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THE $\bar{\partial}$-FORMALISM AND THE C^*-ALGEBRA OF THE BERGMAN n-TUPLE

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0. INTRODUCTION

In this paper, we assign to each bounded subdomain Ω of $\mathbf{C}^n$, an operator theoretical object, the Bergman n-tuple B_Ω. This is the n-tuple of multiplication operators acting on the Bergman space over Ω (i.e., the space of all volume Lebesgue-measure, square-integrable, holomorphic functions on Ω).

Our basic goal is to try to exploit the great wealth of recent results on the geometry of the boundary of pseudoconvex domains Ω, to study the spectral and C^*-algebraic properties of B_Ω. Of course, this program offers a two-way avenue that might also bring some new interesting results to the theory of functions of several complex variables.

We start our preliminary discussion, in Section 1, with a simplification of the formalism of the $\bar{\partial}$-Neumann problem originated with J. J. Kohn and D. Spencer. Our approach is similar to that taken by F.-H. Vasilescu [29].

In Section 2, we initiate the study of the C^*-algebra $C^*(B_\Omega)$ generated by the Bergman n-tuple B_Ω on pseudoregular domains which are not necessarily strongly pseudoconvex. These domains are those on which the Neumann operator is compact (see Definition 1.10). As a consequence of Theorem 2.3 it follows that $C^*(B_\Omega)$ contains the ideal $\mathscr{K}$ of compact operators, and that $C^*(B_\Omega)/\mathscr{K}$ is $*$-isomorphic to the algebra of continuous functions $C(X)$, where $X \subset \partial\Omega$. If X is piecewise smooth, it follows that $X = \partial\Omega$. If, in addition, Ω has the Mergelyan property (see Theorem 2.8), then the spectral algebra $S(B_\Omega)$ (which is the norm closure of the analytic functional calculus of B_Ω on neighborhoods of $\bar{\Omega}$) is a complete set of unitary invariance for biholomorphic equivalence.

The last section of the paper is devoted to proving (see Corollary 3.2) that for any bounded domain Ω, the type of $C^*(B_\Omega)$ and the property that B_Ω be essentially normal are preserved under the proper holomorphic mappings that extend continuously to the boundary. This means, in particular, that if Δ and Ω are two bounded domains in $\mathbf{C}^n$, and $C^*(B_\Delta)$ is either type one or essentially abelian while $C^*(B_\Omega)$ is not, then there cannot exist any proper holomorphic mapping $\varphi: \Delta \to \Omega$ that extends continuously to $\partial\Delta$. This fact might be helpful in trying to construct an example of a proper map between two smoothly bounded domains that may not be smoothly extendable to the boundary. As far as we know, the existence of such an example is still an open problem.

Finally, we would like to mention that most of the results contained in this paper were announced in the special session on C^*-algebras of the A.M.S. Regional Meeting, held at Claremont College, Pomona, California on November 8, 1985.

1. GENERAL FRAMEWORK FOR THE $\bar{\partial}$-FORMALISM

We begin by recalling the main features of the $\bar{\partial}$-formalism. As the reader will notice, such features are also present in many other situations. For example, the space $\mathscr{D}$ introduced in the following definition may play the role of the space of test functions, and $\mathscr{A}(\mathscr{D})$ may serve as the raw model of a typical algebra of pseudodifferential operators acting on $\mathscr{D}$.

DEFINITION 1.1. Let $\mathscr{D}$ be a complex inner product space, with inner product $\langle\cdot,\cdot\rangle$. We say that a linear operator $T_0 : \mathscr{D} \to \mathscr{D}$ has a *formal adjoint* if there exists a linear operator $T_0' : \mathscr{D} \to \mathscr{D}$ such that $\langle T_0 f, g\rangle = \langle f, T_0' g\rangle$, for all $f, g \in \mathscr{D}$. In this case, T_0' is unique and is called the formal adjoint of T_0. The set $\mathscr{A}(\mathscr{D})$ of all linear operators on $\mathscr{D}$ with formal adjoint constitutes a $*$-algebra, with $'$ as involution.

Let $\mathscr{H}$ be the completion of $\mathscr{D}$ with respect to $\langle\cdot,\cdot\rangle$, and let $\mathscr{C}(\mathscr{H})$ be the set of all closed linear extensions of operators in $\mathscr{A}(\mathscr{D})$, i.e., $T \in \mathscr{C}(\mathscr{H})$ if and only if the graph of T is closed in $\mathscr{H} \oplus \mathscr{H}$, and the domain $\mathscr{D}_T$ of T satisfies $\mathscr{D} \subset \mathscr{D}_T \subset \mathscr{H}$, and $T|\mathscr{D} \in \mathscr{A}(\mathscr{D})$. Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded operators on $\mathscr{H}$ so that $\mathscr{C}(\mathscr{H}) \cap \mathscr{L}(\mathscr{H}) = \{T \in \mathscr{C}(\mathscr{H}) : \mathscr{D}_T = \mathscr{H}\}$.

REMARK 1.2. Let $T_0 \in \mathscr{A}(\mathscr{D})$. and let $\mathscr{D}_T = \{f \in \mathscr{H} :$ there exists $C_f > 0$, $|\langle f, T_0' g\rangle| \leq C_f \|g\|$, for all $g \in \mathscr{D}\}$. Given $f \in \mathscr{D}_T$, let $Tf \in \mathscr{H}$ be defined by $\langle Tf, g\rangle = \langle f, T_0' g\rangle$, $g \in \mathscr{D}$. Let $\mathscr{D}_{T'}$ and $T' : \mathscr{D}_{T'} \to \mathscr{H}$ be defined analogously, where one uses T_0 in place of T_0'. Then the following properties are easily obtained:

a) $T, T' \in \mathscr{C}(\mathscr{H})$; T and T' extend T_0 and T_0', respectively; $T_0^* = T'$, $(T_0')^* = T$, so that T_0 and T_0' are closable and their closures satisfy $\bar{T}_0 = T_0^{**} = T'^*$, $\bar{T}_0' = T^*$. In particular, T'^* and T^* are the minimal closed linear extensions of T_0 and T_0', respectively.

b) T and T' are the maximal closed linear extensions of T_0 and T_0', respectively.

c) If $T_0^2 = 0$ then $T_0'^2 = 0$ and $\mathrm{Ran}(T) \subset \mathrm{Ker}(T)$, $\mathrm{Ran}(T^*) \subset \mathrm{Ker}(T^*)$.

Given T in $\mathscr{C}(\mathscr{H})$, we say that T is nilpotent of order two, or briefly nilpotent, if $\mathrm{Ran}(T) \subset \mathrm{Ker}(T)$. On the other hand, we recall that T is said to be self-adjoint if $T = T^*$, in the sense that $\mathscr{D}_T = \mathscr{D}_{T^*}$ and T is symmetric on $\mathscr{D}_T$ (i.e., $\langle Tf, g\rangle = \langle f, Tg\rangle$, $f, g \in \mathscr{D}_T$).

LEMMA 1.3. *Let $T_0 \in \mathscr{A}(\mathscr{D})$ such that $T_0^2 = 0$, and let $T \in \mathscr{C}(\mathscr{H})$ be the maximal closed linear extension of T_0 (as in Remark 1.2). Then:*

a) $\mathscr{H} = [\mathrm{Ker}(T) \cap \mathrm{Ker}(T^*)] \oplus \overline{\mathrm{Ran}(T)} \oplus \overline{\mathrm{Ran}(T^*)}$;

b) *Let $T + T^*: [\mathscr{D}_T \cap \mathscr{D}_{T^*}] \to \mathscr{H}$ be defined by pointwise addition. Then $T + T^*$ is self-adjoint,* $\mathrm{Ker}(T + T^*) = \mathrm{Ker}(T) \cap \mathrm{Ker}(T^*)$, *and* $\overline{\mathrm{Ran}(T + T^*)} = \overline{\mathrm{Ran}(T)} \oplus \overline{\mathrm{Ran}(T^*)}$.

