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ON THE EXISTENCE OF INVARIANT SUBSPACES FOR SOME CONTRACTIONS WITH SPECTRUM DOMINATING AN ARC ON THE UNIT CIRCLE

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1. INTRODUCTION

For $\mathcal{H}$ a separable, infinite dimensional Hilbert space $\mathcal{L}(\mathcal{H})$ denote the algebra of linear bounded operators on $\mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(A)$ the spectrum of A and by $\sigma_{le}(A)$ the left essential spectrum of A. An operator $A \in \mathcal{L}(\mathcal{H})$ is called a *contraction* if $\|A\| \leqslant 1$. For various properties of contractions on Hilbert spaces our standard reference is [11]. For dual algebras and related topics see [1].

In [4], S. Brown, B. Chevreau and C. Pearcy proved that the contractions whose spectrum dominate the unit circle have nontrivial invariant subspaces. The same problem is still open for the contractions whose spectrum dominate an arc on the unit circle.

In the present paper we prove a theorem of existence of nontrivial invariant subspaces for these contractions under an additional hypothesis.

Let $\mathbf{D}$ denote the open unit disc and $\mathbf{T}$ the unit circle. Recall from [8] the following theorem, the form in which we use the techniques developed in [4].

THEOREM 1.1. *Let A be an absolutely continuous contraction (see* Chapter IV *in* [1]*) in $\mathcal{L}(\mathcal{H})$. We make the standard assumptions $\|A\| = 1$, $A \in C_{0\cdot}$ (that is A^n converges strongly to 0 as $n \to \infty$) $\sigma(A) = \sigma_{le}(A)$ and $\sigma(A) \cap \mathbf{D} \neq \emptyset$.*

Denote by Φ the canonical homomorphism $\Phi\colon H^\infty \to \mathcal{L}(\mathcal{H})$, $\Phi(h) = h(A)$. Let $\mathcal{E} \subset H^\infty$ be a weak-closed subspace with the property:*

$$(1.1)\qquad \|f\|_\infty \leqslant M \sup_{z \in \sigma(A) \cap \mathbf{D}} |\hat{f}(z)| \quad \text{for every } f \text{ in } \mathcal{E}$$

(here $\hat{f}$ is the analytic extension of f to $\mathbf{D}$). Then $\Phi(\mathcal{E})$ is weak-closed in $\mathcal{L}(\mathcal{H})$ and has property $(\mathbf{A}_{\aleph_0})$ (see* [1] *for the definition of the property $(\mathbf{A}_{\aleph_0})$).*

We apply this theorem to contractions $A \in \mathscr{L}(\mathscr{H})$ for which $\sigma(A) \cap \mathbf{D}$ dominates an arc $\gamma \subset \mathbf{T}$ with $1 \in \gamma$. (A subset S of $\mathbf{D}$ is dominating for a subset γ of $\mathbf{T}$ if almost every point of γ is a nontangential limit point of S.)

Our approach consists in constructing a subspace $\mathscr{E}$ of H^∞, weak*-closed and with the property (1.1). By Theorem 1.1, for every $\lambda \in \mathbf{D}$ there exist nonzero x and y in $\mathscr{H}$ such that $(f(A)x, y) = f(\lambda)$ for every $f \in \mathscr{E}$.

Let U be the minimal unitary dilation of A, $E(\cdot)$ its spectral measure and $\varphi_{x,y}$ the density of the scalar measure $\mu_{x,y}(\sigma) = (E(\sigma)x, y)$, x and y in $\mathscr{H}$ (the existence of $\varphi_{x,y}$ follows by Theorem 6.4 in Chapter II, § 6 of [11]). Our main result is Theorem 3.4 asserting that if there exists $\lambda \in \mathbf{D}$ such that there exist x and y in $\mathscr{H}$, $x \neq 0, y \neq 0$ with $[x \otimes y] = [C_\lambda]$ in the predual of $\Phi(\mathscr{E})$ and 0 is a Lebesgue point for $\varphi_{x,y}$ ($\varphi_{x,y}$ is defined on $(-\pi, \pi]$) then A has a nontrivial invariant subspace.

2. ε-SUPPORTED SUBSPACES

Let $I \subset (-\pi, \pi]$ be an interval. We denote by γ_I the arc corresponding to I on the unit circle, so $\gamma_I = \{e^{it} \mid t \in I\}$.

DEFINITION 2.1. Let $I \subset (-\pi, \pi]$ be an interval, $\mathscr{E} \subset H^\infty$ a linear subspace, $\varepsilon > 0$. We say $\mathscr{E}$ is *ε-supported on I* (equivalently on γ_I) if and only if

$$\sup_{z \in \mathbf{T} \setminus \gamma_I} |f(z)| \leqslant \varepsilon \|f\|_\infty \quad \text{for every } f \text{ in } \mathscr{E}.$$

$f \in H^\infty$ is ε-supported on I if $\operatorname{span}\{f\}$ is ε-supported on I.

From now on a function in H^∞ and its analytic extension to $\mathbf{D}$ will be denoted by the same symbol.

DEFINITION 2.2. The operators $Q_n\colon H^\infty \to H^\infty$ *and* $S_n\colon H^\infty \to H^\infty$ are defined for $n \geqslant 1$, $n \in \mathbf{N}$ by

$$(Q_n f)(z) = \frac{1}{n} \cdot \sum_{k=1}^{n} e^{\frac{2\pi i k}{n}} f(e^{\frac{2\pi i k}{n}} z)$$

$$(S_n f)(z) = \frac{1}{n} \cdot \sum_{k=1}^{n} f(e^{\frac{2\pi i k}{n}} z)$$

for all $z \in \mathbf{D}$.

