J. OPERATOR THEORY
22(1989), 209–232

UNIFORMLY CONTINUOUS SEMIGROUPS WITH BOUNDED CHARACTERISTIC FUNCTIONS

BRIAN W. McENNIS

1. INTRODUCTION

Let $T(t)$ be a uniformly continuous one-parameter semigroup of operators on a separable Hilbert space $\mathcal{H}$. Thus, for each $t \geqslant 0$, $T(t)$ is a bounded operator on $\mathcal{H}$, $T(t_1)T(t_2) = T(t_1 + t_2)$ for each $t_1, t_2 \geqslant 0$, $T(0) = I$, and $\|T(t) - I\| \to 0$ as $t \to 0^+$. Such a semigroup possesses a bounded infinitesimal generator A, defined as the limit (in norm) of $t^{-1}(T(t) - I)$, as $t \to 0^+$. We can then write $T(t) = \exp(At)$ (See, for example, [2], [4], [6], [7], [13], [14], [15].)

As in [2], we define the following bounded operators on $\mathcal{H}$: $G = A + A^*$, $Q = |G|^{1/2}$, and $J = \operatorname{sgn}(-G)$ (this is the operator S in [2]). We have the relations

$$(1.1) \qquad JQ^2 = -G,$$

$$(1.2) \qquad \frac{d}{dt}(T(t)T(t)^*) = T(t)GT(t)^*,$$

and

$$(1.3) \qquad \frac{d}{dt}(T(t)^*T(t)) = T(t)^*GT(t).$$

A Krein space $\mathcal{G}$ is defined by taking $\mathcal{G}$ to be the space $J\mathcal{H}$, equipped with the indefinite inner product

$$(1.4) \qquad [x, y] = (Jx, y) \quad x, y \in \mathcal{G},$$

where $(\cdot,\cdot)$ denotes the inner product on $\mathcal{H}$. The topology on $\mathcal{G}$ is that which it inherits as a subspace of $\mathcal{H}$. We also define the characteristic function $\Theta(\lambda): \mathcal{G} \to \mathcal{G}$ of the semigroup $T(t)$ by

$$(1.5) \qquad \Theta(\lambda) = I - Q(\lambda - A^*)^{-1}JQ$$

for all complex numbers λ for which $(\lambda - A^*)^{-1}$ is bounded. (Compare this with the characteristic function Θ_A given in [13], p. 358 for a dissipative operator.) We will be assuming that the characteristic function (1.5) is defined and bounded in the right half-plane:

$$\sup\{\|\Theta(\lambda)\| : \operatorname{Re}\lambda > 0\} = C < \infty \tag{1.6}$$

and will prove the following theorem, analogous to [3]:

THEOREM 1.1. *Suppose $T(t)$ is a uniformly continuous semigroup with bounded characteristic function. Then $T(t)$ is similar to a contraction semigroup.*

As in the case of a single operator [3], boundedness of the characteristic function is a sufficient, but not a necessary condition for $T(t)$ to be similar to a contraction semigroup:

EXAMPLE 1.2. Let $\mathscr{H}$ be a two-dimensional Hilbert space, and define $T(t)$ on $\mathscr{H}$ by

$$T(t) = \begin{pmatrix} e^{-t} & e^{-t} - 1 \\ 0 & 1 \end{pmatrix}$$

Then it can be readily checked that $T(t)$ is a semigroup, and that for the operator

$$S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

we have

$$S^{-1}T(t)S = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $T(t)$ is similar to a contraction semigroup. However, $T(t)$ has infinitesimal generator

$$A = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

for which G is invertible and for which $\|(\lambda - A^*)^{-1}\|$ is unbounded as $\lambda \to 0$. Consequently, $\Theta(\lambda)$ is unbounded. □

As in [13], [3], and [8], the characteristic function is studied in the context of a unitary dilation, in this case the dilation constructed by Davis [2]. Since this dilation is given only for uniformly continuous semigroups, we restrict ourselves here to this case. No dilation theory currently exists for arbitrary strongly continuous semigroups, although in recent work by the author [11] a dilation is constructed for a class of holomorphic semigroups. The unboundedness of the infinitesimal generator also creates difficulties in the definition of the characteristic function of an arbitrary strongly continuous semigroup; these problems are avoided here by assuming that the infinitesimal generator is bounded.

The proof of Theorem 1.1 will be developed over the remaining sections, in which the characteristic function is represented as a projection on the dilation space.

Acknowledgement. Much of the research of this paper was done while the author was visiting at the Centre for Mathematical Analysis, at The Australian National University, during 1987. The support of the Centre and of the Ohio State University, for this visit, is gratefully acknowledged.

Some of the results of this paper also appeared in [10].

2. SOME PRELIMINARY RESULTS

Throughout this paper, the adjoint of an operator on a Krein space will always be used in the sense of the indefinite inner product. In particular, we will be making repeated use of the following property of the adjoint on the Krein space $\mathscr{G}$:

$$(QSJQ)^* = QS^*JQ \tag{2.1}$$

for any operator S on $\mathscr{H}$. (See [1], Lemma VI.2.1. Note that the adjoint on the left side of (2.1) is taken in the indefinite inner product of $\mathscr{G}$, whereas the adjoint of S (a Hilbert space operator) on the right side is taken in the Hilbert space inner product of $\mathscr{H}$.)

We present here some elementary properties of the characteristic function which are analogues of similar properties of the characteristic function of a single operator. (In Theorem 2.1 below, the adjoints $\Theta(\mu)^*$ and $\Theta(\lambda)^*$ are, of course, taken with respect to the indefinite inner product of $\mathscr{G}$.)

THEOREM 2.1. *Suppose λ and μ are complex numbers for which $\Theta(\lambda)$ and $\Theta(\mu)$ are defined. Then we have*

$$I - \Theta(\mu)^*\Theta(\lambda) = (\lambda + \bar{\mu})Q(\bar{\mu} - A)^{-1}(\lambda - A^*)^{-1}JQ \tag{2.2}$$

and

$$I - \Theta(\lambda)\Theta(\mu)^* = (\lambda + \bar{\mu})Q(\lambda - A^*)^{-1}(\bar{\mu} - A)^{-1}JQ. \tag{2.3}$$

The characteristic function is purely contractive, i.e. *for every nonzero $a \in \mathscr{G}$, and for all λ with* $\operatorname{Re}\lambda > 0$, *we have*

$$[a, a] - [\Theta(\lambda)a, \Theta(\lambda)a] > 0 \tag{2.4}$$

and

(2.5) $$[a, a] - [\Theta(\lambda)^*a, \Theta(\lambda)^*a] > 0.$$

Proof. From the definition (1.5) we get, using properties (1.1) and (2.1)

$$\Theta(\mu)^*\Theta(\lambda) = I - Q(\lambda - A^*)^{-1}JQ - Q(\bar{\mu} - A)^{-1}JQ - Q(\bar{\mu} - A)^{-1}G(\lambda - A^*)^{-1}JQ.$$

Thus,

$$I - \Theta(\mu)^*\Theta(\lambda) = Q(\bar{\mu} - A)^{-1}[(\bar{\mu} - A) + (\lambda - A^*) + G](\lambda - A^*)^{-1}JQ,$$

proving (2.2), since $G = A + A^*$. (2.3) follows similarly. By putting $\mu = \lambda$ in (2.2) we get

$$[a, a] - [\Theta(\lambda)a, \Theta(\lambda)a] = [(I - \Theta(\lambda)^*\Theta(\lambda))a, a] =$$

$$= 2(\operatorname{Re}\lambda)\|(\lambda - A^*)^{-1}JQa\|^2 \geqslant 0.$$

▨

The strict inequality in (2.4) follows from the observations that $\operatorname{Re}\lambda > 0$ and that JQ is injective on $\mathscr{G}$. (2.5) is proved similarly.

