J. OPERATOR THEORY
23(1990), 125–146

INDEX THEORY AND TOEPLITZ ALGEBRAS ON CERTAIN CONES IN $\mathbf{Z}^2$

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The study of Toeplitz operators in various contexts has been an important application of operator algebras for several years, and there has been much progress lately in studying Toeplitz algebras defined on semigroups of abelian groups. In this paper we use operator algebra techniques to study the index theory of operators in Toeplitz algebras defined on certain cones in $\mathbf{Z}^2$. Specifically, we take two lines in $\mathbf{R}^2$ passing through the origin with slopes α and β, and we form the C^*-algebra $\mathscr{T}^{\alpha,\beta}$ generated by the Toeplitz operators obtained by compressing the translation operators on $\mathbf{Z}^2$ to one of the four cones in $\mathbf{Z}^2$ bounded by these two lines. We shall call such a cone a *quarter-plane*.

The index theory of these Toeplitz operators has been examined in the special case where the cone considered is the first quadrant in $\mathbf{Z}^2$. In [8], the authors form the C^*-algebra $\mathscr{T}^{0,\infty}$ generated by the Toeplitz operators on the first quadrant, and use the fact that this C^*-algebra can be expressed as a tensor product to obtain necessary and sufficient conditions for operators in $\mathscr{T}^{0,\infty}$ to be Fredholm. We extend this result to arbitrary quarter-planes. However, our techniques are somewhat different than those used in [8], since in general $\mathscr{T}^{\alpha,\beta}$ cannot be written as a tensor product.

In [4], homotopy theory is used to give a procedure for finding the index of Fredholm operators in $\mathscr{T}^{0,\infty}$. While this result allows one to compute indices in theory, in practice the necessary homotopies are almost impossible to construct. In this paper, we use cyclic cohomology to construct an index formula that can be used to compute the index of many Fredholm operators in the C^*-algebra generated by the Toeplitz operators on an arbitrary quarter-plane.

This paper is organized as follows. In Section 1, we define most of the relevant C^*-algebras, including the Toeplitz algebra $\mathscr{T}^{\alpha,\beta}$, and we also define several important maps. In Section 2, we establish necessary and sufficient conditions for operators in $\mathscr{T}^{\alpha,\beta}$ to be Fredholm. In Section 3, we use K-theory to show that Fredholm index is a complete stable deformation invariant for the operators in $\mathscr{T}^{\alpha,\beta}$ in the case that at least one of α and β is rational. When α and β are both irrational, we do not know what happens, and we discuss some possibilities. In Section 4, we use cyclic cohomology to construct an index formula that can be used to explicitly compute the Fredholm index of operators in $\mathscr{T}^{\alpha,\beta}$. Finally, in Section 5, we use our results to show that a specific operator in $\mathscr{T}^{\alpha,\beta}$ is Fredholm, and we compute its index.

1. PRELIMINARIES

We first establish some notation and definitions. For each pair of integer (m, n), let $e_{m,n}$ denote the element of $\ell^2(\mathbf{Z}^2)$ that is 1 at (m, n) and zero elsewhere. Also, for each pair of integers (m, n), define the translation operator $M_{m,n}$ on $\ell^2(\mathbf{Z}^2)$ by $(M_{m,n}f)(k, l) = f(m + k, n + l)$. Next, choose real numbers $\alpha < \beta$, and define the following subspaces of $\ell^2(\mathbf{Z}^2)$:

$$\mathscr{H}^{\alpha} = \text{closed span of } \{e_{m,n}: -\alpha m + n \geqslant 0\}$$

$$\mathscr{H}^{\beta} = \text{closed span of } \{e_{m,n}: -\beta m + n \leqslant 0\}$$

$$\mathscr{H}^{\alpha,\beta} = \mathscr{H}^{\alpha} \cap \mathscr{H}^{\beta}.$$

We could also take $\alpha = -\infty$ or $\beta = \infty$ (but not both). All the results we prove in this paper are still true, but many of them require a separate proof. Therefore, for convienence, we will not consider these cases.

Let P^{α} and P^{β} be the orthogonal projections of $\ell^2(\mathbf{Z}^2)$ onto $\mathscr{H}^{\alpha}$ and $\mathscr{H}^{\beta}$, respectively; note that $P^{\alpha}P^{\beta}$ is the orthogonal projection onto $\mathscr{H}^{\alpha,\beta}$. We then define the quarter-plane Toeplitz C^*-algebra

$$\mathscr{T}^{\alpha,\beta} = C^*\text{-algebra generated by } \{P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta} : (m, n) \in \mathbf{Z}^2\}.$$

To study the index theory of $\mathscr{T}^{\alpha,\beta}$, we also need to consider the half-plane Toeplitz C^*-algebras

$$\mathscr{T}^{\alpha} = C^*\text{-algebra generated by } \{P^{\alpha}M_{m,n}P^{\alpha} : (m, n) \in \mathbf{Z}^2\}$$

$$\mathscr{T}^{\beta} = C^*\text{-algebra generated by } \{P^{\beta}M_{m,n}P^{\beta} : (m, n) \in \mathbf{Z}^2\}.$$

We begin by defining maps

$$\rho^{\alpha} : \mathscr{T}^{\alpha} \to L(\mathscr{H}^{\alpha,\beta})$$

$$\rho^{\beta} : \mathscr{T}^{\beta} \to L(\mathscr{H}^{\alpha,\beta})$$

by

$$\rho^\alpha(X) = P^\beta X P^\beta$$

$$\rho^\beta(Y) = P^\alpha Y P^\alpha.$$

These maps are linear, but they are not multiplicative. Also, we cannot immediately assert that the ranges of these maps lie in $\mathscr{T}^{\alpha,\beta}$. For example, consider the operator $X = P^\alpha M_{r,s} M_{p,q} P^\alpha$ in $\mathscr{T}^\alpha$. Then $\rho^\alpha(X) = P^\alpha P^\beta M_{r,s} P^\alpha M_{p,q} P^\alpha P^\beta$, and since the projections P^α and P^β always appear together in operators in $\mathscr{T}^{\alpha,\beta}$, $\rho^\alpha(X)$ is not obviously in $\mathscr{T}^{\alpha,\beta}$. We therefore define the C^*-algebra

$$\mathscr{R}^{\alpha,\beta} = C^*\text{-algebra generated by } \rho^\alpha(\mathscr{T}^\alpha) \text{ and } \rho^\beta(\mathscr{T}^\beta).$$

Clearly $\mathscr{R}^{\alpha,\beta}$ contains $\mathscr{T}^{\alpha,\beta}$, and a dense subalgebra of $\mathscr{R}^{\alpha,\beta}$ consists of operators of the form

$$\sum_{i=0}^{l} c_i P^\alpha P^\beta M_{m_{i0},n_{i0}} \left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij},n_{ij}} \right] P^\alpha P^\beta$$

where the c_i are constants and each Q_{ij} is either P^α, P^β, or $P^\alpha P^\beta$.

We wish to construct algebra homomorphisms from $\mathscr{R}^{\alpha,\beta}$ onto $\mathscr{T}^\alpha$ and $\mathscr{T}^\beta$, but first we need the following lemma:

LEMMA 1.1. *Let $\{(m_i, n_i)\}$ be a finite collection of pairs of integers. Then there exists a pair of integers (p, q) such that, for all i,*

(i) $-\alpha(m_i + p) + (n_i + q) \geqslant 0$ *if and only if* $-\alpha m_i + n_i \geqslant 0$.

(ii) $-\beta(m_i + p) + (n_i + q) \leqslant 0$.

Proof. Choose positive numbers ε and M so that

$$\varepsilon < \min\{-(-\alpha m_i + n_i) : -\alpha m_i + n_i < 0\}$$

$$M > \max\{-\beta m_i + n_i\}, \quad \varepsilon \leqslant M.$$

Then it suffices to show that there exist integers p and q so that $0 \leqslant -\alpha p + q < \varepsilon$ and $-\beta p + q < -M$.

