \quad \forall\, f, g \in D_V, s, t \in \Gamma.$$

Set $\Gamma_1^{-1} = \{t \in \Gamma : -t \in \Gamma_1\}$; if $\Gamma = \Gamma_1 \cup \Gamma_1^{-1}$, condition (a) *may be replaced by*

b) $$\langle U_t V f, V g \rangle = \langle U_t f, g \rangle, \quad \forall f, g \in D_V, t \in \Gamma_1.$$

Proof. Equalities (a) and (b) are obvious whenever such extensions exist. We shall prove the converse in four steps.

i) We start extending U to a unitary representation $U^\wedge$ of Γ. In order to do that we define a Toeplitz form $(B; \Gamma; E)$ by setting

$$B(h, h') = \sum \{\langle U_{s+u}h(s), U_{t+u}h'(t)\rangle_E : s, t \in \Gamma\}$$

with $u \in \Gamma_1$ such that the sets $\{u + \operatorname{supp} h\}$ and $\{u + \operatorname{supp} h'\}$ are both contained in Γ_1. Clearly, B is well defined and invariant under translations. Since $B(v, w) \equiv \equiv \langle v, w\rangle_E$, the minimal unitary dilation $U^\wedge = \{U_t^\wedge : t \in \Gamma\} \subset L(F^\wedge)$ of B extends U. Let $t \in \Gamma$; for $u \in \Gamma_1$ such that $t + u \in \Gamma_1$ it is clear that

$$\langle U_t^\wedge Vf, Vg\rangle_{F^\wedge} = \langle U_{t+u}Vf, U_u Vg\rangle_E = \langle U_t^\wedge f, g\rangle_{F^\wedge}$$

follows from (a). If $\Gamma = \Gamma_1 \cup \Gamma_1^{-1}$ and $t \in \Gamma_1^{-1}$, setting $u = -t$ the same follows from (b). Thus, we may assume that $\Gamma = \Gamma_1$ and that U is a unitary representation of Γ such that (b) holds.

ii) If $f \in D_V$, the operator V' we are looking for should satisfy $V'U_tf = = U_tVf$ for every $t \in \Gamma$. So in the subspace D_1 given by $D_1 = \bigvee\{U_tD_V : t \in \Gamma\}$ we define an isometry V_1 by setting $V_1(U_tf) = U_tVf$; then $U_tD_1 = D_1$ and $U_tV_1d = = V_1U_td$ hold for every $t \in \Gamma$ and every $d \in D_1$. Consequently we assume that $U_tD_V = D_V$ and $U_tVf = VU_tf$ are satisfied whenever $t \in \Gamma$ and $f \in D_V$.

iii) Let G be the orthogonal complement of $D \equiv D_V$ in E; then each U_t commutes with P_D^E and P_G^E. Let $F^\wedge$ be the Hilbert space of all the sequences $\Phi = \{f, g_1, \ldots, g_n, \ldots\}$ such that $f \in D$, every $g_n \in G$ and $\|\Phi\|_{F^\wedge}^2 \equiv \|f\|^2 + + \sum_{n=1}^{\infty} \|g_n\|^2 < \infty$. Clearly $F^\wedge \supset E$ because the last may be identified with $\{\Phi \in F^\wedge : g_n = 0 \text{ if } n > 0\}$. Let $U_t^\wedge$ and $V^\wedge$ be defined in $F^\wedge$ by $U_t^\wedge\Phi = \{U_tf, U_tg_1, \ldots, U_tg_n, \ldots\}$ and $V^\wedge\Phi = \{P_D^EVf, P_G^EVf, g_1, \ldots, g_n, \ldots\}$. Then the unitary group $\{U_t^\wedge : t \in \Gamma\}$ extends U while the isometry $V^\wedge$ extends V to the whole space $F^\wedge$. We have moreover, for any $\Phi \in F^\wedge$,

$$U_t^\wedge V^\wedge\Phi = \{P_D^EU_tVf, P_G^EU_tVf, U_tg_1, \ldots, U_tg_{n-1}, \ldots\} = V^\wedge U_t^\wedge\Phi.$$

iv) Our problem has been reduced to the case where U is a unitary group, $D_V = E$ and $U_tV = VU_t$ for every $t \in \Gamma$, so the result follows from the proof of Proposition (1.6.3) in [27]. Q.E.D.

2) *Proof of Theorem A(1.1).* Let Y be the orthogonal complement of D_U in E and X a Hilbert space such that if we set $E_1 = Y \oplus X$ there exists an isometry $\alpha: E_1 \to X$; replacing E by $D_U \oplus E_1 = E \oplus X$ and U by $U \oplus \alpha$, we may assume $D_U = E$ and apply Theorem (1). Q.E.D.

3) REMARK. The condition in Theorem (1) implies:

$$(\#) \qquad UVf = VUf, \quad \text{for every } f \in D_V \text{ such that } Uf \in D_V,$$

but the converse is not true. For example, set $E = \mathbf{C}^4$ and let $\{e_1, e_2, e_3, e_4\}$ be the canonic basis; if $D_V = \text{span}\{e_1, e_2\}$, $Ve_1 = e_1$, $Ve_2 = e_3$, $D_U = E$, $Ue_1 = e_1$, $Ue_2 = e_4$, $Ue_3 = e_3$ and $Ue_4 = -e_2$, then (#) is satisfied but $\langle UVe_2, Ve_2\rangle = 1 \neq 0 = \langle Ue_2, e_2\rangle$.

4. POSITIVE TOEPLITZ EXTENSIONS

Let $V = \{V_t : t \in \Gamma\}$ be the family of isometries associated to a positive T-K-C form, D_t and R_t the domain and range, respectively, of each V_t; thus, the defect indexes of V_t are the codimensions of D_t and R_t. When Γ is the integer group we have the following result on existence and uniqueness of positive Toeplitz extensions.

1) PROPOSITION. *Let $\{B;\ \mathbf{Z}, \Omega;\ E)$ be a positive* T-K-C *form such that its associated family of isometries V satisfies $V_n = V_1^n|D_n$ for every $n > 0$. Then B can be extended to a positive Toeplitz form, and that extension is unique if and only if at least one of the defect indexes of V_1 equals zero.*

Proof. Let $U \in L(F)$ be a unitary extension of V_1 to a Hilbert space containing H_Ω; since $V_{-n} = V_n^{-1}$ for every $n \in \mathbf{Z}$, $\{U^n : n \in \mathbf{Z}\}$ is a unitary extension of V. The result follows from Lemma 2.2. Q.E.D.

The above proposition has as a special case the fundamental property of the so-called operator-valued Generalized Toeplitz Kernels [7]. In order to extend it we shall consider forms in product groups and their restrictions to the factors. Precisely, when we say that the restriction $\{B_1;\ \Gamma_1, \Omega_1;\ E\}$ of $\{B;\ \Gamma, \Omega;\ E\}$ to Γ_1 is well defined, the following properties are assumed:

i) $\Gamma = \Gamma_0 \times \Gamma_1$, with Γ_0 a group with neutral element e_0;

ii) $\Omega = \{\omega_j^{(0)} \times \omega_j^{(1)} : j = 1, \ldots, n\}$, $e_0 \in \omega_j^{(0)} \subset \Gamma_0$ and $\omega_j^{(1)} \subset \Gamma_1$, for $j = 1, \ldots, n$, $\Omega_1 = \{\omega_1^{(1)}, \ldots, \omega_n^{(1)}\}$;

iii) B_1 is given by $B_1(h, h') = B(H, H')$, $\forall\, h, h' \in A_{\Omega_1}$, where, for any $h \in A_{\Omega_1}$, we put $H(t_0, t_1) = 0$ if $t_0 \neq e_0$ and $H(e_0, t_1) \equiv h(t_1)$.

2) PROPOSITION. *Let $\{B;\ \Gamma_0 \times \mathbf{Z}, \Omega;\ E)$ be a positive* T-K-C *form with the following properties:*

i) *Every $\omega_j \in \Omega$ satisfies $\omega_j = \Gamma_0 \times \omega_j^{(1)}$, with Γ_0 an abelian group and $\omega_j^{(1)} \subset \mathbf{Z}$.*

ii) *The restriction of B to $\mathbf{Z}$ is well defined and its associated family of isometries satisfies $V_n = V^n|D_n$ for every positive n.*

Then the given form has positive Toeplitz extensions.

Proof. Let $\mathscr{V} = \{V_{(t,m)}: t \in \Gamma_0,\ m \in \mathbf{Z}\}$ be the family of isometries associated to B; set $V = V_{(e_0,1)}$ and $U_t = V_{(t,0)}$ for every $t \in \Gamma_0$. From (i) it follows that $\{U_t : t \in \Gamma_0\}$ is a unitary group; clearly $U_t Vf = VU_t f$ holds for every f in D_V. From (ii) we know that $V_{(e_0,m)}$ equals the restriction of V^m to $D_{(e_0,m)}$ for every positive m. Then Theorem 3.1 gives a unitary extension of $\mathscr{V}$, so once more we may apply Lemma 2.2. Q.E.D.