COROLLARY 2.2. *If $T(t)$ is a uniformly continuous contraction semigroup, then its characteristic function is bounded by one.*

Proof. If $T(t)$ is contractive, then A is dissipative, i.e. $G = A + A^* \leqslant 0$. (See, for example, [13], p. 141.) Therefore $J = I$, and the indefinite inner product (1.4) on $\mathscr{G}$ is the same as the Hilbert space inner product. Thus, (2.4) implies $\|\Theta(\lambda)\|^2 \leqslant 1$. ▨

The following example shows that there are semigroups with bounded characteristic function which are not contraction semigroups.

EXAMPLE 2.3. Let $\mathscr{H}$ be a two-dimensional Hilbert space, and define $T(t)$ on $\mathscr{H}$ by

$$T(t) = \begin{pmatrix} e^{-at} & te^{-at} \\ 0 & e^{-at} \end{pmatrix}$$

for some $a > 0$. It is easily checked that $T(t)$ is a semigroup with infinitesimal generator given by

$$A = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}.$$

The numerical range of A is a circle of radius 1/2 centered at $-a$ (cf. [5], p. 112), and the spectrum of A consists of the single point $\{-a\}$. Thus, for $0 < a < 1/2$,

A is not dissipative (its numerical range includes points in the right half-plane), but $(\lambda - A^*)^{-1}$ is uniformly bounded for $\operatorname{Re}\lambda > 0$. Consequently, for $0 < a < 1/2$, $T(t)$ is not a contraction semigroup, but does have bounded characteristic function. ▨

In the course of proving Theorem 1.1, we will be needing the following preliminary estimate.

LEMMA 2.4. *If $T(t)$ has bounded characteristic function, then there is a constant M such that, for all $t \geqslant 0$,*

$$\|QT(t)JQ\| \leqslant M \quad \textit{and} \quad \|QT(t)^*JQ\| \leqslant M. \tag{2.6}$$

Proof. Note that $QT(t)^*JQ$ is the adjoint of $QT(t)JQ$ in the indefinite inner product of $\mathscr{G}$ (see (2.1)). Thus it suffices to prove just the first inequality of (2.6).

We can express $T(t)$ in terms of the resolvent of its infinitesimal generator A by means of the integral

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda - A)^{-1}\, d\lambda, \tag{2.7}$$

where Γ is a positively oriented curve enclosing the spectrum of A (see [6], Theorem 11.3.1; [7], p. 489). The adjoint of $\Theta(\bar{\lambda})$, with respect to the indefinite inner product on $\mathscr{G}$ is $\Theta(\bar{\lambda})^* = I - Q(\lambda - A)^{-1}JQ$ and thus, by (1.6), we can conclude that there is a constant K such that

$$\|Q(\lambda - A)^{-1}JQ\| \leqslant K \quad \text{for } \operatorname{Re}\lambda > 0. \tag{2.8}$$

Since $\|(\lambda - A)^{-1}\| \to 0$ as $|\lambda| \to \infty$, there is a constant R such that the spectrum of A is contained inside the circle $|\lambda| = R$ and

$$\|Q(\lambda - A)^{-1}JQ\| \leqslant K \quad \text{for } |\lambda| = R. \tag{2.9}$$

Let ε be an arbitrary positive number, and take Γ to be the positively oriented curve consisting of the line segment $\operatorname{Re}\lambda = \varepsilon$, $|\lambda| \leqslant R$, and the circular arc $|\lambda| = R$, $\operatorname{Re}\lambda \leqslant \varepsilon$. Note that, by (2.8) and by the assumption on R, the spectrum of A is contained inside Γ. Thus, since the length of Γ is less than $2\pi R$, we obtain from (2.7), (2.8), and (2.9) the estimate

$$\|QT(t)JQ\| \leqslant RK e^{\varepsilon t}. \tag{2.10}$$

Since (2.10) is valid for all $t \geqslant 0$ and for all $\varepsilon > 0$, we obtain the result (2.6). ▨

The semigroup $T(t)^*$ has infinitesimal generator A^*, whose resolvent can be given by the formula

$$(2.11)\qquad (\lambda - A^*)^{-1} = \int_0^\infty e^{-\lambda t} T(t)^* \, dt,$$

valid for Re $\lambda > \beta$, where $\beta = \|A\|$. (See [4], Theorem VIII.1.2; [6], Theorem 11.2.1.) Thus for Re $\lambda > \beta$, the characteristic function can be represented by

$$(2.12)\qquad \Theta(\lambda) = I - \int_0^\infty e^{-\lambda t} QT(t)^* JQ \, dt.$$

When the characteristic function is bounded, we can use Lemma 2.4 to conclude that both sides of (2.12) are defined and holomorphic for Re $\lambda > 0$. Thus we get:

THEOREM 2.5. *If the semigroup $T(t)$ has bounded characteristic function, then the representation* (2.12) *of $\Theta(\lambda)$ is valid for all λ with* Re $\lambda > 0$.

3. FOURIER TRANSFORMS

If $\mathscr{G}$ is a Krein space, and if $\mathbf{R}$ denotes the real numbers, then we denote by $L^p(\mathbf{R}, \mathscr{G})$ the Banach space of (equivalence classes of) functions $f: \mathbf{R} \to \mathscr{G}$ which are strongly measurable and for which

$$(3.1)\qquad \|f\|_p = \left(\int_{-\infty}^{\infty} \|f(t)\|^p dt \right)^{1/p} < \infty, \quad p < \infty.$$

For $p = \infty$, (3.1) is replaced by

$$(3.2)\qquad \|f\|_\infty = \operatorname{ess\,sup}\{\|f(t)\| : t \in \mathbf{R}\}.$$

The subspace of $L^p(\mathbf{R}, \mathscr{G})$ consisting of all functions with support contained in the interval $[0, \infty)$ will be denoted by $L^p(\mathbf{R}^+, \mathscr{G})$. Likewise, $L^p(\mathbf{R}^-, \mathscr{G})$ and $L^p([0, s], \mathscr{G})$ will denote the subspaces of functions supported on $(-\infty, 0]$ and $[0, s]$, respectively.

The space $L^2(\mathbf{R}, \mathscr{G})$ is a Krein space with indefinite inner product defined by

$$(3.3)\qquad [f, g] = \int_{-\infty}^{\infty} [f(t), g(t)] dt \quad f, g \in L^2(\mathbf{R}, \mathscr{G}),$$

where $[f(t), g(t)]$ denotes the indefinite inner product of $\mathscr{G}$.

We will be needing a vector-valued Fubini theorem (see [4], Corollary III.11.15):

THEOREM 3.1. *Suppose that f is a strongly measurable function of two variables which satisfies*

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \|f(u, v)\| \,\mathrm{d}u\right)\mathrm{d}v < \infty, \tag{3.4}$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u, v)\mathrm{d}u\,\mathrm{d}v = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u, v)\mathrm{d}v\,\mathrm{d}u. \tag{3.5}$$ ▨

Consider the Fourier transform of a function $f \in L^1(\mathbf{R}, \mathscr{G})$, defined for $y \in \mathbf{R}$ by

$$F(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}yt} f(t)\mathrm{d}t. \tag{3.6}$$

We have a vector-valued Plancherel theorem (see [14], p. 139):

THEOREM 3.2. *Let f be a function in $L^1(\mathbf{R}, \mathscr{G}) \cap L^2(\mathbf{R}, \mathscr{G})$, and let $F(y)$ be the Fourier transform given by* (3.6). *Then $F \in L^2(\mathbf{R}, \mathscr{G})$ and $\|F\|_2 = \|f\|_2$.* ▨

Consequently, the definition (3.6) can be extended by continuity from $L^1(\mathbf{R}, \mathscr{G}) \cap L^2(\mathbf{R}, \mathscr{G})$ to all $f \in L^2(\mathbf{R}, \mathscr{G})$, and as in the scalar case, this extension is given by

$$F(y) = \underset{n\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}yt} f(t)\,\mathrm{d}t \tag{3.7}$$

where l.i.m. denotes limit in mean, i.e., the strong limit in $L^2(\mathbf{R}, \mathscr{G})$.