We will need the following result from [12] (Lemma 2).

PROPOSITION 2.3. (a) S_n, Q_n *are projections of norm one on* H^∞. *The images of these projectios,* $\operatorname{Ran} Q_n$, $\operatorname{Ran} S_n$ *are isomorphic as Banach spaces with the whole space;*

(b) $Q_n(1) = 0$ *for every* $n \geqslant 2$; $S_n(1) = 1$ *for every* $n \geqslant 1$;

(c) $f \in \operatorname{Ran} Q_n$ *implies* $f(e^{\frac{2\pi i}{n}} z) = e^{-\frac{2\pi i}{n}} f(z)$, $z \in \mathbf{D}$;

(d) *For* $f \in \operatorname{Ran} Q_n$, $F \in H^\infty$ *we have* $Q_n(F \cdot f) = S_n(F)f$;

(e) *Let* $h \in H^\infty$, *such that* h *is* δ*-supported on an interval* I *and* $2\pi/n < |I| < \pi/4$. *Then*

$$\|S_n(h)\|_\infty \leqslant \left(\delta + \frac{|I|}{\pi}\right) \|h\|_\infty$$

$$\|Q_n(h)\|_\infty \leqslant \left(\delta + \frac{|I|}{\pi}\right) \|h\|_\infty$$

where by $|I|$ *we denote the length of the interval* I.

PROPOSITION 2.4. *For every* $\varepsilon \in (0, 1/2)$ *and every interval* $I \subset (-\pi, \pi)$ *there exists a weak*-closed subspace* $\mathscr{E}$ *of* H^∞, ε*-supported on* I, *isomorphic with* $\operatorname{Ran} Q_n$ *as Banach spaces.*

Proof. This is essentially Proposition 2 from [12] but since the construction of $\mathscr{E}$ is important for the next steps we give its proof in full detail.

Let $J_1 \subset (-\pi, \pi)$ be an interval with $|J_1| < \varepsilon$. We construct a function F analytic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$ such that

$$\|F\|_\infty = 1$$

$$|F(z) - 1| < \frac{\varepsilon}{8} \|1 - F\|_\infty \quad \text{for every } z \in \mathbf{T} \setminus \gamma_{J_1} \tag{2.1}$$

$$|F(z)| < \frac{\varepsilon}{3} \quad \text{for every } z \in \gamma_{J_2} \tag{2.2}$$

with $J_2 \subset J_1$ a properly chosen interval.

Consider $f(z) = 1/(z + 2)^k$ with $k \in \mathbf{N}$ chosen such that $|f(z)| < \varepsilon/3$ for every $z \notin \Delta = \{z \in \overline{\mathbf{D}} \mid |z + 1| < \eta\}$, η being a small positive number conveniently chosen.

By observing that $\|f\|_\infty = 1$, $f(-1) = 1$, f is continuous, we infer that there exists an arc γ_J centered at -1, $\gamma_J \subset \Delta$ such that

$$|f(z) - 1| < \frac{\varepsilon}{8} \|f - 1\|_\infty \quad \text{for every } z \in \gamma_J.$$

For $\varphi : \mathbf{D} \to \mathbf{D}$ a conformal mapping we also denote by φ its continuous extension to $\mathbf{D}$.

Let then $\varphi : \mathbf{D} \to \mathbf{D}$ be a conformal mapping such that $\varphi(\mathbf{T}\setminus\gamma_{J_1}) = \gamma_J$, implying $\varphi(\gamma_{J_1}) = \mathbf{T}\setminus\gamma_J$.

Let $\gamma_{J_2} = \varphi^{-1}(\mathbf{T}\setminus(\Delta \cap \mathbf{T}))$. Then $\varphi(\gamma_{J_2}) \subset \mathbf{T}\setminus\gamma_J = \varphi(\gamma_{J_1})$ and this is equivalent to $\gamma_{J_2} \subset \gamma_{J_1}$, that is $J_2 \subseteq J_1$. The function we need will be $F = f \circ \varphi$. Indeed $\|F\|_\infty = 1$; if $z \in \mathbf{T}\setminus\gamma_{J_1}$ then $t = \varphi(z) \in \gamma_J$ and we have

$$|F(z) - 1| = |(f\circ\varphi)(z) - 1| = |f(t) - 1| < \frac{\varepsilon}{8}\|f - 1\|_\infty = \frac{\varepsilon}{8}\|F - 1\|_\infty$$

(the composition operator $f \mapsto f\circ\varphi$ is isometric on H^∞).

If $z \in \gamma_{J_2}$ then $\varphi(z) \notin \Delta$ such that

$$|F(z)| = |f(\varphi(z))| < \frac{\varepsilon}{3}.$$

Let $n \in \mathbf{N}$ be such that $2\pi/n < |J_1|$. (2.1) implies $F - 1$ is $(\varepsilon/8)$-supported on J_1 such that from Proposition 2.3 (b) and (e) we deduce

$$\|S_n(F-1)\|_\infty = \|S_n(F) - 1\|_\infty \leqslant \left(\frac{\varepsilon}{8} + \frac{\varepsilon}{\pi}\right)\|F - 1\|_\infty \leqslant \tag{2.3}$$
$$\leqslant 2\left(\frac{\varepsilon}{8} + \frac{\varepsilon}{\pi}\right) \leqslant \varepsilon < 1$$

and this implies $S_n(F)(z) \neq 0$ for every $z \in \mathbf{D}$ and $\frac{1}{S_n(F)} = \sum_{k=0}^{\infty} [1 - S_n(F)]^k$ is in H^∞ with

$$\left\|\frac{1}{S_n(F)}\right\|_\infty \leqslant \sum_{k=0}^{\infty} \varepsilon^k = \frac{1}{1-\varepsilon}. \tag{2.4}$$

Let $\mathcal{E}_1 = F\cdot\mathrm{Ran}\, Q_n$. We prove first that $\mathcal{E}_1$ is ε-supported on $\mathbf{T}\setminus\gamma_{J_2}$.

