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STABILITY OF THE INDEX OF A FREDHOLM SYMMETRICAL PAIR

C.-G. AMBROZIE

1. INTRODUCTION

The aim of this work is to prove the stability under small perturbations of the index of a Fredholm symmetrical pair (see Definition 2.1), which is an object of the following form:

$$S : \mathrm{D}(S) \subset X/X_0 \to Y/Y_0, \quad T : \mathrm{D}(T) \subset Y/Y_0 \to X/X_0$$

are closed linear operators such that

$$\mathrm{R}(S) \subset \mathrm{N}(T), \quad \mathrm{R}(T) \subset \mathrm{N}(S)\,.$$

$$\dim \mathrm{N}(T)/\mathrm{R}(S) < \infty, \quad \dim \mathrm{N}(S)/\mathrm{R}(T) < \infty.$$

The index of the pair (S,T) is defined by the equality

$$\mathrm{ind}(S,T) := \dim \mathrm{N}(S)/\mathrm{R}(T) - \dim \mathrm{N}(T)/\mathrm{R}(S) \tag{1.1}$$

where $\mathrm{R}(S), \mathrm{N}(S)$ mean the range and the null-space of S, respectively.

We consider X_0, X, Y_0, Y to be closed linear subspaces of a fixed Banach space. Both spaces and operators may be perturbed, with respect to the gap topology [4].

The main result of the present paper (see Theorem 3.1 below) asserts that the index of such a pair is locally constant (i.e. stable under small perturbations).

In this way, it is possible to give an affirmative answer to a problem raised by E. Albrecht and F.-H. Vasilescu in [1]. It was already known (see [2], Proposition 2.10) that the functions $(S,T) \to \dim \mathrm{N}(S)/\mathrm{R}(T)$, $(S,T) \to \dim \mathrm{N}(T)/\mathrm{R}(S)$ (defined on the set of all pairs (S,T) with $ST = 0$, $TS = 0$) are upper semicontinuous; so, the set of Fredholm symmetrical pairs is open (with respect to the gap topology) in that of symmetrical pairs (see Section 2 for precise definitions). Now we can also state the stability of the index.

The property proved here for pairs generalizes the theorem concerning the stability of the index of a complex of Banach spaces (see [1], [2], [5]), as we shall see in Section 4. We obtain some simplifications in the study of semi-Fredholm complexes (the proof of the present results is quite elementary).

With some exceptions, we follow the notation and the terminology from [2] and [4].

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2. NOTATIONS AND DEFINITIONS

We denote by Λ the field of scalars ($\Lambda = \mathbf{R}$ or $\mathbf{C}$).

Unlike in [2] or [4], the direct sum of two Banach spaces X and Y, which is denoted by $X \times Y$, will be endowed with the norm $\|(x,y)\| = \|x\| + \|y\|$ for all $x \in X$ and $y \in Y$.

The family of all closed linear subspaces of a Banach space X will be denoted by $G(X)$.

If $Z \in G(X)$ and $x \in X$, then the symbol $d(x, Z)$ stands for the *distance* from x to Z.

If $Y, Z \in G(X)$, then

$$\delta(Y,Z) := \sup\{d(y,Z)\,;\ y \in Y,\ \|y\| \leqslant 1\},$$

$$\hat{\delta}(Y,Z) := \max\{\delta(Y,Z), \delta(Z,Y)\}.$$

The mapping $\hat{\delta}$, which is equivalent to the Pompeiu-Hausdorff metric on the set of all unit balls of the spaces from $G(X)$, defines the *gap topology* of $G(X)$ [4].

If Y, Z are Banach spaces, then we shall denote by $C(Y,Z)$ the set of all closed linear operators S defined on linear subspaces of Y, with values in Z. The domain of definition of such an S will be designated by $\mathrm{D}(S)$.

Consider the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and let $X_0, X \in G(\mathcal{X})$ and $Y_0, Y \in G(\mathcal{Y})$ be such that $X_0 \subset X$ and $Y_0 \subset Y$. For every $S \in C(X/X_0, Y/Y_0)$, we define:

$$\gamma(S) := \text{the reduced minimum modulus of } S \text{ [4]};$$

$$N_0(S) := \{x \in X\,;\ x + X_0 \in \mathrm{N}(S)\};$$

$$R_0(S) := \{y \in Y\ ;\ y + Y_0 \in \mathrm{R}(S)\};$$

$$G_0(S) := \{(x, y) \in X \times Y\ ;\ x + X_0 \in \mathrm{D}(S),\ S(x + X_0) = y + Y_0\};$$

$$D_0(S) := \{x \in X\ ;\ x + X_0 \in \mathrm{D}(S)\}.$$

If $\tilde{X}_0, \tilde{X} \in G(\mathcal{X})$, $\tilde{X}_0 \subset \tilde{X}$ and $\tilde{Y}_0, \tilde{Y} \in G(\mathcal{Y})$, $\tilde{Y}_0 \subset \tilde{Y}$ then, for any $\tilde{S} \in$ $\in C(\tilde{X}/\tilde{X}_0, \tilde{Y}/\tilde{Y}_0)$, we define

$$\delta_0(S, \tilde{S}) := \delta(G_0(S), G_0(\tilde{S})),$$

$$\hat{\delta}_0(S, \tilde{S}) := \max\{\delta_0(S, \tilde{S}), \delta_0(\tilde{S}, S)\} \qquad ([1], [2]).$$

2.1. Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be fixed Banach spaces and let $X_0, X \in$ $\in G(\mathcal{X})$, $Y_0, Y \in G(\mathcal{Y})$, $X_0 \subset X$, $Y_0 \subset Y$, $S \in C(X/X_0, Y/Y_0)$, $T \in C(Y/Y_0, X/X_0)$, $\mathrm{R}(S) \subset \mathrm{N}(T)$, $\mathrm{R}(T) \subset \mathrm{N}(S)$.

A pair of operators (S, T) as before will be called a *symmetrical pair*. Let $\partial_S(\mathcal{X}, \mathcal{Y})$ denote the family of all symmetrical pairs.

The mapping $\hat{\delta}_0$ defines the gap topology on the set

$$\{S \in C(X/X_0, Y/Y_0)\ ;\ X_0, X \in G(\mathcal{X}),\ Y_0, Y \in G(\mathcal{Y}),\ X_0 \subset X,\ Y_0 \subset Y\}.$$

Indeed, for such an S, $G_0(S) \in G(X \times Y) \subset G(\mathcal{X} \times \mathcal{Y})$. For another one, say $\tilde{S}$, $G_0(\tilde{S}) \in G(\mathcal{X} \times \mathcal{Y})$ and $\hat{\delta}_0(S, \tilde{S})$ is computed in $\mathcal{X} \times \mathcal{Y}$. Therefore, $\hat{\delta}_0$ defines a topology on $\partial_S(\mathcal{X}, \mathcal{Y})$.

We say that $(S, T) \in \partial_S(\mathcal{X}, \mathcal{Y})$ is *Fredholm* if $\dim \mathrm{N}(S)/\mathrm{R}(T)$, $\dim \mathrm{N}(T)/\mathrm{R}(S)$ are finite. In this case we define the *index* of (S, T) by (1.1).

3. A STABILITY RESULT

3.1. Theorem. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and let $(S, T) \in \partial_S(\mathcal{X}, \mathcal{Y})$ be a Fredholm pair. Then there exists an $\varepsilon > 0$ such that if $(\tilde{S}, \tilde{T}) \in \partial_S(\mathcal{X}, \mathcal{Y})$ and $\hat{\delta}_0(S, \tilde{S})$, $\hat{\delta}_0(T, \tilde{T}) < \varepsilon$, then:*

(a) *$(\tilde{S}, \tilde{T})$ is Fredholm;*

(b) $\begin{cases} \dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) \leqslant \dim \mathrm{N}(S)/\mathrm{R}(T), \\ \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) \leqslant \dim \mathrm{N}(T)/\mathrm{R}(S); \end{cases}$

(c) $\mathrm{ind}(\tilde{S}, \tilde{T}) = \mathrm{ind}(S, T)$.