Now let f be a function in $L^p(\mathbf{R}^+, \mathscr{G})$ $(1 \leqslant p \leqslant \infty)$. Then the function $\mathrm{e}^{-\lambda t}f(t)$ is in $L^1(\mathbf{R}^+, \mathscr{G})$ for all complex numbers λ with $\operatorname{Re}\lambda > 0$, and we can define the holomorphic Fourier transform

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t, \quad \operatorname{Re}\lambda > 0. \tag{3.8}$$

We will be needing the following:

THEOREM 3.3. *Suppose* $f \in L^p(\mathbf{R}^+, \mathscr{G})$, $1 \leqslant p \leqslant \infty$, *and that* $\hat{f}(\lambda) = 0$ *for all complex numbers* λ *with* $\operatorname{Re}\lambda > 0$. *Then* $f = 0$.

Proof. The function $g(t) = e^{-t}f(t)$ is in $L^1(\mathbf{R}^+, \mathscr{G})$ and has Fourier transform given by

$$G(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-iyt} e^{-t} f(t)\, dt = \hat{f}(1 + iy) = 0.$$

For any $a \in \mathscr{G}$, the Fourier transform of $(g(t), a)$ is $(G(y), a)$, which is zero. By the uniqueness theorem for the scalar-valued case (see, e.g. [12], p. 187), $(g(t), a) = 0$ for each $a \in \mathscr{G}$ and for almost all t. Since $\mathscr{G}$ is separable, $g(t) = 0$ a.e., and it follows that $f = 0$. ▨

Let $F(\lambda)$ denote a function taking values in $\mathscr{G}$ and holomorphic in the right half-plane $\operatorname{Re}\lambda > 0$, and for $x > 0$ define F_x by $F_x(y) = F(x + iy)$ $(y \in \mathbf{R})$. The Hardy-Lebesgue space $H^2(0, \mathscr{G})$ (cf. [15], p. 163) is defined as the space of all such functions $F(\lambda)$, with $F_x \in L^2(\mathbf{R}, \mathscr{G})$ for all $x > 0$ and

$$(3.9) \qquad \|F\| = \sup\{\|F_x\|_2 : x > 0\} < \infty.$$

For this space, we have a vector-valued Paley-Wiener theorem:

THEOREM 3.4. *If* $f \in L^2(\mathbf{R}^+, \mathscr{G})$, *then* $\hat{f} \in H^2(0, \mathscr{G})$. *Conversely, if* $F \in H^2(0, \mathscr{G})$, *then there is a function* $f \in L^2(\mathbf{R}^+, \mathscr{G})$ *such that* $F = \hat{f}$.

Proof. Suppose $f \in L^2(\mathbf{R}^+, \mathscr{G})$ and let $F = \hat{f}$. Since $(F(\lambda), a)$ is the holomorphic Fourier transform of the scalar function $(f(t), a)$, where $a \in \mathscr{G}$, it follows (see [15], p. 163) that $(F(\lambda), a)$ is holomorphic, and thus $F(\lambda)$ is holomorphic ([6], p. 92). Also note that $F_x(y)$ is the Fourier transform (3.6) of the function $e^{-xt}f(t)$. Since this is a function that is in $L^1(\mathbf{R}^+, \mathscr{G}) \cap L^2(\mathbf{R}^+, \mathscr{G})$, then by Theorem 3.2, $F_x \in L^2(\mathbf{R}, \mathscr{G})$ and

$$(3.10) \qquad \|F_x\|_2 = \|e^{-xt}f(t)\|_2 \leqslant \|f\|_2,$$

proving that F is in $H^2(0, \mathscr{G})$.

For the converse, assume that $F \in H^2(0, \mathscr{G})$. Then, by hypothesis, the set of functions $\{F_x : x > 0\}$ is a bounded subset of $L^2(\mathbf{R}, \mathscr{G})$. Since $L^2(\mathbf{R}, \mathscr{G})$ is a Hilbert space, it is locally sequentially weakly compact, and thus there is a sequence of positive real numbers $x(n)$, converging to zero, and a function $F_0 \in L^2(\mathbf{R}, \mathscr{G})$ such that $F_{x(n)}$ converges to F_0, weakly in $L^2(\mathbf{R}, \mathscr{G})$.

Define a function f by

$$(3.11) \qquad f(t) = \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{iyt} F_0(y)\, dy.$$

A comparison with (3.7) shows that $f \in L^2(\mathbf{R}, \mathscr{G})$. To complete the proof, we need to show that the support of f is in $[0, \infty)$ and that F is the holomorphic Fourier transform of f. This can be done by appealing to the scalar-valued case for the functions $(f(t), a)$ and $(F(\lambda), a)$, for each $a \in \mathscr{G}$. (See, for example, [15], Theorem VI.4.2.) ▨

THEOREM 3.5. *The holomorphic Fourier transform* (3.8) *is a unitary operator from* $L^2(\mathbf{R}^+, \mathscr{G})$ *onto* $H^2(0, \mathscr{G})$.

Proof. By Theorems 3.3 and 3.4, the holomorphic Fourier transform is one-to-one and onto. It therefore suffices to show that it is isometric.

Let f be in $L^2(\mathbf{R}^+, \mathscr{G})$, and let F be its holomorphic Fourier transform. From (3.9) and (3.10), we conclude that $\|F\| \leqslant \|f\|_2$. Comparing (3.11) and (3.7), we see that $f(-t)$ is the Fourier transform (3.7) of F_0, and thus, by Theorem 3.2, $\|f\|_2 = \|F_0\|_2$. But F_0 was defined as the weak limit of functions that are bounded in $L^2(\mathbf{R}, \mathscr{G})$ by $\|F\|$, and thus $\|F_0\|_2 \leqslant \|F\|$. Consequently, $\|f\|_2 = \|F\|$, and the proof is complete. ▨

The above theorem shows that $H^2(0, \mathscr{G})$ is a Hilbert space. Since $L^2(\mathbf{R}^+, \mathscr{G})$ is also a Krein space, with indefinite inner product given by (3.3), we can make $H^2(0, \mathscr{G})$ into a Krein space too, by defining

$$(3.12) \qquad [\hat{f}, \hat{g}] = [f, g], \quad f, g \in L^2(\mathbf{R}^+, \mathscr{G}).$$

4. THE DILATION SPACE

Let us now return to the study of the semigroup $T(t)$ by introducing its unitary dilation, as constructed by Davis [2].