In [9], it is proved that there exist an infinite number of integers p for which there exists an integer q with

$$0 \leqslant \alpha - \frac{q}{p} < \frac{1}{p^2}.$$

Choose such a p so large that

$$p > \max\left\{\frac{1}{\varepsilon}, \frac{2M}{\beta - \alpha}\right\}.$$

Then

$$0 \leqslant -\alpha p + q < \frac{1}{p} < \varepsilon$$

and

$$-\beta p + q = -(\beta - \alpha)p + (-\alpha p + q) < -(\beta - \alpha)p + \varepsilon <$$

$$< -2M + \varepsilon < -M,$$

as desired. ∎

PROPOSITION 1.2. *There exist surjective C^*-algebra homomorphisms*

$$\gamma^\alpha : \mathscr{R}^{\alpha,\beta} \to \mathscr{T}^\alpha$$

$$\gamma^\beta : \mathscr{R}^{\alpha,\beta} \to \mathscr{T}^\beta$$

such that $\gamma^\alpha\rho^\alpha = \mathrm{id}$, $\gamma^\beta\rho^\beta = \mathrm{id}$.

Proof. Let T be an operator in $\mathscr{R}^{\alpha,\beta}$ of the form

$$T = \sum_{i=0}^{l} c_i P^\alpha P^\beta M_{m_{i0},n_{i0}} \left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij},n_{ij}}\right] P^\alpha P^\beta,$$

and define

$$\gamma^\alpha(T) = \sum_{i=0}^{l} c_i P^\alpha M_{m_{i0},n_{i0}} \left[\prod_{j=1}^{k_i} Q^\alpha_{ij} M_{m_{ij},n_{ij}}\right] P^\alpha,$$

where each Q^α_{ij} equals P^α, I, or P^α, depending on whether Q_{ij} equals P^α, P^β, or $P^\alpha P^\beta$. To show that γ^α is well defined and can be extended to an algebra homomorphism on $\mathscr{R}^{\alpha,\beta}$, it suffices to show that $\|\gamma^\alpha(T)\| \leqslant \|T\|$.

Fix $\varepsilon > 0$, and choose f in $\ell^2(\mathbf{Z}^2)$ so that f has finite support, $\|f\| = 1$, and $\|\gamma^\alpha(T)\| \leqslant \|\gamma^\alpha(T)f\| + \varepsilon$. Then since f has finite support,

$$(*) \qquad \left[\prod_{j=N}^{k_i} M_{m_{ij},n_{ij}}\right] f$$

is also finitely supported for all $0 \leqslant i \leqslant l$ and $0 \leqslant N \leqslant k_i$. Next, consider the set

$$S = \{(m, n) \in \mathbf{Z}^2 : e_{m,n} \in \text{range of } (*) \text{ for some } i, N\}.$$

Apply Lemma 1.1 to the pairs of integers in S to obtain a pair of integers (p, q). The first conclusion in Lemma 1.1 implies that

$$M_{p,q}\gamma^{\alpha}(T)f = \gamma^{\alpha}(T)M_{p,q}f.$$

The second conclusion in Lemma 1.1 implies that in the expression for $TM_{p,q}f$, the projection P^{β} is unnecessary each place that it appears. Therefore

$$\gamma^{\alpha}(T)M_{p,q}f = TM_{p,q}f,$$

and, since $M_{p,q}$ is a unitary operator,

$$\|\gamma^{\alpha}(T)\| \leqslant \|\gamma^{\alpha}(T)f\| + \varepsilon =$$

$$= \|M_{p,q}\gamma^{\alpha}(T)f\| + \varepsilon = \|TM_{p,q}f\| + \varepsilon \leqslant$$

$$\leqslant \| TM_{p,q}\| + \varepsilon = \|T\| + \varepsilon.$$

Since ε was arbitrary, $\|\gamma^{\alpha}(T)\| \leqslant \|T\|$, and thus γ^{α} extends to a contractive algebra homomorphism from $\mathscr{R}^{\alpha,\beta}$ to $\mathscr{T}^{\alpha}$. An appropriate modification of Lemma 1.1 implies that there also exists a contractive algebra homomorphism γ^{β} from $\mathscr{R}^{\alpha,\beta}$ to $\mathscr{T}^{\beta}$. Finally, direct computation shows that $\gamma^{\alpha}\rho^{\alpha}$ and $\gamma^{\beta}\rho^{\beta}$ are the identity on $\mathscr{T}^{\alpha}$ and $\mathscr{T}^{\beta}$, respectively, and this implies that γ^{α} and γ^{β} are onto. ▨

Next, we need to examine the half-plane Toeplitz algebras $\mathscr{T}^{\alpha}$ and $\mathscr{T}^{\beta}$. From [2] we have the short exact sequence

$$0 \to \mathscr{C}^{\alpha} \to \mathscr{T}^{\alpha} \xrightarrow{\sigma^{\alpha}} C(\mathbf{T}^2) \to 0,$$

where $\mathbf{T}^2$ denotes the two-torus, $\mathscr{C}^{\alpha}$ is the commutator ideal of $\mathscr{T}^{\alpha}$, and σ^{α} is defined by requiring

$$\sigma^{\alpha}(P^{\alpha}M_{m,n}P^{\alpha}) = \chi_{m,n},$$

where

$$\chi_{m,n}(\theta_1, \theta_2) = e^{im\theta_1}e^{in\theta_2}.$$

This sequence has a linear splitting

$$\xi^{\alpha}: C(\mathbf{T}^2) \to \mathscr{T}^{\alpha}$$

defined by

$$\xi^{\alpha}(\chi_{m,n}) = P^{\alpha}M_{m,n}P^{\alpha}.$$

Similarly, we have a short exact sequence

$$0 \to \mathscr{C}^{\beta} \to \mathscr{T}^{\beta} \xrightarrow{\sigma^{\beta}} C(\mathbf{T}^2) \to 0$$

with linear splitting $\xi^{\beta}: C(\mathbf{T}^2) \to \mathscr{T}^{\beta}$.

We now record several useful relations among the maps we have defined.

LEMMA 1.3. $\sigma^\alpha\gamma^\alpha = \sigma^\beta\gamma^\beta$.

Proof. It suffices to show that $\sigma^\alpha\gamma^\alpha(T) = \sigma^\beta\gamma^\beta(T)$ for operators T of the form

$$T = P^\alpha P^\beta M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j}\right] P^\alpha P^\beta,$$

since these operators have dense linear span in $\mathscr{R}^{\alpha,\beta}$. As in Proposition 1.2, we define Q_j^α to be P^α, I or P^α when Q_j is P^α, P^β, or $P^\alpha P^\beta$, respectively, and we have

$$\sigma^\alpha\gamma^\alpha(T) = \sigma^\alpha\gamma^\alpha\left\{P^\alpha P^\beta M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j}\right] P^\alpha P^\beta\right\} =$$

$$= \sigma^\alpha\left\{P^\alpha M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j^\alpha M_{m_j,n_j}\right] P^\alpha\right\} = \prod_{j=0}^{k} \chi_{m_j,n_j} = \sigma^\beta\gamma^\beta(T).$$ ▨

LEMMA 1.4. $\gamma^\beta\rho^\alpha = \xi^\beta\sigma^\alpha$ *and* $\gamma^\alpha\rho^\beta = \xi^\alpha\sigma^\beta$.

Proof. We will only verify that $\gamma^\beta\rho^\alpha = \xi^\beta\sigma^\alpha$; showing that $\gamma^\alpha\rho^\beta = \xi^\beta\sigma^\alpha$ involves a similar calculation.

