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COMPACT HANKEL OPERATORS ON THE BERGMAN SPACES OF THE UNIT BALL AND POLYDISK IN $\mathbf{C}^n$

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1. INTRODUCTION

Throughout this paper let $n \in \mathbf{N}$ be fixed. For $z, w \in \mathbf{C}^n$ let $\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$, and let $\|z\| = \langle z, z\rangle^{1/2}$. Let $\mathbf{B}_n = \{z \in \mathbf{C}^n : \|z\| < 1\}$ denote the open unit ball, and let $\mathbf{D}^n = \{z \in \mathbf{C}^n : |z_j| < 1 \text{ for } 1 \leqslant j \leqslant n\}$ denote the polydisk in $\mathbf{C}^n$. For Ω either the ball $\mathbf{B}_n$ or the polydisk $\mathbf{D}^n$ let V denote the Lebesgue measure normalized so that Ω has measure 1. For $1 \leqslant p < \infty$ and $f: \Omega \to \mathbf{C}$ Lebesgue measurable let $\|f\|_p = \left(\int_\Omega |f|^p \mathrm{d}V\right)^{1/p}$. The Bergman space $A^p(\Omega)$ is the Banach space of holomorphic functions $f: \Omega \to \mathbf{C}$ such that $\|f\|_p < \infty$. The Bergman space $A^2(\Omega)$ is a Hilbert space; it is a closed subspace of the Hilbert space $L^2(\Omega, \mathrm{d}V)$ with inner product given by $\langle f, g\rangle = \int_\Omega f(z)\,\overline{g(z)}\,\mathrm{d}V(z)$, for $f, g \in L^2(\Omega, \mathrm{d}V)$. Let P denote the orthogonal projection of $L^2(\Omega, \mathrm{d}V)$ onto $A^2(\Omega)$. The map $I - P$ is the orthogonal projection of $L^2(\Omega, \mathrm{d}V)$ onto $A^2(\Omega)^\perp$ [the orthogonal complement of $A^2(\Omega)$ in $L^2(\Omega, \mathrm{d}V)$]. For a function $f \in L^\infty(\Omega, \mathrm{d}V)$, the Hankel operator $H_f : A^2(\Omega) \to A^2(\Omega)^\perp$ is defined by

$$H_f g = (I - P)(fg), \quad g \in A^2(\Omega).$$

It is clear that H_f is a bounded operator for every function $f \in L^\infty(\Omega, \mathrm{d}V)$. In this paper we consider the question of characterizing the functions $f \in L^\infty(\Omega, \mathrm{d}V)$ for which the Hankel operator H_f is a compact operator on $A^2(\Omega)$, for Ω the unit ball or the polydisk in $\mathbf{C}^n$. For $n = 1$ this question was raised by Sheldon Axler in [1]. Sheldon Axler answered a special case of this problem in [2] where he considered conjugate analytic symbols on the unit disk $\mathbf{D}$. Recently the author obtained a complete answer for Axler's question in [6]. In this paper we will extend those results to the Bergman spaces of the unit ball $\mathbf{B}_n$ and the polydisk $\mathbf{D}^n$.

In our characterization of the compact Hankel operators on the Bergman space $A^2(\Omega)$, for Ω the ball $\mathbf{B}_n$ or the polydisk $\mathbf{D}^n$, the Möbius transformations on Ω will play a crucial role. For the unit ball $\mathbf{B}_n$ these Möbius transformations are described in Section 2.2 of [5]: for $\lambda \in \mathbf{B}_n$ let the Möbius transformation $\varphi_\lambda : \mathbf{B}_n \to \mathbf{B}_n$ be defined by

$$\varphi_\lambda(z) = \frac{\lambda - P_\lambda z - (1 - \|\lambda\|^2)^{1/2} Q_\lambda z}{1 - \langle z, \lambda \rangle}, \tag{1}$$

where $P_\lambda z$ is the orthogonal projection of z onto the subspace spanned by λ, and $Q_\lambda z = z - P_\lambda z$ is the projection of z onto the orthogonal complement of this subspace. More explicitly, $P_0 z = 0$, and for $\lambda \neq 0$, $P_\lambda z = \langle z, \lambda \rangle \lambda / \|\lambda\|^2$ $(z \in \mathbf{B}_n)$. In the case of the polydisk $\mathbf{D}^n$, for each $\lambda \in \mathbf{D}^n$ let the Möbius transformation $\varphi_\lambda : \mathbf{D}^n \to \mathbf{D}^n$ be defined by

$$\varphi_\lambda(z) = \left(\frac{\lambda_1 - z_1}{1 - \bar{\lambda}_1 z_1}, \ldots, \frac{\lambda_n - z_n}{1 - \bar{\lambda}_n z_n} \right), \quad z \in \mathbf{D}^n. \tag{2}$$

For $\Omega \subset \mathbf{C}^n$ let $\partial\Omega$ denote the topological boundary of Ω. The statement $\lambda \to \partial\Omega$ will mean that $\lambda \in \Omega$ and the usual distance $\operatorname{dist}(\lambda, \partial\Omega) \to 0$. For the unit ball $\mathbf{B}_n$ this means that $\|\lambda\| \to 1^-$; for the polydisk $\mathbf{D}^n$ the meaning is that $\min\{1 - |\lambda_j| : 1 \leq j \leq n\} \to 0^+$. The main result of this paper states that the Hankel operator H_f is a compact operator on $A^2(\Omega)$, for Ω the unit ball or the polydisk in $\mathbf{C}^n$, if and only if

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \to 0 \quad \text{as} \quad \lambda \to \partial\Omega.$$

In Section 2 we will give some of the preliminaries for both the unit ball and the polydisk in $\mathbf{C}^n$. In Section 3 we will focus in on the unit ball in $\mathbf{C}^n$ and prove our result in this setting. The case of the polydisk in $\mathbf{C}^n$ will then be discussed in Section 4. In Section 5 we will indicate how the results in Section 3 can be extended to Hankel operators on weighted Bergman spaces on the unit ball in $\mathbf{C}^n$. We end the paper with some remarks and an open question in Section 6.

I would like to thank Sheldon Axler for his helpful comments, and the referee for several suggestions, including the proof of Proposition 1.

Recently Dechao Zheng independently obtained the same answer to Axler's question raised in [1]. Using a different method than ours, Zheng also obtained Theorem 5 and Theorem 12 in [9].

