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ON A PROBLEM OF THE PERTURBATION THEORY OF SELFADJOINT OPERATORS IN KREIN SPACES

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INTRODUCTION

Let $\mathcal{H}$ be a Krein space and let A and B be selfadjoint operators in $\mathcal{H}$ whose resolvent sets have a nonempty intersection. Assume that A is definitizable and that the difference of the resolvents of A and B belongs to some Schatten-von Neumann ideal $\mathfrak{S}_p$, $1 \leqslant p \leqslant \infty$.

If, in addition, A is fundamentally reducible and if $p < \infty$, then B is definitizable over $\overline{\mathbb{R}} \setminus c(A)$ (see Definition 1.1), where $c(A)$ is the set of the critical points of A. In particular, B is either a spectral operator in Dunford's sense or an operator with spectral singularities ([18]) and the accumulation points of the set of spectral singularities belong to $c(A)$. This is, in the essence, a result of H. Langer ([13], see also [7]).

In [8] a result of this type is proved without the assuming fundamental reducibility of A. Instead, an assumption on the accumulation points of the nonreal spectrum of B is introduced in that article.

If the difference of the resolvents of A and B is not of finite rank, it is not true, in general, that B is definitizable over open neighbourhoods of the critical points of A (see [9; proof of Proposition 3]).

Assume now, in addition, that ∞ is a critical point of A. The aim of the present paper is to give sufficient conditions for the operator B to be definitizable over an open neighbourhood of ∞. The main results are Theorems 3.6 and 3.10. We restrict ourselvs to operators B which arise from A by perturbations belonging to a slightly more general class than that of relatively compact perturbations. This class is studied in Section 2. Results of this type were proved in [5] but only for positive and fundamentally reducible operators A and definitizable perturbed operators B.

We shall admit that the selfadjoint operator A is definitizable only over a neighbourhood of ∞ and, moreover, that A and B are selfadjoint with respect to different Krein spaces with the same underlying linear space.

From the results of this article one can easily derive criteria for the operator B to be definitizable over a neighbourhood of a finite critical point of A which is no eigenvalue of A. In a subsequent paper Theorem 3.10 will be used for investigating operators connected with perturbed wave equations.

1. NOTATIONS AND PRELIMINARIES

Throughout this paper $(\mathcal{H}, [\cdot\,,\cdot])$ denotes a separable Krein space. All topological notions are understood with respect to some Hilbert norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot\,,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent. For any subspace $\mathcal{K}$ of $\mathcal{H}$, the least upper bound ($\leqslant \infty$) of the dimensions of the subspaces $\mathcal{K}'$ of $\mathcal{K}$ such that $[\cdot\,,\cdot]$ is positive (negative) definite on $\mathcal{K}'$ is denoted by $\kappa_+(\mathcal{K})$ (resp. $\kappa_-(\mathcal{K})$).

Let A be a selfadjoint operator in the Krein space $\mathcal{H}$. Then the spectrum $\sigma(A)$ of A is symmetric with respect to $\mathbb{R}$. We put $\sigma_0(A) := \sigma(A)\setminus \mathbb{R}$ and $R(z;A) := (A-zI)^{-1}$ for every z in the resolvent set $\rho(A)$ of A. By $\sigma_{\rm p,norm}(A)$ we denote the set of all normal eigenvalues of A, i.e. the set of all eigenvalues λ of A such that the root space corresponding to λ is finite-dimensional and has an A-invariant closed complementary subspace $\mathcal{N}_\lambda$ with $\lambda \in \rho(A \mid \mathcal{N}_\lambda)$. A is called *nonnegative* if $\rho(A) \neq \emptyset$ and $[Ax,x]\geqslant 0$ holds for every x in the domain $\mathcal{D}(A)$ of A. A is called *definitizable*, if $\rho(A) \neq \emptyset$ and there exists a polynomial p such that $[p(A)x,x]\geqslant 0$ holds for all $x \in\mathcal{D}(p(A))$. If A is definitizable, $\sigma_0(A)$ is contained in the set of the zeros of any polynomial p with the above-mentioned property.

