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CHOICE SEQUENCES AND SUBISOMETRIC DILATIONS

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1. INTRODUCTION

Let H, H' be two complex Hilbert spaces, $T \in \mathscr{L}(H)$, $T' \in \mathscr{L}(H')$, $A \in \mathscr{L}(H, H')$ contractions such that $AT = T'A$.

In [6], Douglas and Foiaş generalized the notion of minimal isometric dilation of the contraction T, by considering minimal subisometric dilations, i.e. contractions $\tilde{T} \in \mathscr{L}(H \oplus H_1)$,

$$\tilde{T} = \begin{pmatrix} T & 0 \\ D_{T_1^*}\alpha D_T & T_1 \end{pmatrix} \tag{1.1}$$

where $T_1 \in \mathscr{L}(H_1)$ is a contraction such that $T_1^{*n} \to 0$ strongly and $\alpha : \mathscr{D}_T \to \mathscr{D}_{T_1^*}$ is unitary.

Their main interest was to find the class of minimal subisometric dilations, having the uniqueness property (up to isomorphism) and the intertwinning lifting property.

The uniqueness property (up to isomorphism) for a minimal subisometric dilation $\tilde{T} \in \mathscr{L}(H \oplus H_1)$ of T, is to be understood in the sense that for any other minimal subisometric dilation $\tilde{C}$ of T such that C_1 is unitarily equivalent to T_1, it follows that $\tilde{C}$ is isomorphic to $\tilde{T}$. Actually, Theorem 3 in [6] states that in this case T_1 is unitarily equivalent to $S(m) \otimes 1_{\mathscr{D}_T}$ (see definitions below) for some scalar inner function m. A minimal subisometric dilation $\tilde{T} \in \mathscr{L}(H \oplus H_1)$ of T is said to be a minimal uniform m-Jordan dilation if the contraction T_1 in the representation (1.1) is isomorphic to $S(m) \otimes 1_{\mathscr{D}_T}$ for some scalar inner function m.

The contractive dilations $\tilde{T} \in \mathscr{L}(H \oplus H_1)$ and $\tilde{T}' \in \mathscr{L}(H' \oplus H_1')$ of T (resp. T') have the intertwinning lifting property if for any contraction $X \in \mathscr{L}(H, H')$ such that $XT = T'X$ there exists a contraction $\tilde{X} \in \mathscr{L}(H \oplus H_1, H' \oplus H_1')$ such that $\tilde{X}T = \tilde{T}'\tilde{X}$ and $P'\tilde{X} = XP$, where P (resp. P') are the orthogonal projections of $H \oplus H_1$ (resp. $H' \oplus H_1'$) onto H (resp. H').

In Theorem 5 of [6], Douglas and Foiaş proved that if $B_1 \in \mathscr{L}(E_1)$ (resp. $B_1' \in \mathscr{L}(E_1')$) are contractions satisfying $B_1^{*n} \to 0$, $B_1'^{*n} \to 0$ strongly and have the property that for any contractions C (resp. C'), any minimal subisometric dilations $\tilde{C}$ (resp. $\tilde{C}'$) of C (resp. C') such that $C_1 = B_1$ (resp. $C_1' = B_1'$) have the intertwinning lifting property, then B_1 (resp. B_1') are unitarily equivalent to some $S(m) \otimes 1_{\mathscr{D}}$ (resp. $S(m) \otimes 1_{\mathscr{D}'}$).

So the two properties give rise to the same class, the class of uniform Jordan dilations.

This paper extends the results in [1] and [3] within this frame-work.

We prove that if m is a scalar inner function having at least one zero in the unit open disc, then the necessary and sufficient condition for $\mathrm{CID}_m(A)$ (see the notaions below) to be singleton, is that one of the factorizations AT or $T'A$ be regular.

We also describe the set $\mathrm{CID}_m(A)$ by means of choice sequences, when m is a Blaschke product. An algorithm connecting a lifting with its choice sequence is given.

These results are obtained due to the special structure of $S(m)$ in each of the cases considered.

First of all let us recall the basic definitions and results which will be used in the sequel.

For z a complex number in the unit open disc $T_z = (T - z)(1 - \bar{z}T)^{-1}$ is also a contraction and we define (see [2]):

$$(1.2) \qquad \begin{cases} \mathscr{F}_A(T_z) = \mathscr{F}_{A,z} = \{D_A(T - z)h \oplus (1 - |z|^2)^{1/2} D_T h\,;\ h \in H\}^- \\ \mathscr{R}_A(T_z) = \mathscr{R}_{A,z} = (\mathscr{D}_A \oplus \mathscr{D}_T) \ominus \mathscr{F}_{A,z} \end{cases}$$

$$(1.3) \qquad \begin{cases} \mathscr{F}^A(T_z) = \mathscr{F}^{A,z} = \{D_A(1 - \bar{z}T)h \oplus (1 - |z|^2)^{1/2} D_{T'} A h\,;\ h \in H\}^- \\ \mathscr{R}^A(T_z) = \mathscr{R}^{A,z} = (\mathscr{D}_A \oplus \mathscr{D}_{T'}) \ominus \mathscr{F}^{A,z} \end{cases}$$

$$(1.4) \qquad \begin{cases} p_{A,z} = P^{\mathscr{D}_A \oplus \mathscr{D}_T}_{\mathscr{F}_{A,z}} & q_A = P^{\mathscr{D}_A \oplus \mathscr{D}_T}_{\mathscr{D}_T} \\ p^{A,z} = P^{\mathscr{D}_A \oplus \mathscr{D}_{T'}}_{\mathscr{F}^{A,z}} & q^A = P^{\mathscr{D}_A \oplus \mathscr{D}_{T'}}_{\mathscr{D}_{T'}} . \end{cases}$$