Proof. $\mathrm{Ran}(T)$ and $\mathrm{Ran}(T^*)$ are orthogonal, because for every $f, g \in \mathscr{D}$ we have:

$$\langle Tf, T^*g\rangle = \langle T_0f, T_0'g\rangle = \langle T_0^2f, g\rangle = 0.$$

Also, it is clear that $\mathrm{Ker}(T) \cap \mathrm{Ker}(T^*)$ is the orthogonal complement of $\mathrm{Ran}(T) \oplus \mathrm{Ran}(T^*)$ in $\mathscr{H}$, so (a) follows. Now for all $f \in \mathscr{D}_T \cap \mathscr{D}_{T^*}$, we have:

$$\|(T + T^*)f\|^2 = \|Tf\|^2 + \|T^*f\|^2$$

so the last two assertions of (b) are also clear. Since $T + T^*$ is obviously symmetric, to prove that $T + T^*$ is self-adjoint, it remains to show that $\mathscr{D}_{(T+T^*)^*} = \mathscr{D}_{T+T^*} = \mathscr{D}_T \cap \mathscr{D}_{T^*}$. Note that $\mathscr{D}_T = \{f \in \mathscr{H}$: there exists $C_f > 0$, $|\langle f, T^*g\rangle| \leqslant C_f\|g\|$, for all $g \in \mathscr{D}_{T^*}\}$, $\mathscr{D}_{T^*} = \{f \in \mathscr{H}$: there exists $C_f > 0$, $|\langle f, Tg\rangle| \leqslant C_f\|g\|$, for all $g \in \mathscr{D}_T$, and $\mathscr{D}_{(T+T^*)^*} = \{f \in \mathscr{H}$: there exists $C_f > 0$, $|\langle f, (T+T^*)g\rangle| \leqslant C_f\|g\|$, for all $g \in \mathscr{D}_{T+T^*}\}$. Therefore, it is clear that $\mathscr{D}_T \cap \mathscr{D}_{T^*} \subset \mathscr{D}_{(T+T^*)^*}$. On the other hand, let $f \in \mathscr{D}_{(T+T^*)^*}$. Then for every $g \in \mathscr{D}_T$ we write $g = g' + g''$, where $g' \in \mathrm{Ker}(T^*)$ and $g'' \in \overline{\mathrm{Ran}(T)} \subset \mathrm{Ker}(T) \subset \mathscr{D}_T$. This means, in particular, that $Tg'' = 0$ and that $g' \in \mathscr{D}_T$, so that $g' \in \mathscr{D}_T \cap \mathscr{D}_{T^*}$ and $T^*g' = 0$. Thus, $|\langle f, Tg\rangle|^2 = |\langle f, Tg'\rangle|^2 = |\langle f, (T + T^*)g'\rangle|^2 \leqslant C_f^2\|g'\|^2 \leqslant C_f^2(\|g'\|^2 + \|g''\|^2) = C_f^2\|g\|^2$. Hence, $f \in \mathscr{D}_{T^*}$. Substituting $T\,(=T^{**})$ by T^* in the above argument we also obtain that $f \in \mathscr{D}_T$ as desired.

We say that a nilpotent T in $\mathscr{C}(\mathscr{H})$ is solvable if $\mathrm{Ran}(T)$ is closed (or after Lemma 1.3 if $\mathrm{Ran}(T + T^*)$ is closed). Under the above assumptions we let $K \in \mathscr{L}(\mathscr{H})$ be the self-adjoint operator defined by $K \mid \mathrm{Ker}(T + T^*) = 0$, $K \mid \mathrm{Ran}(T + T^*) = (T + T^*)^{-1}$. Let P, Q, and R be the (orthogonal) projections from $\mathscr{H}$ onto $\mathrm{Ker}(T) \cap \mathrm{Ker}(T^*)$, $\mathrm{Ran}(T)$, and $\mathrm{Ran}(T^*)$, respectively.

LEMMA 1.4. *Let $T \in \mathscr{C}(\mathscr{H})$ be nilpotent and solvable. Then*

a) $1 - P = (T + T^*)K$, *and for every* $f \in \mathscr{D}_T \cap \mathscr{D}_{T^*}$, $(1 - P)f = K(T + T^*)f$.

b) $Q = TK$, *and for every* $f \in \mathscr{D}_{T^*}$, $Qf = KT^*f$.

c) $R = T^*K$, *and for every* $f \in \mathscr{D}_T$, $Rf = KTf$.

d) *Given* $f \in \mathrm{Ker}(T) \ominus \mathrm{Ker}(T^*) = \mathrm{Ran}(T)$, *the unique solution of the equation* $Tu = f$ *which is orthogonal to* $\mathrm{Ker}(T)$ *is given by* $u = Kf$.

Proof. (a) is immediate by the definition of K. To show (b), note that $Q \leqslant 1 - P$, and that $QT^* = 0$, while $QT = T$. So, $Q = Q(1 - P) = Q(T + T^*)K = TK$. Also, if $f \in \mathscr{D}_{T^*}$ then $(1 - Q)f \in \mathrm{Ker}(T) \subset \mathscr{D}_{T^*}$ so $Qf \in \mathscr{D}_{T^*}$ and $T^*(1 - Q)f = 0$. Since $\mathrm{Ran}(T) \subset \mathrm{Ker}(T)$, $TQf = 0$, and hence $Qf = (1 - P)Qf = K(T + T^*)\cdot Qf = KT^*Qf = KT^*f$, and (b) follows. The proof of (c) is similar. To prove (d), let $f \in \mathrm{Ker}(T) \ominus \mathrm{Ker}(T^*)$, and let $u = Kf$. Then we have $Tu = TKf = Qf = f$. Further, $Ru = KTu = Kf = u$ so that $u \in \mathrm{Ran}(T^*) = \mathrm{Ker}(T)^{\perp}$, as desired.

LEMMA 1.5. *Let* $T \in \mathscr{C}(\mathscr{H})$ *be nilpotent, then* T *is solvable if and only if* 0 *is not an accumulation point of* $\sigma(T + T^*)$. *Further, the following conditions are equivalent:*

a) $\mathrm{Ran}(T) = \mathrm{Ker}(T)$;

b) T *is solvable and* $\mathrm{Ker}(T) \cap \mathrm{Ker}(T^*) = \{0\}$;

c) $T + T^* \in \mathscr{L}(\mathscr{H})$, *and* $T + T^*$ *is invertible.*

Proof. It follows easily from Lemma 1.3, Lemma 1.4 and the corresponding definitions.

We shall need the following explicit representation of the exterior algebra Λ over $\mathbf{C}^n$. Let $\mathbf{Z}_n^p$ be the set of all strictly increasing p-tuples whose components are in $\{1, \ldots, n\}$. Let $n_p = \mathrm{card}(\mathbf{Z}_n^p) = \binom{n}{p}$, and let Λ^p be the Hilbert space $\mathbf{C}^{n_p}$ whose standard orthonormal basis is indexed by the set $\mathbf{Z}_n^p$ (lexicographically ordered) so that we can identify Λ^1 with $\mathbf{C}^n$, and the jth vector in the standard orthonormal basis of $\mathbf{C}^n$, with $e_{\{j\}}$, $1 \leqslant j \leqslant n$, where $\{j\}$ is the 1-tuple consisting of the positive integer j.

By definition, we let $\mathbf{Z}_n^0 = \emptyset$, and $\Lambda_0 = \mathbf{C}$ so that its standard basis vector is $1 = e_{()}$. Let $\Lambda = \bigoplus_{p=0}^{n} \Lambda^p$, and we choose the orthonormal basis for the Hilbert space Λ induced by that of each direct summand. (Note that $\dim(\Lambda) = 2^n$.) Let $\wedge$ be defined on the above basis of Λ as follows: given $J \in \mathbf{Z}_n^p$, $K \in \mathbf{Z}_n^q$, we let $e_J \wedge e_K = 0$, if $J \cap K \neq \emptyset$. If $J \cap K = \emptyset$, we first let $J \wedge K$ be the increasing $(p + q)$-tuple obtained from $J \cup K$, and we let $\varepsilon_{J \wedge K}$ be the sign of the permutation that takes $J \cup K$ onto $J \wedge K$. We then define $e_J \wedge e_K = \varepsilon_{J \wedge K} e_{J \wedge K}$.

We next extend $\wedge$ linearly to the whole space Λ. Thus, Λ will be considered as both the exterior algebra over $\mathbf{C}^n$ and a 2^n-dimensional Hilbert space.

For $1 \leqslant j \leqslant n$, let S_j in $\mathscr{L}(\Lambda)$ be defined by $S_j\zeta = e_{\{j\}} \wedge \zeta$, $\forall \zeta \in \Lambda$. Then the following anticommutation relations hold:

$$S_jS_k + S_kS_j = 0,$$

and

$$S_jS_k^* + S_k^*S_j = \delta_{jk},$$

$1 \leqslant j, k \leqslant n$.

Given a commuting n-tuple $t = (T_1, \ldots, T_n)$ of operators in $\mathscr{A}(\mathscr{D})$, we let $\partial_t : \mathscr{D} \otimes \Lambda(\mathbf{C}^n) \to \mathscr{D} \otimes \Lambda$ be given by $\partial_t = \sum_{j=1}^{n} T_j \otimes S_j$.

