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PURE STATE EXTENSIONS AND RESTRICTIONS IN O_2

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1. INTRODUCTION

In [7] and [8], Longo and Popa independently asnwered in the affirmative the long-standing question which asked if a factor state of a (separable) C^*-subalgebra B of a C^*-algebra A always extends to a factor state of A (recall that a state φ on a C^*-algebra A is a *factor state* if the von Neumann algebra generated by the image of A under the GNS-representation induced by φ is a factor). Motivated by this result, the authors of the present paper became interested in determining the possible relations between the type of the image factor determined by the state on B and the type of the image factor determined by a factor-state extension on A. We choose as a prototypical example the embedding of the Choi algebra [2] into the Cuntz algebra O_2 [3], consider the unique trace τ on the Choi algebra, which is a factor state whose associated image factor is of type II_1, and constructed in [6] extensions of τ to pure states of O_2.

In Section 2 of this paper, for each nonperiodic sequence $\nu = \{n_i\}$ of positive integers we construct a pure state φ_ν on O_2 which extends τ on the Choi algebra. For two such sequences μ, ν with different tails we show that φ_μ and φ_ν are not unitarily equivalent. Thus, we obtained uncountable inequivalent pure state extensions of τ to O_2. In Section 3, we consider the restriction of all φ_ν's constructed in Section 2 to the diagonal maximal abelian $*$-subalgebra D of the Fermion algebra which is in turn embedded in O_2. We realize that $\varphi_\nu \mid D$ gives rise to a nonperiodic point in the spectrum of D. Hence we are able to identify the unique pure state extensions of those irrational points in the spectrum of D first announced by J. Cuntz in 1982 [4].

2. UNCOUNTABLE PURE STATE EXTENSIONS

In [6] for each increasing sequence of positive integers, $\theta = \{n_i\}$, we constructed a pure state extension φ_θ to O_2 of the tracial state on the Choi subalgebra Ch. In fact, an irreducible representation of O_2 is constructed in the following way. Let

G be the free group on two generators u, v with $u^2 = v^3 = e$, e being the identity element in G. Let L be the left regular representation of G on $\ell^2(G)$. The Choi algebra is isomorphic to C^*-algebra $C^*(L(u), L(v))$ generated by $L(u)$ and $L(v)$. For a given sequence $\theta = \{n_i\}$ of increasing positive integers, we constructed an orthogonal projection P_θ such that $P_\theta + L(u)P_\theta L(u) = 1$ and $P_\theta + L(v)P_\theta L(v^2) + L(v^2)P_\theta L(v) = 1$. O_2 is isomorphic to the C^*-algebra generated by $L(u)$, $L(v)$ and P_θ. The extension φ_θ of the tracial state is simply the vector state on O_2 induced by the vector χ_e, the characteristic function supported on e. In this section we show that each nonperiodic sequence of positive integers $\nu = \{n_i\}$ induces a pure state extension φ_ν to O_2 of the tracial state on the Choi subalgebra in a similar way. Furthermore, sequences ν_1, ν_2 with different tails induce inequivalent pure state extensions to O_2.

We denote by A the subset of G consisting of all words of the following form

$$(uv^{\delta_n}) \dots (uv^{\delta_1})e, \quad \text{where } \delta_i \text{ is either 1 or 2 for } 1 \leqslant i \leqslant n, \text{ or } e.$$

For a fixed $w \in G$, and a nonperiodic sequence $\nu = \{n_i\}$ of positive integers, we denote by $S(\nu, w)$ the set of words of the form

$$(v^\delta u)^{n_p} \dots (v^2u)^{n_2}(vu)^{n_1}w, \quad p = 0, 1, 2, \dots .$$

where δ is 2 for even p and 1 for odd p. Let F_ν be the subset of G consisting of words of the form xy, with $x \in A$ and $y \in S(\nu, v)$. We will show that F_ν satisfies the following two conditions

(2.1) $F_\nu \cap (uF_\nu) = F_\nu \cap (vF_\nu) = \emptyset, \quad F_\nu \cup (uF_\nu) = G, \quad (vF_\nu) \cup (v^2F_\nu) = uF_\nu.$

(2.2) There exists a sequence $\{z_i\}$ of words in u, v, such that $\bigcap_i (z_iF_\nu) = \{e\}$.