5. ON A THEOREM OF KREIN

1) *Proof of Theorem B (1.2).* Let $\{B;\ \mathbf{Z}^2,\ \Omega;\ E\}$ be the T-K-C form given by $\Omega = \{\omega_1\}$, $\omega_1 = \{(m, n) \in \mathbf{Z}^2 : 0 \leqslant m \leqslant a,\ 0 \leqslant n \leqslant b\}$ and

$$B(h, h') = \sum \{\langle k(s-t)h(s), h'(t)\rangle_E : s,\ t \in \mathbf{Z}^2\},$$

which is positive because k is a function of positive type. Let $\mathscr{V} = \{V_{(m,n)} : (m, n) \in \mathbf{Z}^2\}$ be the associated family of isometries. Assume that $k(\cdot\,, 0)$ has only one extension of positive type to $\mathbf{Z}$. Set $U = V_{(1,0)}$ and $V = V_{(0,1)}$. Then (4.1) implies that U is defined in the whole space H_Ω. The result follows from Theorem A(1.1). Q.E.D.

With a similar proof we obtain from Proposition 4.2 the following analogous property.

2) PROPOSITION. *Let Γ be a group, b a fixed natural number, E a Hilbert space and k: $\{(s, n): s \in \Gamma,\ |n| \leqslant b\} \to L(E)$ a function of positive type. Then there exists a function of positive type, K: $\Gamma \times \mathbf{Z} \to L(E)$, that extends k.*

6. ON THE HELSON-SZEGÖ THEOREM

Let $(\Omega, \mathscr{A}, P)$ be a probability space, Γ a LCA group and $X = \{X_t : t \in \Gamma)\} \subset L^2(\Omega, \mathscr{A}, P)$ a stochastic process. Its covariance K: $\Gamma \times \Gamma \to \mathbf{C}$ is given by $K(s, t) = \langle X_s, X_t\rangle = \int_\Omega X_s \bar{X}_t \mathrm{d}P$. When $K(s+u, t+u) = K(s, t)$ for every $s, t, u \in \Gamma$ it is said that X is a weak sense stationary process; in that case, if K is also continuous, X has a spectral measure, i.e., there exists a positive Borel measure μ on the dual group $\Gamma^\wedge$ such that its Fourier transform $\mu^\wedge$ satisfies $K(s, t) \equiv \mu^\wedge(s-t)$. If X is a Gaussian process, a theorem due to Kolmogorov and Rozanov (see [22]) says that the probabilistic dependence between the σ-algebras generated by two subsets $X^{(1)}$ and $X^{(2)}$ of X can be measured by the cosinus of the angle between the subspaces generated by these subsets. When $\Gamma = \mathbf{Z}$, Helson and Szegö [23] solved the problem in prediction theory of characterizing the spectral measure of a weakly stationary process X with positive angle between its "past" and its "future", i.e., the subspaces generated by $\{X_t : t < 0\}$ and $\{X_t : t \geqslant 0\}$, respectively. Concerning that problem we have the following:

1) THEOREM. *Let Γ be a* LCA *group, $X = \{X_t : t \in \Gamma\}$ a weak sense stationary process with spectral measure μ, ω_1 and ω_2 subsets of I such that $\Gamma = \{s - t : s \in \omega_1, t \in \omega_2\}$.*

i) *The following conditions* (a) *and* (b) *are equivalent.*

a) $d\mu = e^u d|\theta|$, *with u a real function $|\theta|$-essentially bounded on $\Gamma^\wedge$ and $|\theta|$ the total variation of a complex Borel measure θ on $\Gamma^\wedge$ such that:*

$$\sup\{|\arg\theta(\beta)| : \beta \subset \Gamma^\wedge, \text{ a Borel set}\} < \pi/2,$$

and $\theta^\wedge(s - t) = 0$, $\forall\, s \in \omega_1,\ t \in \omega_2$;

b) $\exists\, r \in (0, 1)$ *such that the* T-K-C *form* $\{B_r; \Gamma, \Omega; C^2)$ *given by* $\Omega = \{\omega_1, \omega_2\}$ *and*

$$B_r[(h_1, h_2), (h_1', h_2')] = \sum \{\mu^\wedge(s - t) r^{1-|j-k|} h_j(s) \bar{h}_k'(t) : s,\ t \in \Gamma,\ j,\ k = 1, 2\}$$

has a positive Toeplitz extension.

ii) *The preceding conditions imply that the angle between the subspaces generated by $\{X_t ; t \in \omega_1\}$ and $\{X_t : t \in \omega_2\}$ is positive.*

Proof. It is not difficult to see that B_r is positive if and only if the cosinus of the above mentioned angle is bounded by r; thus, (ii) follows from (b).

Assume that there exists a positive Toeplitz extension of B_r. From (1.1d) we obtain a positive matrix $\{\mu_{jk}\}_{1 \leqslant j,k \leqslant 2}$ of measures on $\Gamma^\wedge$ such that $\mu^\wedge(s - t) = r^{-1+|j-k|}\mu^\wedge(s - t)$ holds for every $(s, t) \in \omega_j \times \omega_k$, j, $k = 1, 2$. Set $\theta = \mu - \mu_{12}$; then $\theta^\wedge(s - t) = 0$, $\forall\, (s, t) \in \omega_1 \times \omega_2$. The assumptions in ω_1 and ω_2 imply $r\mu = \mu_{11} = \mu_{22}$. For any Borel set $\beta \subset \Gamma^\wedge$ we have

$$0 \leqslant \mu_{11}(\beta)\mu_{22}(\beta) - |\mu_{12}(\beta)|^2 = (r^2 - 1)\mu^2(\beta) + 2[\mathrm{Re}\,\theta(\beta)]\mu(\beta) - |\theta(\beta)|^2.$$

Since $r < 1$, μ must be $|\theta|$-absolutely continuous; $|\arg\theta(\beta)| \leqslant \alpha$ holds if $\sin\alpha = r$ and $\alpha \in (0, \pi/2)$; also there exist two positive constants m and M such that $m|\theta(\beta)| \leqslant \mu(\beta) \leqslant M|\theta(\beta)|$. Thus, (b) implies (a). The argument is reversible

Q.E.D.

When $\Gamma = \mathbf{Z}$, $\omega_1 = \{m \in \mathbf{Z} : m < 0\}$ and $\omega_2 = \{m \in \mathbf{Z} : m \geqslant 0\}$, Proposition 4.1 says that B_r is positive if and only if it has a positive Toeplitz extension, so (i) and (ii) are equivalent; in that case the theorem of F. and M. Riesz shows that $d\theta = \psi ds$ holds with ds the Lebesgue measure on $\mathbf{T}$ and ψ a function of the Hardy class $H^1(\mathbf{T})$; setting $v = \arg\psi$ we get $d\mu = e^{u+v^\sim} ds$, as is established by the Helson-

-Szegö theorem. Details and related results can be found in [18], [9], [10] and [21]. Conditions (i) and (ii) are also equivalent in the case of half-plane prediction of stationary processes in $\mathbf{Z}^2$, as it is considered in [24]. In fact:

2) THEOREM. *Let Γ_0 be a* LCA *group and $\{X_s : s \in \Gamma_0 \times \mathbf{Z}\}$ a weakly stationary process with spectral measure μ. Then the angle between the subspaces generated by $\{X_{(t,m)} : (t, m) \in \Gamma_0 \times \mathbf{Z}, m < 0\}$ and $\{X_{(t,m)} : (t, m) \in \Gamma_0 \times \mathbf{Z}, m \geqslant 0\}$ is positive if and only if $\mathrm{d}\mu = \mathrm{e}^u \mathrm{d}|\theta|$ holds with u a real valued $|\theta|$-essentially bounded function on $\Gamma_0' \times \mathbf{T}$ such that $\sup\{|\arg \theta(\beta)| : \beta \subset \Gamma_0^{\hat{}} \times \mathbf{T}$, a Borel set$\} < \pi/2$ and $\theta^{\wedge}(t, m) = 0$, $\forall t \in \Gamma_0$, $m \in \mathbf{Z}$ and $m < 0$.*

Proof. Follows from Theorem 1 and Proposition 4.2. Q.E.D.

7. ON THE LIFTING OF THE COMMUTANT

We wish to show here that T-K-C forms allow a general approach to some problems related to the dilation of commutants in the Sz.-Nagy—Foiaş sense, leading in particular to a bidimensional version of the fundamental lifting theorem due to them.

A *commutant* is a set $\{T_1, T_2, X\}$ of two contraction semigroups and an intertwining between them.