Note that $(\partial_t)^2 = 0$ by the above anticommutation relations, and that $\partial_t \in \mathscr{A}(\mathscr{D} \otimes \Lambda) = \mathscr{A}(\mathscr{D}) \otimes \mathscr{L}(\mathbf{C}^{2^n})$. Indeed, $(\partial_t)' = \sum_{j=1}^{n} T_j' \otimes S_j^*$. For example, for $n = 1$,

$$\partial_t = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix};$$

for $n = 2$,

$$\partial_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ T_2 & 0 & 0 & 0 \\ 0 & -T_2 & T_1 & 0 \end{pmatrix};$$

and for $n = 3$,

$$\partial_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -T_2 & T_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -T_3 & 0 & T_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -T_3 & T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_3 & -T_2 & T_1 & 0 \end{pmatrix}.$$

Let $\bar{\partial}_t$ be the maximal extension of ∂_t in $\mathscr{C}[\mathscr{H} \otimes \Lambda]$. Then $\bar{\partial}_t$ is nilpotent. We say that t is right exact (left exact, resp.) if $\bar{\partial}_t$ is solvable, and $\mathrm{Ker}(\bar{\partial}_t) \cap \mathrm{Ker}(\bar{\partial}_t^*) \subset \mathscr{H} \otimes \Lambda^0$ (if $\bar{\partial}_t$ is solvable, and $\mathrm{Ker}(\bar{\partial}_t) \cap \mathrm{Ker}(\bar{\partial}_t^*) \subset \mathscr{H} \otimes \Lambda^n$, resp.). We say that t is exact if it is both left and right exact (or equivalently $\mathrm{Ran}(\bar{\partial}_t) = \mathrm{Ker}(\bar{\partial}_t)$). The notion of exact, left exact and right exact joint spectra of t are defined using the above definitions and they are denoted by $\sigma(t)$, $\sigma_\lambda(t)$ and $\sigma_\rho(t)$, respectively. (Note that $\sigma(t) = \sigma_\lambda(t) \cup \sigma_\rho(t)$, and it follows easily that $\partial\sigma(t) \subset \subset \sigma_\lambda(t) \cap \sigma_\rho(t)$.)

REMARK 1.6. (a) If A_j are commuting operators in $\mathscr{A}(\mathscr{D})$ that can be extended to operators $\tilde{A}_j$ in $\mathscr{L}(\mathscr{H})$, when $1 \leqslant j \leqslant n$, and we let $a = (A_1, \ldots, A_n)$, then $\sigma(a)$ coincides with the standard definition of the joint Taylor spectrum of $\tilde{a} = (\tilde{A}_1, \ldots, \tilde{A}_n)$ (see [29]). However, $\sigma_\lambda(a)$ is generally bigger than the left spectrum $\sigma_1(a)$ of $\tilde{a}$ (see for example [25, §3] for the definition of the left spectrum).

(b) Let $\Omega \subset \mathbf{C}^n$ be a bounded pseudoconvex domain (i.e., a domain of holomorphy) with smooth boundary $\partial\Omega$ [19]. Let $\mathscr{D} = \mathbf{C}_0^\infty(\Omega)$ with the standard $L^2(\Omega)$ inner product $\langle\cdot,\cdot\rangle$ defined by the volume Lebesgue measure λ. Given $1 \leqslant j \leqslant n$ let $T_j : \mathscr{D} \to \mathscr{D}$ be defined by

$$T_j f = \frac{\partial}{\partial \bar{z}_j} f = \frac{1}{2}\left(\frac{\partial}{\partial x_j} f + \mathrm{i}\frac{\partial}{\partial y_j} f\right),$$

where $z_j = x_j + \mathrm{i}y_j$ is the j-th complex coordinate in $\mathbf{C}^n$. Then T_j belongs to $\mathscr{A}(\mathscr{D})$ and $T_j' f = -\dfrac{\partial}{\partial z_j} f = -\dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} f + \mathrm{i}\dfrac{\partial}{\partial y_j} f\right)$, $f \in \mathscr{D}$.

With $t = (T_1, \ldots, T_n)$, we let $\bar{\partial} = \bar{\partial}_t \in \mathscr{C}[L^2(\Omega) \otimes \Lambda]$. Here the elements of $L^2(\Omega) \otimes \Lambda^q$ can be identified with the space of all $(0, q)$ forms with coefficients in $L^2(\Omega)$, and the resulting complex induced by $\bar{\partial}$ is called the Dolbeault complex.

By results of Oka, Kohn, Hörmander, and others (see [14, Chapter 4], [16] and [19]), it follows that $\bar{\partial}$ is right exact and that $\mathrm{Ker}(\bar{\partial}) \cap \mathrm{Ker}(\bar{\partial}^*)$ is the Bergman space $A^2(\Omega)$ of all analytic functions in $L^2(\Omega)$.

DEFINITION 1.7. From now on (unless otherwise specified), Ω will denote a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$. Let P_Ω be the orthogonal projection from $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$. Following the usual terminology, we call P_Ω the *Bergman projection* on Ω. Let K be the operator related to $\bar{\partial}$ according to the definition before Lemma 1.4. We denote by $\tilde{K}$ the restriction to the space of $\bar{\partial}$-closed (0,1) forms in $L^2(\Omega) \otimes \Lambda^1$ of the operator K. We also denote by N the restriction of K^2 to the subspace of all (0, 1) forms with coefficients in $L^2(\Omega)$. The operator $\tilde{K}$ is usually referred to as the *Kohn operator*, while N is called the *Neumann operator*. Following [18], we say that the $\bar{\partial}$-Neumann problem *satisfies the compactness property*, if the norm

$$|||\varphi|||^2 = \|\varphi\|^2 + \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2,$$

$\varphi \in \mathscr{D}_{\bar{\partial}} \cap \mathscr{D}_{\bar{\partial}^*} \cap [L^2(\Omega) \otimes \Lambda^1]$ is compact with respect to the L^2-norm. This means that if $\{\varphi_m\}$ is a bounded sequence in the $|||\cdot|||$-norm then there exists a convergence subsequence in the L^2-norm.

REMARK 1.8. a) As a consequence of Lemma 1.4, we obtain Kohn's formula (see [17]):

$$(1 - P_\Omega)f = \tilde{K}\bar{\partial}f, \quad f \in \mathscr{D}_{\bar{\partial}} \cap [L^2(\Omega) \otimes \Lambda^0].$$

Notice that $L^2(\Omega) = L^2(\Omega) \otimes \Lambda^0 \subset \mathscr{D}_{\bar{\partial}^*}$. It also follows easily that, given a $\bar{\partial}$-closed (0, 1) form α with coefficients in $L^2(\Omega)$, the only function u in $L^2(\Omega)$ that solves the equation $\bar{\partial}u = \alpha$ and is orthogonal to $A^2(\Omega)$ is given by $u = \tilde{K}\alpha$. This is called the canonical solution.

b) It is of great interest (and still an open problem) to find necessary and sufficient conditions for the global regularity of the $\bar{\partial}$-Neumann problem, i.e., the global regularity of the canonical solution of the equation $\bar{\partial}u = \alpha$. This means that if α is a $\bar{\partial}$-closed (0, 1) form with coefficients in $C^\infty(\bar{\Omega})$, then $K\alpha \in C^\infty(\bar{\Omega})$. In [18, page 445], it is shown that if the $\bar{\partial}$-Neumann problem satisfies the compactness property (according to Definition 1.7), then it is globally regular (see also [17, page 139]). The most general sufficient geometric condition for the compactness property of the $\bar{\partial}$-Neumann problem was given in [6], and was called property P. It is also shown in the same paper that pseudoconvex domains with weakly regular boundaries satisfy property P. These domains include the strongly pseudoconvex domains, pseudoconvex domains with real analytic boundary (see [1]), and more generally, domains of finite type (see [6] and [7]).

Although the following proposition might be well known, we were unable to find a reference to it in the existing literature.