Consider an operator T in $\mathscr{T}^\alpha$ that has the form

$$T = \prod_{j=0}^{k} P^\alpha M_{m_j,n_j} P^\alpha.$$

It will suffice to prove that $\gamma^\beta\rho^\alpha(T) = \xi^\beta\sigma^\alpha(T)$ for operators of this form, since these operators have dense linear span in $\mathscr{T}^\alpha$.

$$\gamma^\beta\rho^\alpha(T) = \gamma^\beta\rho^\alpha\left[\prod_{j=0}^{k} P^\alpha M_{m_j,n_j} P^\alpha\right] =$$

$$= \gamma^\beta\left\{P^\beta\left[\prod_{j=0}^{k} P^\alpha M_{m_j,n_j} P^\alpha\right] P^\beta\right\} = P^\beta\left[\prod_{j=0}^{k} M_{m_j,n_j}\right] P^\beta.$$

On the other hand,

$$\xi^\beta\sigma^\alpha(T) = \xi^\beta\sigma^\alpha\left[\prod_{j=0}^{k} P^\alpha M_{m_j,n_j} P^\alpha\right] =$$

$$= \xi^\beta\left[\prod_{j=0}^{k} \chi_{m_{ij},n_{ij}}\right] = P^\beta\left[\prod_{j=0}^{k} M_{m_{ij},n_{ij}}\right] P^\beta.$$ ▨

Finally, we define a linear map

$$\xi^{\alpha,\beta}: C(\mathbf{T}^2) \to \mathscr{T}^{\alpha,\beta} \subset \mathscr{R}^{\alpha,\beta}$$

by requiring

$$\xi^{\alpha,\beta}(\chi_{m,n}) = P^\alpha P^\beta M_{m,n} P^\alpha P^\beta.$$

LEMMA 1.5. $\rho^\alpha \xi^\alpha = \xi^{\alpha,\beta} = \rho^\beta \xi^\beta$.

Proof. It is enough to verify the lemma for the functions $\chi_{m,n}$, since the linear span of these functions is dense in $C(\mathbf{T}^2)$.

$$\rho^\alpha \xi^\alpha(\chi_{m,n}) = \rho^\alpha(P^\alpha M_{m,n} P^\alpha) =$$

$$= P^\alpha P^\beta M_{m,n} P^\alpha P^\beta = \xi^{\alpha,\beta}(\chi_{m,n}).$$

Similarly, $\xi^{\alpha,\beta} = \rho^\beta \xi^\beta$. ▨

2. FREDHOLM OPERATORS IN THE QUARTER-PLANE TOEPLITZ C^*-ALGEBRA

In this section, we establish necessary and sufficient conditions for operators in $\mathscr{R}^{\alpha,\beta}$, and hence $\mathscr{T}^{\alpha,\beta}$, to be Fredholm. Define the ideals

$$\mathscr{I}^\alpha = \ker \gamma^\beta$$

$$\mathscr{I}^\beta = \ker \gamma^\alpha$$

$$\mathscr{I}^{\alpha,\beta} = \mathscr{I}^\alpha \cap \mathscr{I}^\beta.$$

PROPOSITION 2.1. *There exists the following commutative diagram with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}^{\alpha,\beta} & \longrightarrow & \mathscr{I}^\alpha & \longrightarrow & \mathscr{C}^\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}^\beta & \longrightarrow & \mathscr{R}^{\alpha,\beta} & \xrightarrow{\gamma^\alpha} & \mathscr{T}^\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \gamma^\beta} & & \downarrow{\scriptstyle \sigma^\alpha} & & \\
0 & \longrightarrow & \mathscr{C}^\beta & \longrightarrow & \mathscr{T}^\beta & \xrightarrow{\sigma^\beta} & C(\mathbf{T}^2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. We have already shown that $\sigma^\alpha\gamma^\alpha = \sigma^\beta\gamma^\beta$. To show that rest of the diagram commutes, we need only check that γ^α map $\mathcal{I}^\alpha$ to $\mathcal{C}^\alpha$ and γ^β map $\mathcal{I}^\beta$ to $\mathcal{C}^\beta$, since the other maps in the diagram are inclusions.

Take X in $\mathcal{I}^\alpha$. Then Lemma 1.3 and the definition of $\mathcal{I}^\alpha$ imply that $\sigma^\alpha\gamma^\alpha(X) = \sigma^\beta\gamma^\beta(X) = 0$. Thus $\gamma^\alpha(X)$ is in $\ker\sigma^\alpha = \mathcal{C}^\alpha$. Similarly, γ^β takes $\mathcal{I}^\beta$ to $\mathcal{C}^\beta$.

Next, let us verify that the rows and columns of the diagram are exact. The last row and last column are exact by the results in [2], and the definitions of $\mathcal{I}^\alpha$ and $\mathcal{I}^\beta$ make the second row and column exact.

It remains to show that the first row and column are exact. We will show that the first row is exact; the first column is exact by similar reasoning.

Let Y be in $\mathcal{C}^\alpha$. Then we have from Lemma 1.4 that $\gamma^\beta\rho^\alpha(Y) = \xi^\beta\sigma^\alpha(Y) = 0$, so $\rho^\alpha(Y)$ is in $\mathcal{I}^\alpha$ and $\gamma^\alpha\rho^\alpha(Y) = Y$. Therefore the sequence is exact at $\mathcal{C}^\alpha$. Next, take X in $\mathcal{I}^\alpha$ to be in the kernel of γ^α. Then obviously X will be in $\mathcal{I}^{\alpha,\beta}$. Conversely, let X be in $\mathcal{I}^{\alpha,\beta}$. Then X is in $\mathcal{I}^\beta$, so $\gamma^\alpha(X) = 0$, and hence X is in the kernel of γ^α restricted to $\mathcal{I}^\alpha$. Therefore the sequence is exact at $\mathcal{I}^\alpha$. Finally, the sequence is exact at $\mathcal{I}^{\alpha,\beta}$, since the map from $\mathcal{I}^{\alpha,\beta}$ to $\mathcal{I}^\alpha$ is just the inclusion map. ▨

Define the C^*-algebra $\mathcal{S}^{\alpha,\beta}$ by

$$\mathcal{S}^{\alpha,\beta} = \{(T^\alpha, T^\beta) \in \mathcal{T}^\alpha \oplus \mathcal{T}^\beta : \sigma^\alpha(T^\alpha) = \sigma^\beta(T^\beta)\}.$$

Note that $\mathcal{S}^{\alpha,\beta}$ is the pullback of $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ along $C(\mathbf{T}^2)$.

PROPOSITION 2.2. *There exists a short exact sequence*

$$0 \to \mathcal{I}^{\alpha,\beta} \to \mathcal{R}^{\alpha,\beta} \xrightarrow{\gamma} \mathcal{S}^{\alpha,\beta} \to 0,$$

where

$$\gamma(T) = (\gamma^\alpha(T),\ \gamma^\beta(T)).$$

Furthermore, this sequence has a linear splitting

$$\rho : \mathcal{S}^{\alpha,\beta} \to \mathcal{R}^{\alpha,\beta}$$

defined by

$$\rho(T^\alpha, T^\beta) = \rho^\alpha(T^\alpha) + \rho^\beta(T^\beta) - \xi^{\alpha,\beta}\sigma^\beta(T^\beta) =$$

$$= \rho^\alpha(T^\alpha) + \rho^\beta(T^\beta) - \xi^{\alpha,\beta}\sigma^\alpha(T^\alpha).$$

Proof. Let T be in $\mathcal{R}^{\alpha,\beta}$. Then $\sigma^\alpha\gamma^\alpha(T) = \sigma^\beta\gamma^\beta(T)$, so γ maps $\mathcal{R}^{\alpha,\beta}$ into $\mathcal{S}^{\alpha,\beta}$. Next, suppose T is in $\mathcal{I}^{\alpha,\beta}$. Then $\gamma^\alpha(T) = 0$ and $\gamma^\beta(T) = 0$, so T is in $\ker\gamma$. Conversely, suppose that T is in $\ker\gamma$. Then T is in $\ker\gamma^\alpha$ and $\ker\gamma^\beta$, whence T is in $\mathcal{I}^{\alpha,\beta}$. Therefore the sequence is exact at $\mathcal{R}^{\alpha,\beta}$. Clearly the sequence is exact at $\mathcal{I}^{\alpha,\beta}$ since the map from $\mathcal{I}^{\alpha,\beta}$ to $\mathcal{R}^{\alpha,\beta}$ is the inclusion map. To show that the sequence is exact at $\mathcal{S}^{\alpha,\beta}$, it suffices to show that the map ρ defined above is a splitting.

