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ON THE RANGE OF A CLOSED OPERATOR

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1. INTRODUCTION

Let X be a Banach space and let A be a closed operator with domain $\mathrm{D}(A)$ and range $\mathrm{R}(A)$ in X. Assume that 0 is a limit point of the resolvent set $\rho(A)$ of A, and $\|\lambda(\lambda I - A)^{-1}\| = \mathrm{O}(1)$ ($\lambda \to 0$, $\lambda \in \rho(A)$). It is easily seen that if

(1) $y \in \mathrm{R}(A)$,

then y satisfies the condition:

(2) $\|(\lambda I - A)^{-1}y\| = \mathrm{O}(1)$ ($\lambda \to 0$).

It follows from the mean ergodic theorem for pseudo-resolvents [6, p. 217] (or from direct computation) that if y satisfies (2), then $y \in \overline{\mathrm{R}(A)}$. In general, such y does not necessarily belong to $\mathrm{R}(A)$. For instance, let A be the multiplication by the function $m(t) = it$ on the space $X \equiv \{f \in C[0,1]\,;\, f(0) = 0\}$. Then $\|\lambda(\lambda I - A)^{-1}\| = \left\|\frac{\lambda}{\lambda - \mathrm{i}t}\right\|_\infty = \sup\limits_{0 \leqslant t \leqslant 1} |\lambda(\lambda^2 + t^2)^{-1/2}| \leqslant 1$ for $\lambda \in \mathbf{R}\setminus\{0\}$. The function $y(t) = t$ is not contained in $\mathrm{R}(A)$, but $\|(\lambda I - A)^{-1}y\| = \left\|\frac{t}{\lambda - \mathrm{i}t}\right\|_\infty \leqslant 1$ for $\lambda \in \mathbf{R}\setminus\{0\}$.

The purpose of this paper is to discuss possible situations under which the two conditions (1) and (2) are equivalent. Lin and Sine [3] have proved that if T is a contraction on L_1, or if it is a dual operator, then $y \in \mathrm{R}(T - I)$ is equivalent to that $\sup\limits_{n \geqslant 1}\left\|\sum\limits_{j=0}^{n-1} T^j y\right\| < \infty$. In [2], similar results are obtained for generators of (C_0)-semigroups. Motivated by ideas from [3], we prove in Section 2 some general theorems, which, when applied to generators of discrete semigroups and (C_0)-semigroups, extend some theorems in [2] and [3]. As a third application, results for generators of cosine operator functions are also obtained.

2. THE MAIN RESULTS

THEOREM 1. *Assume that* $0 \in \overline{\rho(A)}$ *and* $\|\lambda(\lambda I - A)^{-1}\| = O(1)$ $(\lambda \to 0)$. *If* $(\lambda I - A)^{-1}$ *is weakly compact for some (and hence all)* λ *in* $\rho(A)$, *then* (1) *and* (2) *are equivalent.*

Proof. We only have to prove that (2) implies (1). Let $\lambda_0 \in \rho(A)$. The weak compactness of $(\lambda_0 I - A)^{-1}$ and (2) imply the existence of a sequence $\lambda_n \to 0$ such that $-(\lambda_n I - A)^{-1}\lambda_0(\lambda_0 I - A)^{-1}y$ converges weakly to some z. Hence $-A(\lambda_n I - A)^{-1}\lambda_0(\lambda_0 I - A)^{-1}y = \lambda_0(\lambda_0 I - A)^{-1}y - \lambda_n(\lambda_n I - A)^{-1}\lambda_0(\lambda_0 I - A)^{-1}y$ converges to $\lambda_0(\lambda_0 I - A)^{-1}y$. Then we have $Az = \lambda_0(\lambda_0 I - A)^{-1}y$, by the closedness of A. Let $x = z - (\lambda_0 I - A)^{-1}y$. Then $Ax = Az - A(\lambda_0 I - A)^{-1}y = \lambda_0(\lambda_0 I - A)^{-1}y - A(\lambda_0 I - A)^{-1}y = y$. ▨

Since a bounded operator on a reflexive space is weakly compact, the equivalence of (1) and (2) holds whenever X is reflexive. This is the case in particular for the Lebesgue space $L_p(S, \Sigma, \mu)$, $1 < p < \infty$, where μ is a σ-finite measure. For the case $p = 1$, we need the extra assumption that $\lambda(\lambda I - A)^{-1}$ is a contraction.