Between $\mathscr{F}_{A,z}$ and $\mathscr{F}^{A,z}$ acts the natural unitary operator:

$$\sigma_{A,z} : \mathscr{F}_{A,z} \to \mathscr{F}^{A,z},$$

$$(1.5) \qquad \begin{aligned} &\sigma_{A,z}(D_A(T - z)h \oplus (1 - |z|^2)^{1/2} D_T h) = \\ &= D_A(1 - \bar{z}T)h \oplus (1 - |z|^2)^{1/2} D_{T'} A h, \quad h \in H. \end{aligned}$$

The spaces $\mathscr{R}_{A,0} = \mathscr{R}_A$, $\mathscr{R}^{A,0} = \mathscr{R}^A$ appeared in [1] where the necessary and sufficient condition for the uniqueness of the contractive intertwinning dilation of $\tilde{A}$

was given in terms of regular factorizations. So, the factorization $A \cdot T$ (resp. $T'A$) is regular, if and only if $\mathscr{R}_A = 0$ (resp. $\mathscr{R}^A = 0$).

For $\mathscr{D}$ a Hilbert space, $H^2(\mathscr{D})$ denotes the Hardy space on the unit circle of $\mathscr{D}$-valued functions and S (resp. $S \otimes 1_{\mathscr{D}}$) stand for the shift operators on H^2 (resp. $H^2(\mathscr{D})$).

If m is a scalar inner function, we set:

$$H(m) = H^2 \ominus mH^2, \quad S(m) = P_{H(m)}S|H(m),$$

$$H(m) \otimes \mathscr{D} = H^2(\mathscr{D}) \ominus mH^2(\mathscr{D}), \quad S(m) \otimes 1_{\mathscr{D}} = P_{H(m)\otimes\mathscr{D}}S \otimes 1_{\mathscr{D}}|(H(m) \otimes \mathscr{D})$$

where $P_{H(m)}$ (resp. $P_{H(m)\otimes\mathscr{D}}$) are the orthogonal projections of H^2 (resp. $H^2(\mathscr{D})$) onto $H(m)$ (resp. $H(m) \otimes \mathscr{D}$).

An easy computation yields that the minimal uniform m-Jordan dilation of the contraction T is $T(m) \in \mathscr{L}(H \oplus H(m) \otimes \mathscr{D}_T)$,

$$T(m) = \begin{pmatrix} T & 0 \\ (1 - \bar{m}(0)m)D_T & S(m) \otimes 1_{\mathscr{D}_T} \end{pmatrix}.$$

If P (resp. P') are the orthogonal projections of $H \oplus H(m) \otimes \mathscr{D}_T$ (resp. $H' \oplus H(m) \otimes \mathscr{D}_{T'}$) onto H (resp. H'), we define:

$$\mathrm{CID}_m(A) = \{\tilde{A} \in \mathscr{L}(H \oplus H(m) \otimes \mathscr{D}_T,\ H' \oplus H(m) \otimes \mathscr{D}_{T'};\ \|\tilde{A}\| \leqslant 1,$$

$$\tilde{A}T(m) = T'(m)\tilde{A},\ P'\tilde{A} = AP\}.$$

§ 2

Let z_0 be a complex number in the unit open disc and $m_0(z) = \dfrac{z_0 - z}{1 - \bar{z}_0 z}$, $|z| < 1$.

Then we have:

LEMMA 2.1. *There exists a one-to-one correspondence between* $\mathrm{CID}_{m_0}(A)$ *onto the set of contractions:*

$$\Gamma : \mathscr{R}_{A,z_0} \to \mathscr{R}^{A,z_0}.$$

Proof. We choose as the minimal uniform m_0-Jordan dilation of T, $\tilde{T} \in \mathscr{L}(H \oplus \mathscr{D}_T)$,

$$\tilde{T} = \begin{pmatrix} T & 0 \\ (1 - |z_0|^2)^{1/2}D_T & z_0 \end{pmatrix}.$$

The notation $\tilde{T}'$ is now clear.

If $\tilde{A} \in \mathscr{L}(H \oplus \mathscr{D}_T, H' \oplus \mathscr{D}_{T'})$ is in $\mathrm{CID}_{m_0}(A)$, then:

$$\tilde{A} = \begin{pmatrix} A & 0 \\ X_1 D_A & Y_1 \end{pmatrix},$$

$X_1 : \mathscr{D}_A \to \mathscr{D}_{T'}$, $Y_1 : \mathscr{D}_T \to \mathscr{D}_{T'}$ being contractions.

Using $\tilde{A}\tilde{T} = \tilde{T}'\tilde{A}$ we obtain:

$$X_1 D_A(T - z_0) + (1 - |z_0|^2)^{1/2} Y_1 D_T = (1 - |z_0|^2)^{1/2} D_{T'} A. \tag{2.1}$$

Since A is a contraction, so is $C = (X_1 \quad Y_1) : \begin{matrix} \mathscr{D}_A \\ \oplus \\ \mathscr{D}_T \end{matrix} \to \mathscr{D}_{T'}$ and according to (2.1) and (1.2), $C|\mathscr{F}_{A,z_0} = q^A \sigma_{A,z_0} = C_1$.