Choose (T^α, T^β) in $\mathscr{S}^{\alpha,\beta}$; we must show that $\gamma^\alpha\rho(T^\alpha, T^\beta) = T^\alpha$ and that $\gamma^\beta\rho(T^\alpha, T^\beta) = T^\beta$. Combining Lemma 1.4 and Lemma 1.5, we obtain

$$\gamma^\alpha\rho(T^\alpha, T^\beta) = \gamma^\alpha[\rho^\alpha(T^\alpha) + \rho^\beta(T^\beta) - \xi^{\alpha,\beta}\sigma^\beta(T^\beta)] =$$

$$= \gamma^\alpha\rho^\alpha(T^\alpha) + \gamma^\alpha\rho^\beta(T^\beta) - \gamma^\alpha\xi^{\alpha,\beta}\sigma^\beta(T^\beta) =$$

$$= T^\alpha + \gamma^\alpha\rho^\beta(T^\beta) - \gamma^\alpha\rho^\alpha\xi^\alpha\sigma^\beta(T^\beta) =$$

$$= T^\alpha + \xi^\alpha\sigma^\beta(T^\beta) - \xi^\alpha\sigma^\beta(T^\beta) = T^\alpha.$$

The proof that $\gamma^\beta\rho(T^\alpha, T^\beta) = T^\beta$ is similar. ▨

Before continuing, we make a definition. An operator T in $\mathscr{T}^{\alpha,\beta}$ of the form

$$T = P^\alpha P^\beta M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j}\right]P^\alpha P^\beta,$$

where each Q_j is either P^α, P^β, or $P^\alpha P^\beta$, will be called a *finite product*. Note that the finite products have dense linear span in $\mathscr{R}^{\alpha,\beta}$.

Now, for the finite product T above we define the numbers

$$\lambda^\alpha(T) = -\sum_{j=0}^{k} \min\{-\alpha m_j + n_j, 0\}$$

$$\lambda^\beta(T) = \sum_{j=0}^{k} \max\{-\beta m_j + n_j, 0\}.$$

Note that $\lambda^\alpha(T)$ and $\lambda^\beta(T)$ depend upon the particular representation of T; that is, if T can be expressed in an altternate way as a finite product, then $\lambda^\alpha(T)$ and $\lambda^\beta(T)$ may be different.

The definition of $\lambda^\alpha(T)$ implies that when $-\alpha m + n$ is greater than or equal to $\lambda^\alpha(T)$, $T(e_{m,n}) = (\rho^\beta\gamma^\beta(T))(e_{m,n})$, since then the projection P^α is unnecessary each place it appears in the expression for T. Similarly, when $-\beta m + n$ is less than or equal to $-\lambda^\beta(T)$, $T(e_{m,n}) = (\rho^\alpha\gamma^\alpha(T))(e_{m,n})$.

We now return to the exact sequence in Proposition 2.2, and show that we can identify $\mathscr{J}^{\alpha,\beta}$ as a more familiar object.

PROPOSITION 2.3. *$\mathscr{J}^{\alpha,\beta}$ is the ideal $\mathscr{K}(\mathscr{H}^{\alpha,\beta})$ of compact operators on $\mathscr{H}^{\alpha,\beta}$.*

Proof. We first show that $\mathscr{J}^{\alpha,\beta}$ is contained in $\mathscr{K}(\mathscr{H}^{\alpha,\beta})$. Since ρ is a splitting, every element of $\mathscr{J}^{\alpha,\beta}$ can be written in the form $T - \rho\gamma(T)$ for some T in $\mathscr{R}^{\alpha,\beta}$. Furthermore, since finite products have dense linear span in $\mathscr{R}^{\alpha,\beta}$, to show that $\mathscr{J}^{\alpha,\beta}$ is contained in $\mathscr{K}(\mathscr{H}^{\alpha,\beta})$, it suffices to show that $T - \rho\gamma(T)$ is a finite rank operator for every finite product T. Therefore, let T be a finite product. Then using Lemmas 1.4 and 1.5, we obtain

$$T - \rho\gamma(T) = T - \rho(\gamma^\alpha(T), \gamma^\beta(T)) =$$

$$= T - \rho^\alpha\gamma^\alpha(T) - \rho^\beta\gamma^\beta(T) + \xi^{\alpha,\beta}\sigma^\beta\gamma^\beta(T) =$$

$$= T - \rho^\alpha\gamma^\alpha(T) - \rho^\beta\gamma^\beta(T) + \rho^\alpha\xi^\alpha\sigma^\beta\gamma^\beta(T) =$$

$$= T - \rho^\beta\gamma^\beta(T) - \rho^\alpha\gamma^\alpha(T) + \rho^\alpha\gamma^\alpha\rho^\beta\gamma^\beta(T) =$$

$$= (T - \rho^\beta\gamma^\beta(T)) - \rho^\alpha\gamma^\alpha(T - \rho^\beta\gamma^\beta(T)).$$

Our discussion above yields that $(T - \rho\gamma(T))(e_{m,n}) = 0$ for all $e_{m,n}$ such that $-\alpha m + n \geqslant \lambda^\alpha(T)$. We can also write

$$T - \rho\gamma(T) = (T - \rho^\alpha\gamma^\alpha(T)) - \rho^\beta\gamma^\beta(T - \rho^\alpha\gamma^\alpha(T)),$$

so $-\beta m + n \leqslant -\lambda^\beta(T)$ implies that $(T - \rho\gamma(T))(e_{m,n}) = 0$. Therefore the rank of $T - \rho\gamma(T)$ is bounded by the number of pairs of integers (m, n) that satisfy the following two inequalities:

$$0 \leqslant -\alpha m + n < \lambda^\alpha(T)$$

$$-\lambda^\beta(T) < -\beta m + n \leqslant 0.$$

It is easy to see that since α and β are distinct, there are only finitely many pairs of integers satisfying these two inequalities. Therefore, $T - \rho\gamma(T)$ is a finite rank operator when T is a finite product, and thus for arbitrary X in $\mathscr{R}^{\alpha,\beta}$, $X - \rho\gamma(X)$ is compact.

To show that $\mathscr{I}^{\alpha,\beta}$ contains $\mathscr{K}(\mathscr{H}^{\alpha,\beta})$, it is enough to show that $\mathscr{R}^{\alpha,\beta}$, and hence $\mathscr{I}^{\alpha,\beta}$, is irreducible. That $\mathscr{R}^{\alpha,\beta}$ is irreducible follows from the fact that $\mathscr{T}^{\alpha,\beta}$ is irreducible ([2]) and that $\mathscr{T}^{\alpha,\beta} \subset \mathscr{R}^{\alpha,\beta}$.

PROPOSITION 2.4. $\mathscr{R}^{\alpha,\beta} = \mathscr{T}^{\alpha,\beta}$.

Proof. The proof breaks into two cases, the case where α and β are both rational numbers, and the case where at least one of α and β is irrational.

Case 1: α and β both rational. Write $\alpha = p/q$, where p and q are relatively prime integers, let x and y be arbitrary integers, and consider the expression $P^\alpha M_{x,y}P^\alpha P^\beta$. Now, for every integer m, $M_{mq,mp}$ commutes with P^α, so

$$P^\alpha M_{x,y}P^\alpha P^\beta = M_{-mq,-mp}P^\alpha M_{mq+x,mp+y}P^\alpha P^\beta.$$

Furthermore, if we choose m so that $-\beta(mq + x) + (mp + y)$ is less than zero, then we have

$$P^\alpha M_{x,y} P^\alpha P^\beta = M_{-mq,-mp} P^\alpha P^\beta M_{mq+x,mp+y} P^\alpha P^\beta.$$

Similarly, if we write $\beta = r/s$ for relatively prime integers r and s, then we have

$$P^\beta M_{x,y} P^\alpha P^\beta = M_{-ns,-nr} P^\alpha P^\beta M_{ns+x,nr+y} P^\alpha P^\beta$$

for some integer n.