Now we recall some definitions for a selfadjoint operator A in $\mathcal{H}$ (see [7]). Let Δ_0 be an open real interval. Assume that $\Delta_0 \cap \overline{\sigma_0(A)} = \emptyset$ and that for every compact subset δ_0 of Δ_0 there exists an $m\geqslant 1$ such that

$$(1.1)\qquad \sup\Big\{ |\mathrm{Im}\, z|^m \ \|R(z;A)\| \ : \mathrm{Re}\, z \in \delta_0,\ 0 < |\mathrm{Im}\, z| < \varepsilon\Big\} < \infty,$$

if ε is sufficiently small. Δ_0 is said to be of positive (negative) type with respect to A, if for every nonnegative $f \in C^\infty(\mathbb{R})$ with compact $\operatorname{supp} f \subset \Delta_0$ the operator $f(A)$ (resp. $-f(A)$) (defined, on account of (1.1), by extension of the Riesz-Dunford-Taylor functional calculus; see e.g. [7]) is nonnegative. We say that the interval Δ_0 is of definite type with respect to A, if it is of positive or of negative type. For an equivalent definition of the intervals of definite type see [7; Remark 2.5].

In this paper we restrict our considerations, for convenience, to those selfadjoint operators which satisfy the following condition:

(f_0) $\quad\sigma_0(A)$ has no more than a finite number of nonreal accumulation points.

Obviously, all definitizable selfadjoint operators satisfy the condition (f_0).

Let Δ be an open subset of $\overline{\mathbb{R}}$. Here and in the following $\overline{\mathbb{R}}$ denotes the closure of $\mathbb{R}$ in the complex sphere $\overline{\mathbb{C}}$.

DEFINITION 1.1 ([7; Definition 2.3]). A selfadjoint operator A in $\mathcal{H}$ satisfying the condition (f_0) is called *definitizable over* Δ, if the following holds:

(i) No point of Δ is an accumulation point of $\sigma_0(A)$.

(ii) For every closed subset δ of Δ there exist $m \geqslant 1$ and $M > 0$ such that

$$\|R(z;A)\| \leqslant M(1 + |z|)^{2m-2}|\operatorname{Im} z|^{-m}$$

for all z in a neighbourhood of δ (in $\overline{\mathbb{C}}$) with $z \neq \infty$ and $\operatorname{Im} z \neq 0$.

(iii) For every $t \in \Delta \setminus \{\infty\}$ there exist open intervals of definite type of the form (t', t) and $(t, t''), t' < t < t''$. If $\infty \in \Delta$, then there exist intervals of definite type of the form (t', ∞) and $(-\infty, t'')$.

A selfadjoint operator A is definitizable if and only if it is definitizable over $\overline{\mathbb{R}}$ and $\sigma_0(A)$ consists of a finite number of poles of the resolvent of A.

For an arbitrary selfadjoint operator A in $\mathcal{H}$ satisfying the condition (f_0), $\Delta(A)$ denotes the union of all open subsets Δ of $\overline{\mathbb{R}}$ such that A is definitizable over Δ. A finite point $t \in \Delta(A)$ is called *critical point of* A, if there is no open interval of definite type containing t. $\infty \in \Delta(A)$ is called a critical point of A, if for every pair of intervals (t', ∞) and $(-\infty, t'')$ of definite type one of the intervals is not of positive and the other not of negative type. The set of critical points of A is denoted by $c(A)$. For the spectral function $E(\cdot\,; A)$ and its properties we refer to [7]. We mention only that for every subinterval δ of $\Delta(A)$ whose endpoints belong to $\Delta(A) \setminus c(A)$, $E(\delta; A)$ is defined and is a selfadjoint projection in the Krein space $\mathcal{H}$.

We shall say that an open interval $\Delta_0 \subset \Delta(A)$ is of type $\pi_+(\pi_-)$ if $\kappa_-(E(\delta; A)\mathcal{H}) < \infty$ (resp. $\kappa_+(E(\delta; A)\mathcal{H}) < \infty$) for every compact subinterval δ of Δ_0 such that $E(\delta; A)$ is defined. By $c_\infty(A)$ we denote the set of those critical points t such that there exists no open interval containing t which is of type π_+ or of type π_-. A critical point t is called *regular* if there exists an open deleted neighbourhood δ_0 (in $\Delta(A)$) of t such that the set of the projection $E(\delta)$, where δ runs through all intervals with $\overline{\delta} \subset \delta_0$, is bounded. The set of all regular critical points of A is denoted by $c_r(A)$. The elements of $c_s(A) := c(A) \setminus c_r(A)$ are called *singular critical points*.