Then it follows from Theorem 3.1 and Corollary 3.2 in [6] that φ_ν is a pure state extension of the tracial state to O_2. First we remark that if a subset F of G satisfies (2.1) and $S(\nu, v) \subseteq F$, then Lemmas 3.5, 3.6, 3.9 and Proposition 3.3 in [6] hold. The following lemma will replace the role Lemma 3.7 played in [6].

2.3. Lemma. *Let $\{m_1, \dots, m_k\}$ be a nonempty set of positive integers. Suppose that a subset F of G satisfies* (2.1) *and $S(\nu, v) \subseteq F$. Then, there exists a positive integer p such that $(v^\sigma u)^{n_p} \dots (v^2u)^{n_2}(vu)^{n_1}(v^2u)^{m_k} \dots (v^\delta u)^{m_1}v$, $\delta = 1$ for even k and $\delta = 2$ for odd k, is not in F.*

Proof. Suppose $\delta = 2$. Then $(vu)^{n_1}(v^2u)^{m_k} \dots (v^2u)^{m_1}v$ is not in F by Lemma 3.6 in [6].

Suppose $\delta = 1$. Compare $\{m_1, m_2, \ldots, m_k, n_1, \ldots, n_i, \ldots\}$ with $\{n_1, n_2, \ldots\}$. Let s be the smallest positive integer such that $n_s \neq n_{s+k}$. This existence of such an s is assured by the nonperiodicity of $\{n_i\}$. By Lemma 3.6 in [6],

$$(v^\sigma u)^{n_p} \ldots (vu)^{n_1}(v^2u)^{m_k} \ldots (vu)^{m_1} v \text{ is not in } F \quad \text{for } p = s+1. \qquad \text{Q.E.D.}$$

2.4. PROP SITION. *Suppose that a subset F of G satisfies* (2.1) *and* $S(v, v) \subseteq F$. *Then v is the only word w in u, v such that* $S(v, w) \subseteq F$.

Proof. Case 1. Let w be a reduced word beginning with v^δ, $\delta = 1$ or 2, but different from v. If w ends with v^2 or u, $(vu)^{n_1}w$ is not in F by Lemma 3.9 in [6]. If w ends with v and begins with v^2, then there is a positive integer p such that $(v^\sigma u)^{n_p} \ldots (vu)^{n_1}w$ is not in F by Lemma 2.3. Let w begin and end with v. Then there is a positive integer p such that $(v^\sigma u)^{n_p} \ldots (vu)^{n_1} (v^2u)w$ is not in F by Lemma 2.3. Thus $S(v, w) \nsubseteq F$.

Case 2. Let w be a reduced word beginning with u of length $p, p > 0$. Consider a word $w_1 = (v^2u)^{n_{2p}} \ldots (vu)^{n_1}w$. In its reduced form, w_1 begins with v^2 and if w_1 ends with either u or v^2, then it is not in F by Lemma 3.9 in [6]. If w_1 ends with v, then $w_1 = (v^2u)^{n_k} \ldots (vu)^{n_1}v$ for some k. Consider $(v^\sigma u)^{n_r} \ldots (vu)^{n_{2p+1}}w_1$, denoted by w_2. Since $v = (n_i)$ is not $(2p-k)$-periodic, we can find an integer r such that $n_{r-1} \neq n_{(r-1)-(2p-k)}$. By Lemma 3.6 in [6] w_2 is not in F. Thus $S(v, w) \nsubseteq F$. This completes the proof of this this proposition. Q.E.D.