More precisely, when we speak of a commutant, we assume that the following objects are given:

i) a semigroup Γ_1 with unit element e;

ii) two families $T_j = \{T_j(s) : s \in \Gamma_1\} \subset L(E_j)$ of contractions in Hilbert spaces E_j such that $T_j(e)$ equals the identity operator and $T_j(st) = T_j(s)T_j(t)$ holds for every $s, t \in \Gamma_1$, $j = 1, 2$;

iii) an operator $X \in L(E_1, E_2)$ that intertwines T_1 and T_2, i.e., such that

$$XT_1(s) = T_2(s)X, \quad \forall s \in \Gamma_1.$$

In the special case $\Gamma_1 = \{m \in \mathbf{Z} : m \geqslant 0\}$ we shall sometimes identify T_j with $T_j(1)$.

We are interested in commutants with such properties that lifting problems may be posed. So we state some assumptions and also fix some notations that are to be kept from now on.

Γ_1 will be a sub-semigroup of an abelian group Γ with unit element e, such that every $u \in \Gamma$ can be written as $u = s - t$, $s, t \in \Gamma_1$. For $j = 1, 2$ there exists $U_j = \{U_j(s) : s \in \Gamma\}$, a unitary dilation of T_j to a Hilbert space F_j; that means that U_j is a unitary representation of Γ on $F_j \supset E_j$ such that

$$T_j(s) = P_{E_j}^{F_j} U_j(s) \mid E_j, \quad \forall s \in \Gamma_1;$$

we may assume that the following minimality condition also holds:

$$F_j = \bigvee \{U_j(s)E_j : s \in \Gamma\}.$$

From U_j we get a minimal isometric dilation $W_j \subset L(M_j)$ by setting

$$M_j = \bigvee \{U_j(s)E_j : s \in \Gamma_1\} \quad \text{and} \quad W_j(s) = U_j(s)|M_j, \quad \text{for } s \in \Gamma_1 ;$$

in fact, it is clear that $M_j \supset E_j$, $M_j = \bigvee \{W_j(s)E_j : s \in \Gamma_1\}$ and $T_j(s) = P_{E_j}^{M_j} W_j(s)|E_j$ for every $s \in \Gamma_1$, $j = 1, 2$.

When we consider the *problems of the lifting of the commutant* $\{T_1, T_2, X\}$ we assume that the minimal unitary dilations U_1 and U_2 of T_1 and T_2 are fixed, and we assume also that W_1 and W_2 are the minimal isometric dilations of T_1 and T_2 given as above by U_1 and U_2, respectively. In such conditions, the problems are to determine if X can be lifted to intertwinings between U_1 and U_2, and between W_1 and W_2. That is, we want to know if there exist operators γ and τ such that the following hold:

(1) $\gamma \in L(M_1, M_2)$, $\gamma W_1(s) = W_2(s)\gamma$ for every $s \in \Gamma_1$, $\|\gamma\| = \|X\|$ and $P_{E_2}^{M_2}\gamma = XP_{E_1}^{M_1}$;

(2) $\tau \in L(F_1, F_2)$, $\tau U_1(s) = U_2(s)\tau$ for every $s \in \Gamma$, $\|\tau\| = \|X\|$ and $P_{E_2}^{F_2}\tau|M_1 = XP_{E_1}^{M_1}$, so $P_{E_2}^{F_2}\tau|E_1 = X$.

Note that even in the simpler cases it is not always possible to get τ such that also $P_{E_2}^{F_2}\tau = XP_{E_1}^{F_1}$ ([15]).

If $X = 0$, the problems are trivial. If $X \neq 0$, we may assume by homogeneity that $\|X\| = 1$ and so we do from now on.

We approach these problems by associating to the commutant $\{T_1, T_2, X\}$ and to the minimal unitary dilations U_1 and U_2 a T-K-C form

$$\{B; \Gamma, \Omega; E_1 \times E_2\}$$

which we define setting $\Gamma_1^{-1} = \{s \in \Gamma : -s \in \Gamma_1\}$, $\Omega = \{\Gamma_1, \Gamma_1^{-1}\}$ and

$$B(h, h') = \sum \{\langle g_{jk}(s-t)h_j(s), h_k'(t)\rangle_{E_k} : j, k = 1, 2, \ s, t \in \Gamma\}$$

$$\text{with } g_{jj}(s) = P_{E_j}^{F_j} U_j(s)|E_j, \ s \in \Gamma, \ j = 1, 2, \tag{3}$$

$$g_{12}(s) = XT_1(s), \ s \in \Gamma_1, \quad g_{21}(s) = g_{12}(-s)^*, \ s \in \Gamma_1^{-1}.$$

Let us remark that in the trivial case $T_1 = T_2 = T$, a contraction, and $X = I$, B is given by the function g such that $g(n) = T^n$ if $n \geqslant 0$ and $g(n) = T^{*-n}$ if $n \leqslant 0$, by means of which Sz.-Nagy's theorem on the unitary dilation of contraction is deduced from Naimark's theorem (see [27]).

We start verifying that B is always positive. If $h = (h_1, h_2) \in \Omega$ is given, we set $f_j = \sum \{U_j(s)h_j(s) : s \in \Gamma\}$, $j = 1, 2$; then

$$B(h, h) = \|f_1\|_{F_1}^2 + \|f_2\|_{F_2}^2 + 2\mathrm{Re} \sum \{\langle g_{12}(s-t)h_1(s), h_2(t)\rangle_{E_2} : (s,t)\in\Omega\};$$

from the equalities

$$\langle XT_1(-t)T_1(s)h_1(s), h_2(t)\rangle_{E_2} = \langle XP_{E_1}^{F_1}U_1(s)h_1(s),\ T_2(-t)^*h_2(t)\rangle_{E_2} =$$

$$= \langle XP_{E_1}^{F_1}U_1(s)h_1(s),\ P_{E_2}^{F_2}U_2(t)h_2(t)\rangle_{E_2}$$

it follows that

$$B(h, h) = \|f_1\|_{F_1}^2 + \|f_2\|_{F_2}^2 + 2\mathrm{Re}\langle XP_{E_1}^{F_1}f_1, f_2\rangle_{E_2} \geqslant 0$$

because $\|X\| = 1$.

Call $\mathscr{B}$ the set of all positive Toeplitz extensions of the T-K-C form given by (3). Then:

4) LEMMA. *For every $B' \in \mathscr{B}$ there exist four functions*

$$g'_{jk} : \Gamma \to L(E_j, E_k), \quad j, k = 1, 2,$$

such that

$$g'_{11} \equiv g_{11}, \quad g'_{12}|\Gamma_1 \equiv g_{12}, \quad g'_{21}(s) \equiv g'_{12}(-s)^*, \quad g'_{22} \equiv g_{22}$$

and

$$B'(h, h') = \sum \{\langle g'_{jk}(s-t)h_j(s), h'_k(t)\rangle_{E_k} : j, k = 1, 2, \quad s, t \in \Gamma\}$$

for every $h, h' \in A(\Gamma, E_1 \times E_2)$.

Proof. If $h_1, h'_1 \in A(\Gamma, E_1)$ let $a \in \Gamma_1$ be such that the supports of $S_a h_1$ and $S_a h'_1$ are contained in Γ_1; then $B'([h_1, 0], [h'_1, 0]) = B(S_a[h_1, 0], S_a[h'_1, 0]) = \sum \{\langle g_{11}(s-t)h_1(s), h'_1(t)\rangle_{E_1} : s, t \in \Gamma\}$. The corresponding fact for g_{22} is proved in the same way.

For $t \in \Gamma, v_1 \in E_1$ and $v_2 \in E_2$ set $G_t(v_1, v_2) = B'([S_t v_1, 0], [0, v_2])$ and $h = [S_t v_1, v_2] \in A(\Gamma, E_1 \times E_2)$; then

$$0 \leqslant B'(h, h) = B'([S_t v_1, 0], [S_t v_1, 0]) + G_t(v_1, v_2) + G_t(v_1, v_2)^- + B'([0, v_2], [0, v_2]),$$

so $|G_t(v_1, v_2)|^2 \leqslant \|v_1\|^2_{E_1} \|v_2\|^2_{E_2}$ holds for every $v_1 \in E_1$, $v_2 \in E_2$. Consequently there exists $g'_{12}(t) \in L(E_1, E_2)$ such that $\langle g'_{12}(t) v_1, v_2 \rangle = G_t(v_1, v_2)$. The result follows. Q.E.D.