By the structure of a matrix row contraction, there exists a contraction $C_2 : \mathscr{R}_{A,z_0} \to \mathscr{D}_{C^*}$ such that $C = (C_1 \quad D_{C_1^*} C_2)$. Now it is easy to see that the operator:

$$\tilde{W}_1 : \mathscr{D}_{C_1^*} \to \mathscr{R}^{A,z_0}, \quad \tilde{W}_1 D_{C_1^*} = (1 - p^{A,z_0})|\mathscr{D}_T$$

is unitary and $D_{C^*}\tilde{W}_1^* = q^A|\mathscr{R}^{A,z_0}$. Setting $\Gamma_1 = \tilde{W}_1 C_2$ we obtain:

$$C = (q^A \sigma_{A,z_0} q^A \Gamma_1) : \begin{matrix} \mathscr{F}_{A,z_0} \\ \oplus \\ \mathscr{R}_{A,z_0} \end{matrix} \to \mathscr{D}_{T'}.$$

So:

$$\begin{cases} X_1 = q^A(\sigma_{A,z_0} p_{A,z_0} + \Gamma_1(1 - p_{A,z_0}))|\mathscr{D}_A \\ Y_1 = q^A(\sigma_{A,z_0} p_{A,z_0} + \Gamma_1(1 - p_{A,z_0}))|\mathscr{D}_T. \end{cases} \tag{2.2}$$

By the above lemma, $\mathrm{CID}_{m_0}(A)$ is a singleton if and only if $\mathscr{R}_{A,z_0} = 0$ or $\mathscr{R}^{A,z_0} = 0$, therefore if and only if one of the factorizations AT_{z_0} or $T'_{z_0}A$ is regular. Let us note now that this is equivalent to that one of the factorizations AT or $T'A$ be regular. Indeed, if (U, K) (resp. (U', K')) are the minimal isometric dilations of T (resp. T'), then ([8], Proposition I.4.3) (U_{z_0}, K) (resp. (U'_{z_0}, K')) are the minimal isometric dilations of T_{z_0} (resp. T'_z) and it is obvious that:

$$\mathrm{CID}(A) = \{\tilde{A} \in \mathscr{L}(K, K') ; \|\tilde{A}\| \leqslant 1, \tilde{A}U = U'\tilde{A}, P_{H'}\tilde{A} = AP_H\} =$$

$$= \mathrm{CID}(A, z_0) = \{\tilde{A} \in \mathscr{L}(K, K') ; \|\tilde{A}\| \leqslant 1, U'_{z_0}\tilde{A} = \tilde{A}U_{z_0}, P_{H'}\tilde{A} = AP_H\}.$$

Then $\mathscr{R}_{A,z_0} = 0$ or $\mathscr{R}^{A,z_0} = 0$ is equivalent to $\mathrm{CID}(A, z_0)$ be a singleton which is equivalent to $\mathrm{CID}(A)$ be a singleton which again is equivalent to one of the factorizations AT or $T'A$ be regular, where the first and last statements hold in virtue of Theorem 1.1 in [1].

So we have proved:

COROLLARY 2.2. *$\mathrm{CID}_{m_0}(A)$ is a singleton if and only if one of the factorizations AT or $T'A$ is regular.*

Let now $m = m_1m_2$, m_1, m_2 being two non trivial scalar inner functions.

Since for any $h \in H$, $g_1 \in H(m_1) \otimes \mathscr{D}_T$,

$$\|D_{T(m_1)}(h \oplus g_1)\|^2 = \|D_T h\|^2 + \|g_1\|^2 - \|(1 - \bar{m}_1(0)m_1)D_T h + S(m_1)\otimes 1_{\mathscr{D}_T} g_1\|^2 =$$

$$= \|D_T h + d\|^2, \quad \text{where } d = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t}\bar{m}_1(\mathrm{e}^{\mathrm{i}t})g_1(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t,$$

it follows that the operator:

$$\omega_1 : \mathscr{D}_{T(m_1)} \to \mathscr{D}_T$$

given by:

$$\omega_1 D_{T(m_1)}(h \oplus g_1) = \bar{m}_1(0)D_T h + \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t}\bar{m}_1(e^{\mathrm{i}t})g_1(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t,$$

$h \in H$, $g_1 \in H(m_1) \otimes \mathscr{D}_T$ is unitary.

A simple computation yields that with respect to the decomposition:

$$H \oplus H(m) \otimes \mathscr{D}_T = (H \oplus (H(m_1) \otimes \mathscr{D}_T)) \oplus m_1(H(m_2) \otimes \mathscr{D}_T),$$

$$T(m) = \begin{pmatrix} T(m_1) & 0 \\ m_1(1 - \bar{m}_2(0)m_2)\omega_1 D_{T(m_1)} & Z_1 \end{pmatrix}$$

where $Z_1 : m_1(H(m_2) \otimes \mathscr{D}_T) \to m_1(H(m_2) \otimes \mathscr{D}_T)$, $Z_1(m_1g_2) = m_1(S(m_2) \otimes 1_{\mathscr{D}_T} g_2)$, $g_2 \in H(m_2) \otimes \mathscr{D}_T$.

So $T(m)$ is the minimal m_2-Jordan dilation of $T(m_1)$.

Let now $B \in \mathrm{CID}_m(A)$ and define

$$B_1 = P_{H' \oplus (H(m_1) \otimes \mathscr{D}_{T'})} B|(H \oplus (H(m) \otimes \mathscr{D}_T)).$$

As in the proof of Theorem 4 in [6] one can obtain that:

$$B = \begin{pmatrix} B_1 & 0 \\ * & * \end{pmatrix}$$

and since $P'B = P'B_1 = AP$, it follows that $B_1 \in \mathrm{CID}_{m_1}(A)$.

