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IS AN ISOMETRY DETERMINED BY ITS INVARIANT SUBSPACE LATTICE?

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If A and B are bounded operators on a separable Hilbert space $\mathscr{H}$ and their lattices of invariant subspaces, Lat A and Lat B, are lattice isomorphic, what can be said about the relation between A and B? In such generality this problem seems out of reach, though in a number of specific situations it is quite feasible as well as interesting. If A and B are self-adjoint operators, then the present authors have been able to give necessary and sufficient conditions such that Lat A is isomorphic to Lat B [4]. Specifically, it was shown that two self-adjoint operators have isomorphic invariant subspace lattices if and only if one operator is unitarily equivalent to a generator of the von Neumann algebra generated by the other. Using measure theory and the spectral representation of self-adjoint operators, this condition can be given a concrete interpretation.

This paper looks at the above question for isometries (and, hence, also for unitaries). It will be shown that if U and V are isometries and Lat U is isomorphic to Lat V, then the shift parts in their Wold decompositions are unitarily equivalent and their unitary parts have isomorphic invariant subspace lattices. Consider now two unitary operators U and V with isomorphic invariant subspace lattices. Then U is reductive if and only if V is reductive and, in this case, each operator is unitarily equivalent to a generator of the von Neumann algebra generated by the other. If U and V are not reductive, then their absolutely continuous parts have isomorphic lattices and the singular part of each is unitarily equivalent to a generator of the von Neumann algebra generated by the singular part of the other. This last statement for singular unitary operators as well as the previous statement for reductive unitary operators is actually a consequence of the results for self-adjoint operators [4] and can be specified precisely in terms of the multiplicity structures of the operators concerned. For the absolutely continuous parts of the non-reductive unitaries, if these are canonically decomposed as the direct sum of a bilateral shift of some multiplicity and a reductive unitary, then the multiplicities of the two bilateral shifts are equal and the absolutely continuous reductive part of each is unitarily equivalent to a generator of the von Neumann algebra generated by the absolutely continuous reductive part of the other.

Unfortunately we have not been able to obtain any structural conditions which are both necessary and sufficient for two unitaries to have isomorphic invariant subspace lattices. It follows from our results that if U and V are unitary operators with isomorphic invariant subspace lattices, then V is unitarily equivalent to $\varphi(U)$ for a function φ in $L^{\infty}(\mu)$ such that the polynomials in φ and $\bar{\varphi}$ are weak* dense on $L^{\infty}(\mu)$, where μ is a scalar-valued spectral measure for U. It is not known whether the converse of this statement or some appropriately refined version of it holds.

In particular, let E be the top half of the unit circle and let $F = \{e^{i\theta} : \pi/2 \leqslant \theta \leqslant \pi \text{ or } 3\pi/2 \leqslant \theta \leqslant 2\pi\}$. If $U = M_z \oplus M_z$ on $L^2(\partial \mathbf{D}) \oplus L^2(E)$ and $V = M_z \oplus M_z$ on $L^2(\partial \mathbf{D}) \oplus L^2(F)$, then V is unitarily equivalent to $\varphi(U)$ for a function of the type described above; but it is not known whether $\operatorname{Lat} U \approx \operatorname{Lat} V$. A description of the invariant subspaces of the preceding examples of unitary operators is included in an appendix to this paper. It may be possible to test the validity of the above converse against these operators, but we have not succeeded in doing so.

The first section of the paper contains some preliminary material. Section 2 characterizes the central elements of the invariant subspace lattice of an arbitrary normal operator. In [4], the central elements of an invariant subspace lattice of a self-adjoint operator played a crucial role. In a sense the result in Section 2 shows that the central elements of the invariant subspace lattice of a normal operator cannot play this crucial role since there are so few of them. To a large extent this is the crux of the difficulty in extending the results of [4] to the unitary and normal cases. Section 3 presents the results for lattice isomorphisms of the lattices of isometries that were outlined above.

1. PRELIMINARIES

Generally the terminology and notation of [3] will be followed, but some additions to this source are given in the present section together with several results that are of particular importance here. Throughout, all Hilbert spaces are assumed to be separable and complex.

If $\mathscr{L}_1$ and $\mathscr{L}_2$ are two lattices, then a lattice isomorphism is a bijection $\Theta : \mathscr{L}_1 \to \mathscr{L}_2$ that preserves the join and meet operations. Equivalently, a bijection $\Theta : \mathscr{L}_1 \to \mathscr{L}_2$ is a lattice isomorphism precisely when $\Theta(x) \leqslant \Theta(y)$ if and only if $x \leqslant y$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are isomorphic lattices, this will be denoted by writing $\mathscr{L}_1 \approx \mathscr{L}_2$. The lattice operations on the set of all closed subspaces of a Hilbert space $\mathscr{H}$ are defined by $\mathscr{M} \vee \mathscr{N} \equiv \operatorname{cl}(\mathscr{M} + \mathscr{N})$ and $\mathscr{M} \wedge \mathscr{N} \equiv \mathscr{M} \cap \mathscr{N}$. With these definitions it is easy to see that $\operatorname{Lat} A$, the lattice of invariant subspaces of the bounded operator A, is a complete lattice with a unit (viz., $\mathscr{H}$) and a 0 (viz., (0)).

For a lattice $\mathscr{L}$, an element e of $\mathscr{L}$ is said to be *central* if there is an element e' of $\mathscr{L}$ such that $e \wedge e' = 0$ and $f = (f \wedge e) \vee (f \wedge e')$ for every f in $\mathscr{L}$. In this case the element e' is uniquely determined and is also central with $e'' = e$.

For an operator T, let $P^\infty(T)$ denote the WOT (weak operator topology) -closed algebra generated by T and the identity operator. For a compactly supported measure μ, let $P^\infty(\mu)$ denote the weak* closure of the (analytic) polynomials in $L^\infty(\mu)$. If N is a normal operator and μ is its scalar-valued spectral measure, then the functional calculus $\varphi \mapsto \varphi(N)$ defines an isometric isomorphism between $P^\infty(\mu)$ and $P^\infty(N)$ that is a homeomorphism if $P^\infty(\mu)$ is given its weak* topology and $P^\infty(N)$ is given the WOT.

If A and B are operators, then unitary equivalence between A and B is denoted by $A \cong B$. In [4] it was shown that if N and M are normal operators and there is a unitary operator U such that $U(\operatorname{Lat} N) = \operatorname{Lat} M$, then U^*MU is a WOT generator of $P^\infty(N)$. By identifying $P^\infty(N)$ and $P^\infty(\mu)$ as above, this gives a complete description of when such a phenomenon occurs ([8], [9]). It was also shown in [4] that $\operatorname{Lat} N \approx \operatorname{Lat} N^*$ for any normal operator. In fact, if N is represented as multiplication by some bounded function φ on $L^2(X, \Omega, \mu)$ for some measure space (X, Ω, μ), then the map $\mathscr{M} \mapsto \mathscr{M}^* \equiv \{\bar{f} : f \in \mathscr{M}\}$ defines an isomorphism of $\operatorname{Lat} N$ onto $\operatorname{Lat} N^*$. This leads to the following question.

QUESTION. If N and M are normal operators and $\operatorname{Lat} N \approx \operatorname{Lat} M$, must it be that either M or M^* is unitarily equivalent to a generator of $P^\infty(N)$?

The answer to this question is yes in the case of self-adjoint operators, but remains unanswered in the case of a unitary operator. There is a curious dilemma here. If M and N are normal and either M or M^* is unitarily equivalent to a generator of $P^\infty(N)$, then, as shown above, $\operatorname{Lat} N \approx \operatorname{Lat} M$ and the question above asks whether the converse holds. Now if M and N are unitaries and $\operatorname{Lat} N \approx \approx \operatorname{Lat} M$, then it will be shown in this paper that M is unitarily equivalent to a generator of the von Neumann algebra $W^*(N)$, but it is unknown whether M, or M^*, is unitarily equivalent to a generator of $P^\infty(N)$. To further complicate this matter, it is unknown whether this last fact holds for all normal operators. That is, if $\operatorname{Lat} M \approx \operatorname{Lat} N$, is M unitarily equivalent to a generator of $W^*(N)$?

The following description of the invariant subspaces of the bilateral shift of multiplicity 1 will be needed. This can be found in [6].

1.1. PROPOSITION. *If m is Lebesgue measure on $\partial\mathbf{D}$ and U is multiplication by the independent variable on $L^2(m)$, then $\mathscr{M} \in \operatorname{Lat} U$ if and only if either there is a Borel set Δ contained in $\partial\mathbf{D}$ such that $\mathscr{M} = \{f \in L^2(m) : f = 0$ off $\Delta\}$ or there is a φ in $L^\infty(m)$ such that $|\varphi| = 1$ a.e. and $\mathscr{M} = \varphi H^2$. The spaces of the first type are reducing for U and the spaces of the second type are not.*

1.2. COROLLARY. *If S is the unilateral shift of multiplicity 1 and $\mathscr{M}$ and $\mathscr{N}$ are non-zero elements of $\operatorname{Lat} S$, then $\mathscr{M} \cap \mathscr{N} \neq 0$.*

There is a way to distinguish the reducing subspaces for a unitary operator from its non-reducing subspaces. The next result is well known and its proof is an easy exercise.