2.5. PROPOSITION. *Suppose that a subset F of G satisfies* (2.1) *and* $S(v, v) \subseteq$ $\subseteq F$. *Then F satisfies* (2.2) *as well.*

Proof. See the proof of Proposition 3.10 in [6] with Proposition 3.4 in [6] replaced by Proposition 2.4. Q.E.D.

2.6. PROPOSITION. F_v *satisfies* (2.1).

Proof. The proof of Proposition 3.11 in [6] applies with Lemma 3.7 in [6] replaced by Lemma 2.3, and θ replaced by v. Q.E.D.

Next, we show the irreducible representations Π_μ, Π_v induced by nonperiodic sequences with different tails μ, v are inequivalent. Let F be a subset of G and P_F be the orthogonal projection of $\ell^2(G)$ onto the closed subspace $\mathscr{H}_F$ spanned by the characteristic functions χ_w supported on $\{w\}$ for $w \in F$. In this notation we first show the following lemma.

2.7. LEMMA. *Suppose that F is a subset of G satisfying* (2.1). *If*

$$\bigcap_{p=1}^{\infty} \{(L(u)L(v^2))^{n_1} \ldots (L(u)L(v^\sigma))^{n_p}\mathscr{H}_F\} \neq \{0\},$$

then

$$\bigcap_{p=1}^{\infty} \{(uv^2)^{n_1} \dots (uv^\sigma)^{n_p} F\} \neq \emptyset,$$

where L is the left regular representation.

Proof. It is clear that the following sequence of closed subspaces is decreasing in terms of inclusion, for $(uv^\sigma)F \subseteq F$, $\sigma = 1$ or 2.

$$\mathscr{H}_p = \{(L(u)L(v^2))^{n_1} \dots (L(u)L(v^\sigma))^{n_p} \mathscr{H}_F\}, \quad p = 1, 2, \dots .$$

Each $\mathscr{H}_p$ has an orthonormal basis $\mathscr{B}_p = \{\chi_{\{(uv^2)^{n_1} \dots (uv^\sigma)^{n_p} w\}} : w \in F\}$, $p = 1, 2, \dots$. $\{B_p\}_p$ is again a decreasing sequence of subsets of G. Then it is obvious that, if

$$\bigcap_p \mathscr{H}_p \neq \{0\}, \text{ then } \bigcap_p B_p \neq \emptyset \text{ and hence } \bigcap_{p=1}^{\infty} \{(uv^2)^{n_1} \dots (uv^\sigma)^{n_p} F\} \neq \emptyset.$$

Q.E.D.

2.8. Definition. Let $\mu = \{m_i\}_i$ and $\nu = \{n_i\}_i$ be two nonperiodic sequences of positive integers. μ and ν are said to have a *common tail* if there exist positive integers k_1, k_2 such that $m_{k_1+i} = n_{k_2+i}$ for all $i = 1, 2, 3, \dots$. Otherwise, μ, ν are said to have *different tails.*

2.9. Proposition. *Let F be a subset of G satisfying Condition 2.1 and $S(\nu, v) \subseteq F$ for some nonperiodic sequence of positive integers ν. If there exist a nonperiodic sequence of positive integers $\mu = \{m_i\}$ and an element $w \in G$ such that $S(\mu, w) \subseteq F$, then μ and ν must have a common tail.*

Proof. We observe that if an element z is in F, then $(uv^\sigma)z$ is in F for $\sigma = 1, 2$. Thus it follows from $S(\mu, w) \subseteq F$ that w is in F. By Lemma 3.9 in [6] we know that w in its reduced form can not begin with v or v^2 and end with u or v^2. Thus it remains to check the following two cases. Either w begins with v^σ, $\sigma = 1$ or 2 and ends with v, or w begins with u.