2. A RIGGING GENERATED BY A SELFADJOINT OPERATOR IN A KREIN SPACE AND A CLASS OF PERTURBATIONS

2.1. Let A be a selfadjoint operator in $\mathcal{H}$ with $\rho(A) \neq \emptyset$. We set $R(z) :=$ $:= R(z;A)$, $z \in \rho(A)$.

The following spaces were introduced in [10; §1]. We denote by $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ or, more fully, by $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A)$ the Hilbert space $(\mathcal{D}((A^*A)^{\frac{1}{4}}), (\cdot,\cdot)^{(*)}_{+\frac{1}{2}})$,

$$(x,y)^{(*)}_{+\frac{1}{2}} := ((I+A^*A)^{\frac{1}{4}}x, (I+A^*A)^{\frac{1}{4}}y), \quad x,y \in \mathcal{D}((A^*A)^{\frac{1}{4}}).$$

Here $(\cdot,\cdot)$ is the scalar product corresponding to $\|\cdot\|$ (see Section 1.1). We set $\|x\|^{(*)}_{+\frac{1}{2}} := (x,x)^{(*)\frac{1}{2}}_{+\frac{1}{2}}$. The completion of $\mathcal{H}$ with respect to the norm

$$\|x\|_{-\frac{1}{2}} := \sup\left\{ |[x,y]| \ : \ y \in \mathcal{H}^{(*)}_{+\frac{1}{2}},\ \|y\|^{(*)}_{+\frac{1}{2}} \leqslant 1 \right\}, \quad x \in \mathcal{H},$$

is denoted by $\mathcal{H}_{-\frac{1}{2}}$ or $\mathcal{H}_{-\frac{1}{2}}(A)$. Evidently, the form $[\cdot,\cdot]$ can be extended by continuity to $\mathcal{H}^{(*)}_{+\frac{1}{2}} \times \mathcal{H}_{-\frac{1}{2}}$ and to $\mathcal{H}_{-\frac{1}{2}} \times \mathcal{H}^{(*)}_{+\frac{1}{2}}$. The extended form is denoted in the same way. We have $[x,y] = \overline{[y,x]}$, $x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$, $y \in \mathcal{H}_{-\frac{1}{2}}$. The mapping $\iota : \mathcal{H}_{-\frac{1}{2}} \ni x \longmapsto g_x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$, defined by

$$[y,x] = (y,g_x)^{(*)}_{+\frac{1}{2}}, \quad y \in \mathcal{H}^{(*)}_{+\frac{1}{2}},$$

is an isometry of $\mathcal{H}_{-\frac{1}{2}}$ onto $\mathcal{H}^{(*)}_{+\frac{1}{2}}$.

It is easy to see that for arbitrary $z \in \mathbb{C}$ the operator $A - zI$ can be extended by continuity to a continuous linear operator $(A-zI)^\sim$ from $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ in $\mathcal{H}_{-\frac{1}{2}}$ ([10]). If $z \in \rho(A)$, then $(A-zI)^\sim$ is an isomorphism of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ onto $\mathcal{H}_{-\frac{1}{2}}$. In this case we have $((A-zI)^\sim)^{-1} = \tilde{R}(z)$, where $\tilde{R}(z)$ is the extension by continuity of $R(z)$ to a continuous linear operator of $\mathcal{H}_{-\frac{1}{2}}$ onto $\mathcal{H}^{(*)}_{+\frac{1}{2}}$.

2.2. In the case when A is definitizable over an open subset Δ_0 of $\mathbb{R}$ the spectral projections of A are continuous in $\mathcal{D}(A)$ with respect to the graph norm. Therefore, by interpolation, they map $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ continuously into itself. Hence the spectral function $E(\cdot) := E(\cdot;A)$ can be extended by continuity to a set function $\tilde{E}$ whose values are projections in $\mathcal{H}_{-\frac{1}{2}}$. For any bounded real interval δ such that $E(\delta)$ is defined we have $\tilde{E}(\delta)\mathcal{H}_{-\frac{1}{2}} = E(\delta)\mathcal{H} = E(\delta)\mathcal{H}^{(*)}_{+\frac{1}{2}}$.