Let $z \in \gamma_{J_2}$ and $f \in \mathrm{Ran}\, Q_n$. By Proposition 2.3 (a), (d) and relations (2.2) and (2.4), we have

$$|F(z)f(z)| < \frac{\varepsilon}{3}\|f\|_\infty = \frac{\varepsilon}{3}\left\|\frac{1}{S_n(F)}\, Q_n(F\cdot f)\right\| \leqslant$$
$$\leqslant \frac{\varepsilon}{3(1-\varepsilon)}\|Ff\|_\infty < \varepsilon\|Ff\|_\infty \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right).$$

We show now that $\mathscr{E}_1$ is w*-closed.

Let $(F\cdot f_m)_{m=1}^{\infty}$ be a sequence in $\mathscr{E}_1$, $f_m \in \operatorname{Ran} Q_n$, $\|F\cdot f_m\|_\infty \leqslant 1$ for every $m \geqslant 1$, and $F\cdot f_m \xrightarrow[m\to\infty]{w^*} h$ with $h \in H^\infty$.

It is easy to see that Q_n is w*-continuous so that $Q_n(F\cdot f_m) \xrightarrow[m\to\infty]{w^*} Q_n(h)$ or equivalently $S_n(F)f_m \xrightarrow[m\to\infty]{w^*} Q_n(h)$ and $\|S_n(F)f_m\|_\infty \leqslant 1$ for every $m \geqslant 1$.

By (2.4), $\|f_m\|_\infty \leqslant \|S_n(F)\cdot f_m\|_\infty \cdot \left\|\frac{1}{S_n(F)}\right\|_\infty \leqslant \frac{1}{1-\varepsilon}$ for every $m \geqslant 1$ such that there exists $(f_{m_k})_{k\geqslant 1}$ a subsequence of $(f_m)_{m\geqslant 1}$ such that

$$f_{m_k} \xrightarrow[k\to\infty]{w^*} f.$$

But $f_{m_k} \in \operatorname{Ran} Q_n$ for every $k \geqslant 1$. Q_n is a w*-continuous projection such that $\operatorname{Ran} Q_n$ is a w*-closed subspace in H^∞. It results that $f \in \operatorname{Ran} Q_n$ and $F\cdot f_{m_k} \xrightarrow[k\to\infty]{w^*} F\cdot f$. But $F\cdot f_{m_k} \xrightarrow[k\to\infty]{w^*} h$ so $h = F\cdot f \in \mathscr{E}_1$.

We show now that $\mathscr{E}_1$ is isomorphic with $\operatorname{Ran} Q_n$ by the operator $Q_n \mid \mathscr{E}_1$.

For $f \in \operatorname{Ran} Q_n$, $Q_n(F\cdot f) = f\cdot S_n(F) = g$ with $g \in \operatorname{Ran} Q_n$ and the injectivity of $Q_n \mid \mathscr{E}_1$ results.

We prove the surjectivity of $Q_n \mid \mathscr{E}_1$.

Let $g \in \operatorname{Ran} Q_n$. We have to find $f \in \operatorname{Ran} Q_n$ such that $Q_n(Ff) = g$ or equivalently $f = \frac{1}{S_n(F)} g$. So it must be proved that for $g \in \operatorname{Ran} Q_n$,

$$f = \frac{1}{S_n(F)} g = \left(\sum_{k=0}^{\infty} [1 - S_n(F)]^k\right) g$$

is in $\operatorname{Ran} Q_n$, that is $Q_n(f) = f$. We have

$$Q_n(f) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j C_n^j Q_n[S_n(F)^j \cdot g].$$

But $Q_n(S_n(F)g) = S_n(S_n(F))g = S_n(F)g$ and it follows that $Q_n[S_n(F)^j g] = S_n(F)^j g$, implying $Q_n(f) = f$.

Let $f \in \operatorname{Im} Q_n$, $g = Q_n(F\cdot f) = S_n(F)\cdot f$.

By (2.4) and $\|F\|_\infty = 1$,

$$\|F\cdot f\|_\infty = \left\|g\cdot F\cdot \frac{1}{S_n(F)}\right\|_\infty \leqslant \|g\|_\infty \frac{1}{1-\varepsilon}$$

and we infer that $(Q_n \mid \mathscr{E}_1)^{-1}$ exists and $\|(Q_n \mid \mathscr{E}_1)^{-1}\| < \dfrac{1}{1-\varepsilon}$.

In the end consider $\varphi_1 \colon \mathbf{D} \to \mathbf{D}$ a conformal mapping such that $\varphi_1(\gamma_I) = \mathbf{T} \setminus \gamma_{J_2}$ and the operator $J_{\varphi_1} \colon H^\infty \to H^\infty$, $J_{\varphi_1}(f) = f\circ\varphi_1$. J_{φ_1} is an isometric isomorphism and a w*-homeomorphism. Let $\mathscr{E} = J_{\varphi_1}(\mathscr{E}_1)$. $\mathscr{E}$ satisfies the requirements of the proposition and the proof is completed.