1.3. Proposition. (a) *If U is unitary and $\mathscr{M} \in \operatorname{Lat} U$, then $\mathscr{M}$ is reducing for U if and only if $U\mathscr{M} = \mathscr{M}$.*

(b) *If V is an isometry and $\mathscr{N} \in \operatorname{Lat} V$ such that $V|\mathscr{N}$ is unitary, then $\mathscr{N}$ reduces V and $\mathscr{N} \subseteq \bigcap_{1}^{\infty} \operatorname{ran}(V^n)$.*

It is perhaps worthwhile to mention that the preceding proposition generalizes to normal operators in the following form: If N is normal and $\mathscr{M} \in \operatorname{Lat} N$, then $\mathscr{M}$ reduces N if and only if $N|\mathscr{M}$ is normal.

The proof of the next proposition was shown to us by Professor Ciprian Foiaş.

1.4. Proposition. *If U and V are unilateral shifts and X is an operator that is injective and has dense range such that $XU = VX$, then U and V have the same multiplicity. That is, U and V are unitarily equivalent.*

Proof. Let m = the multiplicity of V and let n = the multiplicity of U. Since $XU = VX$, $U^*X^* = X^*V^*$. Thus $X^*(\ker V^*) \subseteq \ker U^*$. Since X^* is injective, $m \leqslant n$. So if $m = \infty$, the proof is complete. Assume that $m < \infty$ and suppose that $m < n \leqslant \infty$.

The operator V (resp., U) can be represented as multiplication by z on the Hardy space of $\mathbf{C}^m$-valued (resp., $\mathbf{C}^n$-valued) analytic functions, $H^2_{\mathbf{C}^m}$ (resp., $H^2_{\mathbf{C}^n}$), where $\mathbf{C}^n$ denotes the sequence space ℓ^2 if $n = \infty$. Since $X\colon H^2_{\mathbf{C}^n} \to H^2_{\mathbf{C}^m}$ and $XU = VX$, X can be represented as multiplication by an $m \times n$ matrix with entries from H^∞. We wish to perform elementary row operations on the matrix X, where instead of only multiplying by non-zero scalars, we will multiply by non-zero meromorphic functions that are the quotient of two functions from H^∞. Note that because the operator X is injective, a matrix

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} \ldots a_{1m} & a_{1,m+1} \ldots a_{1n} \\ 0 & 1 & a_{23} \ldots a_{2m} & a_{2,m+1} \ldots a_{2n} \\ \cdot & \cdot & \cdot \;\ldots\; \cdot & \cdot \;\;\ldots\; \cdot \\ \cdot & \cdot & \ldots\; \cdot & \cdot \;\;\ldots\; \cdot \\ 0 & 0 & 0 \;\ldots\; 1 & a_{m,m+1} \ldots a_{mn} \end{bmatrix}$$

is obtained by performing such operations, where each entry a_{ij} in this matrix is a meromorphic function that is the quotient of two bounded analytic functions on $\mathbf{D}$. (The obvious interpretation of A is understood when $n = \infty$.) Furthermore, from linear algebra, it is known that except for λ belonging to the countable set consisting of the points in $\mathbf{D}$ that are poles of the entries of A or poles of the entries of the matrices used to perform the row operations, a vector ξ in $\mathbf{C}^n$ satisfies $A(\lambda)\xi = 0$ if and only if $X(\lambda)\xi = 0$. It will now be shown that there is a non-zero bounded analytic function $f\colon \mathbf{D} \to \mathbf{C}^n$ such that $A(\lambda)f(\lambda) = 0$ for all λ in $\mathbf{D}$. Thus $f \in H^2_{\mathbf{C}^n}$ and $Xf = 0$, contradicting the fact that X is injective.

Let $\varphi \in H^\infty$ such that $\varphi \neq 0$ and $a_{ij}\varphi = b_{ij} \in H^\infty$ for $1 \leqslant i < j \leqslant m + 1$. Let $f_j = 0$ for $j > m + 1$ and $f_{m+1} = \varphi^m$. Put

$$f_m = -a_{m,m+1}\varphi^m = -b_{m,m+1}\varphi^{m-1}.$$

Continue in this fashion, defining f_{m-k} inductively for $0 \leqslant k \leqslant m - 1$ by

$$f_{m-k} = -\sum \{a_{m-k,j}f_j : j \geqslant m - k + 1\}.$$

An induction argument shows that each $f_j \in H^\infty$. Hence $f \equiv (f_j)$ is a non-zero bounded analytic function from $\mathbf{D}$ into $\mathbf{C}^n$. It is routine to show that $Af = 0$, completing the proof. ▨

The preceding proposition makes possible a generalization of Corollary 1.2 above to shifts of finite multiplicity.

1.5. PROPOSITION. *If U is a unilateral shift of finite multiplicity n and $\mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{M}_{n+1}$ are invariant subspaces for U such that $\mathcal{M}_i \cap (\mathcal{M}_{i+1} + \ldots + \mathcal{M}_{n+1}) = (0)$ for all i, then $\mathcal{M}_k = (0)$ for at least one k.*

Proof. Assume that $\mathcal{M}_k \neq (0)$ for all k. Since $U | \mathcal{M}_k$ is a unilateral shift for each k, by passing to a subspace of $\mathcal{M}_k$ if necessary, it may be assumed that $U | \mathcal{M}_k$ is a unilateral shift of multiplicity 1 for each k. Let $\mathcal{H} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \ldots \vee \mathcal{M}_{n+1}$ and let $V \equiv U | \mathcal{M}_1 \oplus U | \mathcal{M}_2 \oplus \ldots \oplus U | \mathcal{M}_{n+1}$ on $\mathcal{K} \equiv \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_{n+1}$. Define $X\colon \mathcal{K} \to \mathcal{H}$ by $X(h_1 \oplus h_2 \oplus \ldots \oplus h_{n+1}) = h_1 + h_2 + \ldots + h_{n+1}$. It is easy to check that $XV = (U | \mathcal{H})X$. Also X clearly has dense range and the assumption on the spaces $\mathcal{M}_k$ implies that X is injective. By Proposition 1.4, $U | \mathcal{H}$ and V have the same multiplicity. Therefore $U | \mathcal{H}$ has multiplicity $n + 1$, contradicting the hypothesis that U has multiplicity n. ▨

If $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ are two σ-finite measure spaces, say that a function $\varphi\colon X \to Y$ is an $(X, \mathfrak{A}, \mu)$-$(Y, \mathfrak{B}, \nu)$ *point isomorphism* if there are sets α in $\mathfrak{A}$ and β in $\mathfrak{B}$ such that $\mu(X \setminus \alpha) = 0 = \nu(Y \setminus \beta)$; φ is a bijection from α onto β; for $\Delta \subseteq \alpha$, $\Delta \in \mathfrak{A}$ if and only if $\varphi(\Delta) \in \mathfrak{B}$ and $\mu(\Delta) = 0$ if and only if $\nu(\varphi(\Delta)) = 0$. If $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ are understood (as is the case when μ and ν are compactly supported regular Borel measures on the complex plane), it will be said that φ is a μ-ν point isomorphism. If μ is a compactly supported measure on $\mathbf{C}$, say that a function φ in $L^\infty(\mu)$ is a generator of $L^\infty(\mu)$ as a von Neumann algebra if $L^\infty(\mu)$ is the smallest von Neumann algebra that contains φ and the constant functions. These two concepts are related by the following result.

1.6. PROPOSITION. *If* $(X, \mathfrak{A}, \mu)$ *is a σ-finite measure space,* $\varphi \in L^\infty(\mu)$, *and* $\nu = \mu \circ \varphi^{-1}$, *then* φ *generates* $L^\infty(\mu)$ *as a von Neumann algebra if and only if* φ *is a* μ-ν *point isomorphism.*

Proof. Since neither the definition of $L^\infty(\mu)$ nor its weak* topology change when μ is replaced by an equivalent measure, it may be assumed that $(X, \mathfrak{A}, \mu)$ is a finite measure space. Specifically, it will be assumed that $(X, \mathfrak{A}, \mu)$ is a probability measure space.

First assume that φ is a μ-ν point isomorphism and let α and β be measurable sets as in the definition given above. Define φ^{-1} on β; so φ^{-1} is defined ν-a.e. Define $T: L^\infty(\nu) \to L^\infty(\mu)$ by $Tf = f \circ \varphi$ and define $R: L^\infty(\mu) \to L^\infty(\nu)$ by $Rg = g \circ \varphi^{-1}$. If $f \in L^\infty(\nu)$, then there is a set Δ in $\mathfrak{A}$ contained in α such that $\mu(\alpha \setminus \Delta) = 0$ and $\|f \circ \varphi\|_\infty = \sup\{|f(\varphi(x))| : x \in \Delta\}$. But $\varphi(\Delta)$ is a Borel set having full ν measure. Hence $\|f\|_\infty \geqslant \|f \circ \varphi\|_\infty$. Similarly, $\|g\|_\infty \geqslant \|g \circ \varphi^{-1}\|_\infty$ for every g in $L^\infty(\mu)$. This shows that T is a surjective isometry (with inverse R). Also, if $\{f_i\}$ is a net in $L^\infty(\nu)$ such that $f_i \to 0$ weak* and $h \in L^1(\mu)$, then $\int (f_i \circ \varphi) h \, d\mu = \int f_i (h \circ \varphi^{-1}) \, d\nu \to 0$. Hence T is weak* continuous. Similarly, R is weak* continuous and so T is a weak* homeomorphism. Since polynomials in z and $\bar{z}$ are weak* dense in $L^\infty(\nu)$, it must be that polynomials in φ and $\bar{\varphi}$ are weak* dense in $L^\infty(\mu)$. That is, φ generates $L^\infty(\mu)$ as a von Neumann algebra.