If b is a Borel subset of $\overline{\mathbb{C}}$ symmetric with respect to $\mathbb{R}$ such that $E(b)$ is defined, then

$$E(b)\mathcal{H}^{(*)}_{+\frac{1}{2}}(A) = \mathcal{H}^{(*)}_{+\frac{1}{2}}(A \,|\, E(b)\mathcal{H}), \ \ \tilde{E}(b)\mathcal{H}_{-\frac{1}{2}}(A) = \mathcal{H}_{-\frac{1}{2}}(A \,|\, E(b)\mathcal{H})$$

and, hence,

$$\begin{aligned} \mathcal{H}^{(*)}_{+\frac{1}{2}}(A) &= \mathcal{H}^{(*)}_{+\frac{1}{2}}(A \,|\, E(b)\mathcal{H} \oplus \mathcal{H}^{(*)}_{+\frac{1}{2}}(A \,|\, (I-E(b))\mathcal{H}), \\ \mathcal{H}_{-\frac{1}{2}}(A) &= \mathcal{H}_{-\frac{1}{2}}(A \,|\, E(b)\mathcal{H}) \ + \ \mathcal{H}_{-\frac{1}{2}}(A \,|\, (I-E(b))\mathcal{H}) \end{aligned} \tag{2.1}$$

up to equivalence of norms.

2.3. If $\infty \in \Delta(A)\setminus c_s(A)$, it is convenient to choose the Hilbert scalar product $(\cdot,\cdot)$ on $\mathcal{H}$ in a special way. In this case it is no restriction to assume that there exists a $t_0 > 0$ such that

$$\Delta := \overline{\mathbb{R}} \setminus (-t_0, t_0) \subset \Delta(A), \quad t_0, -t_0 \notin c(A), \tag{2.2}$$

$E([t_0,\infty))$is nonnegative and $E((-\infty,-t_0])$ is either nonnegative or nonpositive. We set $\alpha_0 = 1$ $(\alpha_0 = -1)$ if $E((-\infty,-t_0])$ is nonnegative (resp. not nonnegative) and set

$$\begin{aligned}(x,y) := \alpha_0[E((-\infty,-t_0])x,y] \; + \; [E([t_0,\infty))x,y] \; + \\ + \; ((I-E(\Delta))x,(I-E(\Delta))y)_0, \quad x,y \in \mathcal{H},\end{aligned} \tag{2.3}$$

where $(\cdot,\cdot)_0$ is an arbitrary Hilbert scalar product on $(I-E(\Delta))\mathcal{H}$. Then setting $A_\Delta := A \,|\, E(\Delta)\mathcal{H}$ we obtain

$$\begin{aligned}(x,y)^{(*)}_{+\frac{1}{2}} = ((|A_\Delta|^2+I)^{\frac{1}{4}}E(\Delta)x,(|A_\Delta|^2+I)^{\frac{1}{4}}E(\Delta)y) \; + \\ + \; ((I-E(\Delta))x,(I-E(\Delta))y)_{0,+}, \quad x,y \in \mathcal{H}^{(*)}_{+\frac{1}{2}}, \\ (x,y)_{-\frac{1}{2}} = ((|A_\Delta|^2+I)^{-\frac{1}{4}}E(\Delta)x,(|A_\Delta|^2+I)^{-\frac{1}{4}}E(\Delta)y) \; + \\ + \; ((I-E(\Delta))x,(I-E(\Delta))y)_{0,-}, \quad x,y \in \mathcal{H},\end{aligned} \tag{2.4}$$

where $(\cdot,\cdot)_{0,+}$ and $(\cdot,\cdot)_{0,-}$ are certain Hilbert scalar products on $(I-E(\Delta))\mathcal{H}$ which are equivalent to $(\cdot,\cdot)_0$. The operator

$$\tau_\Delta := (I \; + \; |A_\Delta|^2)^{\frac{1}{4}}E(\Delta) \; + \; (I-E(\Delta)) \tag{2.5}$$

is an isomorphism of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ onto $\mathcal{H}$ and it can be extended by continuity to an isomorphism $\tilde{\tau}_\Delta$ of $\mathcal{H}$ onto $\mathcal{H}_{-\frac{1}{2}}$.