Call $\mathscr{T}$ the set of all the "liftings" τ of X that verify (2). We now show that those liftings exist if and oniy if B has positive Toeplitz extensions, i.e., that $\mathscr{T}$ is non-void if and only if $\mathscr{B}$ is non-void. Moreover:

5) THEOREM. *A bijection from $\mathscr{B}$ to $\mathscr{T}$ is obtained by associating to each $B' \in \mathscr{B}$ the operator $\tau \in \mathscr{T}$ defined as follows: let $U' \subset L(H')$ be the minimal unitary dilation of B'; then F_j may be identified with a closed subsace of H', $U_j(\cdot)$ with $U'(\cdot)|F_j, j \leqslant 1, 2$, and X with $P^{H'}_{E_2}|E_1$; let γ be the restriction $P^{H'}_{M_2}|M_1$ of $P^{H'}_{M_2}$ to M_1 and τ the restriction $P^{H'}_{F_2}|F_1$ of $P^{H'}_{F_2}$ to F_1; then $P^{M_2}_{E_2}\gamma = XP^{M_1}_{E_1}$ and $\|\gamma\| = \|X\|$ while $\tau \in \mathscr{T}$.*

Proof. i) If $B' \in \mathscr{B}$ is given, we define an isometry λ_1 from F_1 to H', such that $\lambda_1 U_1(t) = U'(t)\lambda_1$ for every $t \in \Gamma$, by setting — with notations as in Section 2 — $\lambda_1(U_1(s)v_1) = \Pi(S_s[v_1, 0])$, $s \in \Gamma$ and $v_1 \in E_1$. Thus, we may assume that $F_j \subset H'$ and that $U_j(\cdot) = U'(\cdot)|F_j, j = 1, 2$. From now on v_j is any vector in E_j. Since

$$\langle Xv_1, v_2 \rangle_{E_2} = \langle g_{12}(e)v_1, v_2 \rangle_{E_2} = \langle v_1, v_2 \rangle_{H'}$$

we see that $X = P^{H'}_{E_2}|E_1$, so it is natural to define the operators γ and τ as above. Clearly $P^{F_2}_{E_2}\tau|E_1 = X$, so $1 \geqslant \|\tau\| \geqslant \|X\| = 1$, i.e., $\|\tau\| = 1$. We also have $\|\gamma\| = \|X\|$ and, for every $s \in \Gamma_1$,

$$\langle (P^{M_2}_{E_2}\gamma) W_1(s) v_1, v_2 \rangle_{E_2} = \langle U_1(s) v_1, v_2 \rangle_{H'} = B'(S_s v_1, v_2) =$$

$$= \langle g_{12}(s) v_1, v_2 \rangle_{E_2} = \langle (XP^{M_1}_{E_1}) W_1(s) v_1, v_2 \rangle_{E_2},$$

so $P^{M_2}_{E_2}\gamma = XP^{M_1}_{E_1}$; consequently $P^{F_2}_{E_2}\tau|M_1 = XP^{M_1}_{E_1}$. For any $t, s_1, s_2 \in \Gamma$ it is true that

$$\langle \tau U_1(t) U_1(s_1) v_1, U_2(s_2) v_2 \rangle_{F_2} = \langle U'(s_1) v_1, U'(-t) U'(s_2) v_2 \rangle_{H'} =$$

$$= \langle P^{H'}_{F_2} U_1(s_1) v_1, U_2(t)^* U_2(s_2) v_2 \rangle_{F_2} = \langle U_2(t) \tau U_1(s_1) v_1, U_2(s_2) v_2 \rangle_{F_2},$$

i.e., $\tau U_1(t) = U_2(t)\tau$. Summing up, $\tau \in \mathscr{T}$.

Moreover, the correspondence from $\mathscr{B}$ to $\mathscr{T}$ given by $B' \to \tau$ is injective: in fact, B' is determined by the function g'_{12} as in Lemma 4, which in turn satisfies

$$\langle g'_{12}(s)v_1, v_2\rangle_{E_2} = B'(S_s[v_1, 0], [0, v_2]) = \langle \Pi S_s[v_1, 0], \Pi[0, v_2]\rangle_{H'} =$$

$$= \langle U_1(s)v_1, v_2\rangle_{H'} = \langle P^{F_2}_{E_2}\tau U_1(s)v_1, v_2\rangle_{E_2},$$

that is

$$(\#) \qquad g'_{12}(s) = P^{F_2}_{E_2}\tau U_1(s)|E_1, \quad \forall\, s \in \Gamma.$$

ii) In order to complete the proof we have to show that, if $\tau_0 \in \mathcal{T}$ is given, there exists $B' \in \mathcal{B}$ such that, with the above construction, we let $\tau \equiv P^{H'}_{F_2}|F_1 = \tau_0$. Naturally, we set $g'_{11} \equiv g_{11}$, $g'_{22} \equiv g_{22}$, $g'_{12}(s) \equiv g'_{21}(-s)^* \equiv P^{F_2}_{E_2}\tau_0 U_1(s)|E_1$ and we consider the Toeplitz form $\{B'; \Gamma; E_1 \times E_2\}$ given, for any $h, h' \in A(\Gamma, E_1 \times E_2)$, by

$$B'(h', h') = \sum \{\langle g'_{jk}(s-t)h_j(s), h'_k(t)\rangle_{E_k} : j, k = 1, 2,\ s, t \in \Gamma\}.$$

Straightforward verifications show that $B' \in \mathcal{B}$. Finally, considering (#) we obtain, for any $s, t \in \Gamma$,

$$\langle \tau U_1(s)v_1, U_2(t)v_2\rangle_{F_2} = \langle g'_{12}(s-t)v_1, v_2\rangle_{E_2} = \langle \tau_0 U_1(s)v_1, U_2(t)v_2\rangle_{F_2};$$

consequently, $\tau = \tau_0$. Q.E.D.

Now we want to see that if the set $\mathcal{G}$ of all the operators γ that satisfy (1) is non-void neither are $\mathcal{T}$ and $\mathcal{B}$. We start with some preliminary remarks.

If $W = \{W(s) : s \in \Gamma_1\} \subset L(M)$ is a semigroup of isometries, then Naimark's dilation theorem applied to the function $g : \Gamma \to L(M)$ given by $g(s-t) = W(t)^*W(s)$ $\forall\, s, t \in \Gamma_1$, shows that W has an essentially unique minimal unitary extension $U \subset L(F)$.

Next we consider the lifting problem for a commutant $\{W_1, W_2, \gamma\}$ where $W_j(s)$ is an isometry in M_j for every $s \in \Gamma_1$, $j = 1, 2$. Let $\{\tilde{B}; \Gamma, G; M_1 \times M_2\}$ be the associated T-K-C form; its domain is the space $A_\Omega\{\Gamma, M_1 \times M_2\}$. In order to extend this form we define four functions as follows: $\tilde{g}'_{jj}(u) \equiv \tilde{g}_{jj}(u) = P^{F_j}_{M_j}U_j(u)|M_j$, $u \in \Gamma$ and $j = 1, 2$; $\tilde{g}'_{12}(s-t) \equiv \tilde{g}'_{21}(t-s)^* = W_2(t)^*\gamma W_1(s)$, $s, t \in \Gamma_1$; let $\{\tilde{B}'; \Gamma; M_1 \times M_2\}$ be the Toeplitz form in $A(\Gamma, M_1 \times M_2)$ given by

$$\tilde{B}'(\psi, \psi') = \sum \{\langle \tilde{g}'(s-t)\psi_j(s), \psi'_k(t)\rangle_{M_k} : j, k = 1, 2,\ s, t \in \Gamma\}.$$

Let $\tilde{g}_{12} = \gamma W_1(\cdot)$; since $\tilde{g}'_{12}(s) = \tilde{g}_{12}(s)$ for every $s \in \Gamma_1$, $\tilde{B}'$ extends $\tilde{B}$. For any $\psi = (\psi_1, \psi_2) \in A(\Gamma, M_1 \times M_2)$ let $a \in \Gamma_1$ be such that the supports of $S_a\psi_1$ and $S_a\psi_2$ are contained in Γ_1; then

$$\tilde{B}'(\psi, \psi) = \tilde{B}'(S_a\psi, S_a\psi) = \sum\{\langle \tilde{g}'_{jk}(s-t)S_a\psi_j(s), S_a\psi_k(t)\rangle_{M_k} : j, k = 1, 2, \; s, t \in \Gamma_1\} =$$

$$= \left\|\sum\{W_1(s)S_a\psi_1(s) : s \in \Gamma_1\}\right\|^2_{M_1} + \left\|\sum\{W_2(t)S_a\psi_2(t) : t \in \Gamma_1\}\right\|^2_{M_2} +$$

$$+ 2\mathrm{Re}\langle \gamma \sum\{W_1(s)S_a\psi_1(s) : s \in \Gamma_1\}, \sum\{W_2(t)S_a\psi_2(t) : t \in \Gamma_1\}\rangle_{M_2} \geqslant 0.$$

We have seen that $\tilde{B}'$ is a positive Toeplitz extension of $\tilde{B}$; it follows from Theorem 5 that the commutant $\{W_1, W_2, \gamma\}$ can be lifted to an intertwining between the unitary extensions U_1 and U_2 of W_1 and W_2, respectively. Moreover, we can use the above construction to prove

6) PROPOSITION. *There exists an injective correspondence $\gamma \to B'$ from $\mathscr{G}$ to $\mathscr{B}$ such that if H' is the space of the minimal unitary dilation of B' then M_1 and M_2 may be identified with closed subspaces of H' and γ with $P^{H'}_{M_2}|M_1$. So if $\mathscr{G}$ is non-void neither are $\mathscr{T}$ and $\mathscr{B}$.*

Proof. The restriction $\{B'; \Gamma; E_1 \times E_2\}$ of $\{\tilde{B}'; \Gamma; M_1 \times M_2\}$ to the space $A(\Gamma, E_1 \times E_2)$ is a positive Toeplitz form. It is given in the usual way by the functions $g'_{jk}(u) = P^{M_k}_{E_k}\tilde{g}'_{jk}(u)|E_j \in L(E_j, E_k)$, $j, k = 1, 2$. Since $g'_{jj}(u) = P^{F_j}_{E_j}U_j(u)|E_j = g_{jj}(u)$, $\forall\, u \in \Gamma$, and for every $s \in \Gamma_1$ we have

$$g'_{12}(s) = P^{M_2}_{E_2}\gamma W_1(s)|E_1 = XP^{M_1}_{E_1}W_1(s)|E_1 = XT_1(s) = g_{12}(s),$$

it follows that B' extends B. Thus $B' \in \mathscr{B}$.