Conversely, it is obvious that for any $B_1 \in \mathrm{CID}_{m_1}(A)$, there exists $B \in \mathrm{CID}_m(A)$ such that:

$$B_1 = P_{H' \oplus (H(m_1) \otimes \mathcal{D}_{T'})} B | (H \oplus (H(m) \otimes \mathcal{D}_T)).$$

(The minimal uniform m_2-Jordan dilations $T(m)$ (resp. $T'(m)$) of $T(m_1)$ (resp. $T'(m_1)$) have the intertwinning lifting property.)

Now we can state the main result of this section.

THEOREM 2.3. *Let m be a scalar inner function having at least one zero in the unit open disc.*

Then $\mathrm{CID}_m(A)$ is a singleton if and only if one of the factorizations $A \cdot T$ or $T'A$ is regular.

Proof. Let z_0 be a zero of m in the open unit disc and set $m_0(z) = \dfrac{z_0 - z}{1 - \bar{z}_0 z}$, $|z| < 1$.

Then $m_1 = \dfrac{m}{m_0}$ is also an inner function and $m = m_0 m_1$.

By the discussion above, if $\mathrm{CID}_m(A)$ is a singleton then $\mathrm{CID}_{m_1}(A)$ is a singleton and by Corollary 2.2 this is equivalent to one of the factorizations $A \cdot T$ or $T'A$ be regular.

The converse of this statement is contained in Theorem 4 in [6].

§ 3

We consider in the sequel that m is a Blaschke product, i.e.

$$m(z) = \prod_{n=1}^{N} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}, \quad |z| < 1$$

where $1 \leqslant N \leqslant \infty$.

(We allow a finite number of the z_n to be zero, in which case the factors corresponding to $\dfrac{\bar{z}_n}{|z_n|} \dfrac{z_n - z}{1 - \bar{z}_n z}$ are simply replaced by z.)

Since $S(m)$ has a lower triangular form, Lemma 2.1 suggests how to obtain a labelling of $\mathrm{CID}_m(A)$ using choice sequences.

So, a sequence of contractions $\{\Gamma_n\}_{n=1}^N$ is called an *(A, m)-choice sequence* if $\Gamma_1 : \mathscr{R}_{A,z_1} \to \mathscr{R}^{A,z_1}$, $\Gamma_n : \mathscr{D}_{\Gamma_{n-1}} \to \mathscr{D}_{\Gamma^*_{n-1}}$, $n \geq 2$.

Let:

$$K(m) = H \oplus \bigoplus_{n=1}^{N} \mathscr{D}_T$$

and for $n \geq 1$ define:

$$m'_n = \frac{\bar{z}_n}{|z_n|}\frac{z_n - z}{1 - \bar{z}_n z}, \quad m_n = m'_n m_{n-1}, \quad m_0 = 1,$$

$$H_n = H_{n-1} \oplus \mathscr{D}_T \subseteq K(m), \quad H_0 = H,$$

$$P_n = P_{H_n}^{K(m)}, \quad P_0 = P = P_H^{K(m)},$$

$$T_n \in \mathscr{L}(H_n), \quad T_0 = T,$$

$$T_n = \begin{pmatrix} T_{n-1} & 0 \\ (1 - |z_n|^2)^{1/2}\omega_{n-1}D_{T_{n-1}} & z_n \end{pmatrix}$$

where $\omega_n : \mathscr{D}_{T_n} \to \mathscr{D}_T$ is the sequence of unitary operators defined by:

$$\omega_n D_{T_n}(h_{n-1} \oplus d) = \bar{z}_n \omega_{n-1} D_{T_{n-1}} h_{n-1} - (1 - |z_n|^2)^{1/2} d, \tag{3.1}$$

$h_{n-1} \in H_{n-1}$, $d \in \mathscr{D}_T$, $\omega_0 = 1_{\mathscr{D}_T}$.

Since for any $n \geq 1$, T_n is the minimal uniform m'_n-Jordan dilation of T_{n-1}, there exists $T_N \in \mathscr{L}(K(m))$ such that $P'_n T_N = T_n P_n$, $n \geq 1$, and T_N is the minimal uniform m-Jordan dilation of T.

The notations $K'(m)$, K'_n, P'_n, T'_n, T'_N are now clear.

Let $A_N \in \mathrm{CID}_m(A)$ and define:

$$A_n = P'_n A_N | H_n \quad n \geq 1.$$

By the discussion preceding Theorem 2.3, for any $n \geq 1$, $A_n \in \mathrm{CID}_{m'_n}(A_{n-1})$

Using Lemma 2.1, there exists a contraction $\Gamma_1 : \mathscr{R}_{A,z_1} \to \mathscr{R}^{A,z_1}$ such that:

$$A_1 = \begin{pmatrix} A & 0 \\ X_1 D_A & Y_1 \end{pmatrix}$$

where $X_1 : \mathscr{D}_A \to \mathscr{D}_{T'}$, $Y_1 : \mathscr{D}_T \to \mathscr{D}_{T'}$ are given by (2.2).