2.4. The operator A'. Let A be a selfadjoint operator in $\mathcal{H}$ which is nonnegâtive over a neighbourhood of ∞: There exists a positive number t_0 such that (2.2) holds and $A_\Delta := A \,|\, E(\Delta)\mathcal{H}$ is nonnegative. We define a positive definite scalar product $(\cdot,\cdot)_\Delta$ on $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ by

$$(x,y)_\Delta := [\tilde{A}E(\Delta)x,y] \; + \; ((I-E(\Delta))x,(I-E(\Delta))y), \quad x,y \in \mathcal{H}^{(*)}_{+\frac{1}{2}}.$$

This scalar product is equivalent to $(\cdot,\cdot)^{(*)}_{+\frac{1}{2}}$. Denoting by $|A^{-1}_\Delta|_\Delta$ the modulus of the bounded selfadjoint operator A^{-1}_Δ in $(E(\Delta)\mathcal{H}^{(*)}_{+\frac{1}{2}},(\cdot,\cdot)_\Delta)$ we define

$$\langle x,y\rangle_\Delta := (|A^{-1}_\Delta|_\Delta E(\Delta)x,E(\Delta)y)_\Delta \; + \; ((I-E(\Delta))x,(I-E(\Delta))y), \quad x,y \in \mathcal{H}^{(*)}_{+\frac{1}{2}}.$$

One verifies without difficulty that

$$\langle x,y\rangle_\Delta = \lim_{n\to\infty}[(E([t_0,n)) - E((-n,-t_0]))x,y] + \\ + ((I-E(\Delta))x,(I-E(\Delta))y), \quad x,y\in\mathcal{H}^{(*)}_{+\frac{1}{2}}.$$

For any set Δ_0 with the same properties as Δ, the scalar products $\langle\cdot,\cdot\rangle_\Delta$ and $\langle\cdot,\cdot\rangle_{\Delta_0}$ are equivalent on $\mathcal{H}^{(*)}_{+\frac{1}{2}}$. The completion of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ with respect to $\langle\cdot,\cdot\rangle_\Delta$ is denoted by $\mathcal{H}_A$. By [3; proof of Lemma 2.3] the form $[\cdot,\cdot]$ restricted to $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ can be extended by continuity to $\mathcal{H}_A$ such that $(\mathcal{H}_A,[\cdot,\cdot])$ is a Krein space. In the following $\mathcal{H}_A$ will denote this Krein space. According to [3; Theorem 2.5] the space $\mathcal{H}$ and $\mathcal{H}_A$ coincide if and only if $\infty\notin c_s(A)$.

With the help of the rigging $\mathcal{H}^{(*)}_{+\frac{1}{2}}\subset\mathcal{H}_A\subset\mathcal{H}_{-\frac{1}{2}}$ we define now an operator A' in $\mathcal{H}_A$:

$$\mathcal{D}(A') := \{x\in\mathcal{H}^{(*)}_{+\frac{1}{2}} : \tilde{A}x\in\mathcal{H}_A\}, \quad A'x := \tilde{A}x, \quad x\in\mathcal{D}(A'). \tag{2.6}$$

It easy to see that A' is a selfadjoint operator in $\mathcal{H}_A$ and that

$$\rho(A)\subset\rho(A'). \tag{2.7}$$

We shall show in the following lemma that the operators A and A' are closely connected.

LEMMA 2.1. (i) $\sigma(A)=\sigma(A')$.

(ii) $\overline{\mathbb{R}}\setminus[-t_0,t_0]\subset\Delta(A')$.

(iii) *If E' denotes the spectral function of A', then*

$$(I-E(\overline{\mathbb{R}}\setminus(-s,s))\mathcal{H} = (I-E'(\overline{\mathbb{R}}\setminus(-s,s)))\mathcal{H}_A \tag{2.8}$$

for every real $s>t_0$. A and A' coincide on the space (2.8).

(iv) $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A)=\mathcal{H}^{(*)}_{+\frac{1}{2}}(A')$.

(v) $\infty\notin c_s(A')$.