Now assume that φ generates $L^\infty(\mu)$ as a von Neumann algebra and let Y be the support of ν. Since polynomials in φ and $\bar{\varphi}$ are weak* dense in $L^\infty(\mu)$, they are weakly dense in $L^2(\mu)$. Thus $(X, \mathfrak{A}, \mu)$ is a separable measure space and M_φ is a cyclic normal operator on $L^2(\mu)$ with cyclic vector 1. Since ν is a scalar-valued spectral measure for M_φ, there is a unitary $U: L^2(\nu) \to L^2(\mu)$ such that $U1 = 1$ and $UM_z = M_\varphi U$. An easy algebraic argument together with an application of the Fuglede-Putnam Theorem (see, for example, page 286 of [3]) shows that $Up(z, \bar{z}) = p(\varphi, \bar{\varphi})$ for every polynomial p in z and $\bar{z}$. An approximation argument now demonstrates that $Uf = f \circ \varphi$ for every f in $L^2(\nu)$. In particular, $U\chi_E = \chi_{\varphi^{-1}(E)}$ for every Borel set E. Since U is unitary, the map $E \mapsto \varphi^{-1}(E)$ induces a Boolean algebra isomorphism of the measure algebra for ν onto the measure algebra for μ. By a result due essentially to Halmos and von Neumann [5] (see also Theorem 1.3 of [4]), there is a μ-ν point isomorphism ψ such that $\chi_{\varphi^{-1}(E)} = \chi_{\psi^{-1}(E)}$ in $L^\infty(\mu)$ for every Borel set E. Note that since φ is bounded, Y is a bounded set and hence $\psi \in L^\infty(\mu)$. It will be shown that $\varphi = \psi$ in $L^\infty(\mu)$.

Let $\alpha \in \mathfrak{A}$ having full μ measure such that ψ is bounded and one-to-one on α and for a subset Δ of α, $\Delta \in \mathfrak{A}$ if and only if $\psi(\Delta)$ is Borel and $\mu(\Delta) = 0$ if and only if $\nu(\psi(\Delta)) = 0$. Fix $\varepsilon > 0$. Since ψ is bounded on α, there is a measurable partition $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ and points x_j in α_j such that, for $1 \leqslant j \leqslant n$, $|\psi(x) - \psi(x_j)| < \varepsilon$ for all x in α_j. Let $B_j = \{z \in \mathbf{C} : |z - \psi(x_j)| < \varepsilon\}$. Then $\chi_{\varphi^{-1}(B_j)} = \chi_{\psi^{-1}(B_j)}$ in $L^\infty(\mu)$. Thus, since $\alpha_j \subseteq \psi^{-1}(B_j)$, $\chi_{\alpha_j} = \chi_{\alpha_j}\chi_{\psi^{-1}(B_j)} = \chi_{\alpha_j}\chi_{\varphi^{-1}(B_j)}$. Hence $|\varphi(x) - \psi(x_j)| < \varepsilon$ μ-a.e. on α_j for each j. It follows that $|\varphi(x) - \psi(x)| < 2\varepsilon$, μ-a.e. on α, and therefore $\varphi = \psi$ μ-a.e. as required. ▨

Here the terminology introduced prior to the preceding proposition will be modified a little. If N and M are normal operators with scalar-valued spectral measures μ and ν, respectively, then say that a function φ is a N-M point isomorphism if φ is a μ-ν isomorphism. Since the definition of a point isomorphism depends only on the measure class and not on the individual measure, this definition is unambiguous. The main result of [4] can be rephrased by saying that if A and B are self-adjoint operators, then $\operatorname{Lat} A \approx \operatorname{Lat} B$ if and only if $B \cong \varphi(A)$ where φ is an A-B point isomorphism.

If $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$, then $P_{\mathcal{M}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. A *reductive* operator is one such that every invariant subspace for it is reducing.

2. THE CENTRAL ELEMENTS OF THE INVARIANT SUBSPACE LATTICE OF A NORMAL OPERATOR

The purpose of this section is to prove the following result.

2.1. THEOREM. *If N is a normal operator and $\mathcal{M} \in \operatorname{Lat} N$, then $\mathcal{M}$ is a central element of* $\operatorname{Lat} N$ *if and only if $P_{\mathcal{M}} \in P^\infty(N)$.*

The proof of this theorem will need a few lemmas, the first of which says that one of the two implications in the theorem is valid. Until the completion of the proof of the theorem, N is a normal operator, $\mathcal{M} \in \operatorname{Lat} N$, and $P = P_{\mathcal{M}}$. If $\mathcal{M}$ is central, then $\mathcal{M}'$ denotes the "complementary" subspace appearing in the definition of a central element; it will be shown that $\mathcal{M}$ is reducing and that $\mathcal{M}' = \mathcal{M}^\perp$.

2.2. LEMMA. *If $P \in P^\infty(N)$, then $\mathcal{M}$ is a central element of* $\operatorname{Lat} N$.

Proof. Since $P \in P^\infty(N)$, there is a net of polynomials $\{p_i\}$ such that $p_i(N) \to P$ (WOT). Also $(1 - p_i)(N) \to P^\perp$ (WOT). If $\mathcal{N} \in \operatorname{Lat} N$, then $p_i(N)\mathcal{N} \leqslant \mathcal{N}$ for every i. Thus $P\mathcal{N} \leqslant \mathcal{N}$. Similarly, $P^\perp\mathcal{N} \leqslant \mathcal{N}$. So if $h \in \mathcal{N}$, then $h = Ph + P^\perp h \in P\mathcal{N} + P^\perp\mathcal{N}$. Therefore $\mathcal{N} = (\mathcal{N} \wedge \mathcal{M}) \vee (\mathcal{N} \wedge \mathcal{M}^\perp)$ and so $\mathcal{M}$ is central (with $\mathcal{M}' = \mathcal{M}^\perp$). ▨

2.3. LEMMA. *If* $\mathcal{M}$ *is central and* $\mathcal{N} \in \operatorname{Lat} N$ *such that* $\mathcal{M} \wedge \mathcal{N} = 0$, *then* $\mathcal{N} \leqslant \mathcal{M}'$.

Proof. This is true of any central element of any lattice. ▨

2.4. LEMMA. *If* $\mathcal{M}$ *is a central element of* $\operatorname{Lat} N$ *and* $\mathcal{M}$ *is reducing for* N, *then* $\mathcal{M}' = \mathcal{M}^{\perp}$ *and* $P \in P^{\infty}(N)$.

Proof. Since $\mathcal{M}$ is a reducing subspace for N, $\mathcal{M}^{\perp} \in \operatorname{Lat} N$. By the preceding lemma $\mathcal{M}^{\perp} \leqslant \mathcal{M}'$. If $h \in \mathcal{M}'$, then $h = h_1 + h_2$ with h_1 in $\mathcal{M}$ and h_2 in $\mathcal{M}^{\perp}$. Thus $h_1 = h - h_2 \in \mathcal{M} \cap \mathcal{M}' = 0$. Hence $\mathcal{M}' = \mathcal{M}^{\perp}$.

Now let $\mathcal{N} \in \operatorname{Lat} N$. Since $\mathcal{N} = (\mathcal{N} \wedge \mathcal{M}) \vee (\mathcal{N} \wedge \mathcal{M}')$, it follows that $\mathcal{N} = (\mathcal{N} \cap \mathcal{M}) \oplus (\mathcal{N} \cap \mathcal{M}^{\perp})$. Therefore $P\mathcal{N} = \mathcal{N} \cap \mathcal{M} \leqslant \mathcal{N}$ for every $\mathcal{N}$ in $\operatorname{Lat} N$. Since normal operators are reflexive [7] (see also [2], page 97), $P \in P^{\infty}(N)$. ▨

2.5. LEMMA. *If* N *is reductive and* $\mathcal{M}$ *is a central element of* $\operatorname{Lat} N$, *then* P *is a spectral projection of* N *and* $\mathcal{M}' = \mathcal{M}^{\perp}$.

Proof. If N is reductive, $P^{\infty}(N) = W^{*}(N)$, the von Neumann algebra generated by N. Thus Lemma 2.5 is a special case of Lemma 2.4. ▨

2.6. LEMMA. *If* μ *is a compactly supported measure on* $\mathbf{C}$, *then there exist compact sets* $\{F_n\}$ *such that for every* n, $\operatorname{int} F_n = \emptyset$, $F_n \subseteq F_{n+1}$, $\mathbf{C} \setminus F_n$ *is connected, and* $\mu(F_n) \to \mu(\mathbf{C})$ *as* $n \to \infty$.

Proof. This goes back to Bram [1]. In fact, the sets $\{F_k\}$ in the proof of Theorem 8.14 on page 344 of [2] have these properties. ▨

2.7. LEMMA. *If* $N = \int z \, \mathrm{d}E(z)$ *is the spectral decomposition of* N, *then there are pairwise disjoint Borel sets* $\{\Delta_n\}$ *such that* $1 = \sum_{n=1}^{\infty} E(\Delta_n)$ *and* $N_n \equiv N \mid E(\Delta_n)\mathcal{H}$ *is reductive for every integer* n.