PROPOSITION 1.9. *Let Ω be a smoothly bounded pseudoconvex domain in* $\mathbf{C}^n$. *Then the restriction* $L\colon \mathscr{D}_{\bar{\partial}^*} \cap [L^2(\Omega) \otimes \Lambda^n] \mapsto \mathscr{D}_{\bar{\partial}^*} \cap \mathscr{D}_{\bar{\partial}^*} \cap [L^2(\Omega) \otimes \Lambda^{n-1}]$ *of* K *to the* $(0, n)$*-forms in* $\mathscr{D}_{\bar{\partial}^*}$ *is a compact transformation.*

Proof. Notice that $L^2(\Omega) \otimes \Lambda^n \subseteq \mathscr{D}_{\bar{\partial}}$. Also, by definition, $\bar{\partial}_n Lf = f$ for every $f \in L^2 \otimes \Lambda^n$ and $L^*\bar{\partial}_n^*\varphi = \varphi$ for every $\varphi \in \mathscr{D}_{\bar{\partial}_n^*} = \mathscr{D}_{\bar{\partial}^*} \cap (L^2(\Omega) \otimes \Lambda^n)$, where $\bar{\partial}_n$ is the operator induced by $\bar{\partial}$ from $(0, n-1)$-forms to $(0, n)$-forms. Since $\bar{\partial}_n^*$ is the minimal extension of $\bar{\partial}_n'$ (which is the formal adjoint of $\bar{\partial}_n$), the graph of $\bar{\partial}_n^*$ is the closure of the graph of $\bar{\partial}_n'$ (defined on $\mathscr{D} \otimes \Lambda^n$). But, a straightforward calculation shows that the $\bar{\partial}_n'$-graph norm of every $f \in \mathscr{D} \otimes \Lambda^n$ coincides with the first Sobolev norm $\|f\|_1$ of f (recall that such f's have compact support in Ω). Thus, $\mathscr{D}_{\bar{\partial}_n^*} = H_0^1(\Omega) \otimes \Lambda^n$. Let $V\colon \mathrm{Ran}(\bar{\partial}_n^* \mapsto H_0^1(\Omega) \otimes \Lambda^n)$ be defined by $V\bar{\partial}_n^* f = f$. Since $\bar{\partial}_n^*$ is injective and has closed range, V is a continuous isomorphism. Furthermore, notice that $L^* = JV$, where $J\colon H_0^1(\Omega) \otimes \Lambda^n \mapsto L^2(\Omega) \otimes \Lambda^n$ is the inclusion map. Finally, since J is compact, it follows that L^* is compact, and hence L is also compact, as desired.

DEFINITION 1.10. A (smoothly bounded pseudoconvex) domain Ω on which the Neumann operator is compact will be called *pseudoregular*.

REMARK 1.11. a) It is easy to see that the $\bar{\partial}$-Neumann problem satisfies the compactness property (see Definition 1.7) if and only if the Neumann operator N is compact. From Proposition 1.9, it follows that, for $n = 2$, N is compact if and only if $\tilde{K}$ is compact. Therefore, for $n = 2$, pseudoregularity is equivalent to the compactness of $\tilde{K}$.

b) From results of [16], it follows that $\tilde{K}$ is globally regular if and only if P_Ω is globally regular, i.e. $P_\Omega C^\infty(\bar{\Omega}) \subseteq C^\infty(\bar{\Omega})$. Indeed, the *only if part* is an immediate consequence of Kohn's formula (see Remark 1.8(a)). For the *if part*, let α be a $\bar{\partial}$-closed (0,1)-form with coefficients in $C^\infty(\bar{\Omega})$. By [16], there exists $v \in C^\infty(\bar{\Omega})$ such that $\bar{\partial}v = \alpha$. But, then, by assumption and Kohn's formula, it follows that $\tilde{K}\alpha = (1 - P_\Omega)v \in C^\infty(\bar{\Omega})$, as desired. Thus, pseudoregular domains satisfy condition R, i.e. P_Ω is globally regular, because the compactness property of the $\bar{\partial}$-Neumann problem implies the global regularity of $\tilde{K}$ (see Remark 1.8(b)).

c) As it was pointed out in [17. page 141], there are domains Ω in $\mathbf{C}^2$ that satisfy condition R, for which the compactness property is not satisfied, i.e., they are not pseudoregular.

d) Using the mean value property for holomorphic functions, it is immediate that evaluation of functions in $A^2(\Omega)$ at a point w of Ω produces a bounded linear functional on $A^2(\Omega)$, which can be represented by $K_w \in A^2(\Omega)$. The function $K(z, w) = K_w(z)$ is usually called the Bergman kernel on Ω and it is the reproducing kernel function of $A^2(\Omega)$. We then have:

$$P_\Omega f = \int_\Omega K(z, \omega) f(\omega) d\lambda(\omega), \quad f \in L^2(\Omega).$$

In [15], it is observed that, given $\omega \in \Omega$, and given a function $\varphi_\omega \in C_0^\infty(\Omega)$, which is radially symmetric about ω and such that $\int \varphi_\omega(z) d\lambda(z) = 1$, we have $(P_\Omega \varphi_\omega)(z) = K(z, \omega)$, for every z in Ω. Thus, as observed in [4], if Ω satisfies condition R, then, for every $\omega \in \Omega$, $K(\cdot, \omega) \in C^\infty(\bar{\Omega})$.

PROPOSITION 1.12. *Let Ω be a (smoothly bounded, pseudoconvex) domain in $\mathbf{C}^n$ satisfying condition* R, *and for each ω in Ω, let $k(z; \omega) = K(z, \omega)/\sqrt{K(\omega, \omega)}$. Then $k(\cdot\,; \omega)$ has norm one and tends weakly to zero as ω tends to $\partial\Omega$. In particular, this property holds, if Ω is pseudoregular.*

Proof. It is enough to check the above limits on a total set of elements in $A^2(\Omega)$. Thus, take the total set consisting of K_z, with z in Ω. Then, since $\lim_{\omega \to \partial\Omega} K(\omega, \omega) = \infty$ (see [22]), we have: $(k(\cdot\,; \omega), K_z) = K(z, \omega)/\sqrt{K(\omega, \omega)} \to 0$ as ω tends to $\partial\Omega$.

2. THE C^*-ALGEBRA OF THE BERGMAN n-TUPLE ON PSEUDOREGULAR DOMAINS

In this section, we study the unital C^*-algebra $C^*(B_\Omega)$ generated by the Bergman n-tuple B_Ω on a pseudoregular domain Ω (see Definition 1.10).

DEFINITION 2.1. Given a continuous complex valued function φ in $C(\bar{\Omega})$, let T_φ be the (*Bergman*) *Toeplitz operator* with symbol φ, i.e., T_φ is defined by:

$$T_\varphi f = P_\Omega \varphi f, \quad f \in A^2(\Omega).$$

In particular, B_Ω is the n-tuple whose i-th component is T_{z_i}, where $z_i : \Omega \to \mathbf{C}$ is the i-th coordinate function.

REMARK 2.2. It readily follows (via the Stone-Weierstrass Theorem, the obvious estimate $\|T_\varphi\| \leqslant \|\varphi\|_\infty$, and the fact that $T_{\varphi\psi} = T_\varphi T_\psi$, for $\varphi, \psi \in C(\bar{\Omega})$ such that ψ is holomorphic on Ω) that $C^*(B_\Omega)$ coincides with the C^*-algebra generated by T_φ, $\varphi \in C(\bar{\Omega})$. Thus, the study of $C^*(B_\Omega)$ coincides with the study of the (Bergman) Toeplitz C^*-algebra over Ω.

The following theorem is an improvement of [30], [8], [25], and [11] (see Definition 1.10 and Remark 1.8 part c)).

THEOREM 2.3. *Let Ω be pseudoregular. Then $\mathscr{K}[A^2(\Omega)] \subset C^*(B_\Omega)$, and the following sequence is exact:*

$$0 \to \mathscr{K}[A^2(\Omega)] \to C^*(B_\Omega) \to C(\partial\Omega) \to 0.$$

In particular, the essential spectrum $\sigma_e(B_\Omega)$ of B_Ω coincides with $\partial\Omega$. Further, if $n = 2$, i.e., *$\Omega \subset \mathbf{C}^2$, then $\sigma(B_\Omega) = \bar{\Omega}$, $\sigma_\lambda(B_\Omega) = \partial\Omega$, and $\sigma_p(B_\Omega) = \bar{\Omega}$.*