Case 1. In its reduced form, w begins with v^σ, $\sigma = 1$ or 2 and ends with v. It follows from Lemma 3.6 in [6] that w has to be of the form

$$w = (v^\sigma u)^k (v^{\bar\sigma} u)^{n_p} (v^\sigma u)^{n_{p-1}} \dots (vu)^{n_1} v,$$

where $\bar\sigma = 3 - \sigma$. Hence we have $m_1 + k = n_{p+1}$, $m_i = n_{p+i}$ for $i = 2, 3, \dots$, when $\sigma = 1$, and $k = n_{p+1}$, $m_i = n_{p+1+i}$ for $i = 1, 2, \dots$, when $\sigma = 2$.

Case 2. w begins with u and is of length q.

Consider $w_0 = (v^2u)^{m_{2q}} \ldots (vu)^{m_1}w$. w_0, in its reduced form, must begin with v^2, and hence must end with v by Lemma 3.9 in [6]. By Lemma 3.6 in [6] w_0 must be of the form

$$w_0 = (v^2u)^k(vu)^{n_r} \ldots (vu)^{n_1}v$$

with $k = m_{2q} = n_{r+1}$. Hence $m_{2q+i} = n_{(r+1)+i}$ for $i = 1, 2, \ldots,$ Q.E.D

2.10. PROPOSITION. *Suppose that μ and ν are different nonperiodic sequences of positive, integers with different tails. Then the irreducible representations Π_μ, Π_ν of O_2 induced by them are inequivalent.*

Proof. Suppose that there is a unitary operator T on $\ell^2(G)$ such that $T\Pi_\nu(\cdot)T^* = \Pi_\mu(\cdot)$. Then $TL(w) = L(w)T$ for all $w \in G$ and $TP_{F_\nu} = P_{F_\mu}T$. Equivalently $T(\mathscr{H}_{F_\nu}) \subseteq \mathscr{H}_{F_\mu}$. We let $\mathscr{H}_p = \{(L(u)L(v^2))^{n_1} \ldots (L(u)L(v^\sigma))^{n_p}\mathscr{H}_{F_\nu}\}$ for $p = 1, 2, \ldots$. Since $\chi_v \in \bigcap_{p=1}^{\infty} \mathscr{H}_p$, it follows that

$$\{0\} \neq T\left\{\bigcap_{p=1}^{\infty}\mathscr{H}_p\right\} = \bigcap_p T(\mathscr{H}_p) =$$

$$= \bigcap_p \{(L(u)L(v^2))^{n_1} \ldots (L(u)L(v^\sigma))^{n_p}T(\mathscr{H}_{F_\nu})\} \subseteq$$

$$\subseteq \bigcap_p \{L(u)L(v^2))^{n_1} \ldots (L(u)L(v^\sigma))^{n_p}\mathscr{H}_{F_\mu}\}.$$

Then, by Lemma 2.7 we have

$$\bigcap_{p=1}^{\infty}\{(uv^2)^{n_1} \ldots (uv^\sigma)^{n_p}F_\mu\} \neq \emptyset.$$

Let w be any element in the above nonempty intersection of those subsets. Thus, $(v^\sigma u)^{n_p} \ldots (vu)^{n_1}w \in F_\mu$ for all $p = 1, 2, \ldots$. It follows from Proposition 2.9 and 2.6 that μ and ν have a common tail. Q.E.D.

This setting for extending a tracial state to pure states may seem restrictive. However, a much more general proposition can be obtained as follows.

2.11. PROPOSITION. *For any nontype* I *C^*-algebra A, there is a non-nuclear C^*-subalgebra B of A and a type* II_1 *factor state φ on B such that φ has uncountable pure state extensions to A.*

Proof. By a result of Blackadar (Theorem 2 in [1]) there exist C^*-subalgebras B, W of A with $B \subseteq W$, and a $*$-homomorphism Λ of W onto O_2 with $\Lambda(B) = \mathrm{Ch}$. Let tr be the tracial state on Ch. It can be easily checked that $\mathrm{tr} \circ \Lambda$ is a factor state on B, for example using Theorem 1.4 in [9]. For each pure state extension φ_ν of tr to O_2 constructed in this section, there is a pure state extension $\varphi_\nu \circ \Lambda$ of $\mathrm{tr} \circ \Lambda$ to W, by 2.11.9 in [5]. Then extend $\varphi_\nu \circ \Lambda$ to a pure state ρ_ν on A. Q.E.D.