Proof. Let μ be a scalar-valued spectral measure for N and let $\{F_n\}$ be the sequence of sets obtained in the preceding lemma with $F_0 = \emptyset$. Put $\Delta_n = F_n \setminus F_{n-1}$ for every $n \geqslant 1$. If N_n is defined as in the statement of the lemma, then $\sigma(N_n) \subseteq F_n$. Since $\mathbf{C} \setminus F_n$ is connected and $\operatorname{int} F_n = \emptyset$, Lavrentiev's Theorem implies there is a sequence of analytic polynomials that converges to $\bar{z}$ uniformly on $\sigma(N_n)$. It follows that N_n is a reductive operator. ▨

Proof Theorem 2.1. Assume that $\mathcal{M}$ is central in $\operatorname{Lat} N$. Let $N = \int z \, \mathrm{d}E(z)$ be the spectral decomposition of N and let μ be a scalar-valued spectral measure for N. Let $\{\Delta_n\}$ be the sequence of Borel sets obtained in Lemma 2.7. Put $\mathcal{M}_n = \mathcal{M} \cap E(\Delta_n)\mathcal{H}$ and $\mathcal{M}'_n = \mathcal{M}' \cap E(\Delta_n)\mathcal{H}$. If $\mathcal{N} \in \operatorname{Lat} N_n$, then $\mathcal{N} \in \operatorname{Lat} N$ and $\mathcal{N} \leqslant E(\Delta_n)\mathcal{H}$. Since $\mathcal{N} = (\mathcal{N} \wedge \mathcal{M}) \vee (\mathcal{N} \wedge \mathcal{M}')$, it is easy to see that $\mathcal{N} =$

$= (\mathcal{N} \wedge \mathcal{M}_n) \vee (\mathcal{N} \wedge \mathcal{M}_n')$. That is, $\mathcal{M}_n$ is central in Lat N_n. By Lemma 2.5 there is a Borel set Σ_n that is contained in Δ_n and is such that $\mathcal{M}_n = E(\Sigma_n)\mathcal{H}$ and $\mathcal{M}_n' = E(\Delta_n \setminus \Sigma_n)\mathcal{H}$. Let $\Sigma = \bigcup_1^\infty \Sigma_n$. Since $E(\Sigma_n)\mathcal{H} = \mathcal{M}_n \leqslant \mathcal{M}$ for every n, it follows that $E(\Sigma)\mathcal{H} \leqslant \mathcal{M}$. Similarly $E\left(\bigcup_1^\infty (\Delta_n \setminus \Sigma_n)\right) \leqslant \mathcal{M}'$. But $1 = \sum_1^\infty E(\Delta_n)$ and hence $E\left(\bigcup_1^\infty (\Delta_n \setminus \Sigma_n)\right) = E(\sigma(N) \setminus \Sigma) = E(\Sigma)^\perp$. Thus $E(\Sigma)^\perp\mathcal{H} \leqslant \mathcal{M}'$. So if $h \in \mathcal{M}$ and $h = h_1 + h_2$ with h_1 in $E(\Sigma)\mathcal{H}$ and h_2 in $E(\Sigma)^\perp\mathcal{H}$, then $h_2 = h - h_1 \in \mathcal{M} \cap \mathcal{M}' = 0$; that is, $h \in E(\Sigma)\mathcal{H}$. Therefore $\mathcal{M} = E(\Sigma)\mathcal{H}$ and $\mathcal{M}$ is reducing. By Lemma 2.4, $P \in P^\infty(N)$. ▨

3. ISOMORPHIC INVARIANT SUBSPACE LATTICES OF ISOMETRIES

For an isometry U, let $U = S \oplus U_0$ be the Wold decomposition of U, where S is a unilateral shift of some multiplicity and U_0 is a unitary operator. The unitary operator can be further decomposed uniquely as $U_0 = U_1 \oplus U_s$, where U_1 is absolutely continuous (that is, its spectral measure is absolutely continuous with respect to Lebesgue measure on $\partial\mathbf{D}$) and U_s is singular (again with respect to Lebesgue measure). The operator U_1 can be still further decomposed as $W \oplus U_a$, where W is a bilateral shift of some multiplicity and U_a is an absolutely continuous unitary operator that is reductive. This last decomposition is unique if W has finite multiplicity but is not unique if W has infinite multiplicity since, in the latter case, $W \cong W \oplus V$ for every absolutely continuous unitary operator V. However, it is unique if it is required that $U_a = 0$ when W has infinite multiplicity. These results for U_1 can be achieved by applying multiplicity theory to U_1 (see Theorem 10.1 on page 300 of [3]).

To recapitulate the discussion of the preceding paragraph, for any isometry U, there is a unique decomposition

$$U = U_u \oplus U_b \oplus U_a \oplus U_s,$$

where U_u is a unilateral shift, U_b is a bilateral shift, U_a is a reductive absolutely continuous unitary operator, U_s is a singular unitary operator, and $U_a = 0$ if U_b has infinite multiplicity. This decomposition will be called the *standard decomposition* of U.

The purpose of this section is to prove the following theorem.

3.1. THEOREM. *Let U and V be isometries with standard decompositions $U = U_u \oplus U_b \oplus U_a \oplus U_s$ and $V = V_u \oplus V_b \oplus V_a \oplus V_s$ and assume that* Lat $U \approx$

$\approx$ Lat V. *Then:*

(a) U_u *and* V_u *have the same multiplicity;*

(b) U_b *and* V_b *have the same multiplicity;*

(c) $\mathrm{Lat}(U_b \oplus U_a \oplus U_s) \approx \mathrm{Lat}(V_b \oplus V_a \oplus V_s)$.

Furthermore:

(d) *if* $U_u \oplus U_b = 0$, *then* U *and* V *are reductive unitaries and* $V \cong \varphi(U)$ *for some* U-V *point isomorphism* φ;

(d′) *if* $U_u \oplus U_b \neq 0$, *then*

(i) $\mathrm{Lat}(U_b \oplus U_a) \approx \mathrm{Lat}(V_b \oplus V_a)$;

(ii) $V_a \cong \varphi(U_a)$ *for some* U_a-V_a *point isomorphism* φ;

(iii) $V_s \cong \psi(U_s)$ *for some* U_s-V_s *point isomorphism* ψ.

REMARK. Suppose that U and V are unitaries. It is easy to deduce from Theorem 3.1 that, if Lat $U \approx$ Lat V, then $V \cong \varphi(U)$ for some U-V point isomorphism φ. (Alternatively, apply Proposition 3.13 below and the results of [4].) The converse of this statement is not valid in general. For instance, take V to be the bilateral shift of multiplicity 1 and U to be multiplication by z on $L^2(E)$, where E is the top half of the unit circle. Then $V \cong \varphi(U)$ for the U-V point isomorphism $\varphi(z) = z^2$ but Lat U is not isomorphic to Lat V by Lemma 3.10 below. However Theorem 3.1 suggests a possible more refined converse as follows.

Let U and V be unitaries such that either $U_b = V_b = 0$ and $V \cong \varphi(U)$ for some U-V point isomorphism φ or U_b and V_b have the same non-zero multiplicity and (d′) (ii), (iii) in the statement of Theorem 3.1 hold. Does it follow that Lat $U \approx$ Lat V? The answer is in the affirmative if $U_b = 0$ or if $U_a = 0$, but the situation is unclear when U_b has finite non-zero multiplicity and $U_a \neq 0$. In the Appendix, we describe the invariant subspace lattices of the unitaries of the form $U_b \oplus U_a$, where U_b has multiplicity 1 and U_a is non-zero and cyclic, against which the truth of this converse may possibly be tested.

The proof of Theorem 3.1 will be postponed until some preliminary results are established. Some of these preliminaries have independent interest.

3.2. PROPOSITION. ([10], or Proposition 1.2 of [9]). *Let* μ *be a positive measure on* $\partial\mathbf{D}$ *and let* m *be normalized arc length measure on* $\partial\mathbf{D}$. *Then* $P^\infty(\mu) \neq L^\infty(\mu)$ *if and only if* $m \ll \mu$.

3.3. COROLLARY. *If* U *is a unitary operator, then the following statements are equivalent.*

(a) U *is not reductive.*

(b) *There is a reducing subspace* $\mathcal{M}$ *for* U *such that* $U|\mathcal{M}$ *is unitarily equivalent to a bilateral shift of multiplicity* 1.

(c) *There is an invariant subspace* $\mathcal{M}$ *for* U *such that* $U|\mathcal{M}$ *is a unilateral shift of multiplicity* 1.

Proof. Clearly (b) implies (a) and, by Corollary 1.2, (c) implies (a). The fact that (a) implies (b) follows readily from the preceding proposition. ▨

The next several lemmas will be used to prove Proposition 3.7 below, which characterizes the central elements of the invariant subspace lattice of an isometry. The proof of the first of these lemmas is straightforward from the definitions.