The correspondence $\gamma \to B'$ is injective because g'_{12} determines γ (and conversely); in fact, for any $s_1, s_2 \in \Gamma_1$, $v_1 \in E_1$, $v_2 \in E_2$, we have

$$\langle \gamma W_1(s_1)v_1, W_2(s_2)v_2\rangle_{M_2} = \langle P^{M_2}_{E_2}W_2(s_2)^*\gamma W_1(s_1)v_1, v_2\rangle_{E_2} =$$
$$= \langle g'_{12}(s_1 - s_2)v_1, v_2\rangle_{E_2}.$$

Theorem 5 implies that $M_j \subset F_j$ may be identified with a closed subspace of H'. From

$$\langle \gamma W_1(s_1)v_1, W_2(s_2)v_2\rangle_{M_2} =$$

$$= \langle g'_{12}(s_1 - s_2)v_1, v_2\rangle_{E_2} = B'(S_{s_1}[v_1, 0], S_{s_2}[0, v_2]) = \langle U'(s_1)v_1, U'(s_2)v_2\rangle_{H'} =$$

$$= \langle P^{H'}_{M_2}W_1(s_1)v_1, W_2(s_2)v_2\rangle_{M_2}$$

it follows that $\gamma = P^{H'}_{M_2}|M_1$. Q.E.D.

If $\Gamma = \Gamma_1 \cup \Gamma_1^{-1}$ the correspondence given by Proposition 6 is bijective. In order to prove it we shall use the following:

7) LEMMA. *Let $\{W(s) : s \in \Gamma_1\} \subset L(M)$ be a minimal isometric dilation of the contraction semigroup $\{T(s) : s \in \Gamma_1\} \subset L(E)$. Then, for every $s \in S_1$, it is true that $W(s)^*E \subset E$ and consequently $T(s)^* = W(s)^*|E$.*

Proof. We have to see that if x belongs to the orthogonal complement E' of E in M then $W(s)x$ also belongs to E'. Set $e_0 = P_E^M W(s)x$ and, for any $s_1, \ldots, s_n \in \Gamma_1$, $e_1, \ldots, e_n \in E$, $\varepsilon = x - \sum\{W(s_j)e_j : 1 \leqslant j \leqslant n\}$. If $e \in E$ then

$$\|\varepsilon\| \cdot \|e\| \geqslant |\langle \varepsilon, T(s)^* e\rangle| = |\langle \sum\{T(s_j)e_j : 1 \leqslant j \leqslant n\}, T(s)^* e\rangle| =$$

$$= |\langle W(s) \sum\{W(s_j)e_j : 1 \leqslant j \leqslant n\}, e\rangle|,$$

so $\langle W(s)x, e_0\rangle$ is bounded by $|\langle W(s)\varepsilon, e_0\rangle| + |\langle W(s) \sum\{W(s_j)e_j : 1 \leqslant j \leqslant n\}, e_0\rangle| \leqslant 2\|\varepsilon\| \cdot \|e_0\|$. Since ε can be made arbitrarily small, $e_0 = 0$. Q.E.D.

8) THEOREM. *Let $\Gamma = \Gamma_1 \cup \Gamma_1^{-1}$; if $B' \in \mathscr{B}$ and H' is the space of its minimal unitary dilation, set $\gamma = P_{M_2}^{H'}|M_1$ and $\tau = P_{F_2}^{H'}|F_1$; then $B' \to \gamma$ and $B' \to \tau$ give bijective correspondences from $\mathscr{B}$ to $\mathscr{G}$ and to $\mathscr{T}$, respectively.*

Proof. Theorem 5 and Proposition 6 show that it is enough to prove that $B' \to \gamma$ gives an injective correspondence from $\mathscr{B}$ to $\mathscr{G}$. If γ intertwines W_1 and W_2, then it determines g'_{12}, as we saw in the proof of (6); the injectivity follows. We know already that $\|\gamma\| = \|X\|$ and $P_{E_2}^{M_2}\gamma = XP_{E_1}^{M_1}$, so it only remains to see that

$$(\#) \qquad \langle \gamma W_1(s)W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2} = \langle W_2(s)\gamma W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2}$$

holds for any $s, t_1, t_2 \in \Gamma_1$, $v_1 \in E_1$, $v_2 \in E_2$. Two cases must be considered.

i) $t_2 - s \in \Gamma_1$: the first member of $(\#)$ equals

$$\langle U'(s + t_1)v_1, U'(t_2)v_2\rangle_{H'} =$$

$$= \langle P_{M_2}^{H'} W_1(t_1)v_1, W_2(t_2 - s)v_2\rangle_{M_2} = \langle W_2(s)\gamma W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2}.$$

ii) $s - t_2 \in \Gamma_1$: the second member of $(\#)$ equals

$$\langle U'(s - t_2)\gamma W_1(t_1)v_1, v_2\rangle_{H'} =$$

$$= \langle W_2(s - t_2)\gamma W_1(t_1)v_1, v_2\rangle_{M_2} = \langle P_{E_2}^{M_2}\gamma W_1(t_1)v_1, W_2(s - t_2)^* v_2\rangle_{E_2} =$$

$$= \langle T_2(s - t_2)XT_1(t_1)v_1, v_2\rangle_{E_2} = \langle XT_1(s - t_2 + t_1)v_1, v_2\rangle_{E_2} =$$

$$= \langle g_{12}(s - t_2 + t_1)v_1, v_2\rangle_{E_2} = \langle U'(s - t_2 + t_1)v_1, v_2\rangle_{H'} =$$

$$= \langle P_{M_2}^{H'} W_1(s)W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2} = \langle \gamma W_1(s)W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2}.$$

Q.E.D.

The last theorem implies the following result on the lifting of the commutant.

9) THEOREM. *Let Γ_0 be an abelian group, $Q_j \subset L(E_j)$ a unitary representation of Γ_0, $Y_j \in L(E_j)$ a contraction that commutes with every element of Q_j, $j = 1, 2$, and $X \in L(E_1, E_2)$ such that $XY_1 = Y_2X$ and $XQ_1(t) = Q_2(t)X$, $\forall\, t \in \Gamma_0$.*

i) *If $Q_j \subset L(M_j)$ is a unitary representation of Γ_0 that extends Q_j, $Y'_j \in L(M_j)$ is an isometric dilation of Y_j that commutes with every element of Q'_j and moreover $M_j = \bigvee \{Y_j'^n Q'_j(s)E_j : s \in \Gamma_0,\ n \geq 0\}$, $j = 1, 2$, then there exists $\gamma \in L(M_1, M_2)$ such that $\gamma Y'_1 = Y'_2\gamma$, $\gamma Q'_1(t) = Q'_2(t)\gamma$ for every $t \in \Gamma_0$, $P^{M_2}_{E_2}\gamma = XP^{M_1}_{E_1}$ and $\|\gamma\| = \|X\|$.*

ii) *If $Q''_j \subset L(F_j)$ is a unitary representation of Γ_0 that extends Q_j and $Y''_j \in L(F_j)$ is a unitary dilation of Y_j that commutes with every element of Q''_j, $j = 1, 2$, then there exists $\tau \in L(F_1, F_2)$ such that $\tau Y''_1 = Y''_2\tau$, $\tau Q''_1(t) = Q''_2(t)\tau$ for every $t \in \Gamma_0$, $P^{F_2}_{E_2}\tau | E_1 = X$ and $\|\tau\| = \|X\|$.*

REMARK. Given Q_j, Y_j as before, we can always find Q'_j, Y'_j as in (i) and Q''_j, Y''_j as in (ii). In fact, set $\mathbf{Z}_1 = \{m \in \mathbf{Z} : m \geqslant 0\}$, $\Gamma = \Gamma_0 \times \mathbf{Z}$, $\Gamma_1 = \Gamma_0 \times \mathbf{Z}_1$ and $T_j(t, m) = Q_j(t)Y_j^m$, $\forall\,(t, m) \in \Gamma_1$, $j = 1, 2$. Then the semigroup of contractions $T_j \subset L(E_j)$ has a unitary dilation; that can be proved as in ([27], 1.6.3).