Proof. Evidently, $(I-E(\Delta))\mathcal{H}$ is a Krein space of $\mathcal{H}_A$ contained in $\mathcal{D}(A')$ such that $A'(I-E(\Delta))\mathcal{H}\subset(I-E(\Delta))\mathcal{H}$ and

$$A\,|\,(I-E(\Delta))\mathcal{H} = A'\,|\,(I-E(\Delta))\mathcal{H}. \tag{2.9}$$

Let $\lambda\in\sigma(A)$. As (2.9) is also true for Δ replaced by closed set $\Delta_1\not\ni\lambda$ with the same properties, we get $\lambda\in\sigma(A')$. Hence $\sigma(A)\subset\sigma(A')$. Then (2.7) gives the assertion (i).

Denote by $\mathcal{H}_A(\Delta)$ the orthogonal complement of $(I - E(\Delta))\mathcal{H}$ in $\mathcal{H}_A$. For $x \in \mathcal{D}(A' \mid \mathcal{H}_A(\Delta))$, i.e. for $x \in E(\Delta)\mathcal{H}^{(*)}_{+\frac{1}{2}}$ such that $\tilde{A}x \in \mathcal{H}_A$, we have

$$[A'x, x] = [\tilde{A}x, x] \geqslant 0. \tag{2.10}$$

If $\lambda \in \rho(A_\Delta)$, then $(A - \lambda I)\tilde{}$ maps $E(\Delta)\mathcal{H}^{(*)}_{+\frac{1}{2}}$ isomorphically onto $\tilde{E}(\Delta)\mathcal{H}_{-\frac{1}{2}}$ and, therefore, $\lambda \in \rho(A' \mid \mathcal{H}_A(\Delta))$. It follows that

$$\sigma(A' \mid \mathcal{H}_A(\Delta)) \subset \sigma(A_\Delta) \subset \Delta. \tag{2.11}$$

Then from (2.9), (2.11) and (2.10) we obtain the relation (ii).

By (2.11) we have $0 \in \rho(A' \mid \mathcal{H}_A(\Delta))$. Then on account of (2.10) (for $x \in \mathcal{D}(A' \mid \mathcal{H}_A(\Delta))$), $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A' \mid \mathcal{H}_A(\Delta))$ coincides with the completion of $\mathcal{D}(A' \mid \mathcal{H}_A(\Delta))$ with respect to the norm $x \mapsto [A'x, x]^{\frac{1}{2}}$. On the other hand $\mathcal{D}(A' \mid \mathcal{H}_A(\Delta))$ is dense in $E(\Delta)\mathcal{H}^{(*)}_{+\frac{1}{2}}$ with respect to the norm $x \mapsto [\tilde{A}x, x]^{\frac{1}{2}}$. It follows that $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A' \mid \mathcal{H}_A(\Delta)) = \mathcal{H}^{(*)}_{+\frac{1}{2}}(A \mid E(\Delta)\mathcal{H})$ and, hence, that (iv) holds.

Let $s > t_0$. We set $E_s := (I - E(\overline{\mathbb{R}} \setminus (-s, s)))$ and $E'_s := (I - E'(\overline{\mathbb{R}} \setminus (-s, s)))$. We have $E_s\mathcal{H} \subset \mathcal{D}(A) \subset \mathcal{H}^{(*)}_{+\frac{1}{2}}$ and $E'_s\mathcal{H}_A \subset \mathcal{D}(A') \subset \mathcal{H}^{(*)}_{+\frac{1}{2}}$. Let $x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$. Then

$$E_s x = -\lim_{m\to\infty} (2\pi \mathrm{i})^{-1} \int_{C_{sm}} R(z; A)x \, \mathrm{d}z, \quad E'_s = -\lim_{m\to\infty} (2\pi \mathrm{i})^{-1} \int_{C_{sm}} R(z; A')x \, \mathrm{d}z$$

in $\mathcal{H}$ and $\mathcal{H}_A$, respectively, where C_{sm}, $m = 1, 2, \dots$, denote certain curves in $\rho(A) = \rho(A')$ (see [15; proof of Theorem 3.1]). We have $R(z; A)x = R(z; A')x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$ for every $z \in \rho(A)$, and $z \longmapsto R(z; A)x$ is a continuous function on $\rho(A)$ with values in $\mathcal{H}^{(*)}_{+\frac{1}{2}}$. This implies $E_s x = E'_s x$.