LEMMA 3.4. *If A is any operator, $\mathcal{M}$ is central in* Lat A *with complementary subspace $\mathcal{M}'$, and $\mathcal{N} \in$* Lat A, *then $\mathcal{M} \cap \mathcal{N}$ is central in* Lat$(A \mid \mathcal{N})$ *with complementary subspace $\mathcal{M}' \cap \mathcal{N}$.*

LEMMA 3.5. *If U is a unilateral shift on $\mathcal{H}$, then the only central elements of* Lat U *are* (0) *and $\mathcal{H}$.*

Proof. Let $\mathcal{M}$ be a central element in Lat U with complementary space $\mathcal{M}'$. Since $\mathcal{M} \wedge \mathcal{M}' = (0)$, the result in the case that U has multiplicity one is immediate from Corollary 1.2. Assume that the multiplicity of U is at least two.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be orthogonal elements of Lat U such that both $U \mid \mathcal{H}_1$ and $U \mid \mathcal{H}_2$ have multiplicity one. It follows from Lemma 3.4 and the multiplicity one case that each of $\mathcal{H}_1$ and $\mathcal{H}_2$ is contained in either $\mathcal{M}$ or $\mathcal{M}'$. It will be shown that either $\mathcal{H}_1 \subseteq \mathcal{M}$ and $\mathcal{H}_2 \subseteq \mathcal{M}$ or that $\mathcal{H}_1 \subseteq \mathcal{M}'$ and $\mathcal{H}_1 \subseteq \mathcal{M}'$. If this is not the case, then it may be assumed that $\mathcal{H}_1 \subseteq \mathcal{M}$ and $\mathcal{H}_2 \subseteq \mathcal{M}'$. Now represent $U \mid \mathcal{H}_1$ and $U \mid \mathcal{H}_2$ as multiplication by z on H^2 and, making the appropriate identifications, let

$$\mathcal{N} = \{f \oplus f : f \in H^2\} \subseteq H^2 \oplus H^2 = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Note that $U \mid \mathcal{N}$ is a unilateral shift of multiplicity one. Again from Lemma 3.4 and the multiplicity one case, it must be that $\mathcal{N} \subseteq \mathcal{M}$ or $\mathcal{N} \subseteq \mathcal{M}'$. In the former case, if $f \in \mathcal{H}_2$, then $0 \oplus f = f \oplus f - f \oplus 0 \in \mathcal{N} - \mathcal{H}_1 \subseteq \mathcal{M}$, a contradiction. Similarly, if $\mathcal{N} \subseteq \mathcal{M}'$, a contradiction to the fact that $\mathcal{H}_1 \subseteq \mathcal{M}$ is obtained. Thus either $\mathcal{H}_1 \subseteq \mathcal{M}$ and $\mathcal{H}_2 \subseteq \mathcal{M}$ or $\mathcal{H}_1 \subseteq \mathcal{M}'$ and $\mathcal{H}_2 \subseteq \mathcal{M}'$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ were arbitrary, the lemma follows. ▨

LEMMA 3.6. *If U is a unilateral shift on $\mathcal{H}$ and V is an absolutely continuous reductive unitary operator on $\mathcal{K}$, then the only central subspaces of* Lat$(U \oplus V)$ *are* (0) *and $\mathcal{H} \oplus \mathcal{K}$.*

Proof. Let $\mathcal{M}$ be a central element of Lat$(U \oplus V)$ with complementary subspace $\mathcal{M}'$. By Lemma 3.4, $\mathcal{M} \cap \mathcal{H}$ is central in Lat U. So Lemma 3.5 implies that either $\mathcal{H} \subseteq \mathcal{M}$ or $\mathcal{H} \subseteq \mathcal{M}'$. Assume that $\mathcal{H} \subseteq \mathcal{M}$, the other case being handled by reversing the roles of $\mathcal{M}$ and $\mathcal{M}'$ in the argument below.

Let μ be a scalar-valued spectral measure for V. Since V is reductive and absolutely continuous, there is a Borel subset F of $\partial\mathbf{D}$ carrying μ with $\mu \equiv m$ on F and $m(\partial\mathbf{D} \setminus F) > 0$. Also, by Lemma 3.4 and Corollary 2.3 of [4], $\mathcal{M} \cap \mathcal{K} = \chi_E(V)\mathcal{K}$

for some Borel subset E of F. Let $\mathscr{H}_1$ and $\mathscr{K}_1 \in \operatorname{Lat}(U \oplus V)$ with $\mathscr{H}_1 \subseteq \mathscr{H}$ and $\mathscr{K}_1 \subseteq \mathscr{K}$, and such that $U \mid \mathscr{H}_1 \cong M_z$ on H^2 and $V \mid \mathscr{K}_1 \cong M_z$ on $L^2(F, m)$. Identify $\mathscr{H}_1$ with H^2 and $\mathscr{K}_1$ with $L^2(F, m)$ and consider the subspaces $\mathscr{N}_+$ and $\mathscr{N}_-$ of $\mathscr{H}_1 \oplus \mathscr{K}_1$ defined by

$$\mathscr{N}_+ = \{f \oplus \chi_F f : f \in H^2\},$$

$$\mathscr{N}_- = \{f \oplus -\chi_F f : f \in H^2\}.$$

Both $(U \oplus V) \mid \mathscr{N}_+$ and $(U \oplus V) \mid \mathscr{N}_-$ are unilateral shifts of multiplicity one; hence, by Lemma 3.4 and Lemma 3.5, each of $\mathscr{N}_+$ and $\mathscr{N}_-$ is contained in either $\mathscr{M}$ or $\mathscr{M}'$. But if $\mathscr{N}_+$ and $\mathscr{N}_- \subseteq \mathscr{M}'$, then this would imply that $\mathscr{H} \cap \mathscr{M}' \neq (0)$, contradicting the fact that $\mathscr{H} \subseteq \mathscr{M}$. So it must be that at least one of $\mathscr{N}_+$ and $\mathscr{N}_- \subseteq \mathscr{M}$. Since $\mathscr{H} \subseteq \mathscr{M}$, this implies (continuing the above identification of $\mathscr{H}_1$ and $\mathscr{K}_1$ with H^2 and $L^2(F, m)$, respectively) that $\chi_F f \in \mathscr{M} \cap \mathscr{K}_1$ for f in H^2 (in particular for $f = 1$). Since $\mathscr{M} \cap \mathscr{K} = \chi_E(V)\mathscr{K}$, it follows that $m(F \setminus E) = 0$ and hence that $\mathscr{M} \cap \mathscr{K} = \mathscr{K}$. That is, $\mathscr{K} \subseteq \mathscr{M}$, completing the proof. ∎

3.7. PROPOSITION. *Let U be an isometry with standard decomposition $U = U_u \oplus U_b \oplus U_a \oplus U_s$ on $\mathscr{H} = \mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s$.*

(a) *If $U_u \oplus U_b \neq 0$, then the central elements of* Lat U *are the subspaces of the form $\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a \oplus \chi_E(U_s)\mathscr{H}_s$ or $\chi_E(U_s)\mathscr{H}_s$, where E is a Borel subset of $\partial\mathbf{D}$.*

(b) *If $U_u \oplus U_b = 0$, then U is a reductive unitary operator and the central elements of* Lat U *are $\{\chi_F(U_a)\mathscr{H}_a \oplus \chi_E(U_s)\mathscr{H}_s : E$ and F are Borel subsets of $\partial\mathbf{D}\}$.*

Proof. Part (b) is an immediate consequence of Proposition 3.2 and Corollary 2.3 of [4]. Thus it suffices to prove (a). So suppose that $U_u \oplus U_b \neq 0$ and let μ be a scalar-valued spectral measure for $U_b \oplus U_a \oplus U_s$. Using the fact that

$$P^\infty(U) = \{\varphi(U_u) \oplus \varphi(U_b) \oplus \varphi(U_a) \oplus \psi(U_s) : \varphi \in H^\infty, \ \psi \in L^\infty(\mu_s)\}$$

where μ_s is the singular part of μ and $\varphi \mapsto \varphi(U_u)$ is the usual H^∞ functional calculus for a unilateral shift, it is easy to verify that each subspace described in (a) is a central element of Lat U.

For the converse, fix a central element $\mathscr{M}$ in Lat U with corresponding complementary subspace $\mathscr{M}'$. By Lemma 3.4 and Theorem 2.1, $\mathscr{M} \cap \mathscr{H}_s = \chi_E(U_s)\mathscr{H}_s$ for some Borel subset E of $\partial\mathbf{D}$. Since $\mathscr{H}_s$ is central in Lat U, it readily follows that

$$\mathscr{M} = \mathscr{M} \cap (\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a) \oplus \chi_E(U_s)\mathscr{H}_s.$$

Thus in order to complete the proof it must be shown that $\mathscr{M} \cap (\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a)$ equals either (0) or $\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a$.

If $U_u = 0$, then, by the assumption of (a), $U_b \neq 0$ and $m \ll \mu_a = \mu - \mu_s$, a scalar-valued spectral measure for $U_b \oplus U_a$. Thus $P^\infty(\mu_a) = H^\infty$ and $P^\infty(U_b \oplus \oplus U_a) \cong P^\infty(\mu_a)$ contains no non-trivial idempotents. This shows, by Theorem 2.1, that the central element $\mathcal{M} \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a)$ of $\mathrm{Lat}(U_b \oplus \oplus U_a)$ equals $\mathcal{H}_b \oplus \mathcal{H}_a$ or (0), as required.

For the remainder of the proof, assume that $U_u \neq 0$. Once again, Lemma 3.4 implies that $\mathcal{M} \cap \mathcal{H}_u$ is central in Lat U_u. Thus, by Lemma 3.5, $\mathcal{H}_u \subseteq \mathcal{M}$ or $\mathcal{H}_u \subseteq \mathcal{M}'$.