Proof. a) Let $\bar{U}_j \subset L(\bar{F}_j)$ be a minimal unitary dilation of the semigroup $T_j = \{T_j(t, m) : (t, m) \in \Gamma_1\}$ we just defined. With Γ, Γ_1 as above and $\Omega = \{\Gamma_1, \Gamma_1^{-1}\}$ let $\{B; \Gamma, \Omega; E_1 \times E_2\}$ be the positive T-K-C form associated to the commutant $\{T_1, T_2, X\}$ and the minimal unitary dilations $\bar{U}_1$ and $\bar{U}_2$. Proposition 4.2 shows that this form has a positive Toeplitz extension, i.e., $\mathscr{B} \neq \emptyset$. Then Theorem 5 says that there exists $\bar{\tau} \in L(\bar{F}_1, \bar{F}_2)$ such that $\bar{\tau}\bar{U}_1(s) = \bar{U}_2(s)\bar{\tau}$ for every $s \in \Gamma$, $P^{\bar{F}_2}_{E_2}\bar{\tau} | E_1 = X$, $\|\bar{\tau}\| = \|X\|$.

b) Let $U_j \subset L(F_j)$ be the unitary representation of Γ given by $U_j(t, m) = Q''_j(t)Y_j''^m$ for every $(t, m) \in \Gamma$, $j = 1, 2$. Setting $\bar{F}_j = \bigvee \{U_j(s)E_j : s \in \Gamma\}$ and $\bar{U}_j(s) = U_j(s)|\bar{F}_j$ we can find $\bar{\tau}$ as in (a), so $\tau \in L(F_1, F_2)$ defined by $\tau(x) = 0$ if x belongs to the orthogonal complement of $\bar{F}_1$ in F_1 and $\tau(x) = \bar{\tau}(x)$ if x belongs to $\bar{F}_1$ shows that assertion (ii) is proved.

c) It follows from the previous remark that the semigroup W_j given by $W_j(t, m) = Q'_j(t)V_j'^m$, $(t, m) \in \Gamma_1$, has a minimal unitary extension $U_j \subset L(F_j)$. Clearly, U_j is a unitary dilation of T_j; moreover, from $F_j = \bigvee \{U_j(s)M_j : s \in \Gamma\}$ and $M_j = \bigvee \{W_j(u)E_j : u \in \Gamma_1\}$ it follows that $F_j = \bigvee \{U_j(s)E_j : s \in \Gamma\}$. So we are in the conditions of the lifting problems we have been considering. Since in this case $\Gamma = \Gamma_1 \cup \Gamma_1^{-1}$ and we know from (ii) that $\mathscr{T} \neq \emptyset$, Theorem 8 shows that $\mathscr{G} \neq \emptyset$. Q.E.D.

When Γ_0 is trivial, assertion (i) is Sz.-Nagy—Foiaş lifting theorem.

Let us go back to the general problem of the lifting of a commutant $\{T_1, T_2, X\}$. The dual problem is naturally posed as follows. Set $\tilde{T}(s) = T_j(-s)^*$, $\forall s \in \Gamma_1^{-1}$, we say that $\{\tilde{T}_1, \tilde{T}_2, X^*\}$ is the dual commutant. Clearly $\tilde{U}_j \equiv U_j$ is a minimal unitary dilation of $\tilde{T}_j$ to $\tilde{F}_j \equiv F_j$. Moreover, setting $\tilde{M}_j = \bigvee\{\tilde{U}_j(s)E_j : s \in \Gamma_1^{-1}\}$ and $\tilde{W}_j(s) = \tilde{U}_j(s)|\tilde{M}_j$, $\forall s \in \Gamma_1^{-1}$, we get a minimal isometric dilation of $\tilde{T}_j$. The sets $\mathscr{G}^\sim$, $\mathscr{T}^\sim$ and $\mathscr{B}^\sim$ are naturally defined. Then $\tau \to \tau^*$ gives a bijection from $\mathscr{T}$ to $\mathscr{T}^\sim$; to see this, it is enough to verify the equality $P_{E_1}^{F_1}\tau^*|\tilde{M}_2 = X^*P_{E_2}^{\tilde{M}_2}$. Now, if $s \in \Gamma_1^{-1}$, $v_1 \in E_1$ and $v_2 \in E_2$ it follows that

$$\langle P_{E_1}^{F_1}\tau^* U_2(s)v_2, v_1\rangle_{E_1} = \langle v_2, P_{E_2}^{F_2}\tau U_1(-s)v_1\rangle_{E_2} =$$

$$= \langle v_2, XP_{E_1}^{M_1}U_1(-s)v_1\rangle_{E_2} = \langle v_2, XT_1(-s)v_1\rangle_{E_2} = \langle T_2(-s)^* v_2, Xv_1\rangle_{E_2} =$$

$$= \langle X^* P_{E_2}^{\tilde{M}_2}U_2(s)v_2, v_1\rangle_{E_2}.$$

These remarks show, for example, that:

10) PROPOSITION. *In the problem of the lifting of the commutant, when Γ is a group ordered by Γ_1, if T_2 is a semigroup of isometries or if T_1 is a semigroup of coisometries, then the sets $\mathscr{G}$, $\mathscr{T}$ and $\mathscr{B}$ have each only one element.*

Proof. From (8) and duality it follows that we may assume that T_2 is a semigroup of isometries; then $M_2 \equiv E_2$ so it must be $\gamma = XP_{E_1}^{M_1}$. Thus, it is enough to verify that the last operator belongs to $\mathscr{G}$: if $s, t \in \Gamma_1$ and $v_1 \in E_1$ then

$$(XP_{E_1}^{M_1})W_1(s)[W_1(t)v_1] = XT_1(s+t)v_1 = W_2(s)(XP_{E_1}^{M_1})[W_1(t)v_1].$$

Q.E.D.

8. COMMUTANTS IN Z^2

In order to consider the lifting problems for pairs of contractions we fix the following notation: $\Gamma = Z^2$, $\Gamma_1 = \{(m, n) \in \Gamma : m, n \geq 0\}$, and we consider semigroups $T_j = \{T_j(s) : s \in \Gamma_1\} \subset L(E_j)$ given by $T_j(m, n) = T_j'^m T_j''^n$, where T_j' and T_j'' are commuting contractions in a Hilbert space E_j, $j = 1, 2$. A theorem due to Ando [4] states that there exists a commuting pair of unitary operators $\{U_j', U_j''\} \subset L(F_j)$ such that $F_j = \bigvee\{U_j'^m U_j''^n E_j : m, n \in Z\} \supset E_j$ and $T_j'^m T_j''^n = P_{E_j}^{F_j} U_j'^m U_j''^n|E_j$, $\forall m, n \geq 0$, $j = 1, 2$. If $X \in L(E_1, E_2)$ intertwines T_1' with T_2' and T_1'' with T_2'', we shall study the lifting problems for $\{T_1, T_2, X\}$.

Considering the associated T-K-C form B, and with notations as in Section 2 we have in this case

(1) $D'_{(m,n)} = \{h = (h_1, h_2) \in A_\Omega(\mathbf{Z}^2, E_1 \times E_2) : h_1(k, j) = 0$ if $k < -m$ or $j < -n$, $h_2(k,j) = 0$ if $k > -m$ or $j > -n\}$, with $\Omega = \{\Gamma_1, \Gamma_1^{-1}\}$, and $S_{(m,n)}|D'_{(m,n)} = S^m_{(1,0)}S^n_{(0,1)}|D'_{(m,n)} = S^n_{(0,1)}S^m_{(1,0)}|D'_{(m,n)}$, so

(2) $V_{(m,n)} = V^m_{(1,0)}V^n_{(0,1)}|D_{(m,n)} = V^n_{(0,1)}V^m_{(1,0)}|D_{(m,n)}$, for every $(m, n) \in \mathbf{Z}^2$.

The basic Lemma 2.2 shows that $\mathscr{B}$ is non-void iff there exists a commuting pair of unitary operators that extends $\{V_{(1,0)}, V_{(0,1)}\}$ to a Hilbert space containing H_Ω. Since $V^{-1}_{(1,0)} = V_{(-1,0)}$ and $V^{-1}_{(0,1)} = V_{(0,-1)}$, any sufficient condition for that remains true when (m, n) is replaced by $(-m, -n)$, (n, m) or $(-n, -m)$.

From (1) it is clear that $A_\Omega(\mathbf{Z}^2, E_1 \times E_2) = D'_{(0,1)} + D'_{(0,-1)}$, so

(3) $H_\Omega = D_{(0,1)} \vee D_{(0,-1)}$.