THEOREM 2. *If* $X = L_1(S, \Sigma, \mu)$ *and if* $\|\lambda(\lambda I - A)^{-1}\| \leqslant 1$ *for all small* $\lambda > 0$, *then* (1) *and* (2) *are equivalent.*

Proof. Take a small $\lambda > 0$ such that $\|\lambda(\lambda I - A)^{-1}\| \leqslant 1$ and such that $\|(\alpha I - A)^{-1}y\| \leqslant M$ for all $0 < \alpha \leqslant \lambda$. Let $\operatorname{LIM}_{\alpha \to 0}$ be a Banach limit on the space of bounded functions on $(0, \lambda/2]$. Then we can define a linear functional p on $X^* = L_\infty$ by $p(x^*) = -\operatorname{LIM}_{\alpha\to 0}\langle(\alpha I - A)^{-1}y, x^*\rangle$. p belongs to $X^{**} = L_\infty^* = \text{ba}(S, \Sigma, \mu)$, the space of bounded finitely additive measures (= charges) $\ll \mu$, and $\|p\| \leqslant M$. We have for $x^* \in X^*$

$$[(\lambda(\lambda I - A)^{-1})^{**}p](x^*) = -\operatorname{LIM}_{\alpha\to 0}\langle(\alpha I - A)^{-1}y, (\lambda(\lambda I - A)^{-1})^* x^*\rangle =$$

$$= -\operatorname{LIM}_{\alpha\to 0}\langle\lambda(\lambda I - A)^{-1}(\alpha I - A)^{-1}y, x^*\rangle =$$

$$= -\operatorname{LIM}_{\alpha\to 0}\langle\lambda(\lambda - \alpha)^{-1}[(\alpha I - A)^{-1} - (\lambda I - A)^{-1}]y, x^*\rangle =$$

$$= -\operatorname{LIM}_{\alpha\to 0}\langle(\alpha I - A)^{-1}y, x^*\rangle - \lim_{\alpha\to 0}\langle\alpha(\lambda - \alpha)^{-1}(\alpha I - A)^{-1}y, x^*\rangle +$$

$$+ \lim_{\alpha\to 0}\lambda(\lambda - \alpha)^{-1}\langle(\lambda I - A)^{-1}y, x^*\rangle = p(x^*) + \langle(\lambda I - A)^{-1}y, x^*\rangle.$$

Hence $(\lambda(\lambda I - A)^{-1})^{**}p = p + (\lambda I - A)^{-1}y$.

L_1 can be identified, via the Radon-Nikodym theorem, with $M(S, \Sigma, \mu)$, the subspace of $\mathrm{ba}(S, \Sigma, \mu)$ which consists of all countably additive measures $\ll \mu$. Decompose $p = p_1 + p_2$ with $p_1 \in M(S, \Sigma, \mu)$ and p_2 a pure charge (cf. [7]). Using the contraction assumption and the fact that the norm of an element of $\mathrm{ba}(S, \Sigma, \mu)$ is the sum of the norms of its parts we obtain the estimate:

$$\|p_2\| \geqslant \|(\lambda(\lambda I - A)^{-1})^{**}p_2\| = \|-\lambda(\lambda I - A)^{-1}p_1 + p_1 + (\lambda I - A)^{-1}y + p_2\| =$$

$$= \|-\lambda(\lambda I - A)^{-1}p_1 + p_1 + (\lambda I - A)^{-1}y\| + \|p_2\|,$$

which gives that $p_1 = \lambda(\lambda I - A)^{-1}p_1 - (\lambda I - A)^{-1}y \in \mathrm{D}(A)$ and $Ap_1 = y$. ▨

It has been known that in some respects the local weak*-compactness of the dual space X^* makes the dual operator A^* nicer than A. The following theorem is an example.

