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# ON SOME TRACE CLASS NORM ESTIMATES

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We prove a mediation formula similar to that of Kato ([5], Lemma X.5.10) and use this formula to give a direct proof for the existence of the principal function [2], [4] for almost normal operators.

As a corollary of our approach we obtain a sharper estimate for the principal function. Namely if $T = X + \mathrm{i}Y$ is an almost normal operator acting on a Hilbert space $H$, i.e. $[X, Y] \in C_1(H)$ and $g_T$ is the principal function of $T$ (see below) then it is known ([4]) that $\|g_T\|_1 \leqslant 2\pi\|[X, Y]\|_1$. We prove that if $X_{\mathrm{a}} \oplus X_{\mathrm{s}}$ is the Hilbert space decomposition of $X$ into absolutely continuous and singular part, which corresponds to the decomposition $H = H_{\mathrm{a}} \oplus H_{\mathrm{s}}$, (see [6]) and $P_{\mathrm{a}}(X)$ is the projection onto $H_{\mathrm{a}}$ then

$$\|g_T\|_1 \leqslant 2\pi\|P_{\mathrm{a}}(X)[X, Y]P_{\mathrm{a}}(X)\|_1 .$$

Finally we use this approach to give another proof of the estimate for the scattering operator $S$ given by M. S. Birman in [1] and compute the moments of the trace $\mathrm{tr}(S_t - 1)$ of the difference between the fiber image of the scattering operator and the identity.

Precisely we obtain

$$\int t^n \mathrm{tr}(S_t - 1)\mathrm{d}t = -(2\pi \mathrm{i})\mathrm{tr}(A^n\Omega^+(A, B)C), \quad n \in \mathbf{N},$$

if $B = A + C$, $C$ is nuclear, $A$, $B$ selfadjoint, $\Omega^+(A, B)$ the wave operator.

We first recall some notations. If $X$ is an (unbounded) selfadjoint operator acting on a Hilbert space $H$ and $\xi$ a vector in $H$ then $\mathrm{d}\mu_\xi^X$ (or simply $\mathrm{d}\mu_\xi$ when no confusion is possible) is the measure defined by $\int f\mathrm{d}\mu_\xi = \langle f(X)\xi, \xi\rangle$ for any $f$ in $\mathrm{Bor}(\mathbf{R})$ (the bounded Borel functions on $\mathbf{R}$); $p_\xi^X$ (or simply $p_\xi$) is the projection onto the cyclic subspace of $X$ with respect to $\xi$. Also recall ([7]) that $H_{\mathrm{a}}(X)$ is the Hilbert subspace of those vectors $\xi$ such that $\mathrm{d}\mu_\xi$ is absolutely continuous with respect to the Lesbegue measure $\lambda$ on the real line and $P_a(X)$ is the orthogonal projection onto this subspace.

Our mediation result can now be stated as follows.

PROPOSITION 1. *Let $X$ be an (unbounded) selfadjoint operator acting on $H$ and $\xi$, $\eta$ two vectors in $H_a(X)$ such that $f = \frac{d\mu_\xi}{d\lambda}$, $g = \frac{d\mu_\eta}{d\lambda}$ are essentially bounded Let $\langle \cdot, \xi \rangle \eta$ be the rank one operator determined by $\xi$ and $\eta$ and let $V$ be the partia isometry with initial space $p_{\xi'}$ and final space $p_{\eta'}$ defined by the requirement*

$$V(h(X)\xi') = h(X)\eta', \quad \textit{for } h \textit{ in } \mathrm{Bor}(\mathbf{R}),$$

*where $\xi' = g^{1/2}(X)\xi$, $\eta' = f^{1/2}(X)\eta$. Then the following integral converges weakly and*

$$(0) \qquad \int_{-\infty}^{+\infty} e^{itX}\langle \cdot, \xi \rangle \eta e^{-itX} dt = 2\pi p_{\eta'} f^{1/2}(X) V g^{1/2}(X) p_{\xi'}.$$