2. PRELIMINARIES

Let Ω denote the unit ball $\mathbf{B}_n$ or the polydisk $\mathbf{D}^n$ in $\mathbf{C}^n$. Point evaluation is a bounded linear functional on the Hilbert space $A^2(\Omega)$, thus for every $\lambda \in \Omega$ there exists a unique holomorphic $k_\lambda \in A^2(\Omega)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle, \quad \text{for all } f \in A^2(\Omega).$$

These functions k_λ ($\lambda \in \Omega$) are called the reproducing kernels for $A^2(\Omega)$. For the unit ball and the polydisk in $\mathbf{C}^n$ these reproducing kernels can be computed explicitly (see, for example, Section 1.4 in [4]). For $\Omega = \mathbf{B}_n$ we have:

$$(3) \qquad k_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^{n+1}}, \quad z \in \mathbf{B}_n ;$$

for $\Omega = \mathbf{D}_n$ we have:

$$(4) \qquad k_\lambda(z) = \prod_{j=1}^{n} \frac{1}{(1 - z_j \bar{\lambda}_j)^2}, \quad z \in \mathbf{D}^n .$$

For $g \in L^2(\Omega, \mathrm{d}V)$ and $z \in \Omega$ we have $(Pg)(z) = \langle Pg, k_z \rangle = \langle g, k_z \rangle$, so we get the following formula for the projection Pg:

$$(5) \qquad (Pg)(z) = \int_\Omega g(w) \overline{k_z(w)} \, \mathrm{d}V(w), \quad z \in \Omega.$$

Using this formula for the product fg and for $g = Pg$ we get the following formula for the Hankel operator H_f; for $f \in L^\infty(\Omega, \mathrm{d}V)$ and $g \in A^2(\Omega)$ we have:

$$(6) \qquad (H_f g)(z) = \int_\Omega (f(z) - f(w)) g(w) \overline{k_z(w)} \mathrm{d}V(w), \quad z \in \Omega.$$

We will repeatedly use that every Möbius transformations φ_λ is its own inverse under composition: $(\varphi_\lambda \circ \varphi_\lambda)(z) = z$ for all $z \in \Omega$ (for the unit ball see [5], Section 2.2).

The following proposition gives a formula for the image under H_f of the reproducing kernels k_λ for $\lambda \in \Omega$. This formula will play an important role in our characterization of the compact Hankel operators.

PROPOSITION 1. *Let $f \in L^\infty(\Omega, \mathrm{d}V)$. For each $\lambda \in \Omega$ we have:*

$$(7) \qquad H_f(k_\lambda) = (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) k_\lambda .$$

Proof. Take $f \in L^\infty(\Omega, \mathrm{d}V)$ and let $\lambda \in \Omega$. Let J_λ be the determinant of the complex Jacobian of φ_λ, and let $U_\lambda : L^2(\Omega, \mathrm{d}V) \to L^2(\Omega, \mathrm{d}V)$ be the operator defined by $U_\lambda g = (g \circ \varphi_\lambda) J_\lambda$ for $g \in L^2(\Omega, \mathrm{d}V)$. Then it is readily seen that U_λ is a unitary operator (with inverse U_λ^{-1} given by $U_\lambda^{-1} g = (g \circ \varphi_\lambda)/(J_\lambda \circ \varphi_\lambda)$ for $g \in L^2(\Omega, \mathrm{d}V)$) such that $U_\lambda(A^2(\Omega)) \subset A^2(\Omega)$. It follows that $PU_\lambda = U_\lambda P$. Thus

$$H_f(J_\lambda) = fJ_\lambda - P(fJ_\lambda) = fJ_\lambda - PU_\lambda(f \circ \varphi_\lambda) =$$

$$= fJ_\lambda - U_\lambda P(f \circ \varphi_\lambda) = (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) J_\lambda .$$

Now (7) follows, because k_λ is a constant (depending on λ) multiple of J_λ. ▧

3. THE UNIT BALL IN $\mathbf{C}^n$

In this section we will characterize the bounded measurable functions on $\mathbf{B}_n$ for which the Hankel operator H_f is compact. Combining (3) and (6) we see that in this case H_f is defined as follows:

$$(8) \qquad (H_f g)(z) = \int_{\mathbf{B}_n} \frac{f(z) - f(w)}{(1 - \langle z, w \rangle)^{n+1}} g(w) \, \mathrm{d}V(w), \qquad z \in \mathbf{B}^n .$$

We will frequently make use of the following well-known identity ([5], Section 2.2):

$$(9) \qquad 1 - \|\varphi_\lambda(z)\|^2 = \frac{(1 - \|\lambda\|^2)(1 - \|z\|^2)}{|1 - \langle z, \lambda \rangle^2|}, \qquad (z, \lambda \in \mathbf{B}_n).$$

It will also be convenient to have the following formula available ([5], Section 2.2):

$$(10) \qquad (1 - \langle z, \lambda \rangle)(1 - \langle \varphi_\lambda(z), \lambda \rangle) = (1 - \|\lambda\|^2) \qquad (z, \lambda \in \mathbf{B}_n).$$

For $\lambda \in \mathbf{B}_n$, the substitution $z = \varphi_\lambda(w)$ results in the Jacobian change in measure given by $\mathrm{d}V(z) = J_{\mathbf{R}}\varphi_\lambda(w)\mathrm{d}V(w)$, where $J_{\mathbf{R}}\varphi_\lambda$, the real Jacobian of φ_λ, is given by $J_{\mathbf{R}}\varphi_\lambda(w) = (1 - \|\lambda\|^2)^{n+1}/|1 - \langle w, \lambda \rangle|^{2n+2}$ ([5], Section 2.2). Thus, for a Lebesgue integrable or non-negative Lebesgue measurable function h on $\mathbf{B}_n$ we have the change-of-variable formula:

$$(11) \qquad \int_{\mathbf{B}_n} h(z) \mathrm{d}V(z) = \int_{\mathbf{B}_n} (h \circ \varphi_\lambda)(w) \frac{(1 - \|\lambda\|^2)^{n+1}}{|1 - \langle w, \lambda \rangle|^{2n+2}} \, \mathrm{d}V(w).$$

To show the compactness of H_f we will actually consider the operator $H_f^* H_f$. The following proposition gives a convenient way of representing this operator.

PROPOSITION 2. *Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. Then for $h \in H^\infty(\mathbf{B}_n)$ and $\lambda \in \mathbf{B}_n$:*

$$(H_f^* H_f h)(\lambda) = \int_{\mathbf{B}_n} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \langle \lambda, z \rangle)^{n+1}} h(z) \mathrm{d}V(z).$$

Proof. Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$, $h \in H^\infty(\mathbf{B}_n)$ and fix a point $\lambda \in \mathbf{B}_n$. Then

$$(H_f^* H_f h)(\lambda) = \langle H_f^* H_f h, k_\lambda \rangle = \langle fh - P(fh), H_f k_\lambda \rangle =$$

$$= \langle fh, H_f k_\lambda \rangle \qquad \text{(since } P(fh) \perp H_f k_\lambda)$$

Now, $P(f \circ \varphi_\lambda) \circ \varphi_\lambda \in A^2(\mathbf{B}_n)$, thus $(P(f \circ \varphi_\lambda) \circ \varphi_\lambda)h \in A^2(\mathbf{B}_n)$, so that we have $\langle (P(f \circ \varphi_\lambda) \circ \varphi_\lambda)h, H_f k_\lambda \rangle = 0$. Using this we get

$$(H_f^* H_f h)(\lambda) = \langle fh, H_f k_\lambda \rangle = \langle (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda)h, H_f k_\lambda \rangle =$$

$$= \langle (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda)h, (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda)k_\lambda \rangle = \qquad \text{(by (7))}$$

$$= \int_{\mathbf{B}_n} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \langle \lambda, z \rangle)^{n+1}} h(z) \mathrm{d}V(z).$$

▨

REMARK. We will use Proposition 2 as stated; however, the formula in Proposition 2 holds for all $h \in A^2(\mathbf{B}_n)$: from the discussion that follows it will become clear that for fixed $\lambda \in \mathbf{B}_n$ the linear functional given by the integral in Proposition 2 is bounded on $A^2(\mathbf{B}_n)$, and thus agrees with the bounded linear functional on $A^2(\mathbf{B}_n)$ defined by $h \mapsto (H_f^* H_f h)(\lambda)$.

The following lemma will be used in the proof of Lemma 4.