3. EXTENSIONS OF PURE STATES ON A MASA

Let F_2 be the Fermion algebra embedded in O_2 and D be the diagonal maximal abelian subalgebra of F_2. (See [3] for details.) Let $\nu = \{n_i\}_i$ be a nonperiodic sequence of positive integers and φ_ν be the pure state extension of the tracial state tr on the Choi subalgebra to O_2 induced by ν in Section 2. In this section we show that the restriction of φ_ν to D is a pure state on D. In fact, $\varphi_\nu \mid D$ corresponds to a point in the spectrum $\mathscr{P}(D)$ of D. It is also shown that all such irrational points in $\mathscr{P}(D)$ are restriction of pure states on O_2 constructed from nonperiodic sequences of positive integers in a similar way. Setting up a notation similar to that in Section 2, we denote by $\mathscr{P}_\nu$ the orthogonal projection of $\ell^2(G)$ onto the subspace $\mathscr{H}_\nu$ spanned by the characteristic funcions χ_{w}, $w \in F_\nu$. Let Π_ν be the irreducible representation of O_2 induced by ν in Section 2.

3.1. First of all, $\mathscr{P}(D)$ can be identified with $\{0,1\}^{\mathrm{N}}$. In fact, the support of $S_{n_1} \ldots S_{n_k} S^*_{n_k} \ldots S^*_{n_1}$ is identified with $\{(\varepsilon_i) \in \{0,1\}^{\mathrm{N}} : 1 + \varepsilon_1 = a_1, \ldots, 1 + \varepsilon_k = a_k\}$, where S_1, S_2 are the generators of O_2. A rational point in $\{0,1\}^{\mathrm{N}}$ is an eventually repeating sequece and an irrational point in $\{0,1\}^{\mathrm{N}}$ is a nonrepeating sequence. We now proceed to define an one-to-one map $\mathscr{M}$ from the set of all nonperiodic sequences of positive integers into $\mathscr{P}(D)$. Furthermore we show that the range of $\mathscr{M}$ includes all irrational points in $\{0,1\}^{\mathrm{N}}$.

The generators S_1, S_2 of $\Pi_\nu(O_2)$ are defined by $S_1 = UV\mathscr{P}_\nu + UV^2\mathscr{P}_\nu U$, $S_2 = V\mathscr{P}_\nu + V^2\mathscr{P}_\nu U$, where $U = L(u)$, $V = L(v)$ and L is the left regular representation of G on $\ell^2(G)$. We get $S_1 = US_2$, $\mathscr{P}_\nu U \mathscr{P}_\nu = 0$, $\mathscr{P}_\nu V^\sigma \mathscr{P}_\nu = 0$ for $\sigma = 1$ or 2, and $S_1S_1^* = \mathscr{P}_\nu$, $S_2S_2^* = U\mathscr{P}_\nu U$, $\mathscr{P}_\nu + U\mathscr{P}_\nu U = I$. In addition we also have