REMARK 2.5. It can also be proved (see [12]) that $\mathscr{E}$ is the image of a w*-continuous projection on H^∞, namely $P = J_{\varphi_1}(Q_n \mid \mathscr{E}_1)^{-1} Q_n J_{\varphi_1}^{-1}$.

In the next lemma we colect some easy to prove facts about Q_n.

LEMMA 2.6. *For* $n \geqslant 2$:

$$Q_n(z^{kn+n-1}) = z^{kn+n-1} \quad \textit{for every } k \geqslant 0$$

$$Q_n(z^m) = 0 \quad \textit{for every } m \notin \{(k+1)n - 1 \mid k \geqslant 0\}$$

$$\operatorname{Ran} Q_n \supset \{z^{n-1} f(z^n) \mid f \in H^\infty\}.$$

The next proposition presents the construction of a larger space once we have a sequence of mutually disjoint intervals and the spaces given by Proposition 2.4.

PROPOSITION 2.7. *Let* $(I_n)_{n\geqslant 1}$ *be a sequence of mutually disjoint intervals in* $(-\pi, \pi)$ *and let* $(\varepsilon_n)_{n\geqslant 1}$ *be a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon < 1.$$

For every $n \geqslant 1$ *let* $\mathscr{E}_n$ *be a* w*-*closed subspace* ε_n-*supported on* I_n *and let* $\mathscr{E} = \overline{\operatorname{span}}^{w^*}\{\mathscr{E}_n\}$, *that is the weak*-closed subspace generated by* $\{\mathscr{E}_n\}_n$.

Then $\mathscr{E}$ *is in a natural way isomorphic to*

$$\left(\sum_{n=1}^{\infty} \mathscr{E}_n\right)_\infty = \{(x_n)_n \mid x_n \in \mathscr{E}_n \textit{ for } n \geqslant 1,\ \sup_n \|x_n\| < \infty\}$$

and

$$(2.5) \qquad (1-\varepsilon)\sup_n\|x_n\| \leqslant \left\|\sum_{n=1}^{\infty} x_n\right\| \leqslant (1+\varepsilon)\sup_n\|x_n\|.$$

Proof. We show first that for a series $y = \sum_{n=1}^{\infty} x_n$ with $x_n \in \mathscr{E}_n$ for every $n \geqslant 1$, w*-convergence is equivalent with bounded almost everywhere convergence on $\mathbf{T}$.

Let $y = \sum_{n=1}^{\infty} x_n$ be a series w*-convergent with $x_n \in \mathscr{E}_n$ for every $n \geqslant 1$. Then $x_n \xrightarrow[n\to\infty]{w^*} 0$, so there exists $M > 0$ such that $\|x_n\| \leqslant M$ for every $n \geqslant 1$. Let $e^{it} \in \mathbf{T}$. If $e^{it} \in \gamma_{I_{n_0}}$ then $e^{it} \notin \gamma_{I_n}$ for all $n \neq n_0$ and it results $\sum_{n=1}^{\infty} |x_n(e^{it})| \leqslant$ $\leqslant |x_{n_0}(e^{it})| + \sum_{\substack{n\geqslant 1\\ n\neq n_0}} \varepsilon_n \|x_n\|_\infty < M + M\sum_{n=1}^{\infty}\varepsilon_n = M(1+\varepsilon)$. If $e^{it} \notin \gamma_{I_n}$ for every $n \geqslant 1$ then $\sum_{n=1}^{\infty} |x_n(e^{it})| \leqslant \varepsilon M < M(1+\varepsilon)$.

So the series converges almost everywhere on $\mathbf{T}$ and the partial sums are bounded by $(1+\varepsilon)M$.

Let $\tilde{y}(e^{it}) = \sum_{n=1}^{\infty} x_n(e^{it})$ for $t \in (-\pi, \pi)$.

By Lebesgue Dominated Convergence Theorem

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left[\sum_{n=1}^{N} x_n(e^{it})\right] \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,dt =$$

$$= \int_{-\pi}^{\pi} \frac{\tilde{y}(e^{it})(1-r^2)}{1+r^2-2r\cos(t-\theta)}\,dt \quad \text{for every } r \in [0, 1),\ \theta \in (-\pi, \pi]$$

and it results $\tilde{y}(\lambda) = \sum_{n=1}^{\infty} x_n(\lambda) = y(\lambda)$ for $\lambda \in \mathbf{D}$ (see [7]). The proof above shows that bounded almost everywhere convergence on $\mathbf{T}$ implies w*-convergence for sequences.

The second inequality in (2.5) is already proved so we prove now the first one.

Let $\sum_{n=1}^{\infty} x_n$ be a w*-convergent series.

Let $m \in \mathbf{N}$ be fixed and $e^{it} \in \gamma_{I_m}$. Then $e^{it} \notin \gamma_{I_n}$ for $n \neq m$ and

$$|x_m(e^{it})| = \left|\sum_{n=1}^{\infty} x_n(e^{it}) - \sum_{\substack{n=1\\ n\neq m}}^{\infty} x_n(e^{it})\right| \leqslant$$

$$\leqslant \left\|\sum_{n=1}^{\infty} x_n\right\| + \left(\sum_{n\neq m} \varepsilon_n\right)(\sup_n \|x_n\|) < \left\|\sum_{n=1}^{\infty} x_n\right\| + \varepsilon(\sup_n \|x_n\|).$$

But

$$\|x_m\|_\infty = \sup_{e^{it}\in\gamma_{I_m}} |x_m(e^{it})|$$

such that

$$\|x_m\|_\infty \leqslant \left\|\sum_{n=1}^{\infty} x_n\right\| + \varepsilon(\sup_n \|x_n\|) \quad \text{for every } m \geqslant 1.$$

Then

$$\sup_{n\geqslant 1} \|x_n\| \leqslant \left\|\sum_{n=1}^{\infty} x_n\right\| + \varepsilon(\sup_n \|x_n\|)$$

and the first inequality in (2.5) is proved.