Proof. Only for the purpose of the proof, we use the following terminology: if m, n are integers, $m, n \geqslant 0$, $(S, T) \in \partial_S(\mathcal{X}, \mathcal{Y})$ and $\dim \mathrm{N}(S)/\mathrm{R}(T) = m$, $\dim \mathrm{N}(T)/\mathrm{R}(S) =$

$= n$, then we call (S,T) an (m,n)-*pair* on $(\mathcal{X},\mathcal{Y})$. We have to prove the next assertion, depending on $m,n \geqslant 0$:

$$(A_{m,n})\begin{cases} \text{Let } m,\ n \geqslant 0 \text{ be integers. Then for every pair of Banach spaces } \mathcal{E},\ \mathcal{F} \text{ and} \\ \text{for each } (m,n)\text{-pair } (A,B) \text{ on } (\mathcal{E},\mathcal{F}) \text{ there exists an } \varepsilon > 0 \text{ such that for all} \\ (\tilde{A},\tilde{B}) \in \partial_S(\mathcal{E},\mathcal{F}) \text{ with } \hat{\delta}_0(A,\tilde{A}),\ \hat{\delta}_0(B,\tilde{B}) < \varepsilon, \text{ we have:} \\ \quad \text{(a) } (\tilde{A},\tilde{B}) \text{ is Fredholm;} \\ \quad \text{(b) } \begin{cases} \dim \mathrm{N}(\tilde{A})/\mathrm{R}(\tilde{B}) \leqslant \dim \mathrm{N}(A)/\mathrm{R}(B), \\ \dim \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}) \leqslant \dim \mathrm{N}(B)/\mathrm{R}(A); \end{cases} \\ \quad \text{(c) } \mathrm{ind}(\tilde{A},\tilde{B}) = \mathrm{ind}(A,B). \end{cases}$$

In order to prove $(A_{m,n})$, we shall see that

$$(A_{m,n}) \Rightarrow (A_{m,n+1}).$$

Thanks to the symmetry, we shall also have

$$(A_{m,n}) \Rightarrow (A_{m+1,n}).$$

The previous implications enables as to reduce the problem of proving $(A_{m,n})$ for arbitrary m,n to the case $m = 0,\ n = 0$. Let us note that $(A_{0,0})$ is true as a consequence of Corollary 2.12 from [2] (see also Remark 3.2). Thus, the theorem will be proved, once we have $(A_{m,n}) \Rightarrow (A_{m,n+1})$. It remains to show this implication.

Let $m,n \geqslant 0$ be integers and suppose that $(A_{m,n})$ is true.

Let $\mathcal{X},\mathcal{Y}$ be Banach spaces and let (S,T) be an $(m,n+1)$-pair on $(\mathcal{X},\mathcal{Y})$, $S \in C(X/X_0, Y/Y_0)$, $T \in C(Y/Y_0, X/X_0)$.

Let N be a subspace of Y/Y_0 such that:

$$\mathrm{N}(T) = \mathrm{R}(S) + N,\quad \dim N = n+1,\quad \mathrm{R}(S) \cap N = \{0\}.$$

We can choose vectors $n_0 \in N$ and $y_1 \in Y$ such that:

$$n_0 = y_1 + Y_0,\quad \|n_0\| = 1,\quad \|y_1\| \leqslant 2.$$

Set:

$$Y_1 := \Lambda y_1,\quad \mathcal{E} := \mathcal{X} \times Y_1,\quad \mathcal{F} := \mathcal{Y}.$$

We shall define $(A,B) \in \partial_S(\mathcal{E},\mathcal{F})$ an (m,n)-pair (therefore $\mathrm{ind}(A,B) = \mathrm{ind}(S,T) + 1$), as follows:

$$\mathrm{D}(A) := (D_0(S) \times Y_1)/(X_0 \times \{0\}) \subset (X \times Y_1)/(X_0 \times \{0\}),$$

$$A : \mathrm{D}(A) \to Y/Y_0,$$

$$A(x + X_0, \lambda y_1) := S(x + X_0) + \lambda n_0, \quad x \in D_0(S),\ \lambda \in \Lambda;$$

$$\mathrm{D}(B) := \mathrm{D}(T) \subset Y/Y_0,$$

$$B : \mathrm{D}(B) \to (X \times Y_1)/(X_0 \times \{0\}),$$

$$B(y + Y_0) := (T(y + Y_0), 0), \quad y \in D_0(T).$$

It is easy to verify that $(A, B) \in \partial_S(\mathcal{E}, \mathcal{F})$ and $\dim \mathrm{N}(A)/\mathrm{R}(B) = m$, $\dim \mathrm{N}(B)/\mathrm{R}(A) = n$.

We assumed $(A_{m,n})$ to be true. Consequentely, there exists an $\varepsilon_0 > 0$ such that:

$$\text{(3.1)} \quad \begin{cases} \text{For every pair } (\tilde{A}, \tilde{B}) \in \partial_S(\mathcal{E}, \mathcal{F}) \text{ with } \hat{\delta}_0(A, \tilde{A}),\ \hat{\delta}_0(B, \tilde{B}) < \varepsilon_0, \\ \text{we have (a), (b) and (c) fulfilled.} \end{cases}$$

Let $\varepsilon > 0$ be such that

$$\text{(3.2)} \quad 8\varepsilon + 7\varepsilon^{\frac{1}{4}} < \varepsilon_0, \quad 3\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{4}} < 1.$$

For any $(\tilde{S}, \tilde{T}) \in \partial_S(\mathcal{X}, \mathcal{Y})$ with $\hat{\delta}_0(S, \tilde{S})$, $\hat{\delta}_0(T, \tilde{T}) < \varepsilon$, we shall find $(\tilde{A}, \tilde{B}) \in$ $\in \partial_S(\mathcal{E}, \mathcal{F})$ such that:

$$\text{(3.3)} \quad \hat{\delta}_0(A, \tilde{A}), \hat{\delta}_0(B, \tilde{B}) < \varepsilon_0,$$

$$\text{(3.4)} \quad \begin{gathered} \dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) \leqslant \dim \mathrm{N}(\tilde{A})/\mathrm{R}(\tilde{B}), \\ \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) \leqslant \dim \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}) + 1, \end{gathered}$$

$$\text{(3.5)} \quad \mathrm{ind}(\tilde{A}, \tilde{B}) = \mathrm{ind}(\tilde{S}, \tilde{T}) + 1.$$

From (3.1) and (3.3), we deduce that $(\tilde{A}, \tilde{B})$ is Fredholm and $\mathrm{ind}(\tilde{A}, \tilde{B}) =$ $= \mathrm{ind}(A, B)$; by (3.4), $(\tilde{S}, \tilde{T})$ is also Fredholm. We also know that $\mathrm{ind}(A, B) =$ $= \mathrm{ind}(S, T) + 1$. This will imply (together with (3.5)) that $\mathrm{ind}(\tilde{S}, \tilde{T}) = \mathrm{ind}(S, T)$. Therefore, $(A_{m,n})$ will follow.

It remains to construct a pair $(\tilde{A}, \tilde{B})$ depending on $(\tilde{S}, \tilde{T})$ and satisfying (3.3), (3.4) and (3.5). So let $(\tilde{S}, \tilde{T}) \in \partial_S(\mathcal{X}, \mathcal{Y})$ be with the property

$$\text{(3.6)} \quad \hat{\delta}_0(S, \tilde{S}),\ \hat{\delta}_0(T, \tilde{T}) < \varepsilon.$$

There are two posibilities:

(A) $d(y_1, N_0(\tilde{T})) \leqslant \hat{\delta}_0(T, \tilde{T})^{\frac{1}{2}}$;

(B) $d(y_1, N_0(\tilde{T})) > \hat{\delta}_0(T, \tilde{T})^{\frac{1}{2}}$.

For each of these cases, we shall give a construction for $(\tilde{A}, \tilde{B})$.

Case (A): $d(y_1, N_0(\tilde T)) \leqslant \hat\delta_0(T,\tilde T)^{\frac12}$.