*In particular when $\xi = \eta$ the right hand term of the preceding equality is $f(X)p_\xi$, and when $\eta$ belongs to $p_\xi H$ it is $\frac{d\mu_{\eta,\xi}}{d\lambda}(X)p_\eta$ where $d\mu_{\eta,\xi}$ is defined by $\int h \, d\mu_{\eta,\xi} = \langle h(X)\eta, \xi \rangle$, $h \in \mathrm{Bor}(\mathbf{R})$.*

*Proof.* Denote the right hand term of (0) by $A$ and the left by $B$. Then for any $h$, $k$ in $\mathrm{Bor}(\mathbf{R})$

$$\langle Ah(X)\xi, k(X)\eta \rangle = \int_{-\infty}^{\infty} \langle h(X)\xi, e^{itX}\xi \rangle \langle e^{itX}\eta, k(X)\eta \rangle \, dt =$$

$$= \int_{-\infty}^{\infty} F\left(h \frac{d\mu_\xi}{d\lambda}\right)(t) \overline{F\left(k \frac{d\mu_\eta}{d\lambda}\right)(t)} \, dt = 2\pi \int h\bar{k}fg \, d\lambda.$$

where $F$ is the Fourier transform and we used Parseval formula.

On the other hand

$$\langle Bh(X)\xi, k(X)\eta \rangle = 2\pi \langle f^{1/2}(X) V g^{1/2}(X) h(X)\xi, k(X)\eta \rangle =$$

$$= 2\pi \langle (\bar{k} f^{1/2})(X) f^{1/2}(X) h(X)\eta, \eta \rangle = 2\pi \int h\bar{k}fg \, d\lambda,$$

which completes the proof.

We can now give a direct proof for the following theorem

THEOREM (J. W. Helton and R. Howe [4], R. W. Carey and J. D. Pincus [2]). *Let* $T = X + iY$ *be an almost normal operator in* $L(H)$, i.e. *such that* $C = (1/2)[T^*, T] = i[X, Y]$ *is a trace class operator* $(C \in C_1(H))$. *Then there is a function* $g_T$ *in* $L^1(\mathbf{R}^2, d\lambda)$ *with compact support such that*

$$\mathrm{tr}(i[p(X, Y), q(X, Y)]) = \frac{1}{2\pi}\iint\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}\right) g_T \, dx \, dy$$

*for any polynomials* $p$, $q$ *in the two real variables* $x$, $y$. *(Note that since* $[X, Y]$ *is trace class it does not matter the order in which* $X$, $Y$ *enter in* $p$, $q$ *in the left side member.) Moreover,* $\|g_T\|_1 \leq 2\pi\|[X, Y]\|_1$.

Note that as a corollary of the proof we will obtain

$$\|g_T\|_1 \leq 2\pi\|P_{ac}(X)[X, Y]P_{ac}(X)\|_1.$$

*Proof.* Assume first that $X$ is absolutely continuous (i.e. $H_{ac}(X) = H$), and write $i[X, Y] = \sum \lambda_n \langle \cdot, \xi_n \rangle \xi_n$ where $\xi_n$ is an orthonormal family of vectors in $H$ and $\lambda_n$ real numbers with $\sum |\lambda_n| = \|C\|_1$. Further let $f_n = \frac{d\mu_{\xi_n}}{d\lambda}$, $n \in \mathbf{N}$. Then

$$\int \sum |\lambda_n| f_n(x) \, dx = \sum |\lambda_n| \langle \xi_n, \xi_n \rangle = \|[X, Y]\|_1 \tag{1}$$

and an analogous statement for $Y$ replaced by $Y^p$ for each $p \in \mathbf{N}$.

*Step 1.* Assume that:

$(*)$ $\sum |\lambda_n| f_n$ is bounded; and the analogous condition for $Y^p$ in place of $Y$ for each $p \in \mathbf{N}$.

Let $S_X^{\pm}(Y)$ be the strong limits $\lim_{t \to \pm\infty} e^{itX} Y e^{-itX}$ which exists by the Birman-Kato-Rosenblum theorem (see [7], Theorem XI. 7). Clearly $S_X^{\pm}(Y)$ commutes with $X$ and

$$S_X^{+}(Y) - S_X^{-}(Y) = \int_{-\infty}^{\infty} e^{itX} i[X, Y] e^{-itX} \, dt = 2\pi \sum \lambda_n f_n(X) p_{\xi_n}, \tag{2}$$

where the first equality follows from

$$\frac{d}{dt}(e^{itX} Y e^{-itX}) = e^{itX} i[X, Y] e^{-itX}$$

and the second follows from Proposition 1 and assumption $(*)$.