We prove now that $\mathscr{E}$ is identical to the space $\tilde{\mathscr{E}} = \left\{\sum_{n=1}^{\infty} x_n \,\middle|\, x_n \in \mathscr{E}_n \text{ for all } n \geqslant 1 \text{ and the series is w}^*\text{-convergent}\right\}$.

Then $\mathscr{E}_n \subset \tilde{\mathscr{E}} \subset \mathscr{E}$ for every $n \geqslant 1$ so all we have to prove is that $\tilde{\mathscr{E}}$ is w*-closed. Let $y_k = \sum_{j=1}^{\infty} x_j^k$ be in $\tilde{\mathscr{E}}$, $\|y_k\| \leqslant 1$ for every $k \geqslant 1$ and $y_k \xrightarrow[k\to\infty]{w^*} y$.

By (2.5), $\|x_j^k\|_\infty \leqslant \dfrac{1}{1-\varepsilon}$ for every $j, k \geqslant 1$ and this implies that for every $j \geqslant 1$ there exists $(p_{k,j})_{k\geqslant 1}$ a subsequence in $\mathbf{N}$ such that

$$x_j^{p_{k,j}} \to x_j \in \mathscr{E}_j \quad \text{for } k \to \infty.$$

We choose the sequence $(p_{k,j})_{k\geqslant 1}$ such that $(p_{k,j+1})_{k\geqslant 1}$ is a subsequence of $(p_{k,j})_{k\geqslant 1}$ for all $j \geqslant 1$. Then $\|x_j\| \leqslant \dfrac{1}{1-\varepsilon}$ for every $j \geqslant 1$ so there exists $\tilde{y} = \sum_{j=1}^{\infty} x_j$ and $\tilde{y} \in \tilde{\mathscr{E}}$.

We prove that $\tilde{y}(\lambda) = y(\lambda)$ for every $\lambda \in \mathbf{D}$.

Let $\lambda = re^{i\theta}$ be fixed in $\mathbf{D}$; then

$$|x_n^k(\lambda)| = \left| \int_{-\pi}^{\pi} x_k^n(e^{it}) \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} \,dt \right| \leqslant$$

(2.6)
$$\leqslant \left| \int_{I_n} x_n^k(e^{it}) \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} \,dt \right| + \left| \int_{[-\pi,\pi]\setminus I_n} x_n^k(e^{it}) \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} \,dt \right| \leqslant$$

$$\leqslant \frac{1}{1-\varepsilon} \frac{1+r}{1-r} \cdot |I_n| + \frac{\varepsilon_n}{1-\varepsilon} \quad \text{for every } k \geqslant 1.$$

We used $\dfrac{1}{1+r^2-2r\cos(t-\theta)} \leqslant \dfrac{1}{(1-r)^2}$ for all $t, \theta \in [\leqslant\pi, \pi]$ and $|x_n^k(e^{it})| < \varepsilon_n \|x_n^k\|_\infty \leqslant \dfrac{\varepsilon_n}{1-\varepsilon}$ for every $n, k \geqslant 1$ and every $t \in [-\pi, \pi]\setminus I_n$.

Let $\eta > 0$. Since $\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon < 1$, $\sum_{n=1}^{\infty} |I_n| \leqslant 2\pi$, by (2.6) there exists $N(\eta)$ such that

$$\sum_{j=N+1}^{\infty} |x_j^k(\lambda)| < \eta \quad \text{for every } N \geqslant N(\eta) \text{ and every } k \geqslant 1.$$

Then

(2.7)
$$\left| y_k(\lambda) - \sum_{j=1}^{N} x_j^k(\lambda) \right| < \eta \quad \text{for every } N \geqslant N(\eta) \text{ and every } k \geqslant 1.$$

Let now $N \geqslant N(\eta)$ be fixed. By the choice of the sequences $(p_{k,j})_{k\geqslant 1}$ we infer that

(2.8)
$$\begin{matrix} x_1^{p_{k,N}}(\lambda) \to x_1(\lambda) \\ \vdots \\ x_N^{p_{k,N}}(\lambda) \to x_N(\lambda) \end{matrix} \quad \text{as } k \to \infty$$

and also $y_{p_{n,k}}(\lambda) \to y(\lambda)$ as $k \to \infty$.

From (2.7) and (2.8), $\left| y(\lambda) - \sum_{j=1}^{N} x_j(\lambda) \right| < \eta$ for every $N \geqslant N(\eta)$ and it follows that $y(\lambda) = \sum_{j=1}^{\infty} x_j(\lambda) = \tilde{y}(\lambda)$, the proof of the proposition being thereby accomplished.