LEMMA 3. *Let $p < (2n + 2)/(2n + 1)$. Then:*

$$\sup_{\lambda \in \mathbf{B}_n} \int_{\mathbf{B}_n} \frac{1}{(1 - \|w\|^2)^{p/2} |1 - \langle w, \lambda \rangle|^{np}} \mathrm{d}V(w) < \infty.$$

Proof. In Proposition 1.4.10 of [5] take $t = -p/2$ and put $np = n + 1 + t + c$; the condition $p < (2n + 2)/(2n + 1)$ implies that $c < 0$. ▨

The following lemma gives an estimate that will be used in the proof of Theorem 5, our characterization of compact Hankel operators on the Bergman space $A^2(\mathbf{B}_n)$.

LEMMA 4. *Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. Then there exists a finite positive constant C (depending on f) such that for every $\lambda \in \mathbf{B}_n$:*

$$\int_{\mathbf{B}_n} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^4}{(1-\|z\|^2)^{1/2}\,|1-\langle\lambda, z\rangle|^{n+1}}\,\mathrm{d}V(z) \leqslant \frac{C}{(1-\|\lambda\|^2)^{1/2}}\,\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2^{1/(2n+3)}.$$

Proof. Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. In the integral at the left make the change-of--variable $w = \varphi_\lambda(z)$. Using (9), (10) and (11) we get

$$\int_{\mathbf{B}_n} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^4}{(1-\|z\|^2)^{1/2}\,|1-\langle\lambda, z\rangle|^{n+1}}\,\mathrm{d}V(z) =$$

$$= \frac{1}{(1-\|\lambda\|^2)^{1/2}}\int_{\mathbf{B}_n} |f(\varphi_\lambda(w)) - P(f\circ\varphi_\lambda)(w)|^4\,\frac{1}{(1-\|w\|^2)^{1/2}\,|1-\langle w, \lambda\rangle^n|}\,\mathrm{d}V(w).$$

Put $p = (2n+3)/(2n+2)$, and let $M_{p,n}$ denote the quantity of Lemma 3. Applying Hölder's inequality using conjugate exponents p and $q = 2n+3$ we see

$$\int_{\mathbf{B}_n} |f(\varphi_\lambda(w)) - P(f\circ\varphi_\lambda)(w)|^4\,\frac{1}{(1-\|w\|^2)^{1/2}\,|1-\langle w, \lambda\rangle|^n}\,\mathrm{d}V(w) \leqslant$$

$$\leqslant M_{p,n}^{1/p}\left(\int_{\mathbf{B}_n} |f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)|^{4q}\,\mathrm{d}V\right)^{1/q} \leqslant$$

$$\leqslant M_{p,n}^{1/p}\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2^{1/q}\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_{8q-2}^{4-(1/q)} \leqslant$$

(by Cauchy-Schwarz)

$$\leqslant C\,\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2^{1/(2n+3)}$$

where the last inequality is justified as follows: the Bergman projection P maps the space $L^{8q-2}(\mathbf{B}_n, \mathrm{d}V)$ boundedly onto the Bergman space $A^{8q-2}(\mathbf{B}_n)$ ([5], Theorem 7.1.4), so that there is a finite positive constant B such that $\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_{8q-2} \leqslant$ $\leqslant B\|f\circ\varphi_\lambda\|_{8q-2} \leqslant B\|f\|_\infty$. This completes the proof section of this lemma. ▨

We are now ready to prove the main result of this of the paper.

THEOREM 5. *Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. The following statements are equivalent:*

(a) *H_f is compact;*

(b) $\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2 \to 0$ *as* $\|\lambda\| \to 1^-$.

Proof. Fix a function $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$.

Proof that (a) $\Rightarrow$ (b). By the reproducing property we have $\langle k_\lambda, k_\lambda\rangle = k_\lambda(\lambda)$. Using formula (3) for k_λ it follows at once that $\|k_\lambda\|_2^2 = 1/(1-\|\lambda\|^2)^{n+1}$. If $h \in H^\infty(\mathbf{B}_n)$, then $\langle h, k_\lambda/\|k_\lambda\|_2\rangle = (1-\|\lambda\|^2)^{(n+1)/2}h(\lambda) \to 0$ as $\|\lambda\| \to 1^-$. Since

$H^\infty(\mathbf{B}_n)$ is dense in $A^2(\mathbf{B}_n)$, this shows that $k_\lambda/\|k_\lambda\|_2 \to 0$ weakly as $\|\lambda\| \to 1^-$. Now, suppose that H_f is compact. Since a compact operator maps weakly null sequences to noim null sequences, we have $\|H_f(k_\lambda/\|k_\lambda\|_2)\|_2 \to 0$ as $\|\lambda\| \to 1^-$.

$$\|H_f(k_\lambda/\|k_\lambda\|_2)\|_2^2 = (1 - \|\lambda\|^2)^{n+1} \int_{\mathbf{B}_n} |f(w) - P(f\circ\varphi_\lambda)(\varphi_\lambda(w))|^2 |k_\lambda(w)|^2 \mathrm{d}V(w) =$$

(by (7))

$$= \int_{\mathbf{B}_n} |f(w) - P(f\circ\varphi_\lambda)(\varphi_\lambda(w))|^2 \frac{(1 - \|\lambda\|^2)^{n+1}}{|1 - \langle w, \lambda\rangle|^{2n+2}} \mathrm{d}V(w) =$$

$$= \int_{\mathbf{B}_n} |f(\varphi_\lambda(z)) - P(f\circ\varphi_\lambda)(z)|^2 \mathrm{d}V(z), \qquad \text{(by (11))}$$

hus $\|f\circ\varphi_\lambda - P(f_\lambda\circ\varphi_\lambda)\|_2 = \|H_f(k_\lambda/\|k_\lambda\|_2)\|_2 \to 0$ as $\|\lambda\| \to 1^-$.

Proof that (b) $\Rightarrow$ (a). Suppose that (b) holds. We will show that the operator t$H_f^* H_f$ is compact. We will do this by showing that $H_f^* H_f$ can be approximated — in fthe operator norm — by compact operators. In view of Proposition 2 we define or each number $0 < r < 1$ an operator $S_r : A^2(\mathbf{B}_n) \to L^2(\mathbf{B}_n, \mathrm{d}V)$ by

$$(S_r h)(\lambda) = \chi_{r\mathbf{B}_n}(\lambda) \int_{\mathbf{B}_n} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \langle\lambda, z\rangle)^{n+1}} h(z)\mathrm{d}V(z),$$

for $h \in A^2(\mathbf{B}_n)$, $\lambda \in \mathbf{B}_n$. We claim that S_r is a Hilbert-Schmidt operator. To prove this claim we need to show that the kernel of S_r is square-integrable over $\mathbf{B}_n \times \mathbf{B}_n$. Using Fubini's Theorem we have

$$\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} \chi_{r\mathbf{B}_n}(\lambda) \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^4}{|1 - \langle\lambda, z\rangle|^{2n+2}} \mathrm{d}V(z)\right) \mathrm{d}V(\lambda) =$$

$$= \int_{r\mathbf{B}_n} (1 - \|\lambda\|^2)^{-n-1} \left(\int_{\mathbf{B}_n} |f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^4 \frac{(1 - \|\lambda\|^2)^{n+1}}{|1 - \langle\lambda, z\rangle|^{2n+2}} \mathrm{d}V(z)\right) \mathrm{d}V(\lambda) =$$