Now let

$$T = P^\alpha P^\beta M_{x_0,y_0} \left[\prod_{j=1}^{k} Q_j M_{x_j,y_j} \right] P^\alpha P^\beta$$

be a finite product in $\mathscr{R}^{\alpha,\beta}$. If $Q_j = P^\alpha P^\beta$ for all j, then T is in $\mathscr{T}^{\alpha,\beta}$. Otherwise, let l be the largest integer such that $Q_l \neq P^\alpha P^\beta$; say $Q_l = P^\alpha$. Then the argument above shows we have

$$M_{x_{l-1},y_{l-1}} Q_l M_{x_l,y_l} Q_{l+1} = M_{x_{l-1},y_{l-1}} P^\alpha M_{x_l,y_l} P^\alpha P^\beta =$$

$$= M_{-mq+x_{l-1},-mp+y_{l-1}} P^\alpha P^\beta M_{mq+x_l,mp+y_l} P^\alpha P^\beta$$

for some integer m. Thus we can use this equality to rewrite T so that $Q_l = P^\alpha P^\beta$; similarly, if $Q_l = P^\beta$, then we rewrite so that $Q_l = P^\alpha P^\beta$. There are only finitely many projections Q_j, so by induction we can rewrite T so that T is in $\mathscr{T}^{\alpha,\beta}$. Moreover, since finite products have dense linear span in $\mathscr{R}^{\alpha,\beta}$, we see that $\mathscr{R}^{\alpha,\beta} = \mathscr{T}^{\alpha,\beta}$.

Case 2: α and β not both rational. Now suppose that at least one of α and β is irrational; without loss of generality, suppose that β is irrational. We cannot proceed as we did in Case 1, since there is no translation that commutes with P^β. However, in Case 2 we have an additional fact at our disposal which we shall use shortly.

Since $\gamma^\alpha(P^\alpha P^\beta M_{m,n} P^\alpha P^\beta) = P^\beta M_{m,n} P^\alpha$ for every pair of integers (m, n), we see that γ^α maps $\mathscr{T}^{\alpha,\beta}$ onto $\mathscr{T}^\alpha$. Similarly, γ maps $\mathscr{T}^{\alpha,\beta}$ onto $\mathscr{T}^\beta$. Let $\mathscr{I}$ be the kernel of γ^α restricted to $\mathscr{T}^{\alpha,\beta}$. Then we have the commutative square

$$\begin{array}{ccc} \mathscr{I} & \longrightarrow & \mathscr{T}^{\alpha,\beta} \\ \downarrow{\scriptstyle \gamma^\beta} & & \downarrow{\scriptstyle \gamma^\beta} \\ \mathscr{C}^\alpha & \longrightarrow & \mathscr{T}^\beta, \end{array}$$

where the horizontal maps are inclusions. Now, $\mathscr{I}$ is an ideal in $\mathscr{T}^{\alpha,\beta}$ and γ^β maps $\mathscr{T}^{\alpha,\beta}$ onto $\mathscr{T}^\beta$, so $\gamma^\beta(\mathscr{I})$ is an ideal in $\mathscr{T}^\beta$. Furthermore, the commutativity of the square above implies that $\gamma^\beta(\mathscr{I})$ is an ideal in $\mathscr{C}^\beta$. But since β is irrational, $\mathscr{C}^\beta$ is simple ([6]), so $\gamma^\beta(\mathscr{I})$ is either zero or all of $\mathscr{C}^\beta$. It is easy to show that $\gamma^\beta(\mathscr{I})$ contains non-zero operators, so γ^β maps $\mathscr{I}$ onto $\mathscr{C}^\beta$. Next, [2] implies that $\mathscr{T}^{\alpha,\beta}$ is irreducible, so $\mathscr{I}$ is also irreducible. Therefore $\mathscr{I}$ contains all the compact operators, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{I} & \longrightarrow & \mathscr{C}^\beta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{I}^\beta & \longrightarrow & \mathscr{C}^\beta & \longrightarrow & 0, \end{array}$$

where the first and third vertical maps are identity maps and the middle vertical map is inclusion. Since the first and third maps are isomorphisms, the Five Lemma implies that the middle map is also an isomorphism, so $\mathscr{I} = \mathscr{I}^\beta$. Then we have another commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}^\beta & \longrightarrow & \mathscr{T}^{\alpha,\beta} & \longrightarrow & \mathscr{T}^\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{I}^\beta & \longrightarrow & \mathscr{R}^{\alpha,\beta} & \longrightarrow & \mathscr{T}^\alpha & \longrightarrow & 0, \end{array}$$

where the first and third maps are again identity maps and the middle map is inclusion. A second application of the Five Lemma yields $\mathscr{R}^{\alpha,\beta} = \mathscr{T}^{\alpha,\beta}$. ▨

COROLLARY 2.5. *The following sequence is exact:*

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{T}^{\alpha,\beta} \xrightarrow{\gamma} \mathscr{S}^{\alpha,\beta} \longrightarrow 0,$$

and has a linear splitting $\rho: \mathscr{S}^{\alpha,\beta} \to \mathscr{T}^{\alpha,\beta}$.

This short exact sequence gives an index theorem for operators in $\mathscr{T}^{\alpha,\beta}$:

THEOREM 2.6. *An operator T in $\mathscr{T}^{\alpha,\beta}$ is Fredholm if and only if $\gamma(T)$ is invertible in $\mathscr{S}^{\alpha,\beta}$, or equivalently, if and only if $\gamma^\alpha(T)$ and $\gamma^\beta(T)$ are invertible in $\mathscr{T}^\alpha$ and $\mathscr{T}^\beta$, respectively.*

It should be noted that the exact sequence in Corollary 2.5 remains exact when tensored by $M_n(\mathbf{C})$, so Theorem 2.6 extends to give an index theorem for matrices over $\mathscr{T}^{\alpha,\beta}$.

3. K-THEORY AND DEFORMATION INVARIANTS FOR FREDHOLM OPERATORS IN $\mathscr{T}^{\alpha,\beta}$

In this section, we compute $K_1(\mathscr{S}^{\alpha,\beta})$ using the Meyer-Vietoris sequence in K-theory, and then use our results to show that in the case where at least one of α and β is rational, index is a complete stable deformation invariant for Fredholm operators in $\mathscr{T}^{\alpha,\beta}$.

We begin by discussing the K-theory of the half-plane Toeplitz algebra $\mathcal{T}^\alpha$. If α is rational, then [6] shows that $\mathcal{T}^\alpha \cong \mathcal{T} \otimes C(\mathbf{T})$, where $\mathcal{T}$ denotes the C^*-algebra generated by the unilateral shift. An application of the Künneth formula ([13]) yields that $K_0(\mathcal{T}^\alpha) \cong \mathbf{Z}$, where the copy of $\mathbf{Z}$ comes from the "standard" projections; i.e., the projections p_n that have n ones on the diagonal and the rest of the entries zero. We can also use the Künneth formula to compute $K_1(\mathcal{T}^\alpha) \cong \mathbf{Z}$; if we write $\alpha = p/q$ with p and q relatively prime, then $[P^\alpha M_{q,p} P^\alpha]$ is a generator for $K_1(\mathcal{T}^\alpha)$.

When α is irrational, computing the K-theory of $\mathcal{T}^\alpha$ is somewhat more difficult. It has been shown in [10] and [15] that in this case, $K_1(\mathcal{T}^\alpha) \cong 0$ and $K_0(\mathcal{T}^\alpha) \cong \mathbf{Z}$, where the copy of the integers again corresponds to the standard projections p_n.

Before we compute the K-theory of the symbol algebra $\mathcal{S}^{\alpha,\beta}$, we need to know the K-theory of $C(\mathbf{T}^2)$. First, $K_0(C(\mathbf{T}^2)) \cong \mathbf{Z} \oplus \mathbf{Z}$, where one copy of $\mathbf{Z}$ corresponds to the standard projections, and the other copy of $\mathbf{Z}$ is generated by the projection that corresponds to the complex line bundle over $\mathbf{T}^2$ with Chern class one ([11]). Also, $K_1(C(\mathbf{T}^2)) \cong \mathbf{Z} \oplus \mathbf{Z}$, with the copies of $\mathbf{Z}$ generated by $[\chi_{0,1}]$ and $[\chi_{1,0}]$.