PROPOSITION 2.8. *Let $(I_n)_{n\geqslant 1}$ be a sequence of disjoint intervals in $(-\pi, \pi)$ and $(\varepsilon_n)_{n\geqslant 1}$ a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon < 1/3$. For every $n \geqslant 1$, let X_n be a weak*-closed subspace ε_n-supported on I_n, and let $(f_n)_{n\geqslant 1}$ be a sequence of inner functions. Let $\mathscr{E}_n = f_n X_n$ and $\mathscr{E} = \overline{\text{span}}^{w^*}\{\mathscr{E}_n\}_1^{\infty}$.*

Then $\mathscr{E}_n$ is weak-closed and ε_n-supported on I_n for every $n \geqslant 1$ and $\mathscr{E}$ is $1/2$-supported on $\bigcup_{n=1}^{\infty} I_n$.*

Proof. The assertion about $\mathscr{E}_n$ is obvious. For $\mathscr{E}$ we use Proposition 2.7. Let $f = \sum_{n=1}^{\infty} f_n x_n$ be in $\mathscr{E}$, $x_n \in X_n$ for all $n \geqslant 1$. Let $t \notin \bigcup_{n=1}^{\infty} I_n$. Then $|f(e^{it})| \leqslant \sum_{n=1}^{\infty} \varepsilon_n \|x_n\| \leqslant$ $\leqslant \varepsilon(\sup_n \|x_n\|) = \varepsilon \sup_n \|f_n x_n\| \leqslant \frac{\varepsilon}{1-\varepsilon} \|f\| < (1/2)\|f\|$.

3. AN INVARIANT SUBSPACE THEOREM

We consider A a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$, $\|A\| = 1$, $\sigma(A) = \sigma_{\text{le}}(A)$, $A \in C_0$.

LEMMA 3.1. *Let $\mathscr{E}$ be a weak*-closed subspace of H^∞, ε-supported on an interval I with $0 < \varepsilon < 1$. If $\sigma(A) \cap \mathbf{D}$ dominates γ_I then (1.1) holds.*

Proof. Let $e^{it} \notin \gamma_I$ and $f \in \mathscr{E}$. Then

$$|f(e^{it})| < \varepsilon \|f\|_\infty < \|f\|_\infty$$

so

$$\|f\|_\infty = \sup_{e^{it} \in \gamma_I} |f(e^{it})| = \sup_{z \in \sigma(A) \cap \mathbf{D}} |f(z)| \quad \text{for every } f \in \mathscr{E}$$

and condition (1.1) is fulfilled. The lemma is proved.

Let us suppose now that $\sigma(A) \cap \mathbf{D}$ dominates an arc $\gamma \subset \mathbf{T}$. For simplicity we suppose $1 \in \gamma$.

For $n \geqslant 4$, let $I_n = (\alpha_n, \beta_n)$ be such that

$$\bigcup_{n=4}^{\infty} \gamma_{I_n} \subset \gamma, \quad \beta_n = \frac{1}{3^{2n}}, \quad \alpha_n = \frac{1}{3^{2n+1}}$$

(if necessary we may take $n \geqslant n_0$ to insure $\bigcup_{n \geqslant n_0} \gamma_{I_n} \subset \gamma$). Let $\varepsilon_n = 1/3^{n-2}$ for $n \geqslant 4$.

By a specific use of the construction from Proposition 2.4 we get a certain weak*-closed subspace $X_n \subset H^\infty$, ε_n-supported on I_n and such that we can select from every X_n a certain function g_n such that the sequence $(g_n)_n$ converges to 1 almost everywhere on $\mathbf{T}$.

Let $f(z) = 1/(z+2)$, $J_1^n = (-1/3^n, 1/3^n)$. Then $|J_1^n| = 2/3^n < \varepsilon_n$. $f(-1)^n = 1$ implies that there exists an arc γ_{J_n} centered at -1 such that

$$|1 - f(z)^n| < \frac{\varepsilon_n}{8} \|1 - f^n\|_\infty \quad \text{for every } z \in \gamma_{J_n}. \tag{3.1}$$

Let $\psi_n : \mathbf{D} \to \mathbf{D}$ be a conformal mapping, $\psi_n(1) = 1$ and $\psi_n(\mathbf{T} \setminus \gamma_{J_1^n}) = \gamma_{J_n}$ (for the existence of ψ_n see, for example, Theorem 6 from § 3, Chapter II in [7]).

By (3.1)

$$|1 - (f \circ \psi_n)^n(z)| < \frac{\varepsilon_n}{8} \|1 - f\|_\infty = \frac{\varepsilon_n}{8} \|1 - (f \circ \psi_n)\|_\infty \tag{3.2}$$

for every $z \in \mathbf{T} \setminus \gamma_{J_1^n}$.

We have also

$$|(f \circ \psi_n)^n(1)| = \frac{1}{3^n} < \varepsilon_n/3. \tag{3.3}$$

There exists then an interval $J_2^n \subset (-1/3^{3n}, 1/3^{3n}) \subset J_1^n$ such that

$$|(f \circ \psi_n)^n(z)| < \varepsilon_n/3 \quad \text{for every } z \in \gamma_{J_2^n} \tag{3.4}$$

and

$$|1 - \psi_n(z)| < \frac{1}{3^n} \quad \text{for every } z \in \gamma_{J_2^n}. \tag{3.5}$$

Let $F_n(z) = (f \circ \psi_n)^n(z) = 1/[\psi_n(z) + 2]^n$. Then by (3.2), (3.4)

$$|F_n(z)| < \varepsilon_n/3 \quad \text{for every } z \in \gamma_{J_2^n} \tag{3.6}$$

and

$$|1 - F_n(z)| < \frac{\varepsilon_n}{8} \|1 - F_n\|_\infty \quad \text{for every } z \in \mathbf{T} \setminus \gamma_{J_1^n}.$$

Let $p_n \in \mathbf{N}$ be such that $2\pi/p_n < |J_1^n| = 2/3^n$. Then $p_n > \pi\cdot 3^n$. We choose $p_n = 3^{n+2}$.

From the proof of Proposition 2.4 we see that the space $F_n \cdot \operatorname{Ran} Q_{3^{n+2}}$ is ε_n-supported on $\mathbf{T}\setminus\gamma_{J_2^n}$.