Define a Krein space $\mathscr{K}$ by $\mathscr{K} = \mathscr{H} \oplus L^2(\mathbf{R}, \mathscr{G})$, where $\mathscr{G}$ is the Krein space introduced in Section 1, with indefinite inner product given by (1.4). A vector k in $\mathscr{K}$ will be denoted by $k = \langle h, f \rangle$, where $h \in \mathscr{H}$ and $f \in L^2(\mathbf{R}, \mathscr{G})$; the indefinite inner product on $\mathscr{K}$ is given by

$$(4.1) \qquad [k, k'] = (h, h') + [f, f'], \quad k = \langle h, f \rangle,\ k' = \langle h', f' \rangle,$$

where (h, h') is the Hilbert space inner product on $\mathscr{H}$ and $[f, f']$ is the indefinite inner product (3.3) on $L^2(\mathbf{R}, \mathscr{G})$. The unitary dilation $U(s)$ is defined for $s \geqslant 0$ by $U(s)\langle h, f\rangle = \langle h', f'\rangle$, where

$$h' = T(s)h - \int_0^s T(s-t)JQf(-t)\,dt, \tag{4.2}$$

and

$$\begin{aligned} f' = f'(\tau) = f(\tau - s) &+ \\ + \chi_{[0,s]}(\tau)\left[QT(s-\tau)h - \int_0^{s-\tau} QT(s-t-\tau)JQf(-t)\,dt\right]. \end{aligned} \tag{4.3}$$

Here, and in the sequel, we have adopted the convention (also used in [2]) of using a special symbol τ to denote the independent variable. Thus, for example $f(\tau)$ represents an element of $L^2(\mathbf{R}, \mathscr{G})$, whereas $f(t)$ represents a vector in $\mathscr{G}$. $f(\tau - s)$ in (4.3) is the function obtained by shifting f to the right by s units. We will also be using λ in the same role when discussing functions in $H^2(0, \mathscr{G})$.

In [2] it is shown that the $U(s)$ defined above is a semigroup on $\mathscr{K}$, and that it is a dilation of $T(s)$ which is unitary in the sense of the indefinite inner product, i.e., $U(s)$ is invertible and $[U(s)k, U(s)k'] = [k, k']$ for every $k, k' \in \mathscr{K}$ and $s \geqslant 0$. By defining $U(-s) = U(s)^*$ (where the adjoint is taken in the indefinite inner product), $U(s)$ becomes a unitary group on $\mathscr{K}$, again in the sense of the indefinite inner product.

We will be considering the subspace $\mathscr{K}_+$ of $\mathscr{K}$ given by

$$\mathscr{K}_+ = \{\langle h, f\rangle \in \mathscr{K} : f \in L^2(\mathbf{R}^+, \mathscr{G})\}, \tag{4.4}$$

and the semigroup

$$U^+(s) = U(s)\,|\,\mathscr{K}_+, \qquad \text{for } s \geqslant 0. \tag{4.5}$$

(Note that $\mathscr{K}_+$ is invariant for $U(s)$, for $s \geqslant 0$.) Then Theorem 1.1 will be proved by establishing:

THEOREM 4.1. *$U^+(s)$ is similar to a semigroup of operators which are isometries with respect to a Hilbert space inner product.*

The proof of this theorem, and Theorem 1.1, will be completed in the remaining sections, after investigating some of the structure of $\mathscr{K}_+$.

We consider two subspaces $\mathcal{M}$ and $\mathcal{M}_*$ of $\mathcal{K}_+$, and their Fourier representations Φ and Φ_*, respectively. The simplest to describe is the space $\mathcal{M}$ and its Fourier representation:

$$\mathcal{M} = \{\langle 0, f\rangle \in \mathcal{K} : f \in L^2(\mathbf{R}^+, \mathcal{G})\} \quad \text{and} \quad \Phi\langle 0, f\rangle = \hat{f}, \tag{4.6}$$

where $\hat{f}$ is the holomorphic Fourier transform (3.8). By Theorem 3.5 and (3.12), Φ is a unitary operator from $\mathcal{M}$ onto $H^2(0, \mathcal{G})$, preserving both the Hilbert space and indefinite inner products. Also, $\Phi(U(s)\langle 0, f\rangle)$ is the holomorphic Fourier transform of $f(\tau - s)$, so that

$$\Phi(U(s)m) = \mathrm{e}^{-\lambda s}\Phi m \quad \text{for all } m \in \mathcal{M} \text{ and } s \geqslant 0.$$

In order to parallel the theory for a single operator, we define $\mathcal{M}_*$ in the following way. Take $s > 0$, and let f be a function in $L^2([0, s], \mathcal{G})$. Shift f to the left by s units, so as to get a function in $L^2(\mathbf{R}^-, \mathcal{G})$, and then apply $U(s)$. This gives a vector in the subspace $\mathcal{K}_+$ of $\mathcal{K}$; we define $\mathcal{M}_*$ to be the closed linear span of such vectors, i.e.,

$$\mathcal{M}_* = \bigvee\{U(s)\langle 0, f(\tau + s)\rangle : f \in L^2([0, s], \mathcal{G}),\ s > 0\}. \tag{4.7}$$

The Fourier representation Φ_* of $\mathcal{M}_*$ is densely defined on $\mathcal{M}_*$ by

$$\Phi_*[U(s)\langle 0, f(\tau + s)\rangle] = \hat{f}, \quad s > 0. \tag{4.8}$$

(It is easy to check that this is well-defined, i.e., that $U(s)\langle 0, f(\tau + s)\rangle$ uniquely determines f.) For all $s \geqslant 0$ and for a dense set of vectors $m_* \in \mathcal{M}_*$, namely $m_* = U(u)\langle 0, f(\tau + u)\rangle$, where $f \in L^2([0, u], \mathcal{G})$, $u > 0$, we have

$$\begin{aligned} \Phi_*(U(s)m_*) &= \Phi_*(U(s + u)\langle 0, f(\tau + u)\rangle) = \\ &= \Phi_*(U(s + u)\langle 0, f(\tau - s + s + u)\rangle) = \end{aligned} \tag{4.9}$$

$$= \text{the holomorphic Fourier transform of } f(\tau - s) = \mathrm{e}^{-\lambda s}\,\Phi_* m_*.$$

It follows immediately from the definition (4.8) that Φ_* preserves the indefinite inner products on $\mathcal{M}_*$ and $H^2(0, \mathcal{G})$, since both $U(s)$ and the holomorphic Fourier transform have this property. We can, however, draw no conclusion about the boundedness of Φ_* without (as in [3]) first obtaining a geometric interpretation of the characteristic function. This is done in the following paragraphs.

Suppose $m_* = U(s)\langle 0, f(\tau + s)\rangle \in \mathscr{M}_*$ and $m = \langle 0, g\rangle \in \mathscr{M}$, where $f \in L^2([0, s], \mathscr{G})$ and $g \in L^2([0, N], \mathscr{G})$ for some $s > 0$ and $N > 0$. Then by (4.2) and (4.3), $m_* = \langle h', f'\rangle$, where

$$h' = -\int_0^s T(s-t)JQf(s-t)\mathrm{d}t = -\int_0^s T(t)JQf(t)\,\mathrm{d}t \tag{4.10}$$

and

$$\begin{aligned} f' &= f(\tau) - \chi_{[0,s]}(\tau)\int_0^{s-\tau} QT(s-t-\tau)JQf(s-t)\,\mathrm{d}t = \\ &= f(\tau) - \chi_{[0,s]}(\tau)\int_\tau^s QT(t-\tau)JQf(t)\,\mathrm{d}t. \end{aligned} \tag{4.11}$$

Therefore

$$\begin{aligned} [m_*, m] &= \int_0^s [f'(u), g(u)]\,\mathrm{d}u = \\ &= \int_0^s [f(u), g(u)]\mathrm{d}u - \int_0^s\int_u^s [QT(t-u)JQf(t), g(u)]\,\mathrm{d}t\,\mathrm{d}u = \\ &= \int_0^s [f(t), g(t)]\,\mathrm{d}t - \int_0^s\int_0^t [QT(t-u)JQf(t), g(u)]\,\mathrm{d}u\,\mathrm{d}t = \\ &= \int_0^s [f(t), g(t)]\,\mathrm{d}t - \int_0^s\int_0^t [f(t), QT(t-u)^*JQg(u)]\,\mathrm{d}u\,\mathrm{d}t, \end{aligned} \tag{4.12}$$

using (2.1). The interchange of order of integration in (4.12) is justified by Fubini's theorem, since (using Lemma 2.4)

$$\begin{aligned} \int_0^s\int_u^s |[QT(t-u)JQf(t), g(u)]|\mathrm{d}t\,\mathrm{d}u &\leqslant \\ \leqslant \int_0^s\int_0^s M\|f(t)\|\,\|g(u)\|\,\mathrm{d}t\,\mathrm{d}u &\leqslant \\ \leqslant M\sqrt{s}\|f\|_2\int_0^s \|g(u)\|\mathrm{d}u \leqslant Ms\|f\|_2\|g\|_2 < \infty. \end{aligned}$$