Assume that $D_{(m,1)} = D_{(0,1)}$ holds for $m = 1, 2, \ldots$. Then $D_{(1,1)} = D_{(0,1)}$, so $D_{(0,-1)} = V_{(0,1)}D_{(0,1)} = V_{(0,1)}D_{(1,1)} \subset D_{(1,0)}$, because $S_{(0,1)}D'_{(1,1)} \subset D'_{(1,0)}$. From $D_{(1,0)} \supset D_{(1,1)}$ and (3) we get $H_\Omega = D_{(1,0)}$. It follows from (2) that

$$V^m_{(1,0)}V_{(0,1)}|D_{(0,1)} = V^m_{(1,0)}V_{(0,1)}|D_{(m,1)} = V_{(0,1)}V^m_{(1,0)}|D_{(m,1)} =$$

$$= V_{(0,1)}V^m_{(1,0)}|D_{(0,1)}.$$

We can thus apply Theorem A(1.1) to H_Ω, $V_{(1,0)}$ and $V_{(0,1)}$ from Lemma 2.2 and Theorem 7.5 we get:

4) LEMMA. *Let at least one of the following equalities be true for every $m \geqslant 1$:*

$$D_{(m,1)} = D_{(0,1)}, \quad D_{(1,m)} = D_{(1,0)},$$

$$D_{(-m,-1)} = D_{(0,-1)}, \quad D_{(-1,-m)} = D_{(-1,0)}.$$

Then the commutant $\{T_1, T_2, X\}$ can be lifted to an intertwining between the unitary dilations, i.e., $\mathscr{T} \neq \emptyset$.

For $m, n \geqslant 0$ set $P'_{(m,n)} = \{(h_1, h_2) \in D'_{(m,n)} : h_1 \equiv 0\}$; then $D'_{(m,n)} = \{(h_1, h_2) \in A_\Omega : h_2 \equiv 0\} + P'_{(m,n)}$, so if $P_{(m,n)}$ denotes the closure of $\Pi_\Omega P'_{(m,n)}$ we get:

(5) If $P_{(m,1)} = P_{(0,1)}$ then $D_{(m,1)} = D_{(0,1)}$ for every $m \geqslant 1$. Next, we shall prove that

(6) $P_{(m,1)} = P_{(0,1)}$ if and only if

$$\bigvee\{U_2'^r U_2''^s E_2 : r \leqslant -m,\ s \leqslant -1\} = \bigvee\{U_2'^r U_2''^s E_2 : r \leqslant 0,\ s \leqslant -1\}, \quad m \geqslant 1.$$

For any $(0, h_2) \in P'_{(0,0)}$ we set $\Delta(0, h_2) = \sum\{U_2'^r U_2''^s h_2(r, s) : r, s \leqslant 0\}$. If also

$(0, h_2) \in P'_{(0,0)}$ we have

$$\langle \Delta(0, h_2), \Delta(0, h_2')\rangle_{F_2} =$$

$$= \sum \{\langle U_2'^{r-r'} U_2''^{s-s'} h_2(r, s), h_2'(r', s')\rangle_{F_2} : r, s, r', s' \leqslant 0\} =$$

$$= B([0, h_2], [0, h_2']) = \langle \Pi_\Omega(0, h_2), \Pi_\Omega(0, h_2')\rangle,$$

with the last scalar product in H_Ω, so Δ defines an isometry that carries $P_{(m,n)}$ onto $\bigvee\{U_2'^r U_2''^s E_2 : r \leqslant -m, s \leqslant -n\}$ for every $m, n \geqslant 0$; (6) follows.

Consequently, $P_{(m,1)} = P_{(0,1)}$ for every $m \geqslant 1$ iff

$$U_2'^{-1}(\bigvee\{U_2'^r U_2''^s E_2 : r \leqslant 0, s \leqslant -1\}) = \bigvee\{U_2'^r U_2''^s E_2 : r \leqslant 0, s \leqslant -1\},$$

i.e., iff $E_2 \subset \bigvee\{U_2'^r U_2''^s E_2 : r < 0, s \leqslant 0\}$. Thus, Lemmas 4 and 5 show that assertion (a) of Theorem D(1.4) is proved.

If $W \in L(M)$ is an isometric dilation of the contraction $T \in L(E)$ then $E \subset \bigvee\{W^n E : n > 0\}$ iff T^* is an isometry. In order to prove it we may assume minimality. If T^* is an isometry, for every $x \in E$ we have $\|WT^*x\| \leqslant \|x\| = \|TT^*x\|$ and $P_E^M WT^*x = TT^*x$ so $WT^*x = TT^*x = x$; consequently $WE \supset E$. Conversely, if $E \subset \bigvee\{W^n E : n > 0\}$, for any $x \in E$ and $\varepsilon > 0$ there exist $x_n \in E$, $n = 1, \ldots, m$, such that if we set $y = \sum\{W^n x_n : 1 \leqslant n \leqslant m\}$ then $\|x - y\| < \varepsilon$; since $T^* = W^*|E$ it folows that

$$\|T^*x - W^*y\| = \|W^*(x - y)\| < \varepsilon$$

so

$$\varepsilon > \|TT^*x - TW^*y\| = \|TT^*x - P_E^M y\|;$$

thus $\|TT^*x - x\| \leqslant \|TT^*x - P_E^M y\| + \|P_E^M(y - x)\| < 2\varepsilon$, i.e., $TT^*x = x$ for every $x \in E$. So the following holds.

7) Corollary. *If, with the same notation as in Theorem* D(1.4), *at least one of the operators* $T_1'^*$, $T_1''^*$, T_2', T_2'' *is an isometry, the thesis* (a) *of that theorem is true.*

Remark. If $T_1'^*$ or T_2' is an isometry, then the commutant $\{T_1', T_2', X\}$ has only one lifting. We thought that the last was a sufficient condition for the existence of the lifting in $\mathbf{Z}^2$. But that is not true, as the "rapporteur" of a Note we submitted to the "Comptes Rendus" showed us with the following example.

Let $\{T_1, T_2, T_3\} \subset L(E)$ be Parrott's example of three commuting contractions such that the semigroup T given by $T(m, n, r) = T_1^m T_2^n T_3^r$, $m, n, r \geqslant 0$, has no unitary dilation ([27], 1.9.6). Set $T_1' = T_2' = T_1$, $T_1'' = T_2'' = T_2$, $X = T_3$; from Ando, Ceauşescu and Foiaş uniqueness theorem [5] it follows that $\{T_1', T_2', X\}$ has only one lifting. If $\tau \in \mathcal{T}$ should exist, setting $U_1 = U_1' = U_2'$, $U_2 = U_1'' = U_2''$ and considering the commuting system $\{U_1, U_2, \tau\}$, we would be able to construct a unitary dilation of T.

We now turn to the proof of assertion (b) in (1.4). Assuming that the isometric dilations are minimal, we apply Theorem 7.5. There exists $\tau \in \mathcal{T}$, so $\mathcal{B} \neq \emptyset$ and we can find $B' \in \mathcal{B}$ such that if $U' \subset L(H')$ is its minimal unitary dilation then τ may be identified with $P_{F_2}^{H'}|F_1$ while $\gamma = P_{M_2}^{H'}|M_1$ satisfies $P_{E_2}^{M_2}\gamma = XP_{E_1}^{M_1}$ and $\|\gamma\| = \|X\|$. In order to prove that

$$(\#) \qquad \langle \gamma W_1' a_1, a_2\rangle_{M_2} = \langle W_2'\gamma a_1, a_2\rangle_{M_2}$$

holds for every $a_1 = W_1'^r W_1''^s v_1$, $r, s \geqslant 0$, $v_1 \in E_1$ and $a_2 \in M_2$, the assumption on M_2 shows that it is enough to consider the following cases (i) and (ii).

(i) $a_2 = W_2'^m W_2''^n v_2$, $m > 0$, $n \geqslant 0$ and $v_2 \in E_2$: Theorem 7.5 shows that the first member of (#), $\langle P_{M_2}^{H'} W_1' a_1, W_2'^m W_2''^n v_2\rangle_{M_2}$, equals

$$\langle U' a_1, U'^m U''^n v_2\rangle_{H'} =$$
$$= \langle a_1, U'^{m-1} U''^n v_2\rangle_{H'} = \langle P_{M_2}^{H'} a_1, W_2'^{m-1} W_2''^n v_2\rangle_{M_2} =$$
$$= \langle W_2'\gamma a_1, a_2\rangle_{M_2}.$$

(ii) $a_2 = v_2 \in E_2$: Lemma 7.7 says that

$$\langle \gamma a_1, W_2'^* a_2\rangle_{M_2} = \langle P_{E_2}^{M_2}\gamma a_1, T_2'^* v_2\rangle_{E_2} =$$
$$= \langle XP_{E_1}^{M_1} a_1, T_2'^* v_2\rangle_{E_2} = \langle T_2' XP_{E_1}^{M_1} a_1, v_2\rangle_{E_2} = \langle XT_1' P_{E_1}^{M_1} a_1, v_2\rangle_{E_2} =$$
$$= \langle XT_1' P_{E_1}^{M_1} W_1'^r W_1''^s v_1, v_2\rangle_{E_2} = \langle XT_1'^{r+1} T_1''^s v_1, v_2\rangle_{E_2} =$$
$$= \langle XP_{E_1}^{M_1} W_1' a_1, v_2\rangle_{E_2} = \langle P_{E_2}^{M_2}\gamma W_1' a_1, v_2\rangle_{E_2} = \langle \gamma W_1' a_1, a_2\rangle_{M_2}.$$

Thus, $\gamma W_1' = W_2'\gamma$, while $\gamma W_1'' = W_2''\gamma$ can be proved in the same way.