Let $H = \int_{\sigma(X)}^{\oplus} H_x \, dx$; $X = \int_{\sigma(X)}^{\oplus} M_x \, dx$ be the spectral decomposition of $X$ ([3]) where $M_x$ is a scalar operator in each fiber $H_x$, $x \in \sigma(X)$. If $A$ commutes with $X$, then $A$ has also a decomposition $A = \int_{\sigma(X)}^{\oplus} A_x \, dx$, where $A_x$ is a bounded operator for each $x$. Moreover if $\xi$ is any vector in $H$ then the image $(p_\xi)_x$ of $p_\xi$ in each fiber $H_x$ is an at most one dimensional projection, ([3]).

Hence by (2) we deduce that $S_X^+(Y)_x - S_X^-(Y)_x$ is trace class for $\lambda$-almost all $x \in \sigma(X)$ and

(3) $$\operatorname{tr}(S_X^+(Y)_x - S_X^-(Y)_x) = 2\pi \sum_n \lambda_n f_n(x), \text{ a.e.}$$

(4) $$\|S_X^+(Y)_x - S_X^-(Y)_x\|_1 \leqslant 2\pi \sum_n |\lambda_n| f_n(X), \text{ a.e.}.$$

Consequently for any $n \in \mathbf{N}$ we have

$$\operatorname{tr}(X^n \mathrm{i}[X, Y]) = \operatorname{tr}(X^n \sum \lambda_k \langle \cdot, \xi_k \rangle \xi_k) =$$

$$= \sum \lambda_k \langle X^n \xi_k, \xi_k \rangle = \sum \lambda_k \int x^n f_k(x) \, dx =$$

$$= \frac{1}{2\pi} \int x^n \operatorname{tr}(S_X^+(Y)_x - S_X^-(Y)_x) \, dx.$$

If in the preceding equality we replace $Y$ by $Y^p$, $p \in \mathbf{N}$, then by taking into account that $(S_X^+(Y))^p = S_X^+(Y^p)$ we obtain

(5) $$\operatorname{tr}(X^n \mathrm{i}[X, Y^p]) = \frac{1}{2\pi} \int x^n \operatorname{tr}(S_X^+(Y)_x^p - S_X^-(Y)_x^p) dx$$

for all $p, n \in \mathbf{N}$.

We apply now the theorem of M. G. Krein [6] for the trace class perturbation $S_X^-(Y)_x \to S_X^+(Y)_x$ to obtain the existence of a function $g_x$ in $L^1(\mathbf{R})$ with compact support (the phase shift) such that

(6) $$\|g_x\|_1 \leqslant \|S_X^+(Y)_x - S_X^-(Y)_x\|_1, \text{ a.e.}$$

(7) $$\operatorname{tr}(S_X^+(Y)_x^p - S_X^-(Y)_x^p) = \int p y^{p-1} g_x(y) \, dy.$$

The required function $g_T$ will be as in [2], defined by $g_T(x, y) = g_x(y)$.

Note that using the proof of D. Voiculescu, ([8]), for the existence of the phase shift, one obtains that the assignment $(x, y) \to g_x(y) = g_T(x, y)$ defines a measurable function on $\mathbf{R}^2$. (Indeed one can select a family of finite dimensional projections $\{p_n^x\}$ in $L(H_x)$ depending measurable on $x$ in $\sigma(X)$ which give a diagonalization modulo the Hilbert-Schmidt operators $C_2(H_x)$ for $S_x^-(Y)_x$ for $\lambda$-almost all $x$. The measurability of $g_T$ follows then from the minimax principle which assures us that the eigenvalues of a measurable family of finite dimensional selfadjoint operators $\{A_x\}$, $x \in \sigma(X)$ also depend measurable on $x$).