We set:

$$(3.2)\qquad \begin{cases} \sigma_0 = (1 - q^A)\sigma_{A,z_1} p_{A,z_1}|\mathscr{D}_A, & \tilde{\sigma}_0 = (1 - q^A)\sigma_{A,z_1} p_{A,z_1}|\mathscr{D}_T \\ \sigma_0' = q^A \sigma_{A,z_1} p_{A,z_1}|\mathscr{D}_A, & \tilde{\sigma}_0' = q^A \sigma_{A,z_1} p_{A,z_1}|\mathscr{D}_T, \end{cases}$$

$$(3.3)\qquad \begin{cases} p_0 = (1 - p_{A,z_1})|\mathscr{D}_A, & \tilde{p}_0 = (1 - p_{A,z_1})|\mathscr{D}_T \\ p_0' = (1 - p^{A,z_1})|\mathscr{D}_A, & \tilde{p}_0' = (1 - p^{A,z_1})|\mathscr{D}_{T'}, \end{cases}$$

$$\sigma^{(0)} = \sigma_0 \oplus \tilde{\sigma}_0,$$

$$p^{(0)} = p_0 \oplus \tilde{p}_0.$$

Then as in [2], Lemma 2.2 the operator:

$$\Omega_1 : \mathscr{D}_{A_1} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1},$$

given by:

$$(3.4)\qquad \Omega_1 D_{A_1} = ((\sigma^{(0)} + p_0'^{*}\Gamma_1 p^{(0)}) \oplus D_{\Gamma_1} p^{(0)})(D_A P \oplus 1 - P)$$

is unitary.

The key step in this analysis is the following:

LEMMA 3.1. *There exists the unitary operators:*

$$W_1 : \mathscr{R}_{A_1,z_2} \to \mathscr{D}_{\Gamma_2},$$

$$\tilde{W}_1 : \mathscr{R}^{A_1,z_2} \to \mathscr{D}_{\Gamma_1^*}.$$

Proof. Let:

$$(3.5)\qquad \begin{cases} b_1 = \sigma_0 + p_0'^{*}\Gamma_1 p_0, \\ b_1' = \sigma_0' + \tilde{p}_0'^{*}\Gamma_1 p_0, \\ \tilde{b}_1 = \tilde{\sigma}_0 + p_0'^{*}\Gamma_1 \tilde{p}_0, \\ x_1 = (1 - \bar{z}_1 z_2 + (z_1 - z_2) b_1)^{-1} \end{cases}$$

For any $h \in H$, $d \in \mathscr{D}_T$, let $h_1 = h \oplus d$, $\tilde{h}_1 = (1 - \bar{z}_1 T)^{-1} h \oplus 0 \in H_1$.

Then:

$$\Omega_1 D_{A_1}(T_1 - z_2)(h_1 + \bar{z}_1(T_1 - z_1)\tilde{h}_1) = (z_1 - z_2)\Omega_1 D_{A_1} h_1 +$$

$$+ \Omega_1 D_{A_1}(T_1 - z_1)(h_1 + \bar{z}_1(T_1 - z_2)\tilde{h}_1) =$$

$$= \begin{pmatrix} x_1^{-1} D_A h + (z_1 - z_2) b_1 d \\ (z_1 - z_2) D_{\Gamma_1} p^{(0)}(D_A h \oplus d) \end{pmatrix}$$

and

$$\omega_1 D_{T_1}(h_1 + \bar{z}_1(T_1 - z_1)\tilde{h}_1) = \bar{z}_1 D_T(h + \bar{z}_1(T - z_1)(1 - \bar{z}_1 T)^{-1} h) -$$

$$-(1 - |z_1|^2)^{1/2}(d + \bar{z}_1(1 - |z_1|^2)^{1/2} D_T(1 - \bar{z}_1 T)^{-1} h) = -(1 - |z_1|^2)^{1/2} d$$

where we used in order (3.1), (1.1), (1.4) and (3.2)–(3.5).

Therefore:

$$(\Omega_1 \oplus \omega_1)\mathscr{R}_{A_1, z_2} = \{\Omega_1 D_{A_1}(T_1 - z_2) h_1 \oplus (1 - |z_2|^2)^{1/2} \omega_1 D_{T_1} h_1 ;\ h_1 \in H_1\}^- =$$

$$= \left\{ \begin{pmatrix} x_1^{-1} d_A + (z_1 - z_2) b_1 d \\ (z_1 - z_2) D_{\Gamma_1} p^{(0)}(d_A \oplus d) \\ -(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2} d \end{pmatrix} ; d_A \in \mathscr{D}_A,\ d \in \mathscr{D}_T \right\}.$$

If $r \in \mathscr{R}_{A_1, z_2}$ and $(\Omega_1 \oplus \omega_1) = d_A \oplus d_{\Gamma_1} \oplus d$, $(d_A \in \mathscr{D}_A,\ d_{\Gamma_1} \in \mathscr{D}_{\Gamma_1},\ d \in \mathscr{D}_T)$ then using (3.6) it is easy to see that:

$$(3.7) \qquad \begin{cases} d_A = -(\bar{z}_1 - \bar{z}_2) x_1^* p_0^* D_{\Gamma_1} d_{\Gamma_1} \\ d = \dfrac{-(\bar{z}_1 - \bar{z}_2)}{(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}}((\bar{z}_1 - \bar{z}_2)\tilde{b}_1^* x_1^* p_0^* D_{\Gamma_1} d_{\Gamma_1} - \tilde{p}_0^* D_{\Gamma_1} d_{\Gamma_1}). \end{cases}$$

So:

$$(\Omega_1 \oplus \omega_1)\mathscr{R}_{A_1, z_2} =$$

$$(3.8) \qquad = \left\{ \begin{pmatrix} -(\bar{z}_1 - \bar{z}_2) x_1^* p_0^* D_{\Gamma_1} d_{\Gamma_1} \\ d_{\Gamma_1} \\ \dfrac{-(\bar{z}_1 - \bar{z}_2)}{(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}}((\bar{z}_1 - \bar{z}_2)\tilde{b}_1^* x_1^* p_0^* D_{\Gamma_1} d_{\Gamma_1} - \tilde{p}_0^* D_{\Gamma_1} d_{\Gamma_1} \end{pmatrix} ; d_{\Gamma_1} \in \mathscr{D}_{\Gamma_1} \right\}.$$