Thus we have

$$[m_*, m] = \int_0^s [f(t), g(t) - h(t)]\,\mathrm{d}t, \tag{4.13}$$

where

$$h(t) = \int_0^t QT(t-u)^* JQg(u)\,\mathrm{d}u. \tag{4.14}$$

Note that, for all $t \geqslant 0$, the integrand of (4.14) is in $L^1(\mathbf{R}^+, \mathscr{G})$ since, by Lemma 2.4,

$$\int_0^t \|QT(t-u)^* JQg(u)\|\,\mathrm{d}u \leqslant \int_0^N \|QT(t-u)^* JQg(u)\|\,\mathrm{d}u \leqslant M\sqrt{N}\|g\|_2, \tag{4.15}$$

for $g \in L^2([0, N], \mathscr{G})$. Since the right side of the inequality (4.15) is independent of t, it also follows from (4.14) that $h \in L^\infty(\mathbf{R}^+, \mathscr{G})$.

The function h has the form of a convolution, and so it is to be expected that its holomorphic Fourier transform will be of the form of a product of two functions, one operator-valued and the other vector-valued. In fact

$$\hat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\lambda t} \int_0^t QT(t-u)^* JQg(u)\,\mathrm{d}u\,\mathrm{d}t =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_u^\infty \mathrm{e}^{-\lambda t} QT(t-u)^* JQg(u)\,\mathrm{d}t\,\mathrm{d}u = \tag{4.16}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda(t+u)} QT(t)^* JQg(u)\,\mathrm{d}t\,\mathrm{d}u$$

where the interchange of order of integration in the second line of (4.16) is justified by Fubini's theorem and the estimate (using (4.15))

$$\int_0^\infty \int_0^t \|\mathrm{e}^{-\lambda t} QT(t-u)^* JQg(u)\|\,\mathrm{d}u\,\mathrm{d}t \leqslant$$

$$\leqslant \int_0^\infty \mathrm{e}^{-(\mathrm{Re}\,\lambda)t} M\sqrt{N}\|g\|_2\,\mathrm{d}t < \infty \quad \text{for } \mathrm{Re}\,\lambda > 0.$$

Therefore we have, for $\mathrm{Re}\,\lambda > 0$,

$$\hat{h}(\lambda) = \left(\int_0^\infty e^{-\lambda t} QT(t)^* JQ \, dt\right) \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda u} g(u) \, du = \tag{4.17}$$
$$= (I - \Theta(\lambda))\hat{g}(\lambda),$$

by Theorem 2.5. We are assuming (1.6) that $\Theta(\lambda)$ is bounded for $\mathrm{Re}\,\lambda > 0$. Since $g \in L^2(\mathbf{R}^+, \mathscr{G})$, then $\hat{g} \in H^2(0, \mathscr{G})$, and it follows immediately from the boundedness of Θ and the relation (4.17) that $\hat{h} \in H^2(0, \mathscr{G})$. Thus, by Theorem 3.4, $\hat{h}$ is the holomorphic Fourier transform of a function in $L^2(\mathbf{R}^+, \mathscr{G})$. The uniqueness theorem (Theorem 3.3) then implies that $h \in L^2(\mathbf{R}^+, \mathscr{G})$.

We can now rewrite (4.13) as an inner product in $L^2(\mathbf{R}^+, \mathscr{G})$ and, using (3.12), as an inner product in $H^2(0, \mathscr{G})$:

$$[m_*, m] = [f, g - h] = [\hat{f}, \hat{g} - \hat{h}] = [\hat{f}, \Theta\hat{g}] = \tag{4.18}$$
$$= [\Phi_* m_*, \Theta\Phi m]$$

for all m_* of the form $U(s)\langle 0, f(\tau + s)\rangle$ ($f \in L^2([0, s], \mathscr{G})$, $s > 0$) and for all m of the form $\langle 0, g\rangle$ ($g \in L^2([0, N], \mathscr{G})$, $N > 0$). We are using Θ in (4.18) to represent the bounded operator on $H^2(0, \mathscr{G})$ defined by $(\Theta F)(\lambda) = \Theta(\lambda)F(\lambda)$, for $\mathrm{Re}\,\lambda > 0$. If C is the bound for Θ, as given in (1.6), then for the operator Θ on $H^2(0, \mathscr{G})$ we have $\|\Theta\| \leqslant C$.

(4.18) is valid for a dense set of vectors $m \in \mathscr{M}$. Since Φ is a bounded operator from $\mathscr{M}$ to $H^2(0, \mathscr{G})$ and Θ is a bounded operator on $H^2(0, \mathscr{G})$, (4.18) is in fact valid for all $m \in \mathscr{M}$. In order to extend (4.18) to all $m_* \in \mathscr{M}_*$, it is necessary to first establish the boundedness of Φ_*. This can be done by using an approach that is formally the same as that used in the study of a single operator in [3], and thus the details can be omitted. As in [3], we get the estimates

$$\|\Phi_*\|^2 \leqslant 1 + 2C^2, \quad \|\Phi_*^{-1}\|^2 \leqslant 1 + 2C^2 \tag{4.19}$$

and therefore Φ_* can be extended to a bounded operator mapping $\mathscr{M}_*$ onto $H^2(0, \mathscr{G})$. (4.18) can be extended to give

$$[m_*, m] = [\Phi_* m_*, \Theta\Phi m] \quad \text{for all } m \in \mathscr{M},\ m_* \in \mathscr{M}_*, \tag{4.20}$$

giving the promised representation of Θ as a projection.

Φ_* preserves the indefinite inner products of $\mathcal{M}_*$ and $H^2(0, \mathcal{G})$, and hence its adjoint (with respect to the indefinite inner product) is a Krein space isometry mapping $H^2(0, \mathcal{G})$ onto $\mathcal{M}_*$. This implies that, when Θ is bounded, $\mathcal{M}_*$ is a *regular* subspace of $\mathcal{K}_+$, i.e.

$$\mathcal{K}_+ = \mathcal{M}_* \oplus \mathcal{R}, \tag{4.21}$$

where

$$\mathcal{R} = \{k \in \mathcal{K}_+ : [k, m_*] = 0, \text{ for all } m_* \in \mathcal{M}_*\}. \tag{4.22}$$

(See [8], [9]. Note that (4.22) does not follow automatically from (4.21) when the inner product is indefinite.)