$$S_2S_1S_1^*S_2^* = S_2\mathscr{P}_\nu S_2^* = V\mathscr{P}_\nu V^2,$$

$$(S_2)^2(S_2^*)^2 = S_2U\mathscr{P}_\nu US_2^* = V^2\mathscr{P}_\nu V,$$

$$S_2^3 = V^2\mathscr{P}_\nu S_1 = V^2S_1,$$

$$S_2^2S_1S_1^*(S_2^*)^2 = V^2(UV)\mathscr{P}_\nu(V^2U)V,$$

$$S_2^2S_2S_2^*(S_2^*)^2 = V^2(UV^2)\mathscr{P}_\nu(VU)V.$$

Let $\lambda = (l_1, \ldots, l_k)$, $S_\lambda = S_2^{l_1} S_1^{l_2} \ldots S_\sigma^{l_k}$, $\sigma = 1$ for even k and 2 for odd k. Then $S_2^2 S_\lambda S_\lambda^* (S_2^*)^2 = V^2(UV^2)^{l_1} \ldots (UV^{\bar\sigma})^{l_k} \mathscr{P}_\nu (V^\sigma U)^{l_k} \ldots (VU)^{l_1} V$, where $\bar\sigma = 3 - \sigma$. Hence

$$\varphi_\nu(S_1 S_1^*) = \langle \mathscr{P}_\nu \chi_e, \chi_e \rangle = 0$$

$$\varphi_\nu(S_2 S_2^*) = \langle \mathscr{P}_\nu \chi_u, \chi_u \rangle = 1$$

$$\varphi_\nu(S_2 S_1 S_1^* S_2^*) = \langle \mathscr{P}_\nu \chi_{v^2}, \chi_{v^2} \rangle = 0$$

$$\varphi_\nu(S_2 S_2 S_2^* S_2^*) = \langle \mathscr{P}_\nu \chi_v, \chi_v \rangle = 1$$

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$$\varphi_\nu(S_2^2 S_\lambda S_\lambda^* (S_2^*)^2) = \begin{cases} 0, & \text{if } (v^{\bar\sigma} u)^{l_k} \ldots (vu)^{l_1} v \notin F_\nu \\ 1, & \text{if } (v^{\bar\sigma} u)^{l_k} \ldots (vu)^{l_1} v \in F_\nu . \end{cases}$$

The above calculations exhibit the process of determining any point in $\mathscr{P}(D)$ with the first two digits 1, 1, and the relationship between a nonperiodic sequence ν of positive integers and such a point in $\mathscr{P}(D)$. Now we define a map $\mathscr{M}$ from the set of all nonperiodic sequences of positive integers into a subset of $\mathscr{P}(D)$ consisting of points with the first two digits 1, 1 as follows. Let $\nu = \{n_i\}_i$. Then $\mathscr{M}(\nu) = \varphi_\nu | D$ can be viewed as a point in $\mathscr{P}(D)$ with the first two digits 1, 1 followed by n_1 digits of 1's, then by n_2 digits of 0's, ... etc. For example, if $\nu = \{1, 2, 3, 4, \ldots\}$, then $\varphi_\nu | D$ corresponds to a point in $\{0,1\}^{\mathbf{Z}}$ defined as $\{1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, \ldots\}$. This establishes the one-to-one map $\mathscr{M}$.

If we change the construction of F_ν by substituting $S(\nu, v^2)$ for $S(\nu, v)$, then the corresponding Propositions 2.4, 2.5 and 2.6 can also be shown by the same arguments with v and v^2 interchanged. Then the second digit of the corresponding irrational point $\varphi_\nu | D$ in $\mathscr{P}(D)$ will be 0. If we change the construction of F_ν by substituting $S(\nu, vu)$ for $S(\nu, v)$, then the corresponding Propositions 2.4, 2.5 and 2.6 can also be proved by the same arguments with v and vu interchanged. Thus, the first digit of the corresponding irrational point $\varphi_\nu | D$ in $\mathscr{P}(D)$ will be 0. (It is also easily seen that all irrational points in $\{0,1\}^{\mathbf{N}}$ are included in the range of $\mathscr{M}$.)

In summary, we have proved the following proposition.

3.2. PROPOSITION. *Every nonperiodic point in the spectrum $\mathscr{P}(D)$ of D extends to a pure state φ_ν on O_2 and the restriction of φ_ν to the Choi algebra is the tracial state.*

3.3. REMARK. The proof of Proposition 3.2 also gives the identity of the unique pure state extension to O_2 of each irrational point in $\mathscr{P}(D)$ which was first announced in [4].

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