There are $\tilde y_1 \in N_0(\tilde T)$ and $\tilde v \in \mathrm{N}(\tilde T)$ such that

$$\|y_1 - \tilde y_1\| \leqslant 2\hat\delta_0(T,\tilde T)^{\frac12} \cdot \tilde v = \tilde y_1 + \tilde Y_0. \tag{3.7}$$

Let us define $(\tilde A, \tilde B)$ as follows:

$$\mathrm{D}(\tilde A) := (D_0(\tilde S) \times Y_1)/(\tilde X_0 \times \{0\}) \subset (\tilde X \times Y_1)/(\tilde X_0 \times \{0\}),$$

$$\tilde A : \mathrm{D}(\tilde A) \to \tilde Y/\tilde Y_0,$$

$$\tilde A(\tilde x + \tilde X_0, \lambda y_1) := \tilde S(\tilde x + \tilde X_0) + \lambda \tilde v, \quad \tilde x \in D_0(\tilde S),\ \lambda \in \Lambda;$$

$$\mathrm{D}(\tilde B) := \mathrm{D}(\tilde T) \subset \tilde Y/\tilde Y_0,$$

$$\tilde B : \mathrm{D}(\tilde B) \to (\tilde X \times Y_1)/(\tilde X_0 \times \{0\}),$$

$$\tilde B(\tilde y + \tilde Y_0) := (\tilde T(\tilde y + \tilde Y_0), 0), \quad \tilde y \in D_0(\tilde T).$$

It is easy to verify that $(\tilde A, \tilde B) \in \partial_S(\mathcal{E}, \mathcal{F})$. Let us check that (3.3), (3.4), (3.5) are true for $(\tilde A, \tilde B)$ defined above.

$\underline{\delta_0(B, \tilde B) < \varepsilon_0}$: Let $(y, (x, 0)) \in G_0(B)$ be such that $\|y\| + \|x\| \leqslant 1$. Then $y \in D_0(T)$ and $T(y + Y_0) = x + X_0$, and thus $(y, x) \in G_0(T)$. There exists $(\tilde y, \tilde x) \in G_0(\tilde T)$ such that $\|y - \tilde y\| + \|x - \tilde x\| < \varepsilon < \varepsilon_0$, by (3.2). Then $(\tilde y, (\tilde x, 0)) \in G_0(\tilde B)$ and the assertion follows.

$\underline{\delta_0(\tilde B, B) < \varepsilon_0}$: The estimate follows as above.

$\underline{\delta_0(A, \tilde A) < \varepsilon_0}$: Let $((x, \lambda y_1), y) \in G_0(A)$ be such that $\|x\| + \|\lambda y_1\| + \|y\| \leqslant 1$. In particular, $|\lambda| \leqslant \|\lambda y_1\| \leqslant 1$. We have $x \in D_0(S)$ and $S(x + X_0) = y + Y_0 - \lambda v_0$. Therefore $(x, y - \lambda y_1) \in G_0(S)$ and $\|(x, y - \lambda y_1)\| \leqslant 1$. There exists $(\tilde x, \tilde z) \in G_0(\tilde S)$ such that $\|x - \tilde x\| + \|y - \lambda y_1 - \tilde z\| < \varepsilon$. Then $((\tilde x, \lambda y_1), \tilde z + \lambda \tilde y_1) \in G_0(\tilde A)$ and

$$\|x - \tilde x\| + \|y - \tilde z - \lambda \tilde y_1\| \leqslant \|x - \tilde x\| + \|y - \tilde z - \lambda y_1\| + |\lambda|\,\|y_1 - \tilde y_1\| <$$

$$< \varepsilon + 2\hat\delta_0(T,\tilde T)^{\frac12}|\lambda| \leqslant \varepsilon + 2\varepsilon^{\frac12} < \varepsilon_0,$$

by (3.7), (3.6) and (3.2).

$\underline{\delta_0(\tilde A, A) < \varepsilon_0}$: Let $((\tilde x, \lambda y_1), \tilde y) \in G_0(\tilde A)$ be such that $\|\tilde x\| + \|\lambda y_1\| + \|\tilde y\| \leqslant 1$. In particular, $|\lambda| \leqslant |\lambda|\,\|y_1\| \leqslant 1$. Note that $\tilde x \in D_0(\tilde S)$ and $\tilde S(\tilde x + \tilde X_0) = \tilde y + \tilde Y_0 - \lambda \tilde v$. Hence $(\tilde x, \tilde y - \lambda \tilde y_1) \in G_0(\tilde S)$ and

$$\|(\tilde x, \tilde y - \lambda \tilde y_1\| \leqslant \|\tilde x\| + \|\tilde y\| + |\lambda|\,\|y_1\| + |\lambda|\,\|\tilde y_1 - y_1\| \leqslant$$

$$\leqslant 1 + |\lambda| \cdot 2\hat\delta_0(T,\tilde T)^{\frac12} \leqslant 1 + |\lambda| \cdot 2\varepsilon^{\frac12} \leqslant 1 + 2\varepsilon^{\frac12} \leqslant 2.$$

There exists $(x,y) \in G_0(S)$ such that $\|\tilde{x}-x\| + \|\tilde{y}-\lambda\tilde{y}_1 - z\| < 2\varepsilon$. Then $((x,\lambda y_1), z+\lambda y_1) \in G_0(A)$ and

$$\|\tilde{x}-x\| + \|\tilde{y}-\lambda y_1 - z\| \leqslant \|\tilde{x}-x\| + \|\tilde{y}-\lambda\tilde{y}_1 - z\| + |\lambda|\,\|y_1-\tilde{y}_1\| <$$

$$< 2\varepsilon + 2\hat{\delta}_0(T,\tilde{T})^{\frac{1}{2}} < 2\varepsilon + 2\varepsilon^{\frac{1}{2}} < \varepsilon_0,$$

by (3.7), (3.6) and (3.2).

So we have checked (3.3). Now, we prove (3.4) and (3.5). A brief discussion shows us that

$$1 = \dim \mathrm{R}(\tilde{A})/\mathrm{R}(\tilde{S}) + \dim \mathrm{N}(\tilde{A})/(\mathrm{N}(\tilde{S}) \times \{0\})$$

(we can also apply a general result, namely Lemma 2.7 from [5], concerning the finite dimensional extensions).

If $\dim \mathrm{R}(\tilde{A})/\mathrm{R}(\tilde{S}) = 1$ and $\mathrm{N}(\tilde{A}) = \mathrm{N}(\tilde{S}) \times \{0\}$, then $\tilde{v} \notin \mathrm{R}(\tilde{S})$ (because $\mathrm{R}(\tilde{A}) \neq$ $\neq \mathrm{R}(\tilde{S})$). Note that

$$\dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) = \dim \mathrm{N}(\tilde{A})/\mathrm{R}(\tilde{B}) \leqslant \dim \mathrm{N}(A)/\mathrm{R}(B) = \dim \mathrm{N}(S)/\mathrm{R}(T)$$

and

$$\begin{aligned} \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) &= \dim \mathrm{N}(\tilde{T})/(\mathrm{R}(\tilde{S}) + \Lambda\tilde{v}) + 1 = \\ &= \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{A}) + 1 = \dim \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}) + 1 \leqslant \\ &\leqslant \dim \mathrm{N}(B)/\mathrm{R}(A) + 1 = \dim \mathrm{N}(T)/\mathrm{R}(S). \end{aligned}$$

Moreover

$$\begin{aligned} \mathrm{ind}(\tilde{A},\tilde{B}) &= \dim \mathrm{N}(\tilde{A})/(\mathrm{R}(\tilde{T}) \times \{0\}) - \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{A}) = \\ &= \dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) - \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{A}) = \\ &= \dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) - \dim \mathrm{N}(\tilde{T})/(\mathrm{R}(\tilde{S}) + \Lambda\tilde{v}) = \\ &= \mathrm{ind}(\tilde{S},\tilde{T}) + 1. \end{aligned}$$