Setting:

$$a_1 = \Gamma_1 + (\bar{z}_1 - \bar{z}_2)D_{\Gamma_1^*}p_0' x_1^* p_0^* D_{\Gamma_1}$$

$$y_1 = (1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}(|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 a_1^* a_1)^{-1/2}$$

it is easy to see that the operator:

$$W_1 : \mathscr{R}_{A_1, z_2} \to \mathscr{D}_{\Gamma_1}$$

given by:

$$\tilde{W}_1(\Omega_1^* \oplus \omega_1^*)(d_A \oplus d_{\Gamma_1} \oplus d) = y_1^{-1} d_{\Gamma_1}, \quad d_{\Gamma_1} \in \mathscr{D}_{\Gamma_1}$$

is unitary (where d_A and d are given by (3.7)).

Similarly, one can prove that:

$$(\Omega_1 \oplus \omega_1')\mathscr{R}^{A_1, z_2} =$$

$$= \left\{ \begin{pmatrix} x_1 p_0'^* D_{\Gamma_1^*} d_{\Gamma_1^*} \\ -\dfrac{1}{(1 - \bar{z}_1 z_2)}((z_1 - z_2)D_{\Gamma_1} p_0 x_1 p_0'^* D_{\Gamma_1^*} d_{\Gamma_1^*} + \Gamma_1^* d_{\Gamma_1^*}) \\ \dfrac{-1}{(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}}(-(z_1 - z_2)b_1' x_1 p_0'^* D_{\Gamma_1^*} d_{\Gamma_1^*} + \tilde{p}_0'^* D_{\Gamma_1^*} d_{\Gamma_1^*}) \end{pmatrix} ; d_{\Gamma_1^*} \in \mathscr{D}_{\Gamma_1^*} \right\}$$

and the operator:

$$\tilde{W}_1 : \mathscr{R}^{A_1, z_2} \to \mathscr{D}_{\Gamma_1^*}$$

given by:

$$\tilde{W}_1(\Omega_1^* \oplus \omega_1^*)(x_1 p_0'^* D_{\Gamma_1^*} d_{\Gamma_1^*} \oplus * \oplus *) = \tilde{y}_1^{-1} d_{\Gamma_1^*}, \quad d_{\Gamma_1^*} \in \mathscr{D}_{\Gamma_1^*}$$

is unitary, where:

$$\tilde{y}_1 = (1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}|1 - \bar{z}_1 z_2|(|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 a_1 a_1^*)^{-1/2}.$$

Suppose now that we defined up to $1 \leqslant n < N$ the contractions:

$$\Gamma_k : \mathscr{D}_{\Gamma_{k-1}} \to \mathscr{D}_{\Gamma_{k-1}^*} \qquad 2 \leqslant k \leqslant n - 1$$

and the unitary operators:

$$\Omega_k: \mathscr{D}_{A_k} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_k}.$$

$$W_k: \mathscr{R}_{A_k, z_{k+1}} \to \mathscr{D}_{\Gamma_k},$$

$$\tilde{W}_k: \mathscr{R}^{A_k, z_{k+1}} \to \mathscr{D}_{\Gamma_k^*}, \quad k = \overline{1, (n-1)}.$$

For $k = \overline{1, (n-1)}$ we set:

$$(3.9)_k \qquad \begin{cases} \tilde{\sigma}_k = \Omega_k(1 - q^{A_k})\sigma_{A_k, z_{k+1}} p_{A_k, z_{k+1}} \omega_k^* \\ \sigma_k = \Omega_k(1 - q^{A_k})\sigma_{A_k, z_{k+1}} p_{A_k, z_{k+1}} \Omega_k^* \\ \sigma'_k = \omega'_k q^{A_k} \sigma_{A_k, z_{k+1}} p_{A_k, z_{k+1}} \Omega_k^* \\ \tilde{\sigma}'_k = \omega'_k q^{A_k} \sigma_{A_k, z_{k+1}} p_{A_k, z_{k+1}} \omega_k^*, \end{cases}$$

$$(3.10)_k \qquad \begin{cases} p_k = W_k(1 - p_{A_k, z_{k+1}})\Omega_k^* \\ \tilde{p}_k = W_k(1 - p_{A_k, z_{k+1}})\omega_k^* \\ p'_k = \tilde{W}_k(1 - p^{A_k, z_{k+1}})\Omega_k^* \\ p'_k = \tilde{W}_k(1 - p^{A_k, z_{k+1}})\omega_k'^*. \end{cases}$$

By Lemma 2.1, there exists a contraction $\Gamma_1(A_{n-1}, A_n): \mathscr{R}_{A_{n-1}, z_n} \to \mathscr{R}^{A_{n-1}, z_n}$ such that:

$$(3.11)_n \qquad A_n = \begin{pmatrix} A_{n-1} & 0 \\ X_n \Omega_{n-1} D_{A_{n-1}} & Y_n \end{pmatrix}$$

where:

$$X_n = \omega'_{n-1} q^{A_{n-1}} (\sigma_{A_{n-1}, z_n} p_{A_{n-1}, z_n} + \Gamma_1(A_{n-1}, A_n)(1 - p_{A_{n-1}, z_n}))\Omega_{n-1}^*,$$

$$Y_n = \omega'_{n-1} q^{A_{n-1}} (\sigma_{A_{n-1}, z_n} p_{A_{n-1}, z_n} + \Gamma_1(A_{n-1}, A_n)(1 - p_{A_{n-1}, z_n}))\omega_{n-1}^*.$$