The relation (4.9) can now be extended to all of $\mathcal{M}_*$, using the boundedness of Φ_*, i.e.

$$\Phi_*(U(s)m_*) = e^{-\lambda s}\Phi_* m_* \quad \text{for all } m_* \in \mathcal{M}_* \text{ and } s \geqslant 0. \tag{4.23}$$

Thus the semigroup $\{U(s) | \mathcal{M}_* : s \geqslant 0\}$ on $\mathcal{M}_*$ is similar to the semigroup $\{W(s): s \geqslant 0\}$ of operators on $H^2(0, \mathcal{G})$, where $W(s)$ denotes multiplication by the function $e^{-\lambda s}$. It is readily checked that each $W(s)$ is an isometry on $H^2(0, \mathcal{G})$, with respect to both the indefinite and Hilbert space inner products (it corresponds, via the holomorphic Fourier transform, to the shift to the right by s units on $L^2(\mathbf{R}_+, \mathcal{G})$). Thus we have established:

LEMMA 4.2. *If the characteristic function $\Theta(\lambda)$ is bounded, then the semigroup $U_*(s) = U(s)|\mathcal{M}_*$, $s \geqslant 0$, is similar to a semigroup of isometries on a Hilbert space. $U_*(s)$ can be changed to a semigroup of isometries by renorming M_* with an equivalent Hilbert space norm $|\cdot|$ obtained from the norm of $H^2(0, \mathcal{G})$:*

$$|m_*| = \|\Phi_* m_*\| \quad (m_* \in \mathcal{M}_*). \tag{4.24}$$

5. THE RESIDUAL SPACE

One of our objectives was to prove Theorem 4.1 by showing that $U^+(s) = U(s) | \mathcal{K}_+$ is similar to a semigroup of isometries. We have shown (Lemma 4.2) that $U(s)$ acts this way on a subspace $\mathcal{M}_*$ of $\mathcal{K}_+$; we now need to consider the residual space $\mathcal{R}$, defined by (4.22). In our analysis of the properties of $\mathcal{R}$, we do not initially assume that the characteristic function is bounded, although the boundedness of $\Theta(\lambda)$ will be required later.

We will need the following:

THEOREM 5.1. *$\mathscr{R}$ is invariant for the group $U(s)$.*

Proof. Since $\mathscr{M}_*$ is invariant for $U(s)$, $s \geqslant 0$, it follows immediately from the definition (4.22) that $\mathscr{R}$ is invariant for $U(s)^*$, $s \geqslant 0$, where the adjoint is taken in the indefinite inner product. By the unitary property of $U(s)$, this is equivalent to saying that $\mathscr{R}$ is invariant for $U(s)$, for $s \leqslant 0$.

Let $\mathscr{K}_- = \{\langle 0, f\rangle \in \mathscr{K} : f \in L^2(\mathbf{R}^-, \mathscr{G})\}$. It is not difficult to verify that $\mathscr{K} = \mathscr{K}_- \ominus \mathscr{K}_+$, that $\mathscr{K}_- \ominus \mathscr{M}_*$ is invariant for $U(s)^*$, $s \geqslant 0$, and that

$$\mathscr{R} = \{k \in \mathscr{K} : [k, m] = 0 \text{ for all } m \in \mathscr{K}_- \oplus \mathscr{M}_*\}.$$

From this we can conclude that $\mathscr{R}$ is invariant for $U(s)$, for $s \geqslant 0$. ∎

COROLLARY 5.2. *$\mathscr{R}$ is invariant for the semigroup $U^+(s)$.*

We have a representation of the residual space which is analogous to [8], Theorem 4.2.

THEOREM 5.3. *Let k be a vector in $\mathscr{K}_+$, given by $k = \langle h_0, f\rangle$, where $h_0 \in \mathscr{H}$ and $f \in L^2(\mathbf{R}^+, \mathscr{G})$. Then k is in $\mathscr{R}$ if and only if there is a function $h : [0, \infty) \to \mathscr{H}$ such that*

(5.1) $h(0) = h_0$,

(5.2) $Qh(t) = f(t)$ *for almost all* $t \geqslant 0$, *and*

(5.3) $T(t)h(s) = h(s - t)$ *for all s and t satisfying* $0 \leqslant t \leqslant s$.

The function h and the vector k uniquely determine each other.

Proof. Let us assume the existence of a function h with the properties (5.1), (5.2), and (5.3). If $m_* = U(s)\langle 0, g(\tau + s)\rangle$ for some $g \in L^2([0, s], \mathscr{G})$, $s > 0$, then we have, using (4.10) and (4.11),

$$[k, m_*] = -\int_0^s (h_0, T(t)JQg(t))\, dt + \int_0^s [f(t), g(t)]\, dt -$$

(5.4)

$$-\int_0^s\int_u^s [f(u), QT(t-u)JQg(t)]\, dt\, du.$$

Using Fubini's theorem (along with Lemma 2.4) and (2.1), we can write the third term in (5.4) as

$$-\int_0^s\int_0^t [QT(t-u)^*JQf(u), g(t)]\,du\,dt =$$

$$= -\int_0^s\int_0^t [QT(t-u)^*JQ^2h(u), g(t)]\,du\,dt = \qquad \text{(by (5.2))}$$

$$= \int_0^s\int_0^t [QT(u)^*Gh(t-u), g(t)]\,du\,dt = \qquad \text{(using (1.1))}$$

$$= \int_0^s\int_0^t [QT(u)^*GT(u)h(t), g(t)]\,du\,dt,$$

using (5.3). But, by (1.3), $T(u)^*GT(u)$ is the derivative of $T(u)^*T(u)$, and thus we get for the third term in (5.4)

$$\int_0^s [QT(t)^*T(t)h(t) - Qh(t), g(t)]\,dt =$$

$$= \int_0^s [QT(t)^*h(0) - Qh(t), g(t)]\,dt = \qquad \text{(using (5.3))}$$

$$= \int_0^s (h_0, T(t)JQg(t))\,dt - \int_0^s [f(t), g(t)]\,dt,$$

using (5.1) and (5.2). Comparison with the other two terms in (5.4) yields the result that $[k, m_*] = 0$, and since this is true for a dense set of m_* in $\mathcal{M}_*$, it follows that $k \in \mathcal{R}$.

Conversely, suppose $k \in \mathcal{R}$. Then $k \in \mathcal{K}_+$ and $[k, m_*] = 0$ for all $m_* = U(s)\langle 0, g(\tau + s)\rangle$, with $g \in L^2([0, s], \mathcal{G})$, $s > 0$. We define

$$h(t) = T(t)^* h_0 + \int_0^t T(t-u)^*JQf(u)\,du. \tag{5.5}$$

That $h(0) = h_0$ is immediate from (5.5), but the other two properties of h are not so obvious. Let us first show that $f = Qh$. From (5.4), and the fact that $[k, m_{s}] = 0$, we have

$$\int_0^s [f(t), g(t)]\mathrm{d}t =$$

$$(5.6)\qquad \begin{aligned} &= \int_0^s (JQT(t)^* h_0, g(t))\,\mathrm{d}t + \int_0^s\int_u^s [QT(t-u)^* JQf(u), g(t)]\,\mathrm{d}t\,\mathrm{d}u = \\ &= \int_0^s [QT(t)^* h_0, g(t)]\mathrm{d}t + \int_0^s\int_0^t [QT(t-u)^* JQf(u), g(t)]\,\mathrm{d}u\,\mathrm{d}t = \end{aligned}$$

$$= \int_0^s [Qh(t), g(t)]\,\mathrm{d}t.$$

Since (5.6) is valid for all $g \in L^2([0,s], \mathscr{G})$ and all $s > 0$, we conclude that $f = Qh$.