Proof. We define the map $L\colon C(\partial\Omega) \to \mathscr{L}(A^2(\Omega))$ by $L\varphi = T_\psi$, where ψ is the standard Poisson extension of $\varphi \in C(\partial\Omega)$ to $\bar{\Omega}$. It is easy to check that L is a unital completely positive linear map. We now claim that the self-commutator of $L\varphi$ is compact, i.e. $[(L\varphi)^*, L\varphi] = (L\varphi)^* L\varphi - L\varphi(L\varphi)^* \in \mathscr{K}$, for every φ in $C(\partial\Omega)$. Indeed, let ψ be as above, so that we must show that $[(T_\psi)^*, T_\psi] \in \mathscr{K}$. Let M_ψ be the operator on $L^2(\Omega)$ consisting of multiplication by ψ, and observe that

$$[(T_\varphi)^*, T_\varphi] P_\Omega = A^* A,$$

where $A = A_\psi$ is the operator on $L^2(\Omega)$ given by $A_\psi = (1 - P_\Omega)M_\psi P_\Omega$. Since $C^\infty(\bar{\Omega})$ is dense in $C(\bar{\Omega})$, it suffices to show that $A_\psi \in \mathscr{K}$ for every $\psi \in C^\infty(\bar{\Omega})$. But, notice that $\operatorname{Ran}(M_\psi P_\Omega) \subset \mathscr{D}_{\bar{\partial}}$, so that by Kohn's formula (see Remark 1.8 a)), we have:

$$(1 - P_\Omega)M_\psi P_\Omega = \tilde{K}M_{\bar{\partial}\psi}P_\Omega.$$

Since $\psi \in C^\infty(\bar{\Omega})$, it follows that the operator $M_{\bar{\partial}\psi}$ of multiplication by $\bar{\partial}\psi$ from $L^2(\Omega) \otimes \Lambda^0$ to $L^2(\Omega) \otimes \Lambda^1$ is bounded. Moreover, since $\tilde{K}$ is compact because Ω is pseudoregular, our first claim is established. It is now easy to check that $L(\varphi\eta) - L(\varphi)L(\eta)$ is compact for every $\varphi, \eta \in C(\partial\Omega)$ (see [26], Section 2). We next claim that $T_\psi \in \mathscr{K}[A^2(\Omega)]$ if and only if the restriction φ of ψ to $\partial\Omega$ is zero. Indeed, if $\varphi = 0$, then ψ is the uniform limit of a sequence $\{\psi_m\}$ of continuous functions with compact support in Ω. Since multiplication by ψ_m on $L^2(\Omega)$ produces a compact operator, it follows that the Toeplitz operator with symbol ψ_m is compact, and hence, T_ψ is compact. Now, assume that T_ψ is compact, and let $\zeta \in \partial\Omega$. Also, let $\{p_m\}$ be a sequence of polynomials in two indeterminates such that $\{p_m(z, \bar{z})\}$ converges uniformly to $\psi(z)$ on $\bar{\Omega}$. Further, let $\{\zeta_j\} \subset \Omega$ be a sequence that tends to ζ, and let $f_j(z) = k(z; \zeta_j)$, $z \in \Omega$. By Proposition 1.12, $\{f_j\}$ is a sequence of unit vectors in $A^2(\Omega)$ that tends weakly to zero. Now, given $\varepsilon > 0$, let m be large enough so that $\|\psi - p_m\|_\infty < \varepsilon$. We can write p_m as

$$p_m(\omega, \bar{\omega}) = \sum_{\alpha,\beta} \lambda_{\alpha,\beta,m}\omega^\alpha\bar{\omega}^\beta,$$

where $\lambda_{\alpha,\beta,m} \in \mathbf{C}$ and the sum runs over $|\alpha + \beta|$ less than some positive integer k_m. Since $(T_{\bar{z}})^* = T_z$, and $(T_{\bar{z}}^* f_i)(\omega) = \bar{\omega}f_i(\omega)$, we deduce that

$$\langle T_{(z^\alpha\bar{z}^\beta)}f_j, f_j\rangle = (\zeta_j)^\alpha(\bar{\zeta}_j)^\beta + \langle J_{\alpha,\beta}f_j, f_j\rangle,$$

where $J_{\alpha,\beta}$ is the operator

$$P_\Omega M_{\bar{z}^\alpha}^*(1 - P_\Omega)M_{\bar{z}^\beta}$$

is compact because of our first claim. Therefore there is a compact operator J_m such that, for every $i = 1, 2, \ldots,$ we have:

$$\langle T_\psi f_i, f_i\rangle = \langle T_{(\psi - p_m)}f_i, f_i\rangle + p_m(\zeta_i, \bar{\zeta}_i) + (J_m f_i, f_i).$$

Since $\psi \in C(\bar{\Omega})$ and $\{f_j\}$ tends weakly to zero, we can chose j large enough so that $|\psi(\zeta) - \psi(\zeta_j)| < \varepsilon$, $\|J_m f_j\| < \varepsilon$, and $\|T_\psi f_j\| < \varepsilon$. Thus,

$$|\varphi(\zeta)| \leqslant 2\varepsilon + |p_m(\zeta_j, \bar{\zeta}_j)| \leqslant 4\varepsilon + |\langle J_m f_j, f_j\rangle| < 5\varepsilon.$$

Hence, $\psi(\zeta) = 0$, and our second claim is proved. We now use a standard argument to prove irreducibility of $C^*(B_\Omega)$. Let P be a self-adjoint idempotent on $A^2(\Omega)$ that commutes with the components of B_Ω, and let $h = P(1) \in A^2(\Omega)$. If f and g are any two analytic polynomials, we have: $\langle hf, g\rangle = \langle Pf, g\rangle = \langle f, Pg\rangle = \langle f, hg\rangle$. Since the span of the set of functions of the form $f\bar{g}$, where f and g are analytic polynomials, is dense in $L^2(\Omega)$, it follows that h is real valued, and hence constant. The fact that $P^2 = P$ yields either $P = 0$ or $P = 1$, as desired. Since B_Ω is non-normal and irreducible, we conclude that $\mathscr{K} \subset C^*(B_\Omega)$. Let $\pi: C^*(B_\Omega) \to C^*(B_\Omega)/\mathscr{K}$ be the quotient map. From the first part of the proof, we deduce that $\pi \circ L$ is a $*$-monomorphism from $C(\partial\Omega)$ into $C^*(B_\Omega)/\mathscr{K}$. To prove that it is onto, let $T \in C^*(B_\Omega)$, and let $\{q_m\}$ be a sequence of polynomials in two non-commutative indeterminates such that $\{q_m(B_\Omega, B_\Omega^*)\}$ tends to T. By the first part of this proof, there exists a sequence of compact operators $\{K_m\}$ such that $q_m(B_\Omega, B_\Omega^*) = T_{q_m} + K_m$. Furthermore via a suitable compact perturbation K'_m, we can write $T_{q_m} = L(\varphi_m) + K'_m$, where φ_m is the restriction of $q_m(z, \bar{z})$ to $\partial\Omega$, $m = 1, 2, \ldots$. Therefore, $\|\pi(T - L\varphi_m)\| = \|\pi[T - q_m(B_\Omega, B_\Omega^*)]\| \leqslant \|T - q_m(B_\Omega, B_\Omega^*)\| \to 0$, as $m \to \infty$. It follows that $\{\pi L\varphi_m\}$ forms a Cauchy sequence, and since πL is isometric, $\{\varphi_m\}$ converges to $\varphi \in C(\partial\Omega)$. We conclude that $\pi T = \pi L\varphi$, and ontoness of πL is proved, so that the exactness of the above sequence of C^*-algebras follows. The statements about the different spectra for $n = 2$ follow from [11].

REMARK 2.4. a) With a little more work, one can actually prove that $\mathscr{K} \subset C^*(B_\Delta)$, where Δ is any bounded domain such that $\partial\Delta = \partial(\bar{\Delta})$. Indeed, the irreducibility of $C^*(B_\Delta)$ is proved in the same way as in Theorem 2.3. Now let $\rho(z) = \operatorname{dist}(z, \partial\Delta)$. It follows that $\rho \in C(\bar{\Delta})$, and that $T_\rho \neq 0$, because $\langle T_\rho 1, 1\rangle \neq 0$. Since the restriction of ρ to $\partial\Delta$ is zero, one argues, just as in the proof of Theorem 2.3, to deduce that T_ρ is compact, forcing $\mathscr{K}$ to be contained in $C^*(B_\Delta)$.

b) Using the solvability of the $\bar{\partial}$-Neumann problem on a bounded pseudoconvex domain Ω, and employing an argument that involves the Čech-cohomology of Ω with coefficients in the sheaf generated by $A^2(\Omega)$, one can prove that, if in addition, Ω satisfies the condition of part a) of this remark, then the n-tuple $B_\Omega - \omega$ is left exact for every $\omega \in \Omega$, and it is exact for every $\omega \in \mathbf{C}^n \setminus \bar{\Omega}$ (see [23, Section 4], for a similar argument). Further, one can prove that $\operatorname{Ker}(\bar{\partial}_{B_\Omega - z})^* \cap (\bar{\partial}_{B_\Omega - z})$ is one dimensional for every z in Ω. In particular, this means that B_Ω belongs to the class $A_1(\Omega)$, introduced in [10]. In other words, the kernel $K(z, \omega)$ is a generalized Bergman kernel according to [10, Definition 4.10].