We can now compute the K-theory of $\mathcal{S}^{\alpha,\beta}$. Since $\mathcal{S}^{\alpha,\beta}$ is the pullback of $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ along $C(\mathbf{T}^2)$, we can apply the Mayer-Vietoris sequence in K-theory ([14]) to obtain the following exact diagram:

$$\begin{array}{ccccc} K_0(\mathcal{S}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{T}^\alpha) \oplus K_0(\mathcal{T}^\beta) & \longrightarrow & K_0(C(\mathbf{T}^2)) \\ \uparrow & & & & \uparrow \\ K_1(C(\mathbf{T}^2)) & \longleftarrow & K_1(\mathcal{T}^\alpha) \oplus K_1(\mathcal{T}^\beta) & \longleftarrow & K_1(\mathcal{S}^{\alpha,\beta}). \end{array}$$

Consider the map from $K_0(\mathcal{T}^\alpha) \oplus K_0(\mathcal{T}^\beta)$ to $K_0(C(\mathbf{T}^2))$. The elements of $K_0(\mathcal{T}^\alpha)$ and $K_0(\mathcal{T}^\beta)$ consist of standard projections, and thus the map from $K_0(\mathcal{T}^\alpha) \oplus$ $\oplus K_0(\mathcal{T}^\beta)$ to $K_0(C(\mathbf{T}^2))$ maps onto the standard projections. Combining this fact with the calculation of $K_0(C(\mathbf{T}^2))$ above, we see that $K_1(\mathcal{S}^{\alpha,\beta})$ contains at least a factor of $\mathbf{Z}$. Next, consider the map from $K_1(\mathcal{T}^\alpha) \oplus K_1(\mathcal{T}^\beta)$ to $K_1(C(\mathbf{T}^2))$; there are theree cases. First, if $\alpha = p/q$ and $\beta = r/s$ are rational numbers, then $[P^\alpha M_{q,p} P^\alpha]$ and $[P^\beta M_{s,r} P^\beta]$ are generators of $K_1(\mathcal{T}^\alpha)$ and $K_1(\mathcal{T}^\beta)$, respectively, and these generators map to $[\chi_{q,p}]$ and $[\chi_{s,r}]$, respectively. Moreover, it is easy to check that since α and β are distinct, the map from $K_1(\mathcal{T}^\alpha) \oplus K_1(\mathcal{T}^\beta)$ to $K_1(C(\mathbf{T}^2))$ is injective. Next, if only one of α and β is rational, say α, then $K_1(\mathcal{T}^\beta) \cong 0$ and $K_1(\mathcal{T}^\alpha)$ maps injectively into $K_1(C(\mathbf{T}^2))$. Finally, if α and β are both irrational, then $K_1(\mathcal{T}^\alpha)$ and $K_1(\mathcal{T}^\beta)$ are both zero. Hence in all three cases, the only contribution we have to $K_1(\mathcal{S}^{\alpha,\beta})$ comes from $K_0(C(\mathbf{T}^2))$, so

$$K_1(\mathcal{S}^{\alpha,\beta}) \cong \mathbf{Z}.$$

Let S and T be Fredholm operators in $\mathcal{T}^{\alpha,\beta}$. If S and T have the same index, can they be connected by a path of Fredholm operators in $\mathcal{T}^{\alpha,\beta}$? The next result shows that if one of α and β is rational, S and T can at least be *stably* connected.

THEOREM 3.1. *Let α and β be distinct numbers, at least one of which is rational. Then index is a complete stable deformation invariant for Fredholm operators in* $\mathcal{T}^{\alpha,\beta}$.

Proof. In view of the calculation of $K_1(\mathcal{S}^{\alpha,\beta})$ above, it suffices to exhibit a Fredholm operator in $\mathcal{T}^{\alpha,\beta}$ that has index one.

Without loss of generality, assume that α is rational and write $\alpha = p/q$, where p and q are relatively prime integers and q is positive. Since p and q are relatively prime, there exist integers m and n so that $-pm + qn = 1$. With this in mind, define operators

$$A = P^\alpha P^\beta M_{-q,-p} P^\alpha P^\beta,$$

$$B = P^\alpha P^\beta M_{m,n} P^\alpha M_{-m,-n} P^\alpha P^\beta,$$

$$T = I - (I - A)(I - B).$$

Then

$$\gamma^\alpha(T) = I - (I - \gamma^\alpha(A))(I - \gamma^\alpha(B)),$$

and since $\gamma^\alpha(A)$ is unitary, $\gamma^\alpha(B)$ is a projection, and $\gamma^\alpha(A)$ and $\gamma^\alpha(B)$ commute, $\gamma^\alpha(T)$ is invertible with inverse

$$\gamma^\alpha(T)^{-1} = I - (I - \gamma^\alpha(A)^*)(I - \gamma^\alpha(B)).$$

Moreover, $\gamma^\beta(B) = I$, so

$$\gamma^\beta(T) = I - (I - \gamma^\beta(A))(I - \gamma^\beta(B)) = I = \gamma^\beta(T)^{-1}.$$

Since $\gamma^\alpha(T)$ and $\gamma^\beta(T)$ are both invertible, T is Fredholm by Theorem 2.6.

Now we compute the index of T. Since $\alpha < \beta$ and q is positive, $-\beta q + p < < -\alpha q + p = 0$. Therefore A^* is an isometry. Next, since $-pm + qn = 1$ and q is positive, $-\alpha m + n$ is positive, and therefore B is a projection that is not the identity. Finally, direct computation shows that A and B commute. Hence, it is easy to check that

$$TT^* = I$$

$$T^*T = I - (I - A^*A)(I - B).$$

The first equality implies that $\ker T^* = \{0\}$. Furthermore, we have $\ker T = \ker T^*T$. Now, it is easy to check that $e_{k,l}$ is in the kernel of T^*T if and only if the integers k and l satisfy the inequalities

$$0 \leqslant -\alpha k + l < -\alpha m + n$$

$$0 \geqslant -\beta k + l > -\beta q + p;$$

otherwise, $T^*T(e_{k,l}) = e_{k,l}$.

Consider the first inequality. If we write α as p/q and multiply through by q, we obtain

$$0 \leqslant -pk + ql < -pm + qn = 1.$$

Therefore, $-pk + ql = 0$, or $\alpha k = l$.

Now consider the second inequality. If we substitute αk for l, we obtain

$$0 \geqslant (\alpha - \beta)k > -\beta q + p,$$

or, since $\alpha - \beta$ is negative,

$$0 \leqslant k < \frac{-\beta q + p}{\alpha - \beta} = q.$$

Now, since $l = \alpha k = (p/q)k$, we see that the only integers k and l that satisfy both inequalities are $k = l = 0$. Therefore T is a Fredholm operator, and

$$\operatorname{index} T = \dim\ker T - \dim\ker T^* = 1 - 0 = 1.$$ ▨

Theorem 3.1 only applies in the case when one of α and β is rational, so it is natural to ask what happens in the case where α and β are both irrational. We still have the result $K_1(\mathscr{S}^{\alpha,\beta}) \cong \mathbf{Z}$. However, we have been unable to construct Fredholm operators in $\mathscr{T}^{\alpha,\beta}$ that have nonzero index. It may be that there are Fredholm operators of nonzero index in $\mathscr{T}^{\alpha,\beta}$ and that index is a complete stable deformation invariant here as well. Another possibility is that this case is fundamentally different from the other two and all the Fredholm operators have index zero. If this is indeed true, then the isomorphism $K_1(\mathscr{S}^{\alpha,\beta}) \cong \mathbf{Z}$ would correspond to some kind of "secondary" index. Finally, it is possible that the index theory of $\mathscr{T}^{\alpha,\beta}$ depends upon the relationship between α and β. For example, it could turn out that when α and β are rationally dependent, $\mathscr{T}^{\alpha,\beta}$ contains Fredholm operators with nonzero index, and otherwise all the Fredholm operators have index zero. At present, we do not know which of these possibilities occur, and we hope to resolve this point in the future.