(by (11))

$$= \int_{r\mathbf{B}_n} (1 - \|\lambda\|^2)^{-n-1} \left(\int_{\mathbf{B}_n} |(f\circ\varphi_\lambda)(w) - P(f\circ\varphi_\lambda)(w)|^4 \mathrm{d}V(w)\right) \mathrm{d}V(\lambda) \leqslant$$

$$\leqslant c_r \sup_{\lambda\in r\mathbf{B}_n} \|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_4^4 < \infty,$$

and our caim that S_r is Hilbert-Schmidt is verified. Using Proposition 2 and the definition of S_r we see that for $h \in H^\infty(\mathbf{B}_n)$ and $\lambda \in \mathbf{B}_n$:

$$((H_f^*H_f - S_r)h)(\lambda) = \chi_{\mathbf{B}_n \setminus r\mathbf{B}_n}(\lambda) \int_{\mathbf{B}_n} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \langle \lambda, z \rangle)^{n+1}} h(z) \mathrm{d}V(z).$$

Thus

$$|((H_f^*H_f - S_r)h)(\lambda)|^2 \leqslant \chi_{\mathbf{B}_n \setminus r\mathbf{B}_n}(\lambda) \left(\int_{\mathbf{B}_n} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{|1 - \langle \lambda, z \rangle|^{n+1}} |h(z)| \mathrm{d}V(z) \right)^2 \leqslant$$

$$\leqslant \chi_{\mathbf{B}_n \setminus r\mathbf{B}_n}(\lambda) \left(\int_{\mathbf{B}_n} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{(1 - \|z\|^2)^{1/2} |1 - \langle \lambda, z \rangle|^{n+1}} \mathrm{d}V(z) \right) \times$$

$$\times \left(\int_{\mathbf{B}_n} \frac{(1 - \|z\|^2)^{1/2}}{|1 - \langle \lambda, z \rangle|^{n+1}} |h(z)|^2 \mathrm{d}V(z) \right) \leqslant \qquad \text{(by Cauc Schwarz)}$$

$$\leqslant \frac{C}{(1 - \|\lambda\|^2)^{1/2}} \chi_{\mathbf{B}_n \setminus r\mathbf{B}_n}(\lambda) \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/(2n+3)} \times$$

$$\times \left(\int_{\mathbf{B}_n} \frac{(1 - \|z\|^2)^{1/2}}{|1 - \langle \lambda, z \rangle|^{n+1}} |h(z)|^2 \mathrm{d}V(z) \right) \qquad \text{(by Lemma 4)}$$

Integrating the above inequality and applying Fubini's Theorem we get

$$\|(H_f^*H_f - S_r)h\|_2^2 \leqslant C \sup_{\lambda \in \mathbf{B}_n \setminus r\mathbf{B}_n} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/(2n+3)} \times$$

(12)

$$\times \int_{\mathbf{B}_n} (1 - \|z\|^2)^{1/2} |h(z)|^2 \left(\int_{\mathbf{B}_n} \frac{1}{(1 - \|\lambda\|^2)^{1/2} |1 - \langle \lambda, z \rangle|^{n+1}} \mathrm{d}V(\lambda) \right) \mathrm{d}V(z).$$

The proof of Lemma 4 shows that for every $z \in \mathbf{B}_n$ the inner integral in (12) is bounded by $M_{p,n}^{1/p}(1 - \|z\|^2)^{-1/2}$, so that we get, with a possibly different constant C:

$$\|(H_f^*H_f - S_r)h\|_2^2 \leqslant C \sup_{\lambda \in \mathbf{B}_n \setminus r\mathbf{B}_n} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/(2n+3)} \|h\|_2^2.$$

Since $H^\infty(\mathbf{B}_n)$ we can conclude that

$$(13)\qquad \|H_f^*H_f - S_r\| \leqslant C \sup_{\lambda\in\mathbf{B}_n\setminus r\mathbf{B}_n} \|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2^{1/(4n+6)}.$$

It follows immediately from (13) that $S_r \to H^*H_f$ in operator norm as $r\to 1^-$. Since the S_r are Hilbert-Schmidt, thus compact, it follows that $H_f^*H_f$ is compact, and therefore H_f is compact. ▨

REMARK. For $f\in L^2(\mathbf{B}_n, \mathrm{d}V)$ (so f is not necessarily bounded) the operator H_f can be considered as a densely defined operator. It is clear from the above proof that the boundedness of the function f on $\mathbf{B}_n$ in Theorem 5 can be replaced by the weaker condition that $\sup\{\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_p : \lambda\in\mathbf{B}_n\} < \infty$ for sufficiently large p (in fact, we need $p \geqslant 16n + 22$).

As applications of Theorem 5 we turn to some special cases. First, we recover a result of Coburn [3]. Let $\overline{\mathbf{B}}_n$ denote the closed unit ball in $\mathbf{C}^n$.

COROLLARY 6. *If f is a continuous function on $\overline{\mathbf{B}}_n$ then H_f is compact.*

Proof. It is easily checked that for fixed $z\in\mathbf{B}_n$, $\|\varphi_\lambda(z) - \lambda\| \to 0$ as $\|\lambda\| \to 1^-$. So if f is continuous on $\overline{\mathbf{B}}_n$, then by Lebesgue's Dominated Convergence Theorem we have $\|f\circ\varphi_\lambda - f(\lambda)\|_2 \to 0$ as $\|\lambda\|\to 1^-$. Applying $I - P$ gives $\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_2 \to 0$ as $\|\lambda\|\to 1^-$, and by Theorem 5, H_f is compact. ▨

As a corollary of the proof of Theorem 5 we get the following generalization of a theorem of Sheldon Axler [2].

COROLLARY 7. *Let $f\in A^2(\mathrm{B}_n)$. Then the following statements are equivalent:*

(a) *$H_{\bar f}$ is compact;*

(b) $\|f\circ\varphi_\lambda - f(\lambda)\|_2 \to 0$ *as* $\|\lambda\|\to 1^-$.

Proof. Let $f\in A^2(\mathbf{B}_n)$. We will make use of the fact that for a holomorphic function h on $\mathbf{B}_n$, $P(\bar h) = \overline{h(0)}$. Thus for $\lambda\in\mathbf{B}_n$ we have $P(\bar f\circ\varphi_\lambda) = \overline{(f\circ\varphi_\lambda)(0)} = \bar f(\lambda)$ so that $\|\bar f\circ\varphi_\lambda - P(\bar f\circ\varphi_\lambda)\|_2 = \|f\circ\varphi_\lambda - f(\lambda)\|_2$. Furthermore, if $\|f\circ\varphi_\lambda - f(\lambda)\|_2 \to 0$ as $\|\lambda\|\to 1^-$, then certainly $\sup\{\|f\circ\varphi_\lambda - f(\lambda)\|_2 : \lambda\in\mathbf{B}_n\} < \infty$. We claim that then also $\sup\{\|f\circ\varphi_\lambda - f(\lambda)\|_p : \lambda\in\mathbf{B}_n\} < \infty$ for every $p\in(0,\infty)$, so that by the remark following Theorem 5 this theorem applies to give the equivalence of statements (a) and (b). This claim follows from the theory of Bloch functions (more about this in Section 6), but can also be proved directly as follows. If $g\in A^2(\mathbf{B}_n)$, then repeated differentiation of $g(z) = \int_{\mathbf{B}_n} g(w)(1 - \langle z, w\rangle)^{-(n+1)}\mathrm{d}V(w)$ yields the formula