We now wish to show that $T(t)h(s) = h(s-t)$ for all s and t with $0 \leqslant t \leqslant s$. To this end, consider, for $v \geqslant 0$, the quantity $T(v)GT(v)^* h(s-t)$. By (1.2), this is the derivative, with respect to v, of $T(v)T(v)^* h(s-t)$, and therefore

$$(5.7)\qquad \int_0^t T(v)GT(v)^* h(s-t)\,\mathrm{d}v = (T(t)T(t)^* - I)h(s-t).$$

On the other hand, it is readily seen from (5.5) and the semigroup property that

$$(5.8)\qquad T(v)^* h(s-t) = h(v+s-t) - \int_{s-t}^{v+s-t} T(v+s-t-u)^* JQf(u)\,\mathrm{d}u.$$

Using the fact that $f = Qh$ and $JQ^2 = -G$, we get

$$(5.9)\qquad \begin{aligned} T(v)GT(v)^* h(s-t) &= -T(v)JQf(v+s-t) - \\ &- \int_{s-t}^{v+s-t} T(v)GT(v+s-t-u)^* JQf(u)\mathrm{d}u. \end{aligned}$$

Combining (5.7) and (5.9) gives

$$(I - T(t)T(t)^*)h(s-t) =$$

(5.10)

$$= \int_0^t T(v)JQf(v+s-t)\,dv + \int_0^t \int_{s-t}^{v+s-t} T(v)GT(v+s-t-u)^*JQf(u)\,du\,dv.$$

The second integral in (5.10) can be rewritten, applying Fubini's Theorem and the semigroup property, as

$$\int_{s-t}^{s} \int_{u-s+t}^{t} T(u-s+t)T(v+s-t-u)GT(v+s-t-u)^*JQf(u)\,dv\,du =$$

$$= \int_{s-t}^{s} T(u-s+t) \int_0^{s-u} T(v)GT(v)^*JQf(u)\,dv\,du =$$

(5.11)

$$= \int_{s-t}^{s} T(u-s+t)(T(s-u)T(s-u)^* - I)JQf(u)\,du =$$

$$= T(t) \int_{s-t}^{s} T(s-u)^*JQf(u)\,du - \int_0^t T(u)JQf(u+s-t)\,du.$$

Therefore we get, from (5.10) and (5.11), and by applying (5.8) with $v = t$,

$$h(s-t) = T(t)T(t)^*h(s-t) + T(t)\int_{s-t}^{s} T(s-u)^*JQf(u)\,du = T(t)h(s).$$

It remains to prove the uniqueness assertion. It follows immediately from properties (5.1) and (5.2) that h uniquely determines k. To show that k uniquely determines h, we need to show that any function h satisfying (5.1), (5.2), and (5.3) must necessarily be given by (5.5). This follows, since

$$\int_0^t T(t-u)^* JQf(u)\,\mathrm{d}u = -\int_0^t T(t-u)^* Gh(u)\,\mathrm{d}u = \qquad \text{(by (5.2))}$$

$$= -\int_0^t T(u)^* Gh(t-u)\,\mathrm{d}u = -\int_0^t T(u)^* GT(u)h(t)\,\mathrm{d}u = \qquad \text{(by (5.3)}$$

$$= h(t) - T(t)^* T(t) h(t) = h(t) - T(t)^* h(0),$$

by (1.3) and (5.3). Thus, using (5.1), we obtain (5.5). ▨

Using the representation in Theorem 5.3, we can now derive some properties of the residual space, which are analogous to some of the results obtained for a single operator in [8]. We begin by introducing some terminology.

A subspace of a Krein space is called *positive* (*positive definite*) if $[k, k] \geqslant 0$ ($[k, k] > 0$) for all nonzero k in the subspace. A subspace is called *non-degenerate* if no nonzero vector in the subspace is orthogonal to every vector in the subspace (in the sense of the indefinite inner product).

THEOREM 5.4. *$\mathscr{R}$ is a positive subspace. If the semigroup $T(t)$ is equi-bounded, i.e. if there is a constant M such that*

$$\|T(t)\| \leqslant M \quad \textit{for all } t \geqslant 0,$$

then $\mathscr{R}$ is positive definite.

Proof. Let $k = \langle h_0, f \rangle$ be a vector in $\mathscr{R}$, and let h be the function corresponding to k, given by Theorem 5.3. Then

$$[k, k] = \|h_0\|^2 + [f, f]. \tag{5.12}$$

Since the functions $f_s(\tau) = \chi_{[0,s]}(\tau) f(\tau)$ converge strongly to f in $L^2(\mathbf{R}^+, \mathscr{G})$, as $s \to \infty$, it follows that $[f, f]$ can be obtained as the limit of $[f_s, f_s]$, where

$$\begin{aligned}
[f_s, f_s] &= \int_0^s [f(t), f(t)]\,\mathrm{d}t = \int_0^s [Qh(t), Qh(t)]\,\mathrm{d}t = \\
&= \int_0^s [Qh(s-t), Qh(s-t)]\,\mathrm{d}t = \int_0^s [QT(t)h(s), QT(t)h(s)]\,\mathrm{d}t.
\end{aligned} \tag{5.13}$$

Using (1.1), the integrand of (5.13) can be written in the form

$$(JQT(t)h(s),\ QT(t)h(s)) = -(T(t)^*GT(t)h(s),\ h(s)),$$

which is the derivative, with respect to t, of $-(T(t)^*T(t)h(s),\ h(s))$ (by (1.3)). Thus, by (5.1) and (5.3),

$$\begin{aligned}[f_s,\ f_s] &= \|h(s)\|^2 - \|T(s)h(s)\|^2 = \|h(s)\|^2 - \|h(0)\|^2 = \\ &= \|h(s)\|^2 - \|h_0\|^2.\end{aligned} \tag{5.14}$$

Since the limit of $[f_s, f_s]$ equals $[f, f]$, it follows from (5.12), (5.13), and (5.14) that

$$[k,\ k] = \lim_{s\to\infty} \|h(s)\|^2 \geqslant 0. \tag{5.15}$$

Hence $\mathscr{R}$ is positive.

Now suppose that $T(t)$ is equi-bounded and that $[k,\ k] = 0$. If follows from (5.15) that $h(s) \to 0$ as $s \to \infty$, and thus for all $t \geqslant 0$ and $s \geqslant t$ we have

$$\|h(t)\| = \|T(s-t)h(s)\| \leqslant M\|h(s)\| \to 0 \quad \text{as } s \to \infty.$$

Therefore, $h(t) = 0$ for all $t \geqslant 0$, and hence $k = 0$. Thus, if $T(t)$ is equi-bounded, $\mathscr{R}$ is positive definite. ▨

COROLLARY 5.5. *If $T(t)$ is equi-bounded, then $\mathscr{M}_*$ is non-degenerate.*

COROLLARY 5.6. *For each $k \in \mathscr{R}$, the function h given by Theorem 5.3 is bounded.*

Proof. By (5.15) the limit of $\|h(s)\|$, as $s \to \infty$, exists. Since h has the form (5.5), this implies that h is bounded. ▨

THEOREM 5.7. *If $T(t)^*$ converges strongly to zero as $t \to \infty$, then $\mathscr{R} = \{0\}$, i.e., $\mathscr{M}_* = \mathscr{K}_+$.*

Proof. Suppose $k \in \mathscr{R}$ and let h be the function given by Theorem 5.3. Then for all $h' \in \mathscr{H}$, and for all $t \geqslant 0$ and $s \geqslant t$, we have

$$\begin{aligned}|(h(t),\ h')| &= |(T(s-t)h(s),\ h')| = |(h(s),\ T(s-t)^*h')| \leqslant \\ &\leqslant \|h(s)\|\|T(s-t)^*h'\|.\end{aligned} \tag{5.16}$$

Let $s \to \infty$. Since $h(s)$ is bounded, and since, by assumption, $T(s-t)^*h' \to 0$, it follows from (5.16) that $(h(t),\ h') = 0$ for all $t \geqslant 0$ and for all $h' \in \mathscr{H}$. Thus $h = 0$, and it follows that $k = 0$. We conclude that $\mathscr{R} = \{0\}$, and thus, using the definition (4.22) and [1], Theorem III.6.1, $\mathscr{M}_* = \mathscr{K}_+$. ▨

COROLLARY 5.8. *If a vector* $h' \in \mathscr{H}$ *satisfies* $T(t)^* h' \to 0$ *as* $t \to \infty$, *then* $h' \in \mathscr{M}_*$.