4. CYCLIC COHOMOLOGY AND AN INDEX FORMULA

In Section 2, we established criteria for an operator in $\mathscr{T}^{\alpha,\beta}$ to be Fredholm. In this section, we seek a formula for computing the index of Fredholm operators in $\mathscr{T}^{\alpha,\beta}$. An index formula for Fredholm operators in $\mathscr{T}^{0,\infty}$ was given in [4]; the formula uses the existence of certain operator-valued homotopies in the symbol algebra $\mathscr{S}^{0,\infty}$, and these are in practice difficult to produce. Presumably this result generalizes to arbitrary quarter-plane Toeplitz algebras, but we seek an index formula that is easier to compute. We use Connes' cyclic cohomology to produce such a formula.

Consider the following short exact sequence from Section 2:

$$0 \to \mathscr{K} \to \mathscr{T}^{\alpha,\beta} \to \mathscr{S}^{\alpha,\beta} \to 0.$$

We showed in Proposition 2.2 that this sequence has a linear splitting ρ from $\mathscr{S}^{\alpha,\beta}$ to $\mathscr{T}^{\alpha,\beta} \subset L(\mathscr{H}^{\alpha,\beta})$, and therefore for all X and Y in $\mathscr{S}^{\alpha,\beta}$, $\rho(XY)-\rho(X)\rho(Y)$ is compact. Unfortunately, in this generality, this is the most we can say. For example, it is not true that $\rho(XY) - \rho(X)\rho(Y)$ is always a trace class operator. To ensure that $\rho(XY) - - \rho(X)\rho(Y)$ is trace class, we will restrict our attention to a dense subalgebra $\mathscr{S}^{\alpha,\beta}_{\infty}$ of $\mathscr{S}^{\alpha,\beta}$ which we will define presently.

Let T be a finite product, and define

$$\Lambda(T) = \max\{1, \lambda^{\alpha}(T), \lambda^{\beta}(T)\}.$$

Then define $\mathscr{T}^{\alpha,\beta}_{\infty}$ to be the collection of operators X in $\mathscr{T}^{\alpha,\beta}$ that can be written in in the form

$$X = \sum_{k=0}^{\infty} c_k T_k,$$

where each T_k is a finite product, and where the sequence

$$\{c_k(\Lambda(T_k))^2\}$$

is absolutely summable. Note that in particular the sequence $\{c_k\}$ is absolutely summable, and since each finite product T_k has norm 1, the infinite sum above is well defined.

Proposition 4.1. *$\mathscr{T}^{\alpha,\beta}_{\infty}$ is an algebra.*

Proof. Clearly $\mathscr{T}^{\alpha,\beta}_{\infty}$ is closed under addition and scalar multiplication. The only nonobvious point to check is that $\mathscr{T}^{\alpha,\beta}_{\infty}$ is closed under multiplication.

Let S and T be finite products. Then it is easy to see from the definitions of λ^{α} and λ^{β} that

$$\lambda^{\alpha}(ST) = \lambda^{\alpha}(S) + \lambda^{\alpha}(T)$$

$$\lambda^{\beta}(ST) = \lambda^{\beta}(S) + \lambda^{\beta}(T),$$

and therefore

$$\Lambda(ST) \leqslant \Lambda(S) + \Lambda(T).$$

Now let $S = \sum_{l=0}^{\infty} b_l S_l$ and $T = \sum_{k=0}^{\infty} c_k T_k$ be in $\mathscr{T}_{\infty}^{\alpha,\beta}$. Then

$$ST = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} b_l c_k S_l T_k,$$

and it is a simple computation to check that

$$\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} |b_l c_k| (\Lambda(S_l T_k))^2 < \infty.$$

Therefore, ST is in $\mathscr{T}_{\infty}^{\alpha,\beta}$. ∎

LEMMA 4.2. *Let T be a finite product in $\mathscr{T}^{\alpha,\beta}$, and let $\| \quad \|_1$ denote the trace norm. Then*

$$\|T - \rho\gamma(T)\|_1 \leqslant C\Lambda(T)^2.$$

where C is a constant depending only on α and β.

Proof. Since T is a finite product, we may write T in the form

$$T = P^\alpha P^\beta M_{m_0,n_0} \left[\prod_{j=0}^{k} Q_j M_{m_j,n_j} \right] P^\alpha P^\beta.$$

We first claim that $T - \rho\gamma(T)$ is a partial isometry. To see this, first observe that both T and $\rho\gamma(T)$ are finite products, T and $\rho\gamma(T)$ are partial isometries, and moreover, appropriate subsets of the basis elements $e_{m,n}$ of $\mathscr{H}^{\alpha,\beta}$ serve as bases for the ranges and kernels of T and $\rho\gamma(T)$. Furthermore, T and $\rho\gamma(T)$ are composed of the same translations, so if $T(e_{m,n})$ and $(\rho\gamma(T))(e_{m,n})$ are both nonzero for some $e_{m,n}$, then $T(e_{m,n}) = (\rho\gamma(T))(e_{m,n})$. Therefore, $T - \rho\gamma(T)$ is a partial isometry.

Now, since $T - \rho\gamma(T)$ is a partial isometry, the operator $[T - \rho\gamma(T)][T - \rho\gamma(T)]^*$ is a projection onto the range of $T - \rho\gamma(T)$. Therefore,

$$\|T - \rho\gamma(T)\|_1 = \|(T - \rho\gamma(T))^*\|_1 =$$

$$= \mathrm{trace}\{[T - \rho\gamma(T)]\,[T - \rho\gamma(T)]^*\} = \dim \mathrm{ran}(T - \rho\gamma(T)).$$

The dimension of the range of $T - \rho\gamma(T)$ is no greater than the number of $e_{m,n}$ such that $(T - \rho\gamma(T))(e_{m,n}) \neq 0$, and in fact is actually equal to this number. Now, we proved in Proposition 2.3 that $(T - \rho\gamma(T))(e_{m,n}) \neq 0$ implies that the pair of integers (m, n) satisfies the following inequalities:

$$0 \leqslant -\alpha m + n < \lambda^\alpha(T)$$

$$-\lambda^\beta(T) < -\beta m + n \leqslant 0.$$

We can combine these two inequalities to obtain

$$0 \leqslant (\beta - \alpha)m \leqslant \lambda^{\alpha}(T) + \lambda^{\beta}(T),$$

so the number of different possible values of m that can appear in a solution to the inequalities is bounded by

$$\frac{\lambda^{\alpha}(T) + \lambda^{\beta}(T)}{\beta - \alpha} + 1.$$

We can also add these inequalities to eliminate m. Taking into account the fact that α and β may be positive or negative, we see that the number of possible values of n that can appear in a solution to the inequalities is bounded by

$$\frac{|\beta|\lambda^{\alpha}(T) + |\alpha|\lambda^{\beta}(T)}{\beta - \alpha} + 1.$$

Therefore, the total number of possible solutions, and hence the dimension of the range of $T - \rho\gamma(T)$, is bounded by

$$\left[\frac{\lambda^{\alpha}(T) + \lambda^{\beta}(T)}{\beta - \alpha} + 1\right]\left[\frac{|\beta|\lambda^{\alpha}(T) + |\alpha|\lambda^{\beta}(T)}{\beta - \alpha} + 1\right].$$

We get the bound in the statement of the lemma by recalling the definition on $\Lambda(T)$ as the maximum of 1, $\lambda^{\alpha}(T)$, and $\lambda^{\beta}(T)$. ▨

We now define a dense subalgebra $\mathscr{S}_{\infty}^{\alpha,\beta}$ of $\mathscr{S}^{\alpha,\beta}$ by

$$\mathscr{S}_{\infty}^{\alpha,\beta} = \gamma(\mathscr{T}_{\infty}^{\alpha,\beta}).$$

PROPOSITION 4.3. *Let X and Y be in $\mathscr{S}_{\infty}^{\alpha,\beta}$. Then $\rho(XY) - \rho(X)\rho(Y)$ is trace class.*

Proof. Choose operators S and T in $\mathscr{T}_{\infty}^{\alpha,\beta}$ such that $\gamma(S) = X$ and $\gamma(T) = Y$. First consider the case where S and T are both finite products. Then

$$\rho(XY) - \rho(X)\rho(Y) = \rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T)) =$$

$$= S[T - \rho\gamma(T)] + [S - \rho\gamma(S)]\rho\gamma(T) - [ST - \rho\gamma(ST)].$$

Therefore,

$$\|\rho(XY) - \rho(X)\rho(Y)\|_1 \leqslant \|S\|_\infty\|T - \rho\gamma(T)\|_1 +$$

$$+ \|S - \rho\gamma(S)\|_1\|T\|_\infty + \|ST - \rho\gamma(ST)\|_1,$$

where $\|\ \|_\infty$ denotes the operator norm. Since $\|S\|_\infty = \|T\|_\infty = 1$, the estimate from Lemma 4.2 gives

$$\|\rho(XY) - \rho(X)\rho(Y)\|_1 - C[\Lambda(S)^2 + \Lambda(ST)^2 + \Lambda(T)^2].$$

Thus $\rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T))$ is a trace class operator, and is in fact a finite rank operator.