$$\frac{\partial^{n+1} g}{\partial z^\alpha}(z) = \frac{(2n+1)!}{n!}\int \frac{1}{g(w)\bar w^\alpha(1 - \langle z, w\rangle)^{(2n+2)}}\,\mathrm{d}V(w),$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order $|\alpha| = \alpha_1 + \ldots + \alpha_n = n + 1$. Replacing g by $g - g(z)$, and using change-of-variable formula (11) it follows that

$$\left|\frac{\partial^{n+1}g}{\partial z^\alpha}(z)\right| \leqslant \frac{(2n+1)!}{n!}\int_{\mathbf{B}_n} |g(w) - g(z)| \frac{1}{|1 - \langle z, w\rangle|^{(2n+2)}} \, \mathrm{d}V(w) =$$

$$= \frac{(2n+1)!}{n!} \frac{1}{(1 - \|z\|^2)^{n+1}} \int_{\mathbf{B}_n} |(g \circ \varphi_z)(w) - g(z)| \, \mathrm{d}V(w).$$

Hence $\left|\frac{\partial^{n+1}g}{\partial z^\alpha}(z)\right| \leqslant \frac{(2n+1)!}{n!}(1 - \|z\|^2)^{-(n+1)}\sup\{\|g \circ \varphi_\lambda - g(\lambda)\|_2 : \lambda \in \mathbf{B}_n\}$.

Repeated integration yields that $|g(z) - g(0)| \leqslant C_n \log(1 - \|z\|)^{-1}\sup\{\|g \circ \varphi_\lambda - g(\lambda)\|_2 : \lambda \in \mathbf{B}_n\}$, where C_n is a constant only depending on n. Since $\log(1 - \|z\|)^{-1}$ is p-integrable over $\mathbf{B}_n$ it follows that $\|g - g(0)\|_p \leqslant C\sup\{\|g \circ \varphi_\lambda - g(\lambda)\|_2 : \lambda \in \mathbf{B}_n\}$, where C depends only on n and p. Finally replacing g by $f \circ \varphi_\lambda$ we arrive at $\sup\{\|f \circ \varphi_\lambda - f(\lambda)\|_p : \lambda \in \mathbf{B}_n\} \leqslant C\sup\{\|f \circ \varphi_\lambda - f(\lambda)\|_2 : \lambda \in \mathbf{B}_n\}$, and the claim follows. ▨

COROLLARY 8. *Let $u \in L^2(\mathbf{B}_n, \mathrm{d}V)$ be pluriharmonic. Then the following statements are equivalent:*

(a) *H_u is compact;*

(b) *$\|u \circ \varphi_\lambda - u(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$.*

Proof. Let $u \in L^2(\mathbf{B}_n, \mathrm{d}V)$ be pluriharmonic. Then there is a holomorphic function f on $\mathbf{B}_n$ such that $u = \mathrm{Re} f$ ([5], Theorem 4.4.9). Since the L^2-norm of any holomorphic function on $\mathbf{B}_n$ is controlled by its real part ([5], Theorem 7.1.5) we have $f \in A^2(\mathbf{B}_n)$. Putting $v = \mathrm{Im} f$, we have $H_{iv} = H_{-u}$, because clearly $H_f = 0$. So if H_u is compact, then so is H_{iv}, and thus $H_{\bar{f}} = H_u - H_{iv} = H_{2u}$ is compact. Consequeltly $\|f \circ \varphi_\lambda - f(\lambda)\|_2 \to 0$, and therefore $\|u \circ \varphi_\lambda - u(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$. For the converse, observe that there is a finite constant C such that for all $\lambda \in \mathbf{B}_n$, $\|v \circ \varphi_\lambda - v(\lambda)\|_2 \leqslant C\|u \circ \varphi_\lambda - u(\lambda)\|_2$ ([5], Theorem 7.1.5). So if $\|u \circ \varphi_\lambda - u(\lambda)\|_2 \to 0$, then also $\|v \circ \varphi_\lambda - v(\lambda)\|_2 \to 0$, and hence $\|f \circ \varphi_\lambda - f(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$, from which we conclude that $H_{\bar{f}}$ is compact, thus H_u is compact. ▨

To state another corollary we need to introduce more notation. For $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$ define its Berezin symbol $\tilde{f}$ by

$$\tilde{f}(\lambda) = \langle fk_\lambda/\|k_\lambda\|_2, \; k_\lambda/\|k_\lambda\|_2\rangle =$$

$$= \int_{\mathbf{B}_n} f(z) \frac{(1 - \|\lambda\|^2)^{n+1}}{|1 - \langle w, \lambda\rangle|^{2n+2}} \, \mathrm{d}V(z), \quad \lambda \in \mathbf{B}_n.$$

The following corollary of Theorem 5 generalizes some of Kehe Zhu's results [10].

COROLLARY 9. *Let* $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. *Then the following statements are equivalent:*

(a) H_f *and* $H_{\bar f}$ *are compact;*

(b) $\|f \circ \varphi_\lambda - \tilde f(\lambda)\|_2 \to 0$ *as* $\|\lambda\| \to 1^-$.

Proof. First observe that $\tilde f(\lambda) = P(f \circ \varphi_\lambda)(0) = \overline{P(\bar f \circ \varphi_\lambda)(0)}$ for $\lambda \in \mathbf{B}_n$ (by change-of-variable formula (11) and the definition of the Bergman projection P). We will again make use of the fact that for a holomorphic function h on $\mathbf{B}_n$, $P(\bar h) = \overline{h(0)}$.

Proof (a) $\Rightarrow$ (b). Suppose that both H_f and $H_{\bar f}$ are compact. Since $H_{\bar f}$ is compact we have:

$$\|f \circ \varphi_\lambda - \overline{P(\bar f \circ \varphi_\lambda)}\|_2 = \|\bar f \circ \varphi_\lambda - P(\bar f \circ \varphi_\lambda)\|_2 \to 0 \quad \text{as } \|\lambda\| \to 1^- .$$

Using the boundedness of P as an operator of $L^2(\mathbf{B}_n, \mathrm{d}V)$ onto $A^2(\mathbf{B}_n)$ we get:

$$\|P(f \circ \varphi_\lambda) - \tilde f(\lambda)\|_2 = \|P(f \circ \varphi_\lambda - \overline{P(\bar f \circ \varphi_\lambda)})\|_2 \to 0 \quad \text{as } \|\lambda\| \to 1^- .$$

The compactness of H_f implies that $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$, which combined with the above statement gives that (b) holds.