THEOREM 3. *Let A be a densely defined closed operator such that $0 \in \overline{\rho(A)}$ and $\|\lambda(\lambda I - A)^{-1}\| = \mathrm{O}(1)$ $(\lambda \to 0)$. The two conditions are equivalent:*

(1*) $y^* \in R(A^*)$;

(2*) $\|(\lambda I^* - A^*)^{-1}y^*\| = \mathrm{O}(1)$ $(\lambda \to 0)$.

Proof. If (2*) holds, then there exist $x^* \in X^*$ and $\lambda_n \to 0$ such that $-(\lambda_n I^* - A^*)^{-1}y^* \to x^*$ weakly*. For $x \in D(A)$

$$\langle Ax, x^*\rangle = \lim_{n\to\infty}\langle Ax, -(\lambda_n I^* - A^*)^{-1}y^*\rangle = \lim_{n\to\infty}\langle -(\lambda_n I - A)^{-1}Ax, y^*\rangle =$$

$$= \lim_{n\to\infty}\langle x - \lambda_n(\lambda_n I - A)^{-1}x, y^*\rangle = \langle x, y^*\rangle - \lim_{n\to\infty}\langle x, \lambda_n(\lambda_n I^* - A^*)^{-1}y^*\rangle =$$

$$= \langle x, y^*\rangle.$$

Hence $x^* \in \mathrm{D}(A^*)$ and $A^*x^* = y^*$. ▨

3. GENERATORS OF SEMIGROUPS AND COSINE FUNCTIONS

In this section we apply the above theorems to special operators, namely the generators of discrete semigroups, continuous semigroups, and cosine operator functions.

Let T be a power bounded operator, i.e. $\|T^n\| \leqslant M$, $n = 0, 1, 2, \ldots$. Set $A = T - I$. Then $0 \in \overline{\rho(A)}$ and $\sup_{\lambda>0}\|\lambda(\lambda I - A)^{-1}\| = \sup_{\lambda>1}\|(\lambda - 1)(\lambda I - T)^{-1}\| \leqslant M$.

Let $S_n = \sum_{j=0}^{n-1} T^j$. If $y = Ax = (T - I)x$, then $\sup_{n \geq 1} \|S_n y\| = \sup_{n \geq 1} \|T^n x - x\| \leq$ $\leq (M + 1)\|x\| < \infty$. Next, suppose that $K = \sup_{n \geq 1} \|S_n y\| < \infty$. Then

$$\|(\lambda I - T)^{-1} y\| = \left\| \sum_{n=0}^{\infty} \lambda^{-n-1} T^n y \right\| = \left\| \sum_{n=1}^{\infty} (\lambda^{-n} - \lambda^{-n-1}) S_n y \right\| \leq$$

$$\leq K \sum_{n=1}^{\infty} (\lambda^{-n} - \lambda^{-n-1}) = \frac{K}{\lambda} \leq K$$

for all $\lambda > 1$. Combining these facts with Theorems 2 and 3 we obtain the following two results of Lin and Sine [3, Theorems 7 and 5].

COROLLARY 4. *Let T be a contraction on $L_1(S, \Sigma, \mu)$. Then $f \in L_1$ is of the form $f = (T - I)g$ with $g \in L_1$ if and only if* $\sup_{n \geq 1} \|S_n f\|_1 < \infty$.

COROLLARY 5. *Let T be a power bounded operator on a Banach space X. Then $y^* = T^* x^* - x^*$ for some $x^* \in X^*$ if and only if* $\sup_{n \geq 1} \|S_n^* y^*\| < \infty$.