If $\mathrm{R}(\tilde{A}) = \mathrm{R}(\tilde{S})$ and $\dim \mathrm{N}(\tilde{A})/(\mathrm{N}(\tilde{S}) \times \{0\}) = 1$, then $\tilde{v} \in \mathrm{R}(\tilde{S})$. We also have

$$\begin{aligned} &\dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) \leqslant \dim \mathrm{N}(\tilde{A})/\mathrm{R}(\tilde{B}) \leqslant \\ &\leqslant \dim \mathrm{N}(A)/\mathrm{R}(B) = \dim \mathrm{N}(S)/\mathrm{R}(T), \end{aligned}$$

$$\begin{aligned} &\dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) = \dim \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}) \leqslant \\ &\leqslant \dim \mathrm{N}(B)/\mathrm{R}(A) \leqslant \dim \mathrm{N}(T)/\mathrm{R}(S) \end{aligned}$$

and

$$\operatorname{ind}(\tilde{A},\tilde{B}) = \dim \mathrm{N}(\tilde{A})/(\mathrm{R}(\tilde{T})\times\{0\}) - \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) =$$

$$= \dim \mathrm{N}(\tilde{A})/(\mathrm{N}(\tilde{S})\times\{0\}) + \dim \mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T}) -$$

$$- \dim \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) = 1 + \operatorname{ind}(\tilde{S},\tilde{T}).$$

The discussion concerning the Case (A) is finished.

Case (B): $d(y_1, N_0(\tilde{T})) > \hat{\delta}_0(T,\tilde{T})^{\frac{1}{2}}$.

There are $\tilde{\psi} \in (\tilde{Y}/\tilde{Y}_0)^*$ and $\tilde{y}_2 \in D_0(\tilde{T})$ with:

$$\tilde{\psi}\Big|\mathrm{N}(\tilde{T}) = 0,\quad \tilde{\psi}(\tilde{y}_2 + \tilde{Y}_0) = 1,\quad \|\tilde{\psi}\| \leqslant d(\tilde{y}_2, N_0(\tilde{T}))^{-1},$$
$$d(\tilde{y}_2, N_0(\tilde{T})) > \hat{\delta}_0(T,\tilde{T})^{\frac{3}{4}},\quad \|y_1 - \tilde{y}_2\| < 3\hat{\delta}_0(T,\tilde{T}). \tag{3.8}$$

Let us prove (3.8). First of all, note that $\hat{\delta}_0(T,\tilde{T}) \neq 0$ (otherwise, we should have $y_1 \in N_0(T) = N_0(\tilde{T})$). Since $(y_1, 0) \in G_0(T)$ and $\|y_1\| \leqslant 2$, there exists $(\tilde{y}_2, \tilde{x}_2) \in$ $\in G_0(\tilde{T})$ with

$$\|y_1 - \tilde{y}_2\| + \|\tilde{x}_2\| < 3\hat{\delta}_0(T,\tilde{T}). \tag{3.9}$$

Let $\tilde{y} \in N_0(\tilde{T})$ be arbitrary. Then

$$\|\tilde{y}_2 - \tilde{y}\| \geqslant \|y_1 - \tilde{y}\| - \|\tilde{y}_2 - y_1\| >$$

$$> \|y_1 - \tilde{y}\| - 3\hat{\delta}_0(T,\tilde{T}) \geqslant d(y_1, N_0(\tilde{T})) - 3\hat{\delta}_0(T,\tilde{T}) >$$

$$> \hat{\delta}_0(T,\tilde{T})^{\frac{1}{2}} - 3\hat{\delta}_0(T,\tilde{T}) > \hat{\delta}_0(T,\tilde{T})^{\frac{3}{4}}$$

(the last inequality is true because $3\varepsilon + \varepsilon^{\frac{3}{4}} < \varepsilon^{\frac{1}{2}}$, by condition (3.2); see also (3.6)). Therefore, $d(\tilde{y}_2, N_0(\tilde{T})) > \hat{\delta}_0(T,\tilde{T})^{\frac{3}{4}}$.

Let $\tilde{\varphi} \in \tilde{Y}^*$ be such that $\tilde{\varphi}(\tilde{y}_2) = 1, \tilde{\varphi}\Big|N_0(\tilde{T}) = 0$ and $\|\tilde{\varphi}\| = d(\tilde{y}_2, N_0(\tilde{T}))^{-1}$. Since $\tilde{Y}_0 \subset \mathrm{N}(\tilde{\varphi})$, we can take $\tilde{\psi}(\tilde{y} + \tilde{Y}_0) := \tilde{\varphi}\tilde{y},\ \tilde{y} \in \tilde{Y}$.

Let $(\tilde{A},\tilde{B})$ be as follows:

$$\mathrm{D}(\tilde{A}) := (D_0(\tilde{S}) \times Y_1)/(\tilde{X}_0 \times \{0\}) \subset (\tilde{X} \times Y_1)/(\tilde{X}_0 \times \{0\}),$$
$$\tilde{A} : \mathrm{D}(\tilde{A}) \to \tilde{Y}/\tilde{Y}_0,$$
$$\tilde{A}(\tilde{x} + \tilde{X}_0, \lambda y_1) := \tilde{S}(\tilde{x} + \tilde{X}_0) + \lambda\tilde{y}_2 + \tilde{Y}_0\,;$$

$$\mathrm{D}(\tilde{B}) := \mathrm{D}(\tilde{T}) \subset \tilde{Y}/\tilde{Y}_0,$$
$$\tilde{B} : \mathrm{D}(\tilde{B}) \to (\tilde{X} \times Y_1)/(\tilde{X}_0 \times \{0\}),$$
$$\tilde{B}(\tilde{y} + \tilde{Y}_0) := (\tilde{T}(\tilde{y} + \tilde{Y}_0) - \tilde{\psi}(\tilde{y} + \tilde{Y}_0)\tilde{T}(\tilde{y}_2 + \tilde{Y}_0), 0).$$

It is easy to verify that $(\tilde{A}, \tilde{B}) \in \partial_S(\mathcal{E}, \mathcal{F})$. We have also to check that (3.3), (3.4) and (3.5) hold for $(\tilde{A}, \tilde{B})$ defined as above (i.e. in Case (B)).

$\underline{\delta_0(A, \tilde{A}) < \varepsilon_0}$: Let $((x, \lambda y_1), y) \in G_0(A)$ be such that $\|x\| + \|\lambda y_1\| + \|y\| \leqslant 1$. Then $(x, y - \lambda y_1) \in G_0(S)$ and $\|(x, y - \lambda y_1)\| \leqslant 1$. There exists $(\tilde{x}, \tilde{y}) \in G_0(\tilde{S})$ with $\|x - \tilde{x}\| + \|y - \lambda y_1 - \tilde{z}\| < \varepsilon$. Then $((\tilde{x}, \lambda y_1), \lambda \tilde{y}_2 + \tilde{z}) \in G_0(\tilde{A})$ and

$$\|x - \tilde{x}\| + \|y - \lambda \tilde{y}_2 - \tilde{z}\| \leqslant \|x - \tilde{x}\| + \|y - \lambda y_1 - \tilde{z}\| + |\lambda| \, \|y_1 - \tilde{y}_2\| <$$

$$< \varepsilon + 3|\lambda| \hat{\delta}_0(T, \tilde{T}) \leqslant \varepsilon + 3\varepsilon < \varepsilon_0$$

by (3.6) and (3.2).