Suppose first that $\mathcal{H}_u \subseteq \mathcal{M}$. Then

$$\mathcal{M} \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{H}_u \oplus \mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a).$$

If $\mathcal{H}_b \neq (0)$, then, as above, $P^\infty(U_b \oplus U_a) \cong H^\infty$. Since $\mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a)$ is central in $\mathrm{Lat}(U_b \oplus U_a)$, $\mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{H}_b \oplus \mathcal{H}_a$ or (0). But when $\mathcal{H}_b \neq (0)$, there exists $\mathcal{H}_0$ in Lat U such that $\mathcal{H}_0 \subseteq \mathcal{H}_b$ and $U | \mathcal{H}_0$ is a unilateral shift. Thus $U | (\mathcal{H}_u \oplus \mathcal{H}_0)$ is a unilateral shift, and, by Lemma 3.5 and the assumption that $\mathcal{H}_u \subseteq \mathcal{M}$, it follows that $\mathcal{H}_u \oplus \mathcal{H}_0 \subseteq \mathcal{M}$. Thus $\mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a)$ is non-zero and so $\mathcal{M} \cap (\mathcal{H}_b \oplus \mathcal{H}_a) = (\mathcal{H}_b \oplus \mathcal{H}_a)$. This implies that $\mathcal{M} \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a$ as required (in the case that $\mathcal{H}_u \subseteq \mathcal{M}$ and $\mathcal{H}_b \neq (0)$).

Still assuming that $\mathcal{H}_u \subseteq \mathcal{M}$, consider the case when $\mathcal{H}_b = (0)$. Here $\mathcal{M} \cap (\mathcal{H}_u \oplus \oplus \mathcal{H}_a)$ is central in $\mathrm{Lat}(U_u \oplus U_a)$ and not (0). By Lemma 3.6, $\mathcal{H}_u \oplus \mathcal{H}_a \subseteq \mathcal{M}$. This implies that $\mathcal{M} \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{H}_u \oplus \mathcal{H}_a = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a$, completing the proof in the case that $\mathcal{H}_u \subseteq \mathcal{M}$.

Finally, assume that $\mathcal{H}_u \subseteq \mathcal{M}'$. Then, replacing $\mathcal{M}$ by $\mathcal{M}'$ in the above argument, one concludes that $\mathcal{M}' \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a$. Hence $\mathcal{M} \cap (\mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_a) = (0)$. ▨

It is now time to give a series of lemmas that are special cases of Theorem 3.1, but will be used in its proof.

3.8. Lemma. *If S is the unilateral shift of multiplicity 1, V is an isometry, and* Lat S *and* Lat V *are isomorphic, then $V \cong S$. That is, V is the unilateral shift of multiplicity* 1.

Proof. If Θ: Lat $S \to$ Lat V is a lattice isomorphism and $\mathcal{M}$ is a reducing subspace for V, then $\Theta^{-1}(\mathcal{M}) \wedge \Theta^{-1}(\mathcal{M}^\perp) = 0$. But by Corollary 1.2 this implies that either $\Theta^{-1}(\mathcal{M}) = 0$ or $\Theta^{-1}(\mathcal{M}^\perp) = 0$. Hence V must be an irreducible isometry. But the only irreducible isometry is the unilateral shift of multiplicity 1. ▨

3.9. Lemma. *If U is a reductive unitary operator, V is an isometry, and* Lat $U \approx$ Lat V, *then V is a reductive unitary operator and $V \cong \varphi(U)$ for some U-V point isomorphism φ.*

Proof. Let Θ: Lat $U \to$ Lat V be a lattice isomorphism. Assume that V is not a reductive unitary operator. By the Wold decomposition and Corollary 3.3, there is an invariant subspace $\mathcal{M}$ of V such that $V|\mathcal{M}$ is a unilateral shift of multiplicity 1. But then $\text{Lat}(U|\Theta^{-1}(\mathcal{M})) \approx \text{Lat}(V|\mathcal{M})$. By Lemma 3.8, $U|\Theta^{-1}(\mathcal{M}) \cong V|\mathcal{M}$. But this says that $U|\Theta^{-1}(\mathcal{M})$ is not unitary. By Proposition 1.3, this contradicts the fact that U is reductive. Therefore V is reductive. The remainder of the proposition follows from Theorem 3.2 of [4]. ▨

3.10. LEMMA. *If U is the bilateral shift of multiplicity* 1, *V is an isometry, and* Lat U *and* Lat V *are isomorphic, then $V \cong U$. That is, V is the bilateral shift of multiplicity* 1.

Proof. Let $\Theta: \text{Lat}\, U \to \text{Lat}\, V$ be a lattice isomorphism and let $V = S \oplus W$ on $\mathcal{K} = \mathcal{S} \oplus \mathcal{W}$ be the Wold decomposition of V, where S is a unilateral shift and W is unitary. First note that $\mathcal{W} \neq 0$. Indeed, if $\mathcal{W} = (0)$, then V is a unilateral shift. But there is a reducing subspace $\mathcal{M}$ for U such that $U|\mathcal{M}$ is a reductive unitary operator. By Lemma 3.9, $V|\Theta(\mathcal{M})$ is a reductive unitary operator. But this is a contradiction since unilateral shifts can have no such invariant subspaces. Thus $\mathcal{W} \neq (0)$.

Since $\Theta^{-1}(\mathcal{W}) \in \text{Lat}\, U$, Proposition 1.1 implies that either there is a Borel set Δ contained in $\partial\mathbf{D}$ such that $\Theta^{-1}(\mathcal{W}) = \mathcal{M}_\Delta \equiv \{f \in L^2(m): f = 0 \text{ off } \Delta\}$ or there is a φ in $L^\infty(m)$ such that $|\varphi| = 1$ a.e. and $\Theta^{-1}(\mathcal{W}) = \varphi H^2$. If it were the case that $\Theta^{-1}(\mathcal{W}) = \varphi H^2$, then it would follow that $\text{Lat}(U|\varphi H^2) \approx \text{Lat}\, W$. But $U|\varphi H^2$ is a unilateral shift of multiplicity 1. By Lemma 3.8, this is impossible since W is unitary. Therefore it must be that $\Theta^{-1}(\mathcal{W}) = \mathcal{M}_\Delta$ for some Borel subset of $\partial\mathbf{D}$.

Claim. $\Delta = \partial\mathbf{D}$.

Let $\Delta' \equiv \partial\mathbf{D}\setminus\Delta$ and assume that $m(\Delta') > 0$. Let P be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{S}$ and let $\mathcal{L} = \Theta(\mathcal{M}_{\Delta'}) = \Theta(\mathcal{M}_\Delta^\perp)$. So $\mathcal{L} \wedge \mathcal{W} = 0$ and $\mathcal{L} + \mathcal{W}$ is dense in $\mathcal{K}$. Define $A: \mathcal{L} \to \mathcal{S}$ by $A = P|\mathcal{L}$. So A is a bounded operator that is injective and has dense range. If $L = V|\mathcal{L}$, then L is an isometry and $AL = SA$. Hence $L^*A^* = A^*S^*$. But there is a non-zero vector e in $\mathcal{S}$ such that $S^*e = 0$. Hence $L^*A^*e = 0$ and $A^*e \neq 0$. Hence L is not unitary. By the Wold decomposition there is an invariant subspace $\mathcal{N}$ for V such that $\mathcal{N} \leqslant \mathcal{L}$ and $V|\mathcal{N}$ is a unilateral shift of multiplicity 1. But $\Theta^{-1}(\mathcal{N}) \leqslant \mathcal{M}_{\Delta'}$ and $\text{Lat}(U|\Theta^{-1}(\mathcal{N})) \approx \text{Lat}(V|\mathcal{N})$. By Lemma 3.8, $U|\Theta^{-1}(\mathcal{N})$ is a unilateral shift of multiplicity 1. By Corollary 3.3, it follows that $\Delta' = \partial\mathbf{D}$ and so $\Delta = \emptyset$. Thus $\mathcal{M}_\Delta = (0)$ and so $\mathcal{W} = (0)$, a contradiction. Therefore $m(\Delta') = 0$, or $\Delta = \partial\mathbf{D}$, proving the claim.

Because $\Delta = \partial\mathbf{D}$ and Θ is injective, $\mathcal{K} = \mathcal{W}$. That is, V is unitary.

But $\text{Lat}(U|H^2) \approx \text{Lat}(V|\Theta(H^2))$. So Lemma 3.8 implies that $V|\Theta(H^2) \cong U|H^2$. Hence there is a reducing subspace $\mathcal{N}$ for V such that $\mathcal{N} \geqslant \Theta(H^2)$ and $V|\mathcal{N} \cong U$. But, from Proposition 1.1, either there is a φ in $L^\infty(m)$ with $|\varphi| = 1$ a.e. such that $\Theta^{-1}(\mathcal{N}) = \varphi H^2$ or there is a Borel subset Δ of $\partial\mathbf{D}$ such that $\Theta^{-1}(\mathcal{N}) = \mathcal{M}_\Delta$. If $\Theta^{-1}(\mathcal{N}) = \varphi H^2$, then $\text{Lat}(U|\varphi H^2) \approx \text{Lat}(V|\mathcal{N})$. Since $V|\mathcal{N}$ is unitary, this is a contradiction by Lemma 3.8. So it must be that $\Theta^{-1}(\mathcal{N}) = \mathcal{M}_\Delta$ for some

Borel set. But $\Theta^{-1}(\mathscr{N}) \geqslant H^2$ and so $\Delta = \partial\mathbf{D}$. Therefore $\mathscr{N} = \Theta(L^2) = \mathscr{K}$. That is, $V = V|\mathscr{N} \cong U$. ▨

3.11. LEMMA. *If* $1 \leqslant n \leqslant \infty$, *$U$ is a bilateral shift of multiplicity n, V is an isometry, and* $\operatorname{Lat} U \approx \operatorname{Lat} V$, *then* $V \cong U$.