c) The exact sequence in Theorem 2.3, actually gives rise to an extension τ_Ω of $\mathscr{K}$ by $C(\partial\Omega)$, i.e., τ_Ω is a unital $*$-monomorphism from $C(\partial\Omega)$ into the Calkin algebra. The equivalence class containing τ_Ω up to unitary equivalence is denoted by $[\tau_\Omega]$. Furthermore, the map L (introduced in the proof of Theorem

2.3) is a completely positive lifting of τ_Ω, i.e., $\tau_\Omega = \pi L$, where π is the canonical quotient map into the Calkin algebra. It follows that L is $\mathscr{C}_p$-smooth according to [26] (here, $\mathscr{C}_p$ denotes the pth-Schatten ideal) if and only if $L(|z_i|^2) - L(\bar{z}_i)L(z_i) \in \mathscr{C}_p$, for $1 \leqslant i \leqslant n$. This is the case, with p appropriately large, for the ellipsoidal domains Ω_q, where $q = (q_1, \ldots, q_n)$ is an n-tuple in $\mathbf{Z}_+$. We recall that

$$\Omega_q = \{\zeta \in \mathbf{C}^n : \sum_{1 \leqslant j \leqslant n} |\zeta_j|^{2q_j} < 1\},$$

(see the calculations in [11, Section 5]). The domains Ω_q are pseudoregular because $\partial\Omega_q$ is real analytic (see Remark 1.8 part c)). However, the fact that B_Ω is essentially normal can be proved by a direct calculation, as in [11, Section 5].

THEOREM 2.5. *Let Ω and Ω' be pseudoregular domains in $\mathbf{C}^n$. Then the following conditions are equivalent:*

a) *There exists a homeomorphism $h: \partial\Omega \to \partial\Omega'$ such that $h^*[\tau_\Omega] = [\tau_{\Omega'}]$, where h^* is the homomorphism induced by h at the* Ext-*level.*

b) *There exists a unitary $U: A^2(\Omega) \to A^2(\Omega')$, such that $C^*(B_\Omega) = U^*C^*(B_{\Omega'})U$.*

Proof. If π ad π' are the quotient maps by the ideal of compact operators, and L and L' are the corresponding completely positive liftings of τ_Ω and $\tau_{\Omega'}$, respectively (as in Remark 2.4 (c)), so that $\tau_\Omega = \pi L$, and $\tau_{\Omega'} = \pi L'$. We first prove that b) implies a). Thus, let U be a unitary transformation as in b). Then, $\pi \circ L$ and $\pi \circ (U^*L'(\cdot)U)$ are $*$-isomorphisms from $C(\partial\Omega)$ and $C(\partial\Omega')$ onto the same quotient algebra. Therefore, there exists a homeomorphism $h: \partial\Omega \to \partial\Omega'$, such that, for every f in $C(\partial\Omega')$, we have: $\pi(U^*L'(f)U) = \pi L(f \circ h)$. This means that a) is valid. On the other hand, if a) is satisfied, then there must exist a unitary $U: A^2(\Omega) \to A^2(\Omega')$, such that the last identity holds. Since $\mathscr{K} \subset C^*(B_\Omega)$, and $\mathrm{Ran}(L) \subset C^*(B_\Omega)$, b) easily follows.

REMARK 2.6. As expected, the structure of $C^*(B_\Omega)$ is seldom fine enough to distinguish biholomorphically inequivalent pseudoregular domains. For example, let Ω and Ω' be two different ellipsoidal domains of the type described in Remark 2.4c). Let $h: \Omega \to \Omega'$ be the radial projection. Then, by an easy index argument, $h^*\tau_\Omega = \tau_{\Omega'}$, but Ω and Ω' are not biholomorphically equivalent. In fact, if $\Omega = \Omega_p$ and $\Omega' = \Omega_q$, where the first component (say) of p and q are relatively prime, then there does not exist any proper holomorphic mapping that maps one domain onto the other (see [20]).

DEFINITION 2.7. Let $S(B_\Omega)$ be the norm closure in $C^*(B_\Omega)$ of a set of elements of the form $f(B_\Omega)$, where f is a complex valued function holomorphic in a neighborhood of $\bar{\Omega}$, and $f(B_\Omega)$ is calculated, for instance, using the Bochner-Martinelli kernel (see [28]). In other words, $S(B_\Omega)$, which was called in [27] the spectral algebra of B_Ω, is the closure of the analytic functional calculus of B_Ω.

THEOREM 2.8. *Assume that Ω and Ω' are both pseudoregular domains in $\mathbf{C}^n$ and that they satisfy the Mergelyan property (see* [13]*), i.e., every function in $C^\infty(\bar{\Omega})$ holomorphic on Ω is the uniform limit on $\bar{\Omega}$ of holomorphic functions in a neighborhood of $\bar{\Omega}$. Then the following conditions are equivalent:*

a) *Ω and Ω' are biholomorphically equivalent.*

b) *There exists a unitary $V: A^2(\Omega') \to A^2(\Omega)$ such that $V^*S(B_\Omega)V = S(B_{\Omega'})$.*

Proof. Assume there exists a biholomorphism $\varphi: \Omega \to \Omega'$. Since both Ω and Ω' satisfy condition R (see Remark 1.11), by results of [4], we can extend φ to $\partial\Omega$ so that it becomes a homeomorphism $h: \partial\Omega \to \partial\Omega'$. However, we need a more special unitary transformation than the one provided by Theorem 2.5. Indeed, the map φ induces a unitary transformation $V: A^2(\Omega') \to A^2(\Omega)$, given by $Vg = \det(\varphi')g\circ\varphi$. This is the standard transformation induced by the change of variables. The Bergman kernels are also transformed under the standard formula

$$K_\Omega(\zeta, \omega) = \det(\varphi'(\zeta))K_{\Omega'}(\varphi(\zeta), \varphi(\omega))\overline{\det(\varphi'(\omega))}.$$

We then have: $VB_\Omega V^* = T_\varphi$, where T_φ is the n-tuple of Toeplitz operators on $A^2(\Omega)$ determined by φ. Given a holomorphic function g in a neighborhood of Ω', the function $g\circ\varphi$ is holomorphic on Ω and is in $C^\infty(\bar{\Omega})$. Since Ω has the Mergelyan property, there exists a sequence $\{f_m\}$ of holomorphic functions in neighborhoods of $\bar{\Omega}$ that converges uniformly to $g\circ\varphi$. But, $T_{f_m} = f_m(B_\Omega)$, so that $T_\varphi \in S(B_\Omega)$, and hence $V^*S(B_{\Omega'})V \subset S(B_\Omega)$. Since the other inclusion is obtained by using V^{-1} and φ^{-1} in place of V and φ, we proved that a) implies b). For the opposite implication, let V be a unitary transformation as in b). Further, let π be the quotient map onto $C^*(B_\Omega)/\mathcal{K}$ and let $\tau_\Omega: C(\partial\Omega) \to \pi C^*(B_\Omega)$, as in the proof of Theorem 2.5. Also, let $T = V^*B_{\Omega'}V$. Since $T \in S(B_\Omega)$, by b), $\pi T = \tau_\Omega(h)$, where $h: \partial\Omega \to \partial\Omega'$ is a homeomorphism which is the uniform limit of functions holomorphic in a neighborhood of $\bar{\Omega}$. Therefore, h extends holomorphically to Ω. Using the same argument with V and h replaced by V^{-1} and h^{-1}, we conclude that h is a biholomorphism from Ω onto Ω', and a) follows.

REMARK 2.9. a) We point out that the Mergelyan property of a domain Ω forces $\bar{\Omega}$ to have a system of pseudoconvex neighborhoods. On the other hand, the only other requirement for the proof of Theorem 2.8 to go through is that Ω and Ω' satisfy property R. It would be interesting to know if the Mergelyan property is really necessary for the validity of the conclusion of Theorem 2.8.

b) Of course, there are pseudoconvex domains (whose boundary is not smooth) for which b) does not imply a) in Theorem 2.8. For example, consider $\Omega = \mathbf{D}\times(\mathbf{D}\setminus\{0\})$ and $\Omega' = \mathbf{D}\times\mathbf{D}$, where $\mathbf{D}$ is the open unit disk in $\mathbf{C}$. By the L^2-Riemann extension Theorem (see for instance [2]), one can show that every f in $A^2(\Omega)$ extends uniquely to Ω', and that the extending map is a unitary transformation. It is obvious that statement b) of Theorem 2.8 is valid but a) is not.

c) Among the class of domains Ω for which Theorem 2.8 is valid, we have a natural invariant for the biholomorphic equivalent classes of these domains, provided by the K-Theory of the spectral algebra of B_Ω, i.e., $K_*(S(B_\Omega))$ provides obstruction to biholomorphic equivalence. On the other hand, if the structures of $C^*(B_\Omega)$ and $C^*(B_{\Omega'})$ are different, then Ω and Ω' cannot be biholomorphic. In the next section, we shall see that, in some cases, the structures of such C^*-algebras can be so different that there may not even exist any proper holomorphic mapping from Ω onto Ω'.