$\underline{\delta_0(\tilde{A}, A) < \varepsilon_0}$: Let $((\tilde{x}, \lambda y_1), \tilde{y}) \in G_0(\tilde{A})$ be such that $\|\tilde{x}\| + \|\lambda y_1\| + \|\tilde{y}\| \leqslant 1$. Then $\tilde{x} \in D_0(\tilde{S})$ and $\tilde{S}(\tilde{x} + \tilde{X}_0) = \tilde{y} - \lambda \tilde{y}_2 + \tilde{Y}_0$. Since $(\tilde{x}, \tilde{y} - \lambda \tilde{y}_2) \in G_0(\tilde{S})$ and

$$\|(\tilde{x}, \tilde{y} - \lambda \tilde{y}_2)\| \leqslant \|\tilde{x}\| + \|\tilde{y}\| + \|\lambda \tilde{y}_2\| \leqslant$$

$$\leqslant \|\tilde{x}\| + \|\tilde{y}\| + \|\lambda y_1\| + |\lambda| \, \|\tilde{y}_2 - y_1\| \leqslant$$

$$\leqslant 1 + 3\hat{\delta}_0(T, \tilde{T}) < 1 + 3\varepsilon < 2,$$

there exists $(x, z) \in G_0(S)$ such that

$$\|\tilde{x} - x\| + \|\tilde{y} - \lambda \tilde{y}_2 - z\| < 2\varepsilon.$$

Then $((x, \lambda y_1), \lambda y_1 + z) \in G_0(A)$ and

$$\|\tilde{x} - x\| + \|\tilde{y} - \lambda y_1 - z\| \leqslant \|\tilde{x} - x\| + \|\tilde{y} - \lambda \tilde{y}_2 - z\| + |\lambda| \, \|\tilde{y}_2 - y_1\| \leqslant$$

$$\leqslant 2\varepsilon + 3\hat{\delta}_0(T, \tilde{T}) < 2\varepsilon + 3\varepsilon < \varepsilon_0,$$

by (3.9), (3.6) and (3.2).

$\underline{\delta_0(B, \tilde{B}) < \varepsilon_0}$: Let $(y, (x, 0)) \in G_0(B)$ be such that $\|y\| + \|x\| \leqslant 1$. Then $y \in$ $\in D_0(T)$ and $(y, x) \in G_0(T)$. Hence there exists $(\tilde{y}, \tilde{x}) \in G_0(\tilde{T})$ such that $\|y - \tilde{y}\| +$ $+ \|x - \tilde{x}\| < \varepsilon$. Then

$$(\tilde{y}, (\tilde{x} - \tilde{\psi}(\tilde{y} + \tilde{Y}_0)\tilde{x}_2, 0)) \in G_0(\tilde{B})$$

and

$$\|\tilde{y} - y\| + \|x - \tilde{x} + \tilde{\psi}(\tilde{y} + \tilde{Y}_0)\tilde{x}_2\| \leqslant$$

$$\leqslant \|y - \tilde{y}\| + \|x - \tilde{x}\| + |\tilde{\psi}(\tilde{y} + \tilde{Y}_0)| \, \|\tilde{x}_2\| \leqslant$$

$$\leqslant \varepsilon + \|\tilde{\psi}\| \, \|\tilde{y} + \tilde{Y}_0\| \, \|\tilde{x}_2\| \leqslant \varepsilon + \|\tilde{\psi}\| \, \|\tilde{y}\| \cdot 3\hat{\delta}_0(T, \tilde{T}) \leqslant$$

$$\leqslant \varepsilon + (\|\tilde{y} - y\| + \|y\|) d(\tilde{y}_2, N_0(\tilde{T}))^{-1} \cdot 3\hat{\delta}_0(T, \tilde{T}) \leqslant$$

$$\leqslant \varepsilon + (\varepsilon+1)\hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}} \cdot 3\hat{\delta}_0(T,\tilde{T}) \leqslant \varepsilon + 2\cdot 3\hat{\delta}_0(T,\tilde{T})^{\frac{1}{4}} < \varepsilon + 6\varepsilon^{\frac{1}{4}} < \varepsilon_0$$

by (3.9), (3.6) and (3.2).

$\underline{\delta_0(\tilde{B},B) < \varepsilon_0}$: Let $(\tilde{y},(\tilde{x},0)) \in G_0(\tilde{B})$ be such that $\|\tilde{y}\| + \|\tilde{x}\| \leqslant 1$. Then $\tilde{y} \in D_0(\tilde{T})$ and $\tilde{T}(\tilde{y}+\tilde{Y}_0) - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{T}(\tilde{y}_2+\tilde{Y}_0) = \tilde{x}+\tilde{X}_0$. Since

$$\tilde{T}(\tilde{y} - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{y}_2 + \tilde{Y}_0) = \tilde{x} + \tilde{X}_0,$$

we must have $(\tilde{y} - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{y}_2, \tilde{x}) \in G_0(\tilde{T})$. Note that

$$\|(\tilde{y} - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{y}_2, \tilde{x})\| \leqslant \|\tilde{y}\| + \|\tilde{\psi}\|\,\|\tilde{y}\|\,\|\tilde{y}_2\| + \|\tilde{x}\| \leqslant$$

$$\leqslant \|\tilde{y}\| + \|\tilde{x}\| + \|\tilde{y}\|\hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}}(\|\tilde{y}_2 - y_1\| + \|y_1\|) \leqslant$$

$$\leqslant 1 + \hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}}(3\hat{\delta}_0(T,\tilde{T}) + 2) \leqslant$$

$$\leqslant 1 + 3\hat{\delta}_0(T,\tilde{T})^{\frac{1}{4}} + 2\hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}} \leqslant$$

$$\leqslant 4 + 2\hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}},$$

by (3.8). Then there exists $(y,x) \in G_0(T)$ such that

$$\|\tilde{y} - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{y}_2 - y\| + \|\tilde{x} - x\| <$$

$$< (4 + 2\hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}}) \cdot 2\hat{\delta}_0(T,\tilde{T}) \leqslant$$

$$\leqslant 8\hat{\delta}_0(T,\tilde{T}) + 4\hat{\delta}_0(T,\tilde{T})^{\frac{1}{4}} < 8\varepsilon + 4\varepsilon^{\frac{1}{4}},$$

by (3.6).

Note that

$$(y + \tilde{\psi}(\tilde{y}+\tilde{Y}_0)y_1, (x,0)) \in G_0(B)$$

and

$$\|\tilde{y} - y - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)y_1\| + \|\tilde{x} - x\| \leqslant$$

$$\leqslant \|\tilde{y} - y - \tilde{\psi}(\tilde{y}+\tilde{Y}_0)\tilde{y}_2\| + \|\tilde{x} - x\| + \|\tilde{\psi}(\tilde{y}+\tilde{Y}_0)(\tilde{y}_2 - y_1)\| \leqslant$$

$$\leqslant 8\varepsilon + 4\varepsilon^{\frac{1}{4}} + \|\tilde{\psi}\|\,\|\tilde{y}\|\,\|\tilde{y}_2 - y_1\| \leqslant$$

$$\leqslant 8\varepsilon + 4\varepsilon^{\frac{1}{4}} + \hat{\delta}_0(T,\tilde{T})^{-\frac{3}{4}} \cdot 3\hat{\delta}_0(T,\tilde{T}) \leqslant$$

$$\leqslant 8\varepsilon + 4\varepsilon^{\frac{1}{4}} + 3\varepsilon^{\frac{1}{4}} < \varepsilon_0$$

by (3.8), (3.6) and (3.2).

So, (3.3) is verified in Case (B) too. We have only to prove (3.4) and (3.5).

Since $\tilde{y}_2 \notin N_0(\tilde{T})$, we have $\tilde{y}_2 + \tilde{Y}_0 \notin \mathrm{R}(\tilde{S})$ and so $\mathrm{R}(\tilde{A}) = \mathrm{R}(\tilde{S}) + \Lambda(\tilde{y}_2 + \tilde{Y}_0)$.

Note also that $\mathrm{N}(\tilde{A}) = \mathrm{N}(\tilde{S}) \times \{0\}$ and $\mathrm{R}(\tilde{B}) = \tilde{T}(\mathrm{N}(\tilde{\psi})) \times \{0\}$ (now, $\tilde{\psi} := := \tilde{\psi}\big|\mathrm{D}(\tilde{T})$).

We have

$$\mathrm{N}(\tilde{A})/\mathrm{R}(\tilde{B}) = \mathrm{N}(\tilde{S})/\tilde{T}(\mathrm{N}(\tilde{\psi})) =$$

$$= (\mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T})) \times (\mathrm{R}(\tilde{T})/\tilde{T}(\mathrm{N}(\tilde{\psi}))) = \tag{3.10}$$

$$= (\mathrm{N}(\tilde{S})/\mathrm{R}(\tilde{T})) \times \Lambda$$

where $=$ stands for isomorphism (because $\mathrm{N}(\tilde{T}) \subset \mathrm{N}(\tilde{\psi})$ and there are algebraic isomorphisms $\mathrm{R}(\tilde{T}) = \mathrm{D}(\tilde{T})/\mathrm{N}(\tilde{T})$ and $\tilde{T}(\mathrm{N}(\tilde{\psi})) = \mathrm{N}(\tilde{\psi})/\mathrm{N}(\tilde{T})$).