Proof. Let $\Theta: \operatorname{Lat} U \to \operatorname{Lat} V$ be the hypothesized lattice isomorphism and let $\mathscr{H}$ and $\mathscr{K}$ be the domains of U and V, respectively.

Case 1. n is finite.

Let $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots \oplus \mathscr{H}_n$ such that each $\mathscr{H}_j$ reduces U and $U|\mathscr{H}_j$ is a bilateral shift of multiplicity 1. Let $\mathscr{K}_j = \Theta(\mathscr{H}_j)$. So $V|\mathscr{K}_j$ is an isometry and $\operatorname{Lat}(V|\mathscr{K}_j) \approx \operatorname{Lat}(U|\mathscr{H}_j)$. By Lemma 3.10, $V|\mathscr{K}_j \cong U|\mathscr{H}_j$. In particular, $V|\mathscr{K}_j$ is unitary. Therefore $\mathscr{K}_j$ reduces V and $\mathscr{K}_j \leqslant \operatorname{ran} V$. Therefore $\operatorname{ran} V \geqslant \mathscr{K}_1 \vee \vee \ldots \vee \mathscr{K}_n = \Theta(\mathscr{H}_1) \vee \ldots \vee \Theta(\mathscr{H}_n) = \Theta(\mathscr{H}) = \mathscr{K}$. Thus V is unitary.

Define $X: \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_n \to \mathscr{K}$ by $X(k_1 \oplus \ldots \oplus k_n) = k_1 + \ldots + k_n$. It follows that X is injective and has dense range. Also

$$X(V|\mathscr{K}_1 \oplus \ldots \oplus V|\mathscr{K}_n) = VX.$$

By one of the standard consequences of the Fuglede-Putnam Theorem (see, for example, page 286 of [3]), $V \cong V|\mathscr{K}_1 \oplus \ldots \oplus V|\mathscr{K}_n \cong U$.

Case 2. n is infinite.

Let $\{\mathscr{H}_j\}$ be an increasing sequence of reducing subspaces for U, whose union is dense in $\mathscr{H}$, and such that $U|\mathscr{H}_j$ is a bilateral shift of multiplicity j. Let $\mathscr{K}_j = \Theta(\mathscr{H}_j)$. By Case 1, $V|\mathscr{K}_j \cong U|\mathscr{H}_j$ for every $j \geqslant 1$. Hence $V|\mathscr{K}_j$ is a bilateral shift of multiplicity j and $\mathscr{K}_j$ reduces V. Also $\mathscr{K}_j \subseteq \mathscr{K}_{j+1}$. A routine application of multiplicity theory (for example, Theorem 10.1 on page 300 of [3]) shows that $V|(\mathscr{K}_{j+1} \ominus \mathscr{K}_j)$ is a bilateral shift of multiplicity one. Therefore $V|(\mathscr{K}_{j+1} \ominus \mathscr{K}_j) \cong U|(\mathscr{H}_{j+1} \ominus \mathscr{H}_j)$ for all j. Since $\Theta^{-1}\left(\bigvee_1^\infty \mathscr{K}_j\right) = \mathscr{H}$, it must be that $\bigvee_1^\infty \mathscr{K}_j = \mathscr{K}$. Therefore $U \cong V$. ▨

3.12. LEMMA. *If U is a unitary operator, V is an isometry, and* $\operatorname{Lat} U \approx \operatorname{Lat} V$, *then V is unitary.*

Proof. Let $\Theta: \operatorname{Lat} U \to \operatorname{Lat} V$ be a lattice isomorphism. The case where U is reductive is dealt with in Lemma 3.9. So assume that U is not reductive and decompose U as $U = U_1 \oplus U_0$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_0$, where U_1 is a bilateral shift of multiplicity n, $1 \leqslant n \leqslant \infty$, and U_0 is reductive. Put $\mathscr{K}_1 = \Theta(\mathscr{H}_1)$. So $\operatorname{Lat}(V|\mathscr{K}_1) \approx \operatorname{Lat} U_1$. By Lemma 3.11, $V|\mathscr{K}_1 \approx U_1$. Hence $V|\mathscr{K}_1$ is unitary and $\mathscr{K}_1$ reduces V (by Proposition 1.3). Let $\mathscr{K}_0 = \mathscr{K}_1^\perp$. So $\mathscr{K}_0 \in \operatorname{Lat} V$ and $V_0 = V|\mathscr{K}_0$ is an isometry.

Let P be the orthogonal projection of $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_0$ onto $\mathscr{H}_0$ and let $\mathscr{L} = \Theta(\mathscr{H}_0)$. Define $X: \mathscr{L} \to \mathscr{H}_0$ by $X = P \mid \mathscr{L}$. It is easy to see that X is injective, has dense range, and $X(V \mid \mathscr{L}) = V_0 X$. But $\text{Lat}(V \mid \mathscr{L}) \approx \text{Lat}\, U_0$ and U_0 is reductive. By Lemma 3.9, $V \mid \mathscr{L}$ is a reductive unitary operator.

If V_0 is not unitary, then V_0^* has a non-trivial kernel. Since $(V \mid \mathscr{L})^* X^* = X^* V_0^*$ and X^* is injective, this would imply that $(V \mid \mathscr{L})^*$ has a non-trivial kernel, contradicting the fact that $(V \mid \mathscr{L})$ is unitary. Hence V_0 is unitary. This implies that $V = V_1 \oplus V_0$ is unitary. ∎

The preceding lemma allows a fact to be deduced which has an interest in its own right.

3.13. PROPOSITION. *If U and V are unitary operators and Θ:* $\text{Lat}\, U \to \text{Lat}\, V$ *is a lattice isomorphism, then $\Theta(\mathscr{M})$ is a reducing subspace for V whenever $\mathscr{M}$ is a reducing subspace for U.*

Proof. If $\mathscr{M}$ is a reducing subspace for U, then $U \mid \mathscr{M}$ is unitary and $\text{Lat}(U \mid \mathscr{M}) \approx \text{Lat}(V \mid \Theta(\mathscr{M}))$. By Lemma 3.12, $V \mid \Theta(\mathscr{M})$ is unitary and, therefore, $\Theta(\mathscr{M})$ reduces V. ∎

3.14. LEMMA. *If U is a unilateral shift of multiplicity n, $1 \leqslant n \leqslant \infty$, and V is an isometry such that* $\text{Lat}\, U \approx \text{Lat}\, V$, *then V is a unilateral shift of multiplicity n.*

Proof. Let $\Theta: \text{Lat}\, V \to \text{Lat}\, U$ be a lattice isomorphism. If V is not a unilateral shift, then there is a reducing subspace $\mathscr{M}$ for V such that $V \mid \mathscr{M}$ is unitary. Since $\text{Lat}(U \mid \Theta(\mathscr{M})) \approx \text{Lat}(V \mid \mathscr{M})$, Lemma 3.12 implies that $U \mid \Theta(\mathscr{M})$ is a unitary, contradicting the hypothesis. Therefore V is a unilateral shift.

If U and V have finite multiplicity n and m, respectively, then there are pairwise orthogonal non-zero reducing subspaces for U, $\mathscr{H}_1, \ldots, \mathscr{H}_n$, whose linear span is all of $\mathscr{H}$ (the domain of U). Let $\Theta^{-1}(\mathscr{H}_k) = \mathscr{M}_k$, $1 \leqslant k \leqslant n$. Then $\mathscr{M}_k \cap (\mathscr{M}_{k+1} + \ldots + \mathscr{M}_n) = (0)$ for each k and no $\mathscr{M}_k$ is (0). By Proposition 1.5, $m \geqslant n$. A similar argument shows that $m \leqslant n$. If the multiplicity of either U or V is infinite, then again Proposition 1.5 can be used to conclude that the other shift must have infinite multiplicity. ∎

Proof of Theorem 3.1. Let $\Theta: \text{Lat}\, U \to \text{Lat}\, V$ be a lattice isomorphism and let $U = U_u \oplus U_b \oplus U_a \oplus U_s$ be the standard decomposition of U acting on $\mathscr{H} = \mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s$. Similarly, let $V = V_u \oplus V_b \oplus V_a \oplus V_s$ on $\mathscr{K} = \mathscr{K}_u \oplus \mathscr{K}_b \oplus \mathscr{K}_a \oplus \mathscr{K}_s$. The proof splits into two cases, the first of which is easy to take care of. First assume that $U_u \oplus U_b = 0$. In this case U is a reductive unitary operator. By Lemma 3.9, V is reductive, $V_u = V_b = 0$, and $V \cong \varphi(U)$ for some U-V point isomorphism φ, completing the proof in this case.