Now, let $S = \sum_{l=0}^{\infty} b_l S_l$ and $T = \sum_{k=0}^{\infty} c_k T_k$ be operators in $\mathcal{T}_{\infty}^{\alpha,\beta}$, where the S_l and T_k are finite products. Since ρ and γ are linear,

$$\rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T)) =$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} b_l c_k[\rho(\gamma(S_l T_k)) - \rho(\gamma(S_l))\rho(\gamma(T_k))],$$

and therefore

$$\|\rho(XY) - \rho(X)\rho(Y)\|_1 \leqslant$$

$$\leqslant \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} |b_l c_k| \|\rho(\gamma(S_l T_k)) - \rho(\gamma(S_l))\rho(\gamma(T_k))\|_1 \leqslant$$

$$\leqslant C \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} |b_l c_k| [\Lambda(S_l)^2 + \Lambda(S_l T_k)^2 + \Lambda(T_k)^2].$$

Since $\Lambda(S_l T_k) \leqslant \Lambda(S_l) + \Lambda(T_k)$ for all l and k, the above sum is finite, and therefore $\rho(XY) - \rho(X)\rho(Y)$ is a trace class operator. ▨

Proposition 4.3 shows that we can define a cyclic 1-cocycle τ on $\mathcal{S}_{\infty}^{\alpha,\beta}$ by

$$\tau(X, Y) = \mathrm{Trace}[[\rho(XY) - \rho(X)\rho(Y)] - [\rho(YX) - \rho(Y)\rho(X)]].$$

Now, from [5] we have a bilinear pairing $\langle\, , \rangle$ of K-theory with cyclic cohomology. This pairing has the property that if u is an invertible element in $\mathcal{S}_{\infty}^{\alpha,\beta}$ that represents the class $[u]$ in $K_1(\mathcal{S}_{\infty}^{\alpha,\beta})$, and if $[\tau]$ denotes the class of τ in the cyclic cohomology group $H_{\lambda}^1(\mathcal{S}_{\infty}^{\alpha,\beta})$, then

$$\langle [u], [\tau] \rangle = \mathrm{index}\, \rho(u).$$

We will use this fact to produce our index formula.

Let T be a Fredholm operator in $\mathcal{T}^{\alpha,\beta}$. Then since T and $\rho\gamma(T)$ differ by a compact operator, $\rho\gamma(T)$ is also Fredholm, and T and $\rho\gamma(T)$ have the same index. Therefore, to determine the index of T, it suffices to compute the index of $\rho\gamma(T)$. Combining this observation with the definition of the pairing between K-theory and cyclic cohomology, we obtain the following index formula:

THEOREM 4.4. *Let T in $\mathcal{T}^{\alpha,\beta}$ be a Fredholm operator such that $\gamma(T)$ and $\gamma(T)^{-1}$ are in $\mathcal{S}^{\alpha,\beta}_{\infty}$. Then the index of T is given by the following formula:*

$$\operatorname{Index} T = \operatorname{Trace}[\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T))]. \qquad \square$$

We conclude this section with some remarks. First, the class of Fredholm operators we consider is not the largest class for which this formula will work. However, Theorem 4.4 applies to many cases of interest. Second, this index formula will also give the index of operators in the matrices over $\mathcal{T}^{\alpha,\beta}$. In this case, the trace in the formula is composed with the usual matrix trace. Finally, Theorem 4.4 is true even when α and β are both irrational. However, when α and β are both irrational, it is not known if there are any operators of nonzero index, so in these cases the theorem may turn out to be uninteresting.

5. AN EXAMPLE

We now take a specific operator T in $\mathcal{T}^{\alpha,\beta}$, use Theorem 2.6 to show that T is Fredholm, and then use Theorem 4.4 to compute the index of T.

Let α be rational and negative, and let β be any positive number, rational or irrational. Write $\alpha = p/q$, with p and q relatively prime and $p < 0$, $q > 0$. Also, choose positive integers r and s. Then let

$$T = \frac{1}{2} P^\alpha P^\beta M_{-q,-p} P^\alpha P^\beta + P^\alpha P^\beta M_{s,r} P^\alpha M_{-s,-r} P^\alpha P^\beta.$$

Now,

$$\gamma^\alpha(T) = \frac{1}{2} P^\alpha M_{-q,-p} P^\alpha + P^\alpha M_{s,r} P^\alpha M_{-s,-r} P^\alpha$$

$$\gamma^\alpha(T)^{-1} = 2P^\alpha M_{q,p} P^\alpha - 2P^\alpha M_{q,p} P^\alpha M_{s,r} P^\alpha M_{-s,-r} P^\alpha + \left[\sum_{n=0}^{\infty} (-2)^{-n} P^\alpha M_{-nq,-np} P^\alpha\right] P^\alpha M_{s,r} P^\alpha M_{-s,-r} P^\alpha.$$

and

$$\gamma^\beta(T) = I + \frac{1}{2} P^\beta M_{-q,-p} P^\beta$$

$$\gamma^\beta(T)^{-1} = \sum_{n=0}^{\infty} (-2)^{-n} P^\beta M_{-nq,-np} P^\beta.$$

Both $\gamma^\alpha(T)$ and $\gamma^\beta(T)$ are invertible, so T is Fredholm by Theorem 2.6. Next,

$$\rho(\gamma(T)) = \frac{1}{2} P^\alpha P^\beta M_{-q,-p} P^\alpha P^\beta + P^\alpha P^\beta M_{s,r} P^\alpha M_{-s,-r} P^\alpha P^\beta$$

$$\rho(\gamma(T)^{-1}) = 2P^\alpha P^\beta M_{q,p} P^\alpha P^\beta (I - P^\alpha P^\beta M_{s,r} P^\alpha M_{-s,-r} P^\alpha P^\beta) +$$

$$+ \left[\sum_{n=0}^{\infty} (-2)^{-n} P^\alpha P^\beta M_{-nq,-np} P^\alpha P^\beta \right] P^\alpha P^\beta M_{s,r} P^\alpha M_{-s,-r} P^\alpha P^\beta .$$

Therefore,

$$\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T)) = AB,$$

where

$$A = P^\alpha P^\beta M_{q,p}(I - P^\alpha P^\beta) M_{-q,-p} P^\alpha P^\beta$$

$$B = P^\alpha P^\beta M_{s,r}(I - P^\alpha) M_{-s,-r} P^\alpha P^\beta .$$

Direct computation yields that AB is a finite rank projection, and $e_{k,l}$ is in the range of AB if and only if the pair of integers (k, l) satisfies the inequalities

$$0 \leqslant -\alpha k + l < -\alpha s + r$$

$$0 \geqslant -\beta k + l > -\beta q + p.$$

Theorem 4.4 yields that the index T is the number of pairs of integers (k, l) that satisfy the inequalities above.

To make this example more concrete, let $q = 1$, $p = -1$, $r = 1$, and $s = 2$. Then $\alpha = -1$. Also, let $\beta = \sqrt{2}$. The above inequalities become

$$0 \leqslant k + l < 3$$

$$0 \geqslant -k\sqrt{2} + l > -\sqrt{2} - 1.$$

The only pairs of integers that are solutions to the inequalities are $(0, 0)$, $(1, 0)$, and $(1, 1)$. Therefore, the index of T in this case is 3.

Acknowledgements. This paper represents my thesis work at SUNY Stony Brook, and I would like to thank my advisor Ron Douglas for his help and encouragement.

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Received October 20, 1988.