Proof that (b) $\Rightarrow$ (a). Suppose that $\|f \circ \varphi_\lambda - \tilde f(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$. Again using the boundedness of P it follows that $\|P(f \circ \varphi_\lambda) - \tilde f(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$, thus $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$. By Theorem 5, H_f is compact. Since also $\|\bar f \circ \varphi_\lambda - \tilde{\bar f}(\lambda)\|_2 \to 0$ as $\|\lambda\| \to 1^-$, $H_{\bar f}$ is compact too. ▨

4. THE POLYDISK IN $\mathbf{C}^n$

In this section we will characterize the bounded measurable functions f on the polydisk $\mathbf{D}^n$ for which the Hankel operator H_f is compact. Combining (4) and (6) we see that in this case H_f is defined as follows:

$$(H_f g)(z) = \int_{\mathbf{D}^n} (f(z) - f(w)) \prod_{j=1}^{n} \frac{1}{(1 - z_j \bar w_j)^2} g(w) \mathrm{d}V(w). \tag{14}$$

For $\lambda \in \mathbf{D}^n$, the real Jacobian $J_{\mathbf{R}}\varphi_\lambda$ of φ_λ is the given by $(J_{\mathbf{R}}\varphi_\lambda)(w) = \prod_{j=1}^{n} \frac{(1 - |\lambda_j|^2)^2}{|1 - w_j \bar\lambda_j|^4}$, so that for a Lebesgue integrable or non-negative Lebesgue measurable function h on $\mathbf{D}^n$ we have the change-of-variable formula:

$$\int_{\mathbf{D}^n} g(z)\mathrm{d}V(z) = \int_{\mathbf{D}^n} (h \circ \varphi_\lambda)(w) \prod_{j=1}^{n} \frac{(1 - |\lambda_j|^2)^2}{|1 - w_j \bar\lambda_j|^4} \mathrm{d}V(w). \tag{15}$$

The proof of Proposition 2 carries over to this setting to give the following proposition.

PROPOSITION 10. *Let $f \in L^\infty(\mathbf{D}^n, \mathrm{d}V)$. Then for $h \in H^\infty(\mathbf{D}^n)$ and $\lambda \in \mathbf{D}^n$:*

$$(H_f^* H_f h)(\lambda) = \int_{\mathbf{D}^n} |f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2 \prod_{j=1}^n \frac{1}{(1 - \bar{\lambda}_j z_j)^2} h(z) \mathrm{d}V(z).$$

The following estimate will be used in Theorem 12.

LEMMA 11. *Let $f \in L^\infty(\mathbf{D}^n, \mathrm{d}V)$. Then there exists a finite positive constant C (depending on f) such that for every $\lambda \in \mathbf{D}^n$:*

$$\int_{\mathbf{D}^n} |f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^1 \prod_{j=1}^n \frac{1}{(1 - |z_j|^2)^{1/2} |1 - \lambda_j \bar{z}_j|^2} \mathrm{d}V(z) \leqslant$$

$$\leqslant C \prod_{j=1}^n \frac{1}{(1 - |\lambda_j|^2)^{1/2}} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/6}.$$

Proof. Let $f \in L^\infty(\mathbf{D}^n, \mathrm{d}V)$. In the integral at the left make the change-of-variable $w = \varphi_\lambda(z)$. Using (10) and (11) (for the disk $\mathbf{D}$) as well as (15) we see that the integral at the left is equal to

$$\prod_{j=1}^n \frac{1}{(1 - |\lambda_j|^2)^{1/2}} \int_{\mathbf{D}^n} |f(\varphi_\lambda(w)) - P(f \circ \varphi_\lambda)(w)|^1 \prod_{j=1}^n \frac{1}{(1 - |w_j|^2)^{1/2} |1 - \lambda_j \bar{w}_j|} \mathrm{d}V(w).$$

Let M be the quantity of Lemma 3 for $\mathbf{B}_1 = \mathbf{D}$ and $p = 6/5$. Using A to denote the normalized Lebesgue area measure on $\mathbf{D}$, we have

$$\int_{\mathbf{D}^n} \prod_{j=1}^n \frac{1}{(1 - |w_j|^2)^{3/5} |1 - \lambda_j \bar{w}_j|^{6/5}} \mathrm{d}V(w) =$$

$$= \prod_{j=1}^n \int_{\mathbf{D}} \frac{1}{(1 - |w|^2)^{3/5} |1 - \lambda_j \bar{w}|^{6/5}} \mathrm{d}A(w) \leqslant M^n.$$

As in the proof of Lemma 4, an application of Hölder's inequality (with conjugate exponents 6 and 6/5), the above estimate and the boundedness of the Bergman projection yield

$$\int_{\mathbf{D}^n} |f(\varphi_\lambda(w)) - P(f \circ \varphi_\lambda)(w)|^4 \prod_{j=1}^n \frac{1}{(1 - |w_j|^2)^{1/2} |1 - \lambda_j \bar{w}_j|} \, dV(w) \leqslant$$

$$\leqslant C \, \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/6}.$$

This completes the proof of this lemma. ▨

Now we are ready to prove the main result of this section.

THEOREM 12. *Let $f \in L^\infty(\mathbf{D}^n, dV)$. The following statements are equivalent:*

(a) *H_f is compact;*

(b) $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \to 0$ *as* $\lambda \to \partial(\mathbf{D}^n)$.

Proof. Fix a function $f \in L^\infty(\mathbf{D}^n, dV)$.

Proof that (a) $\Rightarrow$ (b). It follows from (4) that $\|k_\lambda\|_2^2 = \prod_{j=1}^n (1 - |\lambda_j|^2)^{-2}$. If $h \in H^\infty(\mathbf{D}^n)$, then $\langle h, k_\lambda / \|k_\lambda\|_2 \rangle = \prod_{j=1}^n (1 - |\lambda_j|^2) h(\lambda) \to 0$ as $\lambda \to \partial(\mathbf{D}^n)$. Since $H^\infty(\mathbf{D}^n)$ is dense in $A^2(\mathbf{D}^n)$, this shows that $k_\lambda / \|k_\lambda\|_2 \to 0$ weakly as $\lambda \to \partial(\mathbf{D}^n)$. So if H_f is compact, we have that $\|H_f(k_\lambda / \|k_\lambda\|_2)\|_2 \to 0$ as $\lambda \to \partial(\mathbf{D}^n)$. Using (15) and (7) it is easy to verify that $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 = \|H_f(k_\lambda / \|k_\lambda\|_2)\|_2$, whence statement (b) follows.

Proof that (b) $\Rightarrow$ (a). Suppose that (b) holds. We will show that the operator $H_f^* H_f$ is compact. As in the proof of Theorem 5 we will do this by showing that $H_f^* H_f$ can be approximated – in the operator norm – by compact operators. In view of Proposition 10 we define for each $0 < r < 1$ an operator $S_r : A^2(\mathbf{D}^n) \to L^2(\mathbf{D}^n, dV)$ by

$$(S_r h)(\lambda) = \chi_{r\mathbf{D}^n}(\lambda) \int_{\mathbf{D}^n} |f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2 \prod_{j=1}^n \frac{1}{(1 - \lambda_j \bar{z}_j)^2} h(z) \, dV(z),$$

for $h \in A^2(\mathbf{D}^n)$, $\lambda \in \mathbf{D}^n$. Just as in the proof of Theorem 5 it is easy to verify that S_r is a Hilbert-Schmidt operator. Following the same procedure as in the proof of Theorem 5, but using Proposition 10 instead of Proposition 2 and Lemma 11 instead of Lemma 4, we obtain the inequality

$$\|H_f^* H_f - S_r\| \leqslant C \sup_{\lambda \in \mathbf{D}^n \setminus r\mathbf{D}^n} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^{1/6}. \tag{16}$$

It follows immediately from (16) that $S_r \to H_f^* H_f$ in operator norm as $r \to 1^-$. Since the S_r are Hilbert-Schmidt, thus compact, it follows that $H_f^* H_f$ is compact, and therefore H_f is compact. ▨

REMARK. For $f \in L^2(\mathbf{D}^n, \mathrm{d}V)$ (so f is not necessarily bounded) the operator H_f can be considered as a densely defined operator. It is clear from the above proof that the boundedness of the function f on $\mathbf{D}^n$ in Theorem 12 can be replaced by the weaker condition that $\sup\{\|f\circ\varphi_\lambda - P(f\circ\varphi_\lambda)\|_p : \lambda \in \mathbf{D}^n\} < \infty$ for sufficiently large p.