Let $\varphi_n : \mathbf{D} \to \mathbf{D}$ be a conformal mapping such that $\varphi_n(\gamma_{I_n}) = \mathbf{T}\setminus\gamma_{J_2^n}$. Then

$$X_n = (F_n \circ \varphi_n)[(\operatorname{Im} Q_{3^{n+2}})\circ\varphi_n] \tag{3.7}$$

is the desired subspace of H^∞, ε_n-supported on I_n.

LEMMA 3.2. *With φ_n as before*

$$\lim_{n\to\infty} \varphi_n(z)^{3^{n+2}-1} = 1 \quad \textit{for every } z \in \mathbf{T}$$

and

$$\lim_{n\to\infty} 3^n(F_n \circ \varphi_n)(z) = 1 \quad \textit{for every } z \in \mathbf{T}.$$

Proof. Let $z \in \mathbf{T}$. $z \notin \gamma_{I_n}$ for every $n \geqslant n(z)$ such that

$$\varphi_n(z) \in \gamma_{J_2^n} \Leftrightarrow \varphi_n(z) = e^{it} \quad \text{with } |t| < \frac{1}{3^{2n}}.$$

Then $\varphi_n(z)^{3^{n+2}-1} = e^{it(3^{n+2}-1)} \in \gamma(-1/3^n, 1/3^n)$ for every $n \geqslant n(z)$ and this implies $\lim_{n\to\infty} \varphi_n(z)^{3^{n+2}-1} = 1$.

For the second limit let again $z \in \mathbf{T}$ and $n(z) \in \mathbf{N}$ be such that $z \notin \gamma_{I_n}$ for every $n \geqslant n(z)$.

Let $\zeta_n = \varphi_n(z) \in \gamma_{J_2^n}$ for $n \geqslant n(z)$.

By (3.5), $\psi_n(\zeta_n) = 1 - \lambda_n$ with $|\lambda_n| < 1/3^n$. Then $|2 + \psi_n(\zeta_n)| = |3 - \lambda_n| > 3 - |\lambda_n| > 3 - 1/3^n$ such that $3/|2 + \psi_n(\zeta_n)| < 3/(3 - 1/3^n) < 2$. This implies

$$|3^n(F_n\circ\varphi_n)(z) - 1| = \left|\frac{3^n}{[\psi_n(\zeta_n) + 2]^n} - 1\right| = \left|\frac{[1 - \psi_n(\zeta_n)]\sum_{k=0}^{n-1} 3^{n-1-k}[\psi_n(\zeta_n) + 2]^k}{[\psi_n(\xi_n) + 2]^n}\right| \leqslant$$

$$\leqslant |1 - \psi_n(\zeta_n)|\cdot\frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{3}{|\psi_n(\zeta_n) + 2|}\right)^k \leqslant |1 - \psi_n(\zeta_n)|\cdot\frac{1}{3}\cdot\sum_{k=1}^{n} 2^k =$$

$$= |1 - \psi_n(\zeta_n)|\,\frac{1}{3}\cdot 2(2^n - 1) < |1 - \psi_n(\zeta_n)|\cdot\frac{2}{3}\cdot 2^n \leqslant \left(\frac{2}{3}\right)^{n+1},$$

where we used (3.5) again. The lemma is proved.

Let $\mathbf{N} = \bigcup_{k=0}^{\infty} N_k$, where $N_k = \{p_{n,k} \mid n \geq 1\}$ for $k \geq 0$ are infinite, mutual disjoint subsets of $\mathbf{N}$.

For $n \in N_k$ let $f_n(z) = z^k$. $(f_n)_{n \geq 1}$ is a sequence of inner functions and we define $\mathscr{E}_n = f_n \cdot X_n$ where X_n is the subspace defined in (3.7). $\mathscr{E}_n$ is then weak*-closed and ε_n-supported on I_n.

Let $\mathscr{E} = \overline{\operatorname{span}}^{w^*}\{\mathscr{E}_n\}_1^\infty$. We have

PROPOSITION 3.3. *For every $\lambda \in \mathbf{D}$ there exist x and y in $\mathscr{H}$ such that*

$$(f(A)x, y) = f(\lambda) \quad \textit{for every } f \in \mathscr{E}. \tag{3.7}$$

Proof. We apply Lemma 3.1 and Theorem 1.1 to conclude that $\Phi(\mathscr{E})$ has property $(A_{\aleph_0})$. Consider the elements $[C_\lambda]$ in the predual of $\Phi(\mathscr{E})$ such that $\langle [C_\lambda], f(A)\rangle = f(\lambda)$ for every $f \in \mathscr{E}$ (see the proof of Theorem 2.2 from [8]). We find (see [1], Chapter IX for $[C_0]$) nonzero x and y in $\mathscr{H}$ such that $[x \otimes y] = [C_\lambda]$ and this yields (3.7). The proposition is proved.

Let U be the minimal unitary dilation of A and $E(\sigma)$ its spectral measure defined on the Borel sets of $\mathbf{T}$. Then by Theorem 6.4 in § 6, Chapter II of [11] the scalar measures $\mu_{x,y}(\sigma) = (E(\sigma)x, y)$ are absolutely continuous with respect to the Lebesgue measure. Let us denote by $\varphi_{x,y}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(E_t x, y)$ the density of $\mu_{x,y}$.

THEOREM 3.4. *If there exists $\lambda \in \mathbf{D}$ such that there exist x and y in $\mathscr{H}$, $x \neq 0$, $y \neq 0$, with $[x \otimes y] = [C_\lambda]$ in the predual of $\Phi(\mathscr{E})$ and 0 is a Lebesgue point for $\varphi_{x,y}$, then A has a nontrivial invariant subspace.*

Proof. By Lemma 3.2 the sequence $\{3^n (F_n \circ \varphi_n)\varphi_n^{3^{n+2}-1}\}_n$ converges almost everywhere on $\mathbf{T}$ to the function 1. We show now that the convergence is dominated relative to the measure $\mu_{x,y}$.