Now combining (5), (7) we obtain that for all $p$, $n$ in $\mathbf{N}$

$$\text{(8)} \qquad \operatorname{tr}(X^n i[X, Y^p]) = \frac{1}{2\pi}\iint p x^n y^{p-1} g_T(x, y) \mathrm{d}x\, \mathrm{d}y,$$

and from (1), (4), (6) it follows that

$$\text{(9)} \qquad \|g_T\|_1 = \int \|g_x\|_1 \,\mathrm{d}x \leqslant 2\pi \int \sum |\lambda_n| f_n(x) \mathrm{d}x = 2\pi \|C\|_1 .$$

Finally it is clear that (8) implies the statement of the theorem for $p(x) = x^{n+1}$, $q(y) = y^p$ and simple algebraic computations yield the result for all $p, q$ (see e.g. [3]).

Note also for later use that if $\sigma$ is any Borel subset of $\mathbf{R}$, $E(\sigma)$ the spectral measure of $X$ corresponding to $\sigma$ then $T = X + iY_\sigma$ where $Y_\sigma = E(\sigma) Y E(\sigma)$ is also almost normal (since $[X, Y_\sigma] = E(\sigma)[X, Y]E(\sigma)$) and $S_X^\pm(Y) = E(\sigma) S_X^\pm(Y) = S_X^\pm(Y) E(\sigma)$. Hence by the definition of the principal function,

$$\text{(10)} \qquad g_{T_\sigma}(x, y) = \chi_\sigma(x) g_T(x, y), \quad x, y \text{ in } \mathbf{R},$$

where $\chi_\sigma$ is the characteristic function of $\sigma$.

*Step 2.* We still assume that $H_a(X) = H$ but we drop out assumption (*).

Let $[X, Y^p]$ have the expansion $\sum \lambda_n^p \langle \cdot, \xi_n^p \rangle \xi_n^p$ with $\{\xi_n^p\}_n$ orthogonal for each $p$, $f_n^p = \dfrac{\mathrm{d}\mu_{\xi_n^p}}{\mathrm{d}\lambda}$.

From (1) we have that $\int \sum_n |\lambda_n^p| f_n^p \,\mathrm{d}\lambda$ is finite for each $P$ so that there exist an increasing sequence of borelian sets $\sigma_n$, with $\bigcup \sigma_k = \sigma(X)$, and such that condition (*) is fulfilled when restricted to $\sigma_k$.

Let $Y_k = Y_{\sigma_k}$, $g_k = g_{T_k}$, $T_k = X + iY_k$. By (10) we have

$$g_k(x, y) = g_l(x, y)\, \chi_{\sigma_k}(x) \quad \text{for all } x, y \text{ and } k \leqslant l$$

and by (9)

$$\|g_k\|_1 \leqslant 2\pi\|[X, Y_k]\|_1 \leqslant 2\pi\|[X, Y]\|_1, \quad \text{for all } k \in \mathbf{N}.$$

Hence $\{g_k\}$ converges in $L^1(\mathbf{R})$ to a function $g$ (with compact support) such that $\|g\|_1 \leqslant 2\pi\|[X, Y]\|_1$.

Since $\mathrm{tr}(X^n \mathrm{i}[X, Y_k^p])$ converges to $\mathrm{tr}(X^n \mathrm{i}[X, Y^p])$ for fixed $p$, $n$, it follows by the preceding step that

$$\mathrm{tr}(X^n \mathrm{i}[X, Y^p]) = \frac{1}{2\pi}\iint p x^n y^{p-1} g(x, y) \mathrm{d}x\, \mathrm{d}y, \quad p, n \in \mathbf{N}.$$

Hence $g$ is the required function for $T = X + \mathrm{i}Y$.

*Step 3.* Assume now that $T = X + \mathrm{i}Y$ where $X$ has a decomposition $X = X_a \oplus X_s$ into absolutely continuous and singular parts with nontrivial $X_s$. Let $H = H_a \oplus H_s$ be the corresponding decomposition of the space and

$$Y = \begin{pmatrix} Y_{aa} & Y_{as} \\ Y_{sa} & Y_{ss} \end{pmatrix}$$

the corresponding decomposition of $Y$.