Next, we consider the case that A is the infinitesimal generator of a (C_0)-semigroup $\{T(t) ; t \geq 0\}$ of operators. Assume that $\|T(t)\| \leq M$ for all $t \geq 0$. Then A is a densely defined closed operator such that $\lambda \in \rho(A)$ and $\|\lambda(\lambda I - A)^{-1}\| \leq M$ for all $\lambda > 0$, by the Hille-Yosida theorem. It is known that if $T(t)$ is compact for all $t > 0$, then $(\lambda I - A)^{-1}$ is compact for all $\lambda > 0$ (cf. [1, p. 189]). Hence Theorems 1 and 2 justify the equivalence of (1) and (2) in the next corollary.

COROLLARY 6. *Let A be the generator of a (C_0)-semigroup $T(\cdot)$ with $\|T(t)\| \leq M$, $t \geq 0$. If $T(t)$ is compact for all $t > 0$, or if $X = L_1(S, \Sigma, \mu)$ and $M = 1$, then each of the conditions* (1), (2) *is equivalent to*

(3) $\sup_{t > 0} \left\| \int_0^t T(s) y \, ds \right\| < \infty.$

Proof. It remains to show "(1) ⇒ (3)" and "(3) ⇒ (2)". If $y = Ax$, then

$$\left\| \int_0^t T(s) y \, ds \right\| = \left\| \int_0^t T(s) A x \, ds \right\| = \|T(t)x - x\| \leq (M + 1)\|x\|$$

and hence (3) holds. If (3) holds with $K = \sup_{t>0} \left\| \int_0^t T(s)y\mathrm{d}s \right\|$, then

$$\|(\lambda I - A)^{-1}y\| = \left\| \int_0^\infty e^{-\lambda t}T(t)y\mathrm{d}t \right\| = \left\| \lambda \int_0^\infty e^{-\lambda t} \left(\int_0^t T(s)y\mathrm{d}s \right) \mathrm{d}t \right\| \leqslant$$

$$\leqslant K\lambda \int_0^\infty e^{-\lambda t}\mathrm{d}t = K$$

for all $\lambda > 0$. Hence (2) holds. ▨

In a similar way we easily derive the next corollary from Theorem 3.

COROLLARY 7. *If A is the generator of a uniformly bounded (C_0)-semigroup $T(\cdot)$, then each of the conditions* (1*), (2*) *is equivalent to*

(3*) $\sup_{t>0} \left\| \int_0^t T^*(s)y^*\mathrm{d}s \right\| < \infty,$

where the integral is in the sense of W*-*Riemann integration.*

In the rest we give an application to the generator A of a strongly continuous cosine operator function $\{C(t)\ ;\ t \in \mathbf{R}\}$. By definition $C(\cdot)$ is continuous in the strong operator topology and satisfies: $C(0) = I$, $C(s+t) + C(s-t) = 2C(s)C(t)$, $s, t \in \mathbf{R}$, and $A := C''(0)$. Assume that $\|C(t)\| \leqslant M$ for all $t \in \mathbf{R}$. It is known from the generation theorem (cf. [4]) that A is a densely defined closed operator such that $\lambda \in \rho(A)$ and $\lambda(\lambda^2 I - A)^{-1} = \int_0^\infty e^{-\lambda t}C(t)x\mathrm{d}t$ for all $\lambda > 0$. Clearly, $\|\lambda(\lambda I - A)^{-1}\| \leqslant M$ for all $\lambda > 0$.

The associated sine function $S(\cdot)$ is defined by $S(t)x := \int_0^t C(s)x\mathrm{d}s\ (x \in X)$. It is uniformly continuous. $S(t)$ is compact on an interval of positive length if and only if $S(t)$ is compact for every $t \in \mathbf{R}$, and also iff $(\lambda I - A)^{-1}$ is compact for some (and hence every) λ in $\rho(A)$ (see [5]). Now, we can apply Theorems 1 and 2 to justify the equivalence of (1) and (2) in the following corollary.