For this, consider the system of sets $K_n = (-1/3^{2^n}, 1/3^{2^n})$ converging to 0 in $(-\pi, \pi)$.

Since $\beta_n - \alpha_n = 2/3^{2^n+1} > (1/4)\ (2/3^{2^n}) = (1/4)|K_n|$ for $n \geq 4$ we conclude by [10], p. 206, that, in the hypothesis that 0 is a Lebesgue point for $\varphi_{x,y}$,

$$\lim_{n\to\infty} \frac{\int_{I_n} |\varphi_{x,y}(t)|\mathrm{d}t}{I_n} = |\varphi_{x,y}(0)|.$$

Then there exists $M > 0$ such that

$$3^{2^n+1}\int_{I_n} |\varphi_{x,y}(t)|\mathrm{d}t \leqslant M \quad \text{for every } n \geqslant 4.$$

This implies

$$3^n \int_{I_n} |\varphi_{x,y}(t)|\, dt \leqslant M \cdot \frac{1}{3^{n+1}} \quad \text{for every } n \geqslant 4$$

and then

$$\sum_{n=3}^{\infty} 3^n \int_{I_n} |\varphi_{x,y}(t)|\, dt \quad \text{is finite.} \tag{3.8}$$

Let us define

$$g(z) = \begin{cases} 1 & \text{for } z \in \mathbf{T} \setminus \bigcup_{n \geqslant 4} \gamma_{I_n} \\ 3^n & \text{for } z \in \gamma_{I_n},\ n \geqslant 4. \end{cases} \tag{3.9}$$

Then, by (3.8), $\int_{-\pi}^{\pi} g(e^{it})|\varphi_{x,y}(t)|\, dt$ is finite and since for $z \notin \gamma_{I_n}$ we have $\varphi_n(z) \in \in \gamma_{J_2^n}$ such that

$$|F_n(\varphi_n(z))| \leqslant \frac{1}{3^n} \quad \text{and} \quad |F_n(\varphi_n(z))| \leqslant 1 \quad \text{for every } z \in \mathbf{T},$$

it results

$$|3^n F_n[\varphi_n(z)]\varphi_n(z)^{3^{n+2}-1}| \leqslant g(z) \quad \text{for every } z \in \mathbf{T} \text{ and } n \geqslant 4.$$

Then, by the Lebesgue Dominated Convergence Theorem

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} e^{ikt} F_{p_{n,k}}[\varphi_{p_{n,k}}(e^{it})] \cdot 3^{p_{n,k}} \varphi_{p_{n,k}}(e^{it})^{3^{p_{n,k}+2}-1} \varphi_{x,y}(t) dt =$$

$$= \int_{-\pi}^{\pi} e^{ikt} \varphi_{x,y}(t) dt \quad \text{for every } k \geqslant 0.$$

This is equivalent to

$$\lim_{n \to \infty} (A^k F_{p_{n,k}}[\varphi_{p_{n,k}}(A)] 3^{p_{n,k}} \varphi_{p_{n,k}}(A)^{3^{p_{n,k}+2}-1} x, y) = (A^k x, y)$$

for every $k \geqslant 0$. But since $[x \otimes y] = [C_\lambda]$, the last equality becomes

$$\lim_{n \to \infty} \lambda^k 3^{p_{n,k}} F_{p_{n,k}}[\varphi_{p_{n,k}}(\lambda)] \varphi_{p_{n,k}}(\lambda)^{3^{p_{n,k}+2}-1} = (A^k x, y) \tag{3.10}$$

for every $k \geqslant 0$.

We remark that the function g defined in (3.9) is also integrable with respect to the measure defined by the density $\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{1}{|1 - \overline{\lambda}e^{it}|^2}$. Indeed, if $\lambda = re^{i\theta}$,

$$\int_{I_n} \frac{1}{1 + r^2 - 2r\cos(t - \theta)}\,\mathrm{d}t \leqslant \frac{1}{(1 - r)^2}\,|I_n| < \frac{1}{3^{2^n}(1 - r)^2}.$$

Therefore we apply again the Dominated Convergence Theorem to conclude that $\lim\limits_{n\to\infty} 3^n F_n[\varphi_n(\lambda)]\varphi_n(\lambda)^{3^{n+2}-1} = 1$. Then

$$\lim_{n\to\infty} \lambda^k F_{p_{n,k}}[\varphi_{p_{n,k}}(\lambda)]\varphi_{p_{n,k}}(\lambda)^{3^{p_{n,k}+2}-1} = \lambda^k \tag{3.11}$$

for every $k \geqslant 0$.

(3.10) and (3.11) yield $(A^k x, y) = \lambda^k$ for every $k \geqslant 0$ such that we have $(x, y) = 1$ and

$$((A - \lambda)^k x, y) = 0 \quad \text{for every } k \geqslant 1.$$

Then either λ is an eigenvalue of A or $\bigvee\limits_{k\geqslant 0}(A - \lambda)^{k+1}x$, the closed linear subspace generated by $\{(A - \lambda)^{k+1}x \mid k \geqslant 0\}$ in $\mathscr{H}$ is a nontrivial invariant subspace fo A—λI. But A—λI and A have the same invariant subspaces so we conclude that in either case A has a nontrivial invariant subspace. The theorem is proved.

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