Setting $\Gamma_n = \tilde{W}_{n-1}\Gamma_1(A_{n-1}, A_n)W_n^*: \mathscr{D}_{\Gamma_{n-1}} \to \mathscr{D}_{\Gamma_{n-1}^*}$ we obtain:

$$(3.12)_n \qquad \begin{cases} X_n = (\sigma'_{n-1} + \tilde{p}'^*_{n-1}\Gamma_n p_{n-1}) \\ Y_n = (\tilde{\sigma}'_{n-1} + \tilde{p}'^*_{n-1}\Gamma_n \tilde{p}_{n-1}). \end{cases}$$

Again by Lemma 2.2 in [2] the operator:

$$\Omega_n : \mathscr{D}_{A_n} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}$$

given by:

$$\Omega_n D_{A_n} = ((\sigma^{(n-1)} + p'^{*}_{n-1}\Gamma_n p^{(n-1)}) \oplus$$

(3.13)$_n$

$$\oplus D_{\Gamma_n} p^{(n-1)})(\Omega_{n-1} D_{A_{n-1}} P_{n-1} \oplus 1 - P_{n-1})|H_n$$

is unitary, where

$$\sigma^{(n-1)} := \sigma_{n-1} \oplus \tilde{\sigma}_{n-1}, \quad p^{(n-1)} = p_{n-1} \oplus \tilde{p}_{n-1}, \quad \sigma'^{(n-1)} = \sigma'_{n-1} \oplus \tilde{\sigma}'_{n-1}.$$

Then, as in Lemma 3.1 one can obtain that:

$$(\Omega_n \oplus \omega_n)\mathscr{R}_{A_n, z_{n+1}} = \{d_n \oplus d_{\Gamma_n} \oplus d;\ d_n = -(\bar{z}_n - \bar{z}_{n+1}) x_n p^{*}_{n-1} D_{\Gamma_n} d_{\Gamma'_n},$$

(3.14)$_n$

$$d = (z_n - z_{n+1})\alpha_n^{-1}(\tilde{p}_{n-1} D_{\Gamma_n} d_{\Gamma_n} - (\bar{z}_n - \bar{z}_{n+1})\tilde{b}^{*}_n x^{*}_n p^{*}_{n-1} D_{\Gamma_n} d_{\Gamma_n}), \quad d_{\Gamma_n} \in \mathscr{D}_{\Gamma_n}\}$$

$$(\Omega_n \oplus \omega'_n)\mathscr{R}^{A_n, z_{n-1}} = \Big\{d_n \oplus d_{\Gamma_n} \oplus d';\ d_n = x_n p'^{*}_{n-1} D_{\Gamma^{*}_n} d_{\Gamma^{*}_n},\ d_{\Gamma_n} = -\frac{1}{1 - \bar{z}_n z_{n+1}} a^{*}_n d_{\Gamma^{*}_n},$$

(3.15)$_n$

$$d' = \alpha_n^{-1}((z_n - z_{n+1}) b'_n x_n p'^{*}_{n-1} D_{\Gamma^{*}_n} d_{\Gamma^{*}_n} - \tilde{p}'^{*}_{n-1} D_{\Gamma^{*}_n} d_{\Gamma^{*}_n}), \quad d_{\Gamma^{*}_n} \in \mathscr{D}_{\Gamma^{*}_n}\Big\}$$

and the operators:

$$W_n : \mathscr{R}_{A_n, z_{n+1}} \to \mathscr{D}_{\Gamma_n}$$

$$\tilde{W}_n : \mathscr{R}^{A_n, z_{n-1}} \to \mathscr{D}_{\Gamma^{*}_n}$$

$$W_n(\Omega^{*}_n \oplus \omega^{*}_n)(* \oplus d_{\Gamma_n} \oplus *) = y_n^{-1} d_{\Gamma_n}, \quad d_{\Gamma_n} \in \mathscr{D}_{\Gamma_n}$$

$$\tilde{W}_n(\Omega^{*}_n \oplus \omega'^{*}_n)(x_n p'^{*}_{n-1} D_{\Gamma^{*}_n} d_{\Gamma^{*}_n} \oplus * \oplus *) = \tilde{y}_n^{-1} d_{\Gamma^{*}_n} d_{\Gamma^{*}_n} \in \mathscr{D}_{\Gamma^{*}_n}$$

are unitary where:

$$b_n = \sigma_{n-1} + p'^*_{n-1}\Gamma_n p_{n-1}$$

$$b'_n = \sigma'_{n-1} + \tilde{p}'^*_{n-1}\Gamma_n p_{n-1}$$

$$\tilde{b}_n = \tilde{\sigma}_{n-1} + p'^*_{n-1}\Gamma_n \tilde{p}_{n-1}$$

$$\tilde{b}'_n = \tilde{\sigma}'_{n-1} + \tilde{p}'^*_{n-1}\Gamma_n \tilde{p}_{n-1}$$

$$x_n = (1 - \bar{z}_n z_{n+1} + (z_n - z_{n+1})b_n)^{-1}$$

$$\alpha_n = (1 - |z_n|^2)^{1/2}(1 - |z_{n+1}|^2)^{1/2}$$

$$a_n = \Gamma_n + (\bar{z}_n - \bar{z}_{n+1})D_{\Gamma_n^*} p'_{n-1} x_n^* p_{n-1}^* D_{\Gamma_n}$$

$$y_n = \alpha_n(|1 - \bar{z}_n z_{n+1}|^2 - |z_n - z_{n+1}|^2 a_n^* a_n)^{-1/2}$$

$$\tilde{y}_n = \alpha_n |1 - \bar{z}_n z_{n+1}|(|1 - \bar{z}_n z_{n+1}|^2 - |z_n - z_{n+1}|^2 a_n a_n^*)^{-1/2}.$$