First note that due to [8] there is a diagonal selfadjoint operator $D$ and a trace class $E$ with norm $\|E\|_1$ as small as we want such that $X_s = D + E$. Also note that if $S$ is almost normal and $E'$ is trace class then $S' = S + E'$ is also almost normal and $\mathrm{tr}[S^{*m}, S^n] = \mathrm{tr}[S'^{*m}, S'^n]$ for all $n$. It follows that in order to prove the theorem we may assume that $X_s$ itself is diagonal.

Since a commutator involving a diagonal operator has always null diagonal and hence null trace (if it is nuclear) it follows that for $p, n \in \mathbf{N}$,

$$\mathrm{tr}(X^n[X, Y^p]) = \frac{1}{n+1}\mathrm{tr}([X^{n+1}, Y^p]) =$$

$$= \frac{1}{n+1}\mathrm{tr}([X_a^{n+1}, P_a Y^p P_a]) = \mathrm{tr}(X_a^n[X, P_a Y^p P_a]).$$

Since $S_{X_a}^{\pm}(P_a Y P_a) = S_{X_a}^{\pm}(Y)$ and hence $S_{X_a}^{\pm}(P_a Y^p P_a) = (S_{X_a}^{\pm}(P_a Y P_a))^p$ for $p$ in $\mathbf{N}$, it follows that the arguments from the preceding step go trough also into this case yielding a principal function $g_T$ which satisfies the estimate

$$\|g_T\|_1 \leqslant 2\pi\|[X_a, P_a Y P_a]\|_1.$$

This ends the proof.

We turn now to another circle of ideas concerning the scattering matrix.

Let $A$ be an (unbounded) selfadjoint operator, $B = A + C$ a selfadjoint trace class perturbation of $A$, and $S = \Omega^+(A, B)\Omega^-(B, A)$ the scattering matrix, where the wave operators $\Omega^\pm(B, A)$ are defined as the strong operator topology limits

$$\operatorname*{so-lim}_{t\to\pm\infty} e^{itB}e^{-itA}P_a(A).$$

The existence of the preceding limits is due to the theorem of Birman-Kato-Rosenblum quoted before. Clearly $S$ commutes with $A$, so in the decomposition $A_a = A|H_a(A) = \int_{\sigma(A_a)}^{\oplus} M_t\, dt$ ([3]) we have

$$S = \int_{\sigma(A_a)}^{\oplus} S_t\, dt.$$

We will give a proof for the following theorem

THEOREM (M. S. Birman, [1]). *The scatttering amplitude $S_t - 1$ is trace class for $\lambda$-almost all $t$ in $\sigma(A_a)$ and*

$$\int_{\sigma(A_a)} \|S_t - 1\|_1\, dt \leqslant 2\pi\|C\|_1.$$

*Proof.* Let us write $C = \sum \lambda_n\langle\cdot, \xi_n\rangle\xi_n$ with $\xi_n$ orthonormal vectors, $\sum |\lambda_n| = \|C\|_1$. Let $\xi_n^0 = P_a(A)\xi_n$ and $\eta_n^0 = \Omega^+(A, B)\xi_n$. Let $f_n = \frac{d\mu_{\xi_n^0}}{d\lambda}$, $g_n = \frac{d\mu_{\eta_n^0}}{d\lambda}$.

Clearly we have

$$\int \sum |\lambda_n| f_n^{1/2}(t) g_n^{1/2}(t)dt \leqslant \sum |\lambda_n| \left(\int f_n(t)dt\right)^{1/2}\left(\int g_n(t)dt\right)^{1/2} =$$

(12)

$$= \sum |\lambda_n| \langle\xi_n^0, \xi_n^0\rangle\langle\eta_n^0, \eta_n^0\rangle \leqslant \|C\|_1.$$

Hence arguing as in Step 2 in the proof of the preceding theorem we may assume that