The following corollary shows that in case the dimension of the polydisk is larger than 1 for pluriharmonic symbols the Hankel operator is compact only for trivial reasons.

COROLLARY 13. *Let $n > 1$, and let $u \in L^2(\mathbf{D}^n, \mathrm{d}V)$ be pluriharmonic. Then H_u is compact if and only if u is constant.*

Proof. Let $u \in L^2(\mathbf{D}^n, \mathrm{d}V)$ be pluriharmonic, and suppose that H_u is compact. Since $\mathbf{D}^n$ is simply-connected, there is a holomorphic function f on $\mathbf{D}^n$ such that $u = \mathrm{Re} f$. Since the L^2-norm of any holomorphic function on $\mathbf{D}^n$ is controlled by its real part we have $f \in A^2(\mathbf{D}^n)$. As in the proof of Corollary 8, also $H_{\bar f}$ is compact. Consequently $\|f\circ\varphi_\lambda - f(\lambda)\|_2 \to 0$ as $\lambda \to \partial(\mathbf{D}^n)$. For a holomorphic function $g: \mathbf{D}^n \to \mathbf{C}$ it is easily seen that $\frac{\partial g}{\partial z_k}(0) = 2\int_{\mathbf{D}^n} g(z)\bar z_k \mathrm{d}V(z)$. Applying this identity to $g = f\circ\varphi_\lambda - f(\lambda)$ we get $(|\lambda_k|^2 - 1)\frac{\partial f}{\partial z_k}(\lambda) = 2\int_{\mathbf{D}^n} ((f\circ\varphi_\lambda)(z) - f(\lambda))\bar z_k \mathrm{d}V(z)$, and by Cauchy-Schwarz inequality, $(1 - |\lambda_k|^2)\left|\frac{\partial f}{\partial z_k}(\lambda)\right| \leqslant 2\|f\circ\varphi_\lambda - f(\lambda)\|_2$. Thus $(1 - |\lambda_k|^2)\left|\frac{\partial f}{\partial z_k}(\lambda)\right| \to 0$ as $\lambda \to \partial(\mathbf{D}^n)$, for each $k = 1, \ldots, n$. Timoney has shown that f is necessarily constant ([8], Proposition 4.1). Hence u is constant. ▨

For $f \in L^\infty(\mathbf{D}^n, \mathrm{d}V)$ define its Berezin symbol $\tilde f$ by

$$\tilde f(\lambda) = \langle fk_\lambda/\|k_\lambda\|_2,\ k_\lambda/\|k_\lambda\|_2\rangle =$$

$$= \int_{\mathbf{D}^n} f(z) \prod_{j=1}^{n} \frac{(1 - |\lambda_j|^2)^2}{|1 - \bar\lambda_j z_j|^4}\, \mathrm{d}V(z), \quad \lambda \in \mathbf{D}^n.$$

As a corollay we get the following result.

COROLLARY 14. *Let* $f \in L^\infty(\mathbf{D}^n; \mathrm{d}V)$. *Then the following statements are equivalent:*

(a) H_f *and* $H_{\bar{f}}$ *are compact;*

(b) $\|f \circ \varphi_\lambda - \tilde{f}(\lambda)\|_2 \to 0$ *as* $\lambda \to \partial(\mathbf{D}^n)$.

5. WEIGHTED BERGMAN SPACES ON THE UNIT BALL IN $\mathbf{C}^n$

In this section we indicate how Theorem 5 can be extended to weighted Bergman spaces on $\mathbf{B}_n$.

Fix $-1 < \alpha < \infty$. Let V_α denote the measure given by $\mathrm{d}V_\alpha(z) = c_\alpha(1 - \|z\|^2)^\alpha \mathrm{d}V(z)$, for $z \in \mathbf{B}_n$ where $c_\alpha = \Gamma(n + \alpha + 1)/(n!\Gamma(\alpha + 1))$; the coefficient c_α is for normalization: $V_\alpha(\mathbf{B}_n) = 1$. For a Lebesgue measurable function $f: \mathbf{B}_n \to \mathbf{C}$ let $\|f\|_{2,\alpha} = \left(\int_{\mathbf{B}_n} |f(z)|^2 \mathrm{d}V_\alpha(z)\right)^{1/2}$. The weighted Bergman space $A^{2,\alpha}(\mathbf{B}_n)$ is the set of all holomorphic functions f on $\mathbf{B}_n$ such that $\|f\|_{2,\alpha} < \infty$. The space $A^{2,\alpha}(\mathbf{B}_n)$ is a closed subspace of the Hilbert space $L^2(\mathbf{B}_n, \mathrm{d}V_\alpha)$ with inner product given by $\langle f, g\rangle_\alpha = \int_{\mathbf{B}_n} f(z)\overline{g(z)}\mathrm{d}V_\alpha(z)$, for $f, g \in L^2(\mathbf{B}_n, \mathrm{d}V_\alpha)$. Let P_α denote the orthogonal projection of $L^2(\mathbf{B}_n, \mathrm{d}V_\alpha)$ onto $A^{2,\alpha}(\mathbf{B}_n)$. For $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$, the Hankel operator $H_f: A^{2,\alpha}(\mathbf{B}_n) \to A^{2,\alpha}(\mathbf{B}_n)^\perp$ is defined by

$$H_f g = (I - P_\alpha)(fg), \quad g \in A^{2,\alpha}(\mathbf{B}_n).$$

For $\lambda \in \mathbf{B}_n$ define the function $k_{\alpha;\lambda}$ by

$$k_{\alpha;\lambda}(z) = \frac{1}{(1-\langle z, \lambda\rangle)^{n+\alpha+1}}, \quad z \in \mathbf{B}_n. \tag{17}$$

In Section 7.1 of [5] it is shown that these functions $k_{\alpha;\lambda}$ ($\lambda \in \mathbf{B}_n$) are the reproducing kernels for $A^{2,\alpha}(\mathbf{B}_n)$, $f(\lambda) = \langle f, k_\lambda\rangle_\alpha$, for all $f \in A^{2,\alpha}(\mathbf{B}_n)$.