There is also an isomorphism

$$L : \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}) \to \mathrm{N}(\tilde{T})/\mathrm{R}(\tilde{S}) \tag{3.11}$$

defined as follows: if $\xi \in \mathrm{D}(\tilde{B}) \subset \tilde{Y}/\tilde{Y}_0$ and $\tilde{B}\xi = 0$, then

$$\xi - \tilde{\psi}(\xi)(\tilde{y}_2 + \tilde{Y}_0) \in \mathrm{N}(\tilde{T}).$$

If $\xi \in \mathrm{R}(\tilde{A})$, we have

$$\xi - \tilde{\psi}(\xi)(\tilde{y}_2 + \tilde{Y}_0) = \tilde{S}\eta + \lambda\tilde{y}_2 + \tilde{Y}_0 - \tilde{\psi}(\tilde{S}\eta + \lambda\tilde{y}_2 + \tilde{Y}_0)(\tilde{y}_2 + \tilde{Y}_0) = \tilde{S}\eta \in \mathrm{R}(\tilde{S}).$$

Then we set

$$L(\xi + \mathrm{R}(\tilde{A})) := \xi - \tilde{\psi}(\xi)(\tilde{y}_2 + \tilde{Y}_0) + \mathrm{R}(\tilde{S}),\ \xi + \mathrm{R}(\tilde{A}) \in \mathrm{N}(\tilde{B})/\mathrm{R}(\tilde{A}).$$

It is easy to verify that L is injective and surjective. The assertions (3.10) and (3.11) readily imply (3.4) and (3.5). The proof of the theorem is complete.

3.2. REMARK. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $X_0, X \in G(\mathcal{X})$, $Y_0, Y \in G(\mathcal{Y})$, $X_0 \subset \subset X, Y_0 \subset Y, S \in C(X/X_0, Y/Y_0), T \in C(Y/Y_0, X/X_0), \mathrm{R}(S) = \mathrm{N}(T), \mathrm{R}(T) = \mathrm{N}(S)$. Then, there is an $\varepsilon > 0$ such that if $(\tilde{S}, \tilde{T}) \in \partial_S(\mathcal{X}, \mathcal{Y})$ and $\hat{\delta}_0(S, \tilde{S}), \hat{\delta}_0(T, \tilde{T}) < \varepsilon$, it results $\mathrm{R}(\tilde{S}) = \mathrm{N}(\tilde{T}), \mathrm{R}(\tilde{T}) = \mathrm{N}(\tilde{S})$. To see this, we could invoke Corollary 2.12 from [2], which refers to a more general problem and whose proof is essentially based on a quite laborious aproximation method (see also [1] and [5]). For the convenience of the reader, we shall give a short proof of the (much simpler) remark stated above.

We therefore consider $(\tilde{S}, \tilde{T}) \in \partial_S(\mathcal{X}, \mathcal{Y})$ with $\tilde{S} \in C(\tilde{X}/\tilde{X}_0, \tilde{Y}/\tilde{Y}_0)$, $\tilde{T} \in \in C(\tilde{Y}/\tilde{Y}_0, \tilde{X}/\tilde{X}_0)$, where $\tilde{X}, \tilde{X}_0 \in G(\mathcal{X})$, $\tilde{Y}, \tilde{Y}_0 \in G(\mathcal{Y}), \tilde{X}_0 \subset \tilde{X}, \tilde{Y}_0 \subset \tilde{Y}$. Let $S_0, \tilde{S}_0$ be the injective operators obtained from $S, \tilde{S}$, factoring through the null-spaces. Then:

$$S_0 \in C(X/N_0(S), N_0(T)/Y_0),$$

$$S_0 \text{ is injective with closed range,}$$

$$\tilde{S}_0 \in C(\tilde{X}/N_0(\tilde{S}), N_0(\tilde{T})/\tilde{Y}_0),$$

$$G_0(S_0) = G_0(S), \quad G_0(\tilde{S}_0) = G_0(\tilde{S})$$

and

$$\delta(N_0(S), N_0(\tilde{S})) \leqslant (1 + 2\gamma(T)^{-1})\delta_0(T, \tilde{T})$$

(the last assertion follows from Lemma 3.4 (a) with: $N_0(S) = R_0(T)$ and $\overline{R_0(\tilde{T})} \subset$ $\subset N_0(\tilde{S})$). By applying Proposition 3.3 below for $S_0, \tilde{S}_0$, we obtain that $\mathrm{R}(\tilde{S}_0)$ is closed (if $\hat{\delta}_0(S, \tilde{S})$ and $\hat{\delta}_0(T, \tilde{T})$ are small enough). Therefore, $R_0(\tilde{S})$ is closed. But we also have

$$\delta(N_0(\tilde{T}), R_0(\tilde{S})) \leqslant C(S,T)(\delta_0(\tilde{T}, T) + \delta_0(S, \tilde{S})),$$

where $C(S,T) > 0$ depends only on S and T. Indeed, we can write

$$\delta(N_0(\tilde{T}), R_0(\tilde{S})) \leqslant \delta(N_0(\tilde{T}), N_0(T)) + \delta(N_0(T), R_0(\tilde{S})) +$$

$$+ \delta(N_0(\tilde{T}), N_0(T))\delta(N_0(T), R_0(\tilde{S})) \leqslant$$

$$\leqslant (1 + 2\gamma(T)^{-1})\delta_0(\tilde{T}, T) + (1 + 2\gamma(S)^{-1})\delta(S, \tilde{S}) +$$

$$+ (1 + 2\gamma(T)^{-1})(1 + 2\gamma(S)^{-1})\delta_0(\tilde{T}, T)\delta_0(S, \tilde{S}),$$

by Lemma 3.4, since $N_0(T) = R_0(S)$, where we have used the estimate

$$\delta(X, Y) \leqslant \delta(X, Z) + \delta(Z, Y) + \delta(X, Z)\delta(Z, Y)$$

valid for any linear closed subspaces X, Y, Z of an arbitrary Banach space.

Because of the inclusion $R_0(\tilde{S}) \subset N_0(\tilde{T})$, a well-known lemma of Riesz shows us that $R_0(\tilde{S}) = N_0(\tilde{T})$ (if $\hat{\delta}_0(S, \tilde{S})$, $\hat{\delta}_0(T, \tilde{T})$ are small enough). Consequentely, $\mathrm{R}(\tilde{S}) = \mathrm{N}(\tilde{T})$.

3.3. PROPOSITION. *Let $A \in C(X/X_1, Y_1/Y_0)$ be injective with closed range and $\tilde{A} \in C(\tilde{X}/\tilde{X}_1, \tilde{Y}_1/\tilde{Y}_0)$ with $X, X_1, \tilde{X}, \tilde{X}_1 \in G(\mathcal{X})$, $Y, Y_1, \tilde{Y}, \tilde{Y}_1 \in G(\mathcal{Y})$. If $\delta_0(\tilde{A}, A)$, $\delta(X_1, \tilde{X}_1)$ are sufficiently small, then $\tilde{A}$ is also injective with closed range (see also* [1] *for a version of this result).*