(**) $$\sum |\lambda_n|(f_n(t))^{1/2}(g_n(t))^{1/2}$$

is bounded. Finally let $V_n$ be the isometry associated to $\xi_n^0$, $\eta_n^0$ and $A$ by Proposition 1. We have

$$S - \mathrm{Id}_{H_a(A)} = \Omega^+(A, B)\Omega^-(B, A) - \Omega^+(A, B)\Omega^+(B, A) =$$

$$= -\Omega^+(A, B)\,(\Omega^+(B, A) - \Omega^-(B, A)) =$$

$$= -\Omega^+(A, B) \int_{-\infty}^{\infty} e^{itB} iCe^{-itA} P_a(A)\,dt =$$

$$= (-i) \int_{-\infty}^{\infty} e^{itA}(\Omega^+(A, B)CP_a(A))e^{-itA}\,dt =$$

$$= (-i) \int_{-\infty}^{\infty} e^{-itA}(\sum \lambda_n \langle \cdot , \xi_n^0\rangle \eta_n^0)e^{-itA}\,dt =$$

$$= 2\pi(-i) \sum \lambda_n P_{\eta_n'} f_n^{1/2}(X) V_n g_n^{1/2}(X) P_{\xi_n'},$$

where $\eta_n' = f_n^{1/2}(X)\eta_n^0$, $\xi_n' = g_n^{1/2}(X)\xi_n^0$.

Since clearly the image of $V_n$ in each fiber of $H = \int_{\sigma(A_a)}^{\oplus} H_t\,dt$ is an at most rank one isometry it follows that $S_t - 1$ is trace class for almost all $t$ and

$$\int \|S_t - 1\|_1\,dt \leqslant 2\pi \int \sum |\lambda_n| f_n^{1/2}(t) g_n^{1/2}(t)\,dt \leqslant \|C\|_1$$

where we used (12). This completes the proof.

We have also the following description of $\mathrm{tr}(S_t - 1)$.

PROPOSITION. *Let $f(t) = \mathrm{tr}(S_t - 1)$ for $\lambda$ almost all $t$. Then for each $n \in \mathbf{N}$*

$$\int t^n f(t)\mathrm{d}t = (-2\pi i)\mathrm{tr}(A^n\Omega^+(A, B)C).$$

*Proof.* First we note that in the conditions of Proposition 1, if $X$ is an arbitrary selfadjoint operator with $H_a(X) = H$, and with decomposition

$X = \int^{\oplus} M_t \, \mathrm{d}t$, and if

$$\Gamma = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tX} \langle \cdot , \xi \rangle \eta \mathrm{e}^{-\mathrm{i}tX} \, \mathrm{d}t$$

has the decomposition $\Gamma = \int_{\sigma(X)} \Gamma_t \, \mathrm{d}t$, then

$$\mathrm{tr}(\Gamma_t) = 2\pi \frac{\mathrm{d}\mu_{\eta,\xi}}{\mathrm{d}\lambda}(t) \quad \text{for } \lambda \text{ almost all } t. \tag{13}$$

Indeed both sides of (13) do not change if we replace $\eta$ by its image onto the cyclic projection of $X$ generated by $\xi$ and hence in this case by Proposition 1

$$\Gamma = 2\pi p_\eta \frac{\mathrm{d}\mu_{\eta,\xi}}{\mathrm{d}\lambda}(X).$$

Formula (13) is an obvious consequence of this.
Hence for $\lambda$-almost all $t$

$$\mathrm{tr}(S_t - 1) = -(2\pi\mathrm{i}) \sum \lambda_k \frac{\mathrm{d}\mu_{\eta_k^0,\xi_k^0}}{\mathrm{d}\lambda}(t)$$

so that for all $n \in \mathbf{N}$

$$t^n \mathrm{tr}(S_t - 1) = -(2\pi\mathrm{i})\left( \sum \lambda_k \int t^n \mathrm{d}\mu_{\eta_k^0,\xi_k^0}(t) \right) =$$

$$= -(2\pi\mathrm{i}) \sum \lambda_k \langle A^n \eta_k^0, \xi_k^0 \rangle = -(2\pi\mathrm{i}) \mathrm{tr}(A^n \sum \lambda_k \langle \cdot , \xi_k^0 \rangle \eta_k^0) =$$

$$= -(2\pi\mathrm{i}) \mathrm{tr}(A^n \Omega^+(A, B) C P_{\mathrm{a}}(A)) = -(2\pi\mathrm{i}) \mathrm{tr}(A^n \Omega^+(A, B) C).$$

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