Thus it may be assumed that $U_u \oplus U_b \neq 0$. The result of the previous paragraph applied to Θ^{-1} shows that $V_u \oplus V_b \neq 0$. By Lemma 3.7, $\mathscr{H}_s$ is a central element of Lat U; hence $\Theta(\mathscr{H}_s) = \mathscr{N}$ is central in Lat V. Using Lemma 3.7 again as well as Lemma 3.9, it follows that $V|\mathscr{N}$ is unitary and therefore $\mathscr{N} \leqslant \mathscr{K}_s$. A similar argument shows that $\Theta^{-1}(\mathscr{K}_s) \leqslant \mathscr{H}_s$. Therefore $\Theta(\mathscr{H}_s) = \mathscr{K}_s$ and Lat $U_s \approx$ Lat V_s. By Lemma 3.9, $V_s \cong \varphi_s(U_s)$ for some U_s-V_s point isomorphism φ_s. This proves part (d′) (iii).

By Lemma 3.12, $V|\Theta(\mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s)$ is unitary. But then Proposition 1.3 implies that $\Theta(\mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s) \subseteq \bigcap_1^\infty \operatorname{ran} V^n = \mathscr{K}_b \oplus \mathscr{K}_a \oplus \mathscr{K}_s$. Similarly, $\Theta^{-1}(\mathscr{K}_b \oplus \mathscr{K}_a \oplus \mathscr{K}_s) \subseteq \mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s$. Thus

$$\Theta(\mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s) = \mathscr{K}_b \oplus \mathscr{K}_a \oplus \mathscr{K}_s \tag{3.15}$$

and part (c) follows.

From Proposition 3.7 it is seen that the central element of Lat U complementary to $\mathscr{H}_s$ is $\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a$. Likewise, $\mathscr{K}_u \oplus \mathscr{K}_b \oplus \mathscr{K}_a$ is complementary to $\mathscr{K}_s$ in Lat V. Since $\Theta(\mathscr{H}_s) = \mathscr{K}_s$, it follows that

$$\Theta(\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a) = \mathscr{K}_u \oplus \mathscr{K}_b \oplus \mathscr{K}_a. \tag{3.16}$$

Note that $\mathscr{H}_b \oplus \mathscr{H}_a = (\mathscr{H}_u \oplus \mathscr{H}_b \oplus \mathscr{H}_a) \cap (\mathscr{H}_b \oplus \mathscr{H}_a \oplus \mathscr{H}_s)$ and that a similar result holds in $\mathscr{K}$. Since Θ is a lattice isomorphism, one can deduce from (3.15) and (3.16) that

$$\Theta(\mathscr{H}_b \oplus \mathscr{H}_a) = \mathscr{K}_b \oplus \mathscr{K}_a. \tag{3.17}$$

This proves part (d′) (i).

Now consider $\mathscr{H}_u$ and let m be the multiplicity of U_u. Put $\mathscr{N} = \Theta(\mathscr{H}_u)$. By Lemma 3.14, $V|\mathscr{N}$ is a unilateral shift of multiplicity m. Let Q be the projection of $\mathscr{K}$ onto $\mathscr{K}_u$ and define $X: \mathscr{N} \to \mathscr{K}_u$ by $X = Q|\mathscr{N}$. It is easy to check that X is injective with dense range and, since $QV = VQ$, that $X(V|\mathscr{N}) = V_u X$. By Proposition 1.4, $V|\mathscr{N} \cong V_u$. This proves part (a).

Put $U_0 = U_b \oplus U_a$ and $V_0 = V_b \oplus V_a$ and note that, in light of (3.17), Θ induces an isomorphism of Lat U_0 onto Lat V_0; this induced isomorphism will also be denoted by Θ. Let n and k be the multiplicities of U_b and V_b, respectively, and let $\mathscr{M} = \Theta(\mathscr{H}_b)$. Since $\mathscr{H}_b$ reduces U_0, $\mathscr{M}$ reduces V_0. Also $V_0|\mathscr{M}$ is a bilateral shift of multiplicity n. It follows that $n \leqslant k$. By interchanging U_0 and V_0 in the preceding argument, it follows that $n = k$, giving part (b).

If $n = k = \infty$, then $U_a = 0 = V_a$ and the proof is complete. Assume that $0 \leqslant n = k < \infty$. (The case $n = k = 0$ corresponds to the absence of the summands U_b and V_b.) But $V_0 = V_b \oplus V_a \cong V_0|\mathscr{M} \oplus V_0|\mathscr{M}^\perp$ and both $V_0|\mathscr{M}$ and V_b are bilateral shifts with the same finite multiplicity n. Hence, because $n < \infty$, a repeated application of Proposition 10.6 on page 302 of [3] implies that $V_0|\mathscr{M}^\perp \cong V_a$.

Now $\mathcal{M} \vee \Theta(\mathcal{H}_a) = \Theta(\mathcal{H}_b) \vee \Theta(\mathcal{H}_a) = \mathcal{K}_b \oplus \mathcal{K}_a$ and $\mathcal{M} \wedge \Theta(\mathcal{H}_a) = (0)$. Moreover $\Theta(\mathcal{H}_a)$ reduces V_0. Thus an application of Proposition 2.4 of [4] yields that $V_0 | \Theta(\mathcal{H}_a) \cong V_0 | \mathcal{M}^{\perp}$. Hence $V_a \cong V_0 | \Theta(\mathcal{H}_a)$. Since $\mathrm{Lat}(V_0 | \Theta(\mathcal{H}_a)) \approx \approx \mathrm{Lat}(U_0 | \mathcal{H}_a)$, $\mathrm{Lat}\, U_a \approx \mathrm{Lat}\, V_a$. But U_a and V_a are reductive unitaries and so Theorem 3.2 of [4] implies part (d') (ii), completing the proof. ▨

4. APPENDIX: THE INVARIANT SUBSPACES OF A CERTAIN ABSOLUTELY CONTINUOUS UNITARY OPERATOR

Fix a Borel subset E of the unit circle and assume that $m(E) > 0$ and $m(\partial\mathbf{D} \setminus E) > > 0$. (As before m denotes normalized Lebesgue measure on $\partial\mathbf{D}$.) Let U_0 be the operator defined by multiplication by z on $L^2(E)$ and let W be the bilateral shift on $L^2 \equiv L^2(\partial\mathbf{D})$. Put $U = W \oplus U_0$. In this appendix the characterization of $\mathrm{Lat}\, U$ will be given (without proof).

There is a part of multiplicity theory that is useful in studying this unitary operator U. In particular, the commutant of U, $\{U\}'$, can be represented as the set of all 2×2 matrices (φ_{ij}) such that $\varphi_{11} \in L^\infty(\partial\mathbf{D})$ and $\varphi_{ij} \in L^\infty(E)$ when i and j are not both 1. Note that the reducing subspaces of U correspond to the projections in $\{U\}'$. So to characterize $\mathrm{Lat}\, U$ it suffices to characterize the non-reducing subspaces belonging to $\mathrm{Lat}\, U$.

Say that a partial isometry on $L^2 \oplus L^2(E)$ is *special* if it belongs to $\{U\}'$ and its initial space contains $L^2 \oplus (0)$. Say that a partial isometry X is *analytic* if X is special and $X(H^2 \oplus L^2(E)) \subseteq H^2 \oplus L^2(E)$.

4.1. THEOREM. *A subspace $\mathcal{M}$ of $L^2 \oplus L^2(E)$ is invariant for U but not reducing if and only if there is a special partial isometry X such that $\mathcal{M} = X(H^2 \oplus \oplus L^2(E))$. If X and Y are special partial isometries, then $X(H^2 \oplus L^2(E)) \leqslant Y(H^2 \oplus \oplus L^2(E))$ if and only if there is an analytic partial isometry Z such that $X = YZ$.*

To complete this circle of ideas, it should be mentioned that, given a non-reducing subspace $\mathcal{M}$ in $\mathrm{Lat}\, U$, there is a special partial isometry X such that $\mathcal{M} = = X(H^2 \oplus L^2(E))$ with X of the form

$$X = \begin{bmatrix} f_0 & \bar{g}_0\chi_\sigma \\ g_0 & -\bar{f}_0\chi_\sigma + \chi_\tau \end{bmatrix}$$

where $\sigma, \tau \subseteq E$ with $\sigma \cap \tau = \emptyset$; $|f_0|^2 + |g_0|^2 = 1$ on $\partial\mathbf{D}$; $g_0 = 0$ on $\tau \cup (\partial\mathbf{D} \setminus E)$; and $g_0 \neq 0$ pointwise a.e. on σ. Also a partial isometry Z is analytic if and only if it has the form

$$Z = \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$

where $\varphi_{11} \in H^\infty$, $|\varphi_{11}|^2 + |\varphi_{21}|^2 = 1$ on $\partial\mathbf{D}$, φ_{21} vanishes off E, and $|\varphi_{22}| = \chi_\rho$ for some Borel subset ρ of E.

Using these ideas, it can be shown that if X and Y are special partial isometries, then $X(H^2 \oplus L^2(E)) = Y(H^2 \oplus L^2(E))$ if and only if $Y = XZ$, where Z has the form

$$Z = \begin{bmatrix} \alpha & 0 \\ 0 & \varphi \end{bmatrix}$$

with α in $\mathbf{C}$, $|\alpha| = 1$, and $|\varphi| = \chi_\rho$, where ρ is a Borel subset of E for which $X^*X = 1 \oplus \chi_\rho$.

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