COROLLARY 8. *Let A be the generator of a strongly continuous cosine operator function $C(\cdot)$, with $\|C(t)\| \leqslant M$, $t \in \mathbf{R}$. If $S(t)$ is compact for every $t \in \mathbf{R}$, or if*

$X = L_1(S, \Sigma, \mu)$ *and* $M = 1$, *then each of conditions* (1), (2) *is equivalent to*

$$(4)\ \sup_{t>0} \left\| \int_0^t S(s)y\,ds \right\| < \infty.$$

Proof. It remains to show "(1) ⇒ (4)" and "(4) ⇒ (2)". If $y = Ax$, then $\left\| \int_0^t S(s)y\,ds \right\| = \left\| \int_0^t S(s)Ax\,ds \right\| = \|C(t)x - x\| \leqslant (M + 1)\|x\|$ and hence (4) holds. Finally, we use integration by parts to write

$$(\lambda^2 I - A)^{-1}y = \lambda^{-1} \int_0^\infty e^{-\lambda t}C(t)y\,dt = \int_0^\infty e^{-\lambda t}S(t)y\,dt = \lambda \int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)y\,ds \right) dt,$$

from which it is easy to see that (4) implies (2). ▨

COROLLARY 9. *If A is the generator of a uniformly bounded, strongly continuous cosine operator function $C(\cdot)$, then each of the conditions* (1^*), (2^*) *is equivalent to*

$$(4^*)\ \sup_{t>0} \left\| \int_0^t S^*(s)y^*\,ds \right\| < \infty.$$

Proof. In view of Theorem 3, we only have to show "(1^*) ⇒ (4^*)" and "(4^*) ⇒ ⇒ (2^*)". If $y^* = A^*x^*$, then for all $x \in X$

$$\left| \left\langle x, \int_0^t S^*(s)y^*\,ds \right\rangle \right| = \left| \left\langle \int_0^t S(s)x\,ds,\ y^* \right\rangle \right| = \left| \left\langle A \int_0^t S(s)x\,ds,\ x^* \right\rangle \right| =$$

$$= |\langle C(t)x - x,\ x^* \rangle| \leqslant (M + 1)\|x\|\ \|x^*\|,$$

and hence $\left\| \int_0^t S^*(s)y^*\,ds \right\| \leqslant (M + 1)\|x^*\|$ for all $t > 0$.

Finally, since for all $x \in X$

$$\langle x, (\lambda^2 I^* - A^*)^{-1}y^* \rangle = \langle (\lambda^2 I - A)^{-1}x,\ y^* \rangle = \left\langle \lambda \int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)x\,ds \right) dt,\ y^* \right\rangle =$$

$$= \left\langle x,\ \lambda \int_0^\infty e^{-\lambda t}\left(\int_0^t S^*(s)\,y^*\,ds \right) dt \right\rangle,$$

it follows that for all $\lambda > 0$

$$\|(\lambda^2 I^* - A^*)^{-1} y^*\| = \left\| \lambda \int_0^\infty e^{-\lambda t} \left(\int_0^t S^*(s) y^* ds \right) dt \right\| \leqslant$$

$$\leqslant \sup_{t>0} \left\| \int_0^t S^*(s) y^* ds \right\| .$$

That is, (4*) implies (2*). ▨

CONCLUDING REMARK. Krengel and Lin [2] have given direct proofs for the equivalence of (1) and (3) in Corollary 6, and the equivalence of (1*) and (3*) in Corollary 7. By similar arguments, one can also work out direct proofs for the equivalence of (1) and (4) in Corollary 8, and the equivalence of (1*) and (4*) in Corollary 9. An advantage of Theorems 1, 2, and 3 is that they serve to provide a unified treatment for all these similar results (Corollaries 4—9).

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