So Theorem D(1.4) has been proved.

APPENDIX: ON THE EXTENSION OF ISOMETRIES

In this paper we have systematically applied the basic Lemma 2.2 to obtain positive Toeplitz extensions of T-K-C forms from unitary extensions of isometries. But since both things are equivalent the lemma can be applied in the other way, as it is shown in the following example.

Let $V = \{V_t : t \in \mathbf{R},\ t \geqslant 0\}$ be a family of isometries such that the domain D_t and the range R_t of each V_t are all closed subspaces of the same Hilbert space H. Assume that the following properties hold.

i) Generalized semigroup property: V_0 is the identity in H and, if $x \in D_{t_1+t_2}$, then $x \in D_{t_1}$, $V_{t_1}x \in D_{t_2}$ and $V_{t_1+t_2}x = V_{t_2}V_{t_1}x$.

ii) Existence of incoming and outgoing subspaces: there exist E_1 and E_2, closed subspaces of H such that $E_1 \subset D_t$ and $E_2 \subset R_t$ for all $t > 0$ and also $H = [\bigvee\{V_tE_1 : t > 0\}] \vee [\bigvee\{V_t^{-1}E_2 : t > 0\}]$. (Note that $V_tE_1 \subset D_s$ and $V_t^{-1}E_2 \subset R_s$ hold for all $t, s \geqslant 0$.)

iii) Continuity: $\langle V_tx_1, x_1' + x_2'\rangle$, $\langle x_1 + x_2, V_t^{-1}x_2'\rangle$ depend continuously on $t \geqslant 0$ for any $x_1, x_1' \in E_1$, $x_2, x_2' \in E_2$.

We shall prove that the above conditions ensure the existence of a continuous unitary group U that extends V.

Set $\Gamma_1 = \{t \in \mathbf{R} : t \geqslant 0\}$ and $\Omega = \{\Gamma_1, \Gamma_1^{-1}\}$. Let $\{B; \mathbf{R}, \Omega; E_1 \times E_2\}$ be given by $B(h, h') = \sum\{\langle K_{jk}(s - t)h_j(s), h_k'(t)\rangle_{E_k} : j, k = 1, 2,\ s, t \in \mathbf{R}\}$, with

$$K_{11}(t) = P^H_{E_1}V_t|E_1 \text{ if } t \geqslant 0 \quad\text{and}\quad K_{11}(t) = K_{11}(-t)^* \quad \text{if } t \leqslant 0,$$

$$K_{12}(t) = K_{21}(-t)^* = P^H_{E_2}V_t|E_1 \quad \text{if } t \geqslant 0,$$

$$K_{22}(t) = P^H_{E_2}V_{-t}^{-1}|E_2 \quad \text{if } t \leqslant 0 \quad\text{and}\quad K_{22}(t) = K_{22}(-t)^* \quad \text{if } t \geqslant 0.$$

It is easy to see that B is positive. Then it can be proved [15] that it has a positive Toeplitz extension $\{B'; \mathbf{R}; E_1 \times E_2\}$ such that $B'(S_tv, w)$ depends continuously on $t \in \mathbf{R}$ for every $v \in E_1$ and $w \in E_2$. Now, setting $\Delta(h_1, h_2) = \sum\{V_th_1(t) : t \geqslant 0\} + \sum\{V_{-t}^{-1}h_2(t) : t \leqslant 0\}$, we obtain an isometry from H_Ω onto H. For $t \leqslant 0$ set $V_t = V_{-t}^{-1}$; it can be seen that $\{\Delta^* V_t \Delta : t \in \mathbf{R}\}$ is the family of isometries associated to B. From Lemma 2.2 our assertion follows.

If every V_t has domain H, that is, if V is an ordinary continuous semigroup of isometries, we may put $E_1 = H$ and $E_2 = \{0\}$, so the above result includes a classical theorem of Cooper [17], which is also a particular case of the unitary dilation theory, as Sz.-Nagy and Foiaş have shown [27].

REFERENCES

1. Adamjan, V. M.; Arov, D. Z.; Krein, M. G., Infinite Hankel matrices and Carathéodory-Fejér and Schur problems, *Functional Anal. Appl.*, 2(1968), 1–17.
2. Adamjan, V. M.; Arov, D. Z.; Krein, M. G., Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takaji problem, *Mat. Sb.*, **15**(1971), 31–37.
3. Adamjan, V. M.; Arov, D. Z.; Krein, M. G., Infinite Hankel block-matrices and related continuation problems, *Izv. Akad. Nauk. Armjan. SSR Ser. Mat.*, **6**(1971), 87–112.
4. Ando, T., On a pair of commutative contractions, *Acta Sci. Math. (Szeged)*, **24**(1963), 88–90.
5. Ando, T.; Ceauşescu, Z.; Foiaş, C., On intertwining dilations. II, *Acta Sci. Math. (Szeged)*, **39**(1977), 3–14.

6. AROCENA, R.; COTLAR, M., Generalized Toeplitz kernels and Adamjan-Arov-Krein moment problems, in *Operator Theory: Adv. and Appl.*, 4(1982), pp. 37–55.
7. AROCENA, R.; COTLAR, M., Dilations of generalized Toeplitz kernels and some vectorial moment and weighted problems, in *Springer Lecture Notes in Math.*, **908**(1982), 169–188.
8. AROCENA, R.; COTLAR, M., Generalized Toeplitz kernels, Hankel forms and Sarason's commutation theorem, *Acta Cient. Venezolana*, 33(1982), 89–98.
9. AROCENA, R., A refinement of the Helson-Szegö theorem and the determination of the extremal measures, *Studia Math.*, **71**(1981), 203–221.
10. AROCENA, R., On generalized Toeplitz kernels and their relation with a paper of Adamjan, Arov and Krein, in *North Holland Math. Studies*, **86**(1984), 1–22.
11. AROCENA, R., On the parametrization of Adamjan, Arov and Krein, *Publ. Math. Orsay*,, 83-02(1983), 7–23.
12. AROCENA, R., Generalized Toeplitz kernels and dilations of intertwining operators, *Integral Equation Operator Theory*, 6(1983), 759–778.
13. AROCENA, R., Scattering functions, linear systems, Fourier transforms of measures and unitary dilations to Krein spaces, *Publ. Math. Orsay*, **85**-02(1985), 1–57.
14. AROCENA, R., Naimark's theorem, linear systems and scattering operators, *J. Funct. Anal.*, 69(1986), 281–288.
15. AROCENA, R., Generalized Toeplitz kernels and dilations of intertwining operators (II): the continuous case, *Acta Sci. Math. (Szeged)*, to appear.
16. BEREZANSKI, JU. M., *Expansions of self-adjoint operators in eigenfunctions*, Amer. Math. Soc., vol. 17, Providence, R. I., English Translation, 1968.
17. COOPER, J. L. B., One parameter semigroup of isometric operators in Hilbert space, *Ann. of Math.*, **48**(1947), 827–842.
18. COTLAR, M.; SADOSKY, C., On the Helson-Szegö theorem and a related class of modified Toeplitz kernels, *Proc. Sympos. Pure Math.*, 35(1979), 384–407.
19. COTLAR, M.; SADOSKY, C., Majorized Toeplitz forms and weighted inequalities with general norms, in *Springer Lecture Notes in Math.*, **908**(1982), 139–168.
20. DEVINATZ, A., On the extension of positive definite forms, *Acta Math.*, **102**(1959), 109–134.
21. DOMINGUEZ, M., Completely regular continuous pocesses, *Univ. Central de Venezuela*, preprint, 1984.
22. DYM, H.; MC KEAN, H. P., *Gaussian processes, function theory and the inverse spectral problem*, Academic Press, 1976.
23. HELSON, H.; SZEGÖ, G., A problem in prediction theory, *Ann. Mat. Pura Appl.*, **51**(1960), 107–138.
24. KOREZLIOGLU, H.; LOUBATON, PH., Prediction des procesus stationnaires au sens large sur Z^2 relativement aux demi-plans, *C.R. Acad. Sci. Paris. Ser. I. Math.*, **301**(1985), 27–30.
25. KREIN, M. G., Sur le problème du prolongement des fonctions Hermitiennes positives et continues, *Dokl. Akad. Nauk SSSR*, 26(1940), 17–22.
26. MLAK, V., Unitary dilations in case of ordered groups, *Ann. Polon. Math.*, **17**(1965), 321–328.
27. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators in Hilbert space*, North Holland, 1970.

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