3. PROPER HOLOMORPHIC MAPPINGS

We recall that given two domains Δ and Ω in $\mathbf{C}^n$, a proper holomorphic mapping $\varphi: \Delta \to \Omega$, is a holomorphic function under which preimages of compact subsets of Ω are compact in Δ. It then follows (see [24], Chapter 15) that φ is an open map onto Ω. Further, if $\mathscr{X} \subset \Omega$ is the zero variety of the complex Jacobian $\det(\varphi')$ of φ and $\mathscr{Y} = \varphi(\mathscr{X})$, then card$\{\varphi^{-1}(\omega)\}$ is a fixed finite number m for every $\omega \in \Omega \diagdown \mathscr{Y}$. The number m is called the multiplicity or total branching order of φ.

THEOREM 3.1. *Let Δ and Ω be two bounded domains and let $\varphi: \Delta \to \Omega$ be a proper holomorphic mapping. Let T_φ be the Toeplitz n-tuple on $A^2(\Delta)$ with symbol φ. Then there exists a subspace $\mathscr{M}$ of $A^2(\Delta)$ which is reducing for T_φ, and such that the restriction of T_φ to $\mathscr{M}$ is unitarily equivalent to B_Ω.*

Proof. Let $\mathscr{X} \subset \Delta$ and $\mathscr{Y} \subset \Omega$ be as above. Then there exists an open subset Ω' of $\Omega \diagdown \mathscr{Y}$ which is a union of open balls whose closures are disjoint and contained in $\Omega \diagdown \mathscr{Y}$, and such that the (Lebesgue) measure $\lambda(\Omega \diagdown \Omega')$ of the complement of Ω' in Ω is zero. Since each component of Ω' (i.e., an open ball) is simply connected and it is a subset of $\Omega \diagdown \mathscr{Y}$, there are holomorphic cross sections $\psi_j: \Omega' \to \Delta$ of the covering map $\varphi: \varphi^{-1}(\Omega') \to \Omega'$, $1 \leqslant j \leqslant m$, such that the open sets $\Delta_j = \psi_j(\Omega')$ are disjoint. Here, m is the multiplicity of φ. Further, if we let $\varphi^{-1}(\Omega') = \Delta'$, then we claim that $\lambda(\Delta \diagdown \Delta') = 0$. Indeed, notice that $\Delta' = \bigcup_{j=1}^{m} \Delta_j$. Then, by the change of variables formula, we deduce that, for every function f in $L^2(\Omega)$, we have:

$$\int_\Delta |f \circ \varphi(z)|^2 |\det \varphi'(z)|^2 d\lambda(z) =$$

$$= m \int_\Omega |f(\omega)|^2 d\lambda(\omega).$$

Taking f to be the characteristic function of $\Omega \diagdown \Omega'$, our claim is established. Now, we construct a unitary transformation U from $A^2(\Omega') \otimes \mathbf{C}^m$ onto $A^2(\Delta')$, defined by $U(f_1, \ldots, f_m)(z) = f_j \circ \varphi(z) \det[\varphi(z)]$, $z \in \Delta_j$, $1 \leqslant j \leqslant m$. Also, we define an isometry $D: A^2(\Omega) \to A^2(\Omega) \otimes \mathbf{C}^m$, by the diagonal map $Df = (f, \ldots, f)/\sqrt{m}$, $f \in A^2(\Omega)$. It readily follows that every function in the range of the isometry UD can be extended holomorphically to a function in $A^2(\Delta)$, so that the extending map V is unitary. Indeed, note that given f in $A^2(\Omega)$, $\det(\varphi')f \circ \varphi \in A^2(\Delta)$, and that for every $z \in \Delta$, we have: $(VUDf)(z) = (1/\sqrt{m})\,[\det(\varphi')f \circ \varphi](z)$. It is also easy to see that U^* and D^* are given by $U^*g = \det(\psi_j')g \circ \psi_j$, $1 \leqslant j \leqslant m$, $g \in A^2(\Delta')$, and $D^*(f_1, \ldots, f_m) = (1/\sqrt{m}) \cdot \sum_{1 \leqslant j \leqslant m} f_j$. Let R be the final projection of the isometry $W = VUD$, i.e., $R = WW^*$. Then for every f in $A^2(\Delta)$, we have: $(Rf)(z) = (1/m) \sum_{1 \leqslant j \leqslant m} f \circ \psi_j \circ \varphi(z)$, $z \in \Delta'$. Notice that the right hand side also makes sense for any z in $\Delta \setminus \mathscr{X}$, as long as we choose appropriate holomorphic cross sections of φ. Also, the value of the right hand side at z does not depend on that choice, so that the function Rf is well defined by the right hand side of the above equation on $\Delta \setminus \mathscr{X}$. Further, by the L^2-version of the Riemann Extension Theorem (see [2]), the defining expression of Rf can be extended holomorphically to Δ. We next show that the space $\mathscr{M} = \operatorname{Ran}(R)$ satisfies the required properties. To this end we prove that R commutes with T_φ. Indeed, let $z \in \Delta'$, and $f \in A^2(\Delta)$. Then,

$$(T_\varphi Rf)(z) = \varphi(z)\,(1/m) \sum f^2(\psi_j \cdot \varphi(z)) =$$

$$= \sum (\varphi f) \circ \psi_j \circ \varphi(z).$$

Therefore, $T_\varphi R = RT_\varphi$. In order to complete the proof of the theorem it suffices to show that the restriction of the n-tuple T_φ, acting on $A^2(\Delta')$, to $\mathscr{N} = \operatorname{Ran}(V^*RV)$ is unitarily equivalent to B_Ω. But, notice that UD is a unitary transformation from $A^2(\Omega)$ onto $\mathscr{N}$ so we need only check that UD intertwines $T_\varphi|\mathscr{N}$ and B_Ω. Indeed, for every $z \in \Delta'$, and every $f \in A^2(\Omega)$, we have:

$$(UDB_\Omega f)(z) = \det(\varphi'(z))(B_\Omega f) \circ \varphi(z) =$$

$$= \det(\varphi')(z)\varphi(z)f \circ \varphi(z) = (T_\varphi UDf)(z),$$

as desired.

COROLLARY 3.2. *Let Δ and Ω be bounded domains and let $\varphi: \Delta \to \Omega$ be a proper holomorphic mapping that extends continuously to the boundary $\partial\Delta$. Then, if $C^*(B_\Delta)$ is a type* I *C^*-algebra then $C^*(B_\Omega)$ is also of type* I. *If, in addition, B_Δ is essentially normal,* i.e., *$C^*(B_\Delta)/\mathscr{K}$ is abelian, then the same property holds for B_Ω.*

Proof. Since $\varphi \in C(\bar{\Delta})$, we deduce that the components of T_φ are in $C^*(B_\Delta)$. Thus, if $C^*(B_\Delta)$ is type I, then $C^*(T_\varphi)$ is also type I (see [21]), and if B_Δ is essentially normal, then T_φ is also essentially normal. Now, the corollary is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let Δ and Ω be smoothly bounded pseudoconvex domains in* $\mathbf{C}^n$, *and assume that Δ satisfies condition* R. *If $C^*(B_\Omega)$ is not type* I, *but $C^*(B_\Delta)$ is type* I, *then there cannot exist any proper holomorphic mapping $\varphi: \Delta \to \Omega$. The same conclusion holds, if B_Δ is essentially normal but B_Ω is not.*

Proof. Assume, by way of contradiction, there is a proper holomorphic mapping $\varphi: \Delta \to \Omega$. Then, by [3], or [12], φ can be extended smoothly to the boundary $\partial\Delta$. In particular, $\varphi \in C(\bar{\Delta})$, and Corollary 3.2 applies.