Proof. Suppose $k \in \mathscr{R}$. Then, as in (5.16), we have

$$[k, h'] = (h(0), h') \leqslant \|h(s)\| \|T(s)^* h'\| \to 0 \quad \text{as } s \to \infty.$$

Thus h' is orthogonal to $\mathscr{R}$ (in the indefinite inner product), and so $h' \in \mathscr{M}_*$. ▨

Theorem 5.7 does not in general have a converse: it is possible to have $\mathscr{R} = \{0\}$ when $T(t)^*$ does not converge strongly to zero.

EXAMPLE 5.9. Let $T(t)$ be the semigroup introduced in Example 1.2. Clearly $T(t)^*$ does not converge to zero as $t \to \infty$: we will show, however, that its residual space is zero.

It is clear that $T(t)$ is equi-bounded and thus, by Theorem 5.4, $\mathscr{R}$ is positive definite. Suppose $k = \langle h_0, f \rangle \in \mathscr{R}$, and let h be its associated function, given by Theorem 5.3. Note that Q is an invertible 2×2 matrix, and that $f = Qh$. Thus, since $f \in L^2(\mathbf{R}^+, \mathscr{G})$, it follows that $h \in L^2(\mathbf{R}^+, \mathscr{G})$. Since, by (5.15), the limit of $\|h(s)\|$ exists, this limit must therefore be zero. By (5.15), we have $[k, k] = 0$, and therefore since $\mathscr{R}$ is positive definite, we conclude that $\mathscr{R} = \{0\}$. ▨

Let us now reintroduce the condition (1.6) that the characteristic function $\Theta(\lambda)$ be bounded. We demonstrated at the end of Section 4 that the subspace $\mathscr{M}_*$ is regular (4.21) when the characteristic function is bounded. We include this, and more, in the following:

THEOREM 5.10. *If* $\Theta(\lambda)$ *is bounded, then* $\mathscr{M}_*$ *and* $\mathscr{R}$ *are regular subspaces of* $\mathscr{K}_-$. *With the inner product* $[\cdot,\cdot]$, $\mathscr{R}$ *is a Hilbert space, and the intrinsic topology* (i.e., *the topology derived from the norm* $|x| = [x, x]^{1/2}$) *coincides with the strong topology that* $\mathscr{R}$ *inherits from* $\mathscr{K}$.

Proof. The regularity of both $\mathscr{M}_*$ and $\mathscr{R}$ was established when (4.21) was proved. Since a regular subspace is a Krein space (this follows from [1], Theorem V. 3.4), and since the indefinite inner product on $\mathscr{R}$ is positive, then $\mathscr{R}$ must by a Hilbert space. The intrinsic and strong topologies coincide by virtue of [1], Theorem V.5.2. ▨

6. PROOFS OF THEOREMS 1.1 AND 4.1

Let us assume throughout this section that the characteristic function is bounded. Then, by Lemma 4.2, we know that $\mathscr{M}_*$ can be renormed by (4.24) so as to make the semigroup $U_*(s) = U(s) | \mathscr{M}_*$, $s \geqslant 0$, a semigroup of isometries on a Hilbert space. We also have, as immediate consequences of Theorems 5.1 and 5.10, and the basic unitary property of $U(s)$ (see Section 4), the following:

LEMMA 6.1. *If the characteristic function* $\Theta(\lambda)$ *is bounded, then the group* $U_{\mathscr{R}}(s) = U(s)|\mathscr{R}$ *is similar to a unitary group on a Hilbert space.* $U_{\mathscr{R}}(s)$ *can be changed to a unitary group by renorming* $\mathscr{R}$ *with the equivalent Hilbert space norm* $|\cdot|$ *obtained from the inner product* $[\cdot,\cdot]$:

$$|r| = [r, r]^{1/2} \quad (r \in \mathscr{R}). \tag{6.1}$$

▨

We can now complete the proof of Theorem 4.1. Since $\mathscr{M}_*$ is regular (Theorem 5.10), every $k \in \mathscr{K}_+$ has a unique representation in the form $k = m_* + r$, with $m_* \in \mathscr{M}_*$ and $r \in \mathscr{R}$. Renorm $\mathscr{K}_+$ with the norm, derived from (4.24) and (6.1)

$$|k|^2 = \|\Phi_* m_*\|^2 + [r, r]. \tag{6.2}$$

Using Lemmas 4.2 and 6.1, we can conclude that this norm is equivalent to the original norm on $\mathscr{K}_+$ (cf. [9], Section 6), and the same lemmas show that $U^+(s)$ is isometric with respect to this norm, proving Theorem 4.1. All that remains is to supply a proof of Theorem 1.1.

Let P denote the projection on $\mathscr{K}$ defined by $P\langle h, f\rangle = \langle h, 0\rangle$; P is the orthogonal projection onto $\mathscr{H}$, with respect to both the indefinite and Hilbert space inner products on $\mathscr{K}$. The dilation property of $U(s)$ may be described as $T(s) = PU^+(s)\,|\,\mathscr{H}$, where $U^+(s) = U(s)\,|\,\mathscr{K}_+$. Note that, by (4.4) and (4.6), $\mathscr{K}_+ = \mathscr{H} \oplus \mathscr{M}$, and $\mathscr{M}$ is invariant for $U^+(s)$. Thus, $\mathscr{H}$ is invariant for $U^+(s)^*$, and we may write the dilation property as

$$T(s)^* = U^+(s)^*\,|\,\mathscr{H}. \tag{6.3}$$

Since $U^+(s)$ is similar to a semigroup of Hilbert space isometries (Theorem 4.1), it follows from (6.3). that $T(s)^*$ is similar to a contraction semigroup. Thus $T(t)$ is similar to a contraction semigroup, and Theorem 1.1 is proved.

REFERENCES

1. BOGNÁR, J., *Indefinite inner product spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
2. DAVIS, CH., Dilation of uniformly continuous semi-groups, *Rev. Roumaine Math. Pures Appl.*, **15**(1970), 975–983.
3. DAVIS, CH.; FOIAŞ, C., Operators with bounded characteristic function and their J-unitary dilation, *Acta Sci. Math. (Szeged)*, **32**(1971), 127–139.
4. DUNFORD, N.; SCHWARTZ, J. T., *Linear operators. Part I: General theory*, Wiley-Interscience, New York, London, Sydney, 1957.
5. HALMOS, P. R., *A Hilbert space problem book*, van Nostrand, New York, Toronto, London, Melbourne, 1967.

6. HILLE, E; PHILLIPS, R. S., *Functional analysis and semi-groups*, American Mathematical Society, Providence, 1957.
7. KATO, T., *Perturbation theory for linear operators*, Second edition, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
8. MCENNIS, B. W. Characteristic functions and dilations of noncontractions, *J. Operator Theory*, 3(1980), 71–87.
9. MCENNIS, B. W., The distortion coefficient of an operator with bounded characteristic function, *Indiana Univ. Math. J.*, 30(1981), 65–71.
10. MCENNIS, B. W., The characteristic function of a uniformly continuous semigroup, *Proc. Centre for Mathematical Analysis*, Australian National Univ., vol. 15, 1987.
11. MCENNIS, B. W., Dilations of holomorphic semigroups (preprint), Centre for Mathematical Analysis, Australian National University, CMA-R36-87.
12. RUDIN, W., *Real and complex analysis*, McGraw-Hill, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1966.
13. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, London, 1970.
14. VAN CASTEREN, J. A., *Generators of strongly continuous semigroups*, Pitman, Boston, London, Melbourne, 1985.
15. YOSIDA, K., *Functional analysis*, Sixth edition, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

BRIAN W. McENNIS
Centre for Mathematical Analysis,
The Australian National University,
Canberra, A.C.T. 2601,
Australia.

and

Department of Mathematics,
The Ohio State University,
Marion, Ohio 43302,
U.S.A.

Received December 2, 1987; revised December 26, 1988.