Combining (6) and (17) we see that in this case H_f is defined as follows:

$$(H_f g)(z) = \int_{\mathbf{B}_n} \frac{f(z) - f(w)}{(1 - \langle z, w\rangle)^{n+\alpha+1}} g(w)\mathrm{d}V_\alpha(w), \quad z \in \mathbf{B}_n. \tag{18}$$

Using identity (11) it is easy to verify that for $\lambda \in \mathbf{B}_n$ the substitution $z = \varphi_\lambda(w)$ results in the Jacobian change in measure given by $\mathrm{d}V(z) = J_{\mathbf{R}}\varphi_\lambda(w)\mathrm{d}V(w)$, where $J_{\mathbf{R}}\varphi_\lambda$, the real Jacobian of φ_λ, is given by $J_{\mathbf{R}}\varphi_\lambda(w) = (1 - \|\lambda\|^2)^{n+\alpha+1}/|1 -$

$-\langle w, \lambda\rangle|^{2n+2\alpha+2}$. Thus, for a non-negative Lebesgue measurable function h on $\mathbf{B}_n$ we have the change-of-variable formula:

$$\int_{\mathbf{B}_n} h(z)\mathrm{d}V_\alpha(z) = \int_{\mathbf{B}_n} (h \circ \varphi_\lambda)(w) \frac{(1-\|\lambda\|^2)^{n+\alpha+1}}{|1-\langle w, \lambda\rangle|^{2n+2\alpha+2}} \mathrm{d}V_\alpha(w). \tag{19}$$

From the proof of Proposition 1 we see that for $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$ and $\lambda \in \mathbf{B}_n$:

$$H_f(k_{\alpha;\lambda}) = (f - P_\alpha(f \circ \varphi_\lambda) \circ \varphi_\lambda) k_{\alpha;\lambda}. \tag{20}$$

If furthermore $h \in H^\infty(\mathbf{B}_n)$, then the same proof as for Proposition 2 shows that:

$$(H_f^* H_f h)(\lambda) = \int_{\mathbf{B}_n} \frac{|f(z) - P_\alpha(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1-\langle \lambda, z\rangle)^{n+\alpha+1}} h(z)\mathrm{d}V_\alpha(z). \tag{21}$$

The estimate given in Lemma 4 is extended by the estimate in the following lemma.

LEMMA 15. *Let $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. Then there exists a finite positive constadt C (depending on f) such that for every $\lambda \in \mathbf{B}_n$:*

$$\int_{\mathbf{B}_n} \frac{|f(z) - P_\alpha(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{(1-\|z\|^2)|1-\langle \lambda, z\rangle|^{n+\alpha+1}} \mathrm{d}V_\alpha(z) \leqslant$$

$$\leqslant \frac{C}{(1-\|\lambda\|^2)^{(\alpha+1)/2}} \|f \circ \varphi_\lambda - P_\alpha(f \circ \varphi_\lambda)\|_{2,\alpha}^{(\alpha+1)/(2n+2\alpha+3)}.$$

Since the proof of Lemma 15 is similar to that of Lemma 4, we will only give the main ingredients. Lemma 3 needs to be replaced by

$$\sup_{\lambda \in \mathbf{B}_n} \int_{\mathbf{B}_n} \frac{1}{(1-\|w\|^2)^{p(\alpha+1)/2}|1-\langle \lambda, w\rangle|^{np}} \mathrm{d}V_\alpha(w) < \infty$$

which holds for all $p < (2n+2\alpha+2)/(2n+\alpha+1)$ — this follows from Proposition 1.4.10 of [5] with $t = \alpha - p(\alpha+1)/2$. As in the proof of Lemma 4, after a change-of-variables, apply Hölder's inequality, now with conjugate exponents $p = (2n+2\alpha+3)/(2n+\alpha+2)$ and $q = (2n+2\alpha+3)/(\alpha+1)$. That the Bergman projection P_α is a bounded linear operator on $L^r(\mathbf{B}_n, \mathrm{d}V_\alpha)$ for every $r > 1$ is easily shown by adapting the proof of Theorem 7.1.4 in [5] (defining h by $h(z) = (1-\|z\|^2)^{-(\alpha+1)/rs}$ where $s = r/(r-1)$ is the conjugate exponent of r).

We are now in a position to prove the following theorem.

THEOREM 16. *Let* $-1 < \alpha < \infty$ *and* $f \in L^\infty(\mathbf{B}_n, \mathrm{d}V)$. *The following statements are equivalent:*

(a) H_f *is compact;*

(b) $\|f \circ \varphi_\lambda - P_\alpha(f \circ \varphi_\lambda)\|_{2,\alpha} \to 0$ *as* $\|\lambda\| \to 1^-$.

Since the proof of Theorem 16 is similar to that of Theorem 5, we omit it. It should be clear that also Theorem 12 can be shown to hold for Hankel operators on weighted Bergman spaces on the polydisk $\mathbf{D}^n$ where the weights $(1 - |z_j|^2)^{\alpha_j}$ may even differ on each of the factors. We will not go through the trouble of introducing more notation to formally state this.

6. REMARKS AND AN OPEN QUESTION

In this section we make some remarks and discuss an open question related to the results in Theorems 5 and 12.

(1) As before, let Ω denote the unit ball or the polydisk in $\mathbf{C}^n$. For $f \in L^2(\Omega, \mathrm{d}V)$ (so f is not necessarily bounded) we can consider H_f as an operator $A^2(\Omega) \to A^2(\Omega)^\perp$ densely defined by $H_f g = (I - P)(fg)$, $g \in H^\infty(\Omega)$. It is possible that even for unbounded f the operator H_f is bounded. The question is to find necessary and sufficient conditions of f for the operator H_f to be bounded.

For conjugate holomorphic functions on the unit ball or the polydisk in $\mathbf{C}^n$ the answer is known. Let $\mathscr{B}(\Omega)$ denote the Bloch space on Ω defined as follows: for $\Omega = \mathbf{B}_n$ this is the set of all holomorphic functions $f\colon \mathbf{B}_n \to \mathbf{C}$ such that

$$\sup\{(1 - \|z\|^2)\|(\nabla f)(z)\| : z \in \mathbf{B}_n\} < \infty$$

where $\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ is the analytic gradient of f; for $\Omega = \mathbf{D}^n$ it is the set of all holomorphic functions $f : \mathbf{D}^n \to \mathbf{C}$ such that

$$\max_{1 \leqslant j \leqslant n} \sup\left\{(1 - |z_j|^2)\left|\frac{\partial f}{\partial z_j}(z_1, \ldots, z_n)\right| : (z_1, \ldots, z_n) \in \mathbf{D}^n\right\} < \infty.$$

The definitions given above are equivalent to the ones given by Timoney who in [7] showed that these spaces are Möbius-invariant: if $f \in \mathscr{B}(\Omega)$ and $\lambda \in \Omega$, then $f \circ \varphi_\lambda \in \mathscr{B}(\Omega)$. It is easy to verify that the method used in Axler's paper [2] extends to the unit ball $\mathbf{B}_n$ and the polydisk $\mathbf{D}^n$ and gives: the operator $H_{\bar{f}}$ is bounded if and only if $f \in \mathscr{B}(\Omega)$.

It is clear from its proof that Proposition 1 holds for $f \in L^2(\Omega, dV)$. It follows from the proofs of Theorem 5 and Theorem 12 that

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|^2 = \|H_f(k_\lambda / \|k_\lambda\|_2)\|_2$$

for $\lambda \in \Omega$. From this formula we get a necessary condition of f for the operator H_f to be bounded:

$$\sup_{\lambda \in \Omega} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 < \infty.$$

I conjecture that this condition is also sufficient. Note that for conjugate holomorphic functions this condition is both necessary and sufficient, because $\sup_{\lambda \in \Omega} \|\overline{f} \circ \varphi_\lambda - P(\overline{f} \circ \varphi_\lambda)\|_2 = \sup_{\lambda \in \Omega} \|f \circ \varphi_\lambda - f(\lambda)\|_2 < \infty$ if and only if $f \in \mathscr{B}(\Omega)$.

(2) A careful analysis of the proofs of Theorems 5 and 12 shows that the results generalize to domains Ω of the form $\Omega = \prod_{j=1}^{m} \mathrm{B}_{n_j}$ where the Möbius transformations are defined in the obvious way.

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