Remark 3.4. a) From Theorem 2.3, it follows that if Δ is pseudoregular, then B is essentially normal. Thus, if Δ is pseudoregular, but Ω is not, it is very likely that a conclusion similar to that of Corollary 3.3 holds. Of course, this would be the case if the converse to the statement at the beginning of this paragraph were true. We suspect that this is the case but we are unable to prove it at the time of writing.

b) As we already saw in Remark 2.4 c), $C^*(B_\Delta)$ and $C^*(B_\Omega)$ can be $*$-isomorphic and both B_Δ and B_Ω can be essentially normal, but there may not exist any proper holomorphic mapping that maps one domain onto the other. However, in view of Corollary 3.2, it is still of significant interest to construct examples of bounded domains Ω such that B_Ω is not essentially normal, or furthermore, such that $C^*(B_\Omega)$ is not of type I. When Ω is not pseudoconvex, this is not difficult to do. For example, if Ω is a complete Reinhardt domain, and its logarithmic convex hull is different from Ω, then B_Ω is not essentially normal (see [11]). Further, it was shown, in [9], that if $\Omega = \Omega_{\varepsilon,\delta}$ is the union of two polydisks in $\mathbf{C}^2$ of multiradii $(1, \varepsilon)$ and $(\delta, 1)$, with $\delta \leqslant 1$, $\varepsilon \leqslant 1$, then $C^*(B_\Omega)$ is type I if and only if $\log(\varepsilon)/\log(\delta)$ is rational. On the other hand, if Ω is pseudoconvex, but $\partial\Omega$ is not smooth, again there are many examples of the above phenomenon. For instance, if Ω is a polydisk in $\mathbf{C}^n$, then B_Ω is easily seen not to be essentially normal (see, for example, [11]). Also, if $\Delta_{\varepsilon,\delta}$ is the logarithmic convex hull of the domain $\Omega_{\varepsilon,\delta}$ mentioned above, then a refinement of the techniques used in [9] shows that an analogous result to the one about the domains $\Omega_{\varepsilon,\delta}$ quoted previously, also holds for the domains $\Delta_{\varepsilon,\delta}$.

c) We recall (see [2, Section 5]) that a proper holomorphic mapping $\varphi: \Delta \to \Omega$ can be factored, if and only if the following property is satisfied:

There exists a subgroup $G_\varphi \subset$ AUT(Δ) *of the group of (holomorphic) automorphisms of Δ such that $\varphi\circ\gamma(\zeta) = \varphi(\zeta)$, for every $\zeta \in \Delta$ and every $\gamma \in G_\varphi$; also, $\varphi^{-1}\varphi(\zeta) = \bigcup_\gamma \{\gamma(\zeta)\}$ for every $\zeta \in \Delta$, where the union runs over all $\gamma \in G_\varphi$.*

The existence of G_φ gives rise to a canonical factorization $\varphi = \tilde{\varphi}\,\eta$, where $\eta: \Delta \to \Delta/G_\varphi$ is the quotient map and $\tilde{\varphi}: \Delta/G_\varphi \to \Omega$ is a biholomorphism. An immediate consequence of the next theorem is the fact that when $\varphi: \Delta \to \Omega$ is a proper holomorphic mapping from a pseudoconvex domain Δ onto a pseudoconvex domain Ω, and the above property holds, then T_φ is in the class $A_m(\Omega)$ introduced in [10] (see Remark 2.4 (b)), where m is the multiplicity of φ.

The following theorem is a strengthening of Theorem 3.1, in the case that φ is factorable.

THEOREM 3.5. *Let Δ and Ω be bounded domains in $\mathbf{C}^n$, and let $\varphi\colon \Delta \to \Omega$ be a proper holomorphic mapping. If the property of Remark* 3.4 c) *holds, then T_φ is unitarily equivalent to $B_\Omega \otimes I_m$, where m is the multiplicity of φ.*

Proof. It is clear (see [2], page 168) that $\mathrm{Ord}(G_\varphi) = m$. We let $G_\varphi = \{\gamma_1, \ldots, \gamma_m\}$, and proceeding as in the proof of Theorem 3.1, we shall define (orthogonal) projetions $R_1, \ldots, R_m$ in $\mathscr{L}[A^2(\Delta)]$ whose sum is the identity on $A^2(\Delta)$, and reduce T_φ. To this end we first point out two important properties of the group G_φ:

$$(*) \qquad |\det(\gamma')| = 1, \quad \text{for all } \gamma \in G_\varphi$$

$$(**) \qquad \sum_{0 \leqslant j \leqslant m-1} (\det(\gamma'))^j = 0, \quad \text{for all } I \neq \gamma \in G_\varphi .$$

The above properties are verified easily using, several times, the change of variables formula (see the beginning of the proof of Theorem 3.1). For $1 \leqslant i \leqslant m$, and $f \in A^2(\Delta)$, we define

$$R_i f = (1/m) \sum_{1 \leqslant j \leqslant m} [\det(\gamma_j')]^{i-1} f \circ \gamma_j .$$

Using the chain rule, it readily follows that $(R_i)^2 = R_i$. Also, employing $(*)$ and the change of variables formula, we deduce that $(R_i)^* = R_i$. Furthermore, it is easy to see, from $(**)$, that $\sum_j R_j = I$. Therefore, the projections R_i must be orthogonal, i.e., $R_i R_j = 0$, $i \neq j$. Notice that R_1 coincides with the projection R defined in the proof of Theorem 3.1. In fact, if ψ_j, $1 \leqslant j \leqslant m$, are local inverses for the mapping φ, defined in neighborhoods of Ω outside of the variety $\mathscr{Y}$ of critical values of φ, we have the following alternative expression for R_i on $\Delta \setminus \mathscr{X}$ (here $\mathscr{X}$ is the zero variety of $\det(\varphi')$):

$$(***) \qquad (R_i f)(z) = (1/m) \sum_{1 \leqslant j \leqslant m} [\det(\psi_j \circ \varphi)'(z)]^{i-1} f \circ \psi_j \circ \varphi(z).$$

As in the proof of Theorem 3.1, we point out that, for $z \in \Delta \setminus \mathscr{X}$, the righ thand side of $(***)$ is independent of the choice of local inverses. Using the L^2-version of the Riemann Extension Theorem (see [2]), the right hand side of $(***)$ can be extended uniquely to a function in $A^2(\Delta)$, and the extending map is unitary. Finally, it is easy to check, employing the same arguments as in the proof of Theorem 3.1, that $T_\varphi R_i = R_i T_\varphi$, and that $T_\varphi \mid \mathrm{Ran}(R_i)$ is unitarily equivalent to B_Ω, $1 \leqslant i \leqslant m$. This completes the proof of the theorem.

REMARK 3.6. Given a proper holomorphic mapping $\varphi\colon \Delta \to \Omega$, it would be interesting to know under what less restrictive conditions the conclusion of Theorem 3.5 holds. Observe that if the open sets Δ' and Ω' and the unitary transformation

$U: A^2(\Omega')\otimes \mathbf{C}^m \to A^2(\Delta')$ are as in the proof of Theorem 3.1, then $UT_\varphi U^* = B_\Omega \otimes \otimes I_m$. In the proof of Theorem 3.5, we were able to extend U to a unitary transformation from $A^2(\Omega)\otimes \mathbf{C}^m \to A^2(\Delta)$. In general, if Γ is a simply connected domain in $\Omega\setminus\mathscr{Y}$, where $\mathscr{Y}$ is the variety of critical values of φ, then we can define a generalized Bergman kernel $K_\varphi: \Gamma\times\Gamma \to \mathscr{L}(\mathbf{C}^m)$ (in the sense of [10]), so that its canonical model (i.e., the multiplication n-tuple by Z on the Hilbert space generated by K_φ, see [10, Section 3]) is unitarily equivalent to T_φ. Indeed, we let $K_\varphi(\zeta, \omega)$ be the operator whose representing matrix on the standard basis of $\mathbf{C}^m$ is given by

$$(K_\varphi(\zeta, \omega))_{p,q} = (\det\psi_p')(\zeta)K(\psi_q(\zeta), \psi_q(\omega))\overline{\det(\psi_q')}(\omega),$$

where ψ_p, $1 \leqslant p \leqslant m$, are holomorphic cross sections of φ on Γ, and K is the Bergman kernel on Δ. Notice that the trace of K_φ can be extended to $\Omega\times\Omega$. In fact, if K' denotes the Bergman kernel on Ω, we obtain the transformation law for the Bergman kernels under the mapping φ (see [2]), namely:

$$K'(z, w) = \operatorname{trace}(K_\varphi)(z, w), \quad z, w \text{ in } \Omega.$$

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