Proof. Since A is injective with closed range, there is a $C > 0$ such that

$$C\|x + X_1\| \leqslant \|A(x + X_1)\|, \quad x \in D_0(A). \tag{3.12}$$

Let $\varepsilon > \delta_0(\tilde{A}, A) + \delta(X_1, \tilde{X}_1)$. If $\delta_0(\tilde{A}, A)$ and $\delta(X_1, \tilde{X}_1)$ are sufficiently small, we may also assume that

$$2(3 + C^{-1})\varepsilon < 1. \tag{3.13}$$

Let $\tilde{x} \in D_0(\tilde{A})$, $\tilde{x}_1 \in \tilde{X}_1$, $\tilde{y}_1 \in \tilde{Y}_1$ and $\tilde{y}_0 \in \tilde{Y}_0$ be such that

$$\|\tilde{x} - \tilde{x}_1\| \leqslant 2\|\tilde{x} + \tilde{X}_1\|,$$

$$\tilde{A}(\tilde{x} + \tilde{X}_1) = \tilde{y}_1 + \tilde{Y}_0, \tag{3.14}$$

$$\|\tilde{y}_1 - \tilde{y}_0\| \leqslant 2\|\tilde{y}_1 + \tilde{Y}_0\|.$$

Since $(\tilde{x} - \tilde{x}_1, \tilde{y}_1 - \tilde{y}_0) \in G_0(\tilde{A})$, there exists $(x, y_1) \in G_0(A)$ with

$$\|\tilde{x} - \tilde{x}_1 - x\| + \|\tilde{y}_1 - \tilde{y}_0 - y_1\| \leqslant \varepsilon(\|\tilde{x} - \tilde{x}_1\| + \|\tilde{y}_1 - \tilde{y}_0\|). \tag{3.15}$$

A standard calculation gives the estimate

$$d(v, \tilde{X}_1) \leqslant d(v, X_1) + 2\delta(X_1, \tilde{X}_1)\|v\|, \quad v \in \mathcal{X}. \tag{3.16}$$

Indeed, for arbitrary $v \in \mathcal{X}$ and $\alpha > 0$, we can choose $v_1 \in X_1$ with $\|v - v_1\| \leqslant (1 + \alpha)d(v, X_1)$ and $\|v_1\| \leqslant \|v_1 - v\| + \|v\| \leqslant (1 + \alpha)d(v, X_1) + \|v\| \leqslant (2 + \alpha)\|v\|$. Then there is $v_2 \in \tilde{X}_1$ with $\|v_1 - v_2\| \leqslant (\delta(X_1, \tilde{X}_1) + \alpha)\|v_1\| \leqslant (\delta(X_1, \tilde{X}_1) + \alpha)(2 + \alpha)\|v\|$. Consequently, we can write

$$\|v - v_2\| \leqslant \|v - v_1\| + \|v_1 - v_2\| \leqslant (1 + \alpha)d(v, X_1) + (\delta(X_1, \tilde{X}_1) + \alpha)(2 + \alpha)\|v\|$$

$$d(v, \tilde{X}_1) \leqslant \|v - v_2\| \leqslant (1 + \alpha)d(v, X_1) + (\delta(X_1, \tilde{X}_1) + \alpha)(2 + \alpha)\|v\|.$$

Letting $\alpha \to 0$, we obtain (3.16).

Let $a := \varepsilon(\|\tilde{x} - \tilde{x}_1\| + \|\tilde{y}_1 - \tilde{y}_0\|)$. Then, using (3.12), (3.14), (3.15) and (3.16), we have:

$$\|\tilde{x} + \tilde{X}_1\| = \|\tilde{x} - \tilde{x}_1 + \tilde{X}_1\| \leqslant \|\tilde{x} - \tilde{x}_1 + X_1\| + 2\varepsilon\|\tilde{x} - \tilde{x}_1\| \leqslant$$

$$\leqslant \|\tilde{x} - \tilde{x}_1 + X_1\| + 2a \leqslant 2a + \|\tilde{x} - \tilde{x}_1 - x + X_1\| + \|x + X_1\| \leqslant$$

$$\leqslant 3a + C^{-1}\|y_1\| \leqslant 3a + C^{-1}\|y_1 - \tilde{y}_1 + \tilde{y}_0\| + C^{-1}\|\tilde{y}_1 - \tilde{y}_0\| \leqslant$$

$$\leqslant (3 + C^{-1})a + 2C^{-1}\|\tilde{y}_1 + \tilde{Y}_0\| \leqslant$$

$$\leqslant 2(3 + C^{-1})(\|\tilde{x} + \tilde{X}_1\| + \|\tilde{y}_1 + \tilde{Y}_0\|)\varepsilon + 2C^{-1}\|\tilde{y}_1 + \tilde{Y}_0\|.$$

In this way we have obtained

$$\|\tilde{x} + \tilde{X}_1\|(1 - 2(3 + C^{-1})\varepsilon) \leqslant (2(3 + C^{-1})\varepsilon + 2C^{-1})\|\tilde{y}_1 + \tilde{Y}_0\|,$$

from which we derive, by (3.13), the existence of a constant $c' > 0$ such that

$$\|\tilde{x} + \tilde{X}_1\| \leqslant c'\|\tilde{A}(\tilde{x} + \tilde{X}_1)\|.$$

Therefore A is injective, with closed range.

3.4. LEMMA. *Let* $S \in C(X/X_0, Y/Y_0)$, $\overline{R(S)} = R(S)$, $\tilde{S} \in C(\tilde{X}/\tilde{X}_0, \tilde{Y}/\tilde{Y}_0)$. *Then:*

(a) $\delta(R_0(S), \overline{R_0(\tilde{S})}) \leqslant (1 + 2\gamma(S)^{-1})\delta_0(S, \tilde{S})$,

(b) $\delta(N_0(\tilde{S}), N_0(S)) \leqslant (1 + 2\gamma(S)^{-1})\delta_0(\tilde{S}, S)$.

Proof. A similar result is stated in [2] (see [2], Lemma 2.9). Since the proof of that result is omitted in [2] and our estimates are slightly different (due to the fact that we use another norm on a Cartesian product of Banach spaces), we give here full details.

Let $\varepsilon > \delta_0(S, \tilde{S})$, let $y \in R_0(S)$ and let $x \in D_0(T)$ be such that $S(x+X_0) = y+Y_0$. Then $\|y\| \geqslant \gamma(S)d(x, N_0(S))$. Let $x_1 \in N_0(S)$ with $\|x - x_1\| \leqslant 2\|y\|\gamma(S)^{-1}$. Since $(x - x_1, y) \in G_0(S)$, there exists $(\tilde{x}, \tilde{y}) \in G_0(\tilde{S})$ such that

$$\|x - x_1 - \tilde{x}\| + \|y - \tilde{y}\| \leqslant \varepsilon(\|x - x_1\| + \|y\|).$$

But $\tilde{y} \in R_0(\tilde{S})$ and

$$\|y - \tilde{y}\| \leqslant \varepsilon(\|y\| + 2\|y\|\gamma(S)^{-1}) = \varepsilon(1 + 2\gamma(S)^{-1})\|y\|.$$

Letting $\varepsilon \to \delta_0(S, \tilde{S})$, we obtain (a).

Let $\varepsilon > \delta_0(\tilde{S}, S)$ and let $\tilde{x} \in D_0(\tilde{S})$ be with $\tilde{S}(\tilde{x} + \tilde{X}_0) = 0$. Since $(\tilde{x}, 0) \in G_0(\tilde{S})$ there exists $(x, y) \in G_0(S)$ such that $\|\tilde{x} - x\| + \|y\| \leqslant \varepsilon\|\tilde{x}\|$. Since

$$\varepsilon\|\tilde{x}\| \geqslant \|y\| \geqslant \|y + Y_0\| \geqslant \gamma(S)d(x, N_0(S))$$

there exists $x_1 \in N_0(S)$ such that $\|x - x_1\| \leqslant 2\varepsilon\|\tilde{x}\|\gamma(S)^{-1}$. Then we have

$$\|\tilde{x} - x_1\| \leqslant \|\tilde{x} - x\| + \|x - x_1\| \leqslant \varepsilon\|\tilde{x}\| + 2\varepsilon\|\tilde{x}\|\gamma(S)^{-1} =$$

$$= \varepsilon(1 + 2\gamma(S)^{-1})\|\tilde{x}\|.$$

Letting $\varepsilon \to \delta_0(\tilde{S}, S)$, we obtain (b).