Using $(3.9)_{n-1}$, $(3.10)_{n-1}$ and $(3.12)_n - (3.15)_n$ one can obtain:

$$(3.16)_n \qquad \begin{cases} p_n = (p_i^{(n)})_{i=1,2} \\ \tilde{p}_n = (z_n - z_{n+1})\alpha_n^{-1} y_n D_{\Gamma_n} p^{(n-1)}(1 - (z_n - z_{n+1})x_n \tilde{b}_n) \\ p'_n = (p_i'^{(n)})_{i=1,2} \\ \tilde{p}'_n = \alpha_n^{-1}\tilde{y}_n D_{\Gamma_n^*} p'^{(n-1)}((\bar{z}_n - \bar{z}_{n+1})x_n^* b_n'^* - 1) \end{cases}$$

$$(3.17)_n \qquad \begin{cases} \sigma_n = (\sigma_{ij}^{(n)})_{i,j=1,2} \\ \tilde{\sigma}_n = (\sigma_i^{(n)})_{i=1,2} \\ \sigma'_n = (\sigma_i'^{(n)})_{i=1,2} \\ \tilde{\sigma}'_n = (\sigma'^{(n-1)} + \tilde{p}'^*_{n-1}\Gamma_n p^{(n-1)})(-(z_n - z_{n+1})x_n \tilde{b}_n + 1) + \\ + (\bar{z}_n - \bar{z}_{n+1})\alpha_n^{-1}((z_n - z_{n+1})b'_n x_n p'^*_{n-1} - \tilde{p}'^*_{n-1})D_{\Gamma_n^*} a_n y_n \tilde{p}_n \end{cases}$$

where:

$$p_1^{(n)} = -(z_n - z_{n+1})y_n D_{\Gamma_n} p_{n-1} x_n$$

$$p_2^{(n)} = y_n$$

$$p_1'^{(n)} = \tilde{y}_n D_{\Gamma_n^*} p'_{n-1} x_n^*$$

$$p_2'^{(n)} = -\frac{1}{1 - \bar{z}_n z_{n+1}} \tilde{y}_n D_{\Gamma_n^*} a_n$$

$$\sigma_{11}^{(n)} = ((\bar{z}_n - \bar{z}_{n+1}) + (1 - z_n \bar{z}_{n+1}) b_n) x_n + (\bar{z}_n - \bar{z}_{n+1}) x_n p'^*_{n-1} D_{\Gamma_n^*} a_n y_n p_1^{(n)}$$

$$\sigma_{12}^{(n)} = (\bar{z}_n - \bar{z}_{n+1}) x_n p'^*_{n-1} D_{\Gamma_n^*} a_n y_n p_2^{(n)}$$

$$\sigma_{21}^{(n)} = (1 - z_n \bar{z}_{n+1}) D_{\Gamma_n} p_{n-1} x_n + (\bar{z}_n - \bar{z}_{n+1})(1 - z_n \bar{z}_{n+1}) \alpha_n^{-2} (1 - a_n^* a_n) y_n p_1^{(n)}$$

$$\sigma_{22}^{(n)} = (\bar{z}_n - \bar{z}_{n+1})(1 - z_n \bar{z}_{n+1}) \alpha_n^{-2} (1 - a_n^* a_n) y_n p_2^{(n)}$$

$$\tilde{\sigma}_1^{(n)} = -\alpha_n x_n \tilde{b}_n + (\bar{z}_n - \bar{z}_{n+1}) x_n p'^*_{n-1} D_{\Gamma_n^*} a_n y_n \tilde{p}_n$$

$$\tilde{\sigma}_2^{(n)} = (1 - z_n \bar{z}_{n+1}) \alpha_n^{-1} D_{\Gamma_n} p^{(n-1)} ((z_n - z_{n+1}) x_n \tilde{b}_n - 1) +$$

$$+ (\bar{z}_n - \bar{z}_{n+1})(1 - z_n \bar{z}_{n+1}) \alpha_n^{-2} (1 - a_n^* a_n) y_n \tilde{p}_n$$

$$\sigma_1'^{(n)} = -\alpha_n b'_n x_n + (\bar{z}_n - \bar{z}_{n+1}) \alpha_n^{-1} ((z_n - z_{n+1}) b'_n x_n p'^*_{n-1} - \tilde{p}'^*_{n-1}) D_{\Gamma_n^*} a_n y_n p_1^{(n)}$$

$$\sigma_2'^{(n)} = (\bar{z}_n - \bar{z}_{n+1}) \alpha_n^{-1} ((z_n - z_{n+1}) b'_n x_n p'^*_{n-1} - \tilde{p}'^*_{n-1}) D_{\Gamma_n^*} a_n y_n p_2^{(n)}$$

So we obtain:

THEOREM 3.2. *The formulae* (3.2)–(3.4), $(3.12)_n$, $(3.13)_n$, $(3.16)_n$, $(3.17)_n$ *give a one-to-one correspondence between* $\mathrm{CID}_m(A)$ *onto the set of* (A, m) *choice sequences.*

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