4. AN APPLICATION

In this section we shall present a theorem concerning the stability of the index of a complex of Banach spaces, due to E. Albrecht and F.-H.Vasilescu ([1], [2], [5]), as a consequence of the result in Section 3 (see Theorem 4.3 below). In fact, the concept of symmetrical pair was introduced by the above mentioned authors in connection with the study of semi-Fredholm complexes (see [1]).

4.1. DEFINITION. *A complex of Banach spaces* is a sequence of the form

$$\cdots X^p \xrightarrow{\alpha^p} X^{p+1} \xrightarrow{\alpha^{p+1}} \cdots$$

where X^p is a Banach space and $\alpha^p \in C(X^p, X^{p+1})$ such that $\mathrm{R}(\alpha^p) \subset \mathrm{N}(\alpha^{p+1})$ (for all $p \in \mathbb{Z}$).

A natural treatment of the problem studied in [2] is provided by the following particular definitions.

Let $\mathcal{X}$ be a fixed Banach space. A family $\alpha = (\alpha^p)_{p\in\mathbb{Z}}$, where $\alpha^p \in C(X^p/X_0^p, X^{p+1}/X_0^{p+1})$, $X_0^p, X^p \in G(\mathcal{X})$, $X_0^p \subset X^p$ and $\alpha^{p+1}\alpha^p = 0$ will be called a *complex* in $\mathcal{X}$. The set of all complexes in $\mathcal{X}$ will be denoted by $\partial(\mathcal{X})$.

The complex α is said to be *semi-Fredholm* if at least one of the functions

$$\mathbb{Z} \ni k \mapsto \dim \mathrm{N}(\alpha^{2k})/\mathrm{R}(\alpha^{2k-1})$$

$$\mathbb{Z} \ni k \mapsto \dim \mathrm{N}(\alpha^{2k+1})/\mathrm{R}(\alpha^{2k})$$

is finite and has finite support, and, moreover, we have $\inf\{\gamma(\alpha^p)\,;\, p \in \mathbb{Z}\} > 0$. Then, we define the *index* of α by the equality

$$\operatorname{ind}\alpha := \sum_{p\in\mathbb{Z}} (-1)^p \dim \mathrm{N}(\alpha^p)/\mathrm{R}(\alpha^{p-1}).$$

If, in adition, $\operatorname{ind}\alpha$ is finite, then α will be called *Fredholm*.

We say that $(S,T) \in \partial_S(\mathcal{X},\mathcal{Y})$ ($\mathcal{X},\mathcal{Y}$ Banach spaces) is *semi-Fredholm* if $\mathrm{R}(S)$, $\mathrm{R}(T)$ are closed and at least one of the numbers $\dim \mathrm{N}(S)/\mathrm{R}(T)$, $\dim \mathrm{N}(T)/\mathrm{R}(S)$ is finite. Then,

$$\operatorname{ind}(S,T) := \dim \mathrm{N}(S)/\mathrm{R}(T) - \dim \mathrm{N}(T)/\mathrm{R}(S).$$

The following "reduction" result has been proved in [1] (see Theorem 5.5 from [1]).

4.2. THEOREM. *Let us consider a complex α of Banach spaces of the form*

$$\cdots X^p \xrightarrow{\alpha^p} X^{p+1} \xrightarrow{\alpha^{p+1}} \cdots$$

Then there exist two Banach spaces $\mathcal{X}_0$ and $\mathcal{X}_1$, and a symmetrical pair of densely defined operators $(S_0,S_1) \in \partial_S(\mathcal{X}_0,\mathcal{X}_1)$ with the following properties:

(1) $\inf\{\gamma(\alpha^p)\,;p \in \mathbb{Z}\} = \min\{\gamma(S_0), \gamma(S_1)\}$;

(2) *The complex α is semi-Fredholm (Fredholm) if and only if (S_0,S_1) is semi--Fredholm (Fredholm), and in this case* $\operatorname{ind}\alpha = \operatorname{ind}(S_0,S_1)$.

Let us outline the proof of Theorem 4.2, since we need some details in the following.

With no loss of generality we may assume that $\overline{D(\alpha^p)} = X^p$ for all $p \in \mathbb{Z}$. Let

$$\mathcal{X}_0 := \bigoplus_{k\in\mathbb{Z}} X^{2k}, \quad \mathcal{X}_1 := \bigoplus_{k\in\mathbb{Z}} X^{2k+1}$$

where the direct sum is endowed with the ℓ^2-norm. One defines the operator

$$S_0(\bigoplus_{k\in\mathbb{Z}} x_{2k}) := \bigoplus_{k\in\mathbb{Z}} \alpha^{2k} x_{2k}$$

on the linear space

$$D(S_0) := \left\{ \bigoplus_{k\in\mathbb{Z}} x_{2k} \in \mathcal{X}_0 \,;\, \sum_{k\in\mathbb{Z}} \|\alpha^{2k} x_{2k}\|^2 < \infty \right\}$$

One can see that $S_0 \in C(\mathcal{X}_0, \mathcal{X}_1)$.

Similarly, one defines the operator

$$S_1(\bigoplus_{k\in\mathbb{Z}} x_{2k+1}) := \bigoplus_{k\in\mathbb{Z}} \alpha^{2k+1} x_{2k+1}$$

on the linear space

$$D(S_1) := \left\{ \bigoplus_{k\in\mathbb{Z}} x_{2k+1} \in \mathcal{X}_1 \,;\, \sum_{k\in\mathbb{Z}} \|\alpha^{2k+1} x_{2k+1}\|^2 < \infty \right\}.$$

and one has $S_1 \in C(\mathcal{X}_1, \mathcal{X}_0)$.

To verify (1) and (2) is then a simple matter.

Let us state now the theorem of stability of the index for Fredholm complexes (the original proof, which is also valid for semi-Fredholm complexes, can be found in [2]).

4.3. THEOREM. *Let $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathcal{X})$ be a Fredholm complex. Then there exists an $\varepsilon > 0$ such that if $\tilde{\alpha} = (\tilde{\alpha}^p)_{p\in\mathbb{Z}} \in \partial(\mathcal{X})$ and $\sup\{\hat{\delta}_0(\alpha^p, \tilde{\alpha}^p)\,;\, p \in \mathbb{Z}\} < \varepsilon$, then $\tilde{\alpha}$ is also Fredholm, $\dim N(\tilde{\alpha}^p)/R(\tilde{\alpha}^{p-1}) \leqslant \dim N(\alpha^p)/R(\alpha^{p-1})$ for all $p \in \mathbb{Z}$, and $\operatorname{ind}\tilde{\alpha} = \operatorname{ind}\alpha$.*

Proof. From Theorems 4.2 and 3.1 we draw easily the conclusion about the equality of the indices. For the upper semicontinuity of $\dim N(\alpha^p)/R(\alpha^{p-1})$, we must apply directly Proposition 2.10 from [2] (strictly speaking, Theorem 3.1 allows only to state the semicontinuity of some numbers of the form $\sum \dim N(\alpha^p)/R(\alpha^{p-1})$). In order to derive the estimates of Theorem 4.3 from Theorem 3.1, we make the remark that

$$\mathrm{R}(\tilde{S}_0) = \bigoplus_{k\in\mathbb{Z}} \mathrm{R}(\tilde{\alpha}^{2k}), \quad \mathrm{R}(\tilde{S}_1) = \bigoplus_{k\in\mathbb{Z}} \mathrm{R}(\tilde{\alpha}^{2k+1})$$

(see the outline of proof of Theorem 4.2). Then, $\mathrm{R}(\tilde{\alpha}^p)$, $p \in \mathbb{Z}$ are still closed, if $\sup\{\hat{\delta}_0(\alpha^p, \tilde{\alpha}^p)\,;\ p \in \mathbb{Z}\}$ is small enough. Now the estimates of Lemma 3.4 allows us to apply a theorem due to Fainshtein and Shul'man [3] and to obtain also the inequalities of Theorem 4.3. We omit the details.

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C.-G. AMBROZIE
Departament of Mathematics,
INCREST,
Bd. Păcii 220, 79622 Bucharest,
Romania.

Present address:

The Institute of Mathematics
of The Romanian Academy
P.O. Box 1-764, 70700 Bucharest,
Romania.

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