Assume now that $z \in E_s\mathcal{H} = E_s(E_s\mathcal{H})$. Then there exists a $u \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$ such that $z = E_s u$. It follows that $z = E'_s u$, i.e. $z \in E'_s\mathcal{H}_A$. Similarly we obtain $E'_s\mathcal{H} \subset E_s\mathcal{H}$. Hence (2.8) holds. By the definition of A' the operators A and A' coincide on the space (2.8).

It is easy to see that $\langle \cdot, \cdot \rangle_\Delta$ can be extended by continuity to $\mathcal{H}_{-\frac{1}{2}} \times \mathcal{H}^{(*)}_{+\frac{1}{2}}$ and to $\mathcal{H}^{(*)}_{+\frac{1}{2}} \times \mathcal{H}_{-\frac{1}{2}}$ and that $\langle \tilde{A}x, y \rangle_\Delta = \langle x, \tilde{A}y \rangle_\Delta$ for $x, y \in E(\Delta)\mathcal{H}^{(*)}_{+\frac{1}{2}}$. It follows that A' restricted to $(\mathcal{H}_A(\Delta), \langle \cdot, \cdot \rangle_\Delta)$ is selfadjoint. Hence $\infty \notin \mathrm{c_s}(A')$.

2.5. Let A given as in Section 2.1 and let $R(z) := R(z; A)$, $z \in \rho(A)$. We set $\mathcal{L}^{(A)} := \mathcal{L}(\mathcal{H}^{(*)}_{+\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}})$. For an operator $W \in \mathcal{L}^{(A)}$ the adjoint with respect to the duality $[\cdot, \cdot]$ is denoted by W^+:

$$[Wx, y] = [x, W^+ y], \quad x, y \in \mathcal{H}^{(*)}_{+\frac{1}{2}}.$$

It is easy to see that this adjoint is compatible with the usual Krein space adjoint. An operator $V \in \mathcal{L}^{(A)}$ is called $[\cdot,\cdot]$-symmetric ($[\cdot,\cdot]$-nonnegative) if $\operatorname{Im}[Vx,x] = 0$ (resp. $[Vx,x] \geqslant 0$) for every $x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$.

Let $V \in \mathcal{L}^{(A)}$ be compact. From the fact that ιV (see Section 2.1) is compact in $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ it follows that V can be written in the form

$$V = \sum_{j=1}^{\nu} s_j[\cdot, e_j]f_j, \quad 1 \leqslant \nu \leqslant \infty, \tag{2.12}$$

where $s_j > 0$, $\lim_{j\to\infty} s_j = 0$ if $\nu = \infty$, $e_j, f_j \in \mathcal{H}_{-\frac{1}{2}}$ such that $[\iota e_i, e_j] = [\iota f_i, f_j] = \delta_{ij}$. The sum in (2.12) converges with respect to the norm of $\mathcal{L}^{(A)}$. If, in addition, V is $[\cdot,\cdot]$-symmetric, we have

$$V = \sum_{j=1}^{\nu} s_j \varepsilon_j [\cdot, f_j] f_j, \quad 1 \leqslant \nu \leqslant \infty, \tag{2.13}$$

where s_j and f_j are as above and $\varepsilon_j = \pm 1$.

2.6. Making use of the rigging $\mathcal{H}^{(*)}_{+\frac{1}{2}} \subset \mathcal{H} \subset \mathcal{H}_{-\frac{1}{2}}$ we introduce now a class of perturbations of A. These perturbations could also be defined in terms of sesquilinear forms. We use the notation of [12].

DEFINITION 2.2. Assume that the operator $Z \in \mathcal{L}^{(A)}$ can be written as a sum $Z = Z_1 + Z_2$, $Z_1, Z_2 \in \mathcal{L}^{(A)}$, such that the following holds:

(i) There exists a $z \in \mathbb{C}$ such that $(A - zI)^{\sim} + Z_1$ is an isomorphism of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ onto $\mathcal{H}_{-\frac{1}{2}}$.

(ii) Z_2 is the extension by continuity of a bounded operator in $\mathcal{H}$. Then the restriction of $\tilde{A} + Z$ to

$$\mathcal{D}(A \dot{+} Z) := \left\{ x \in \mathcal{H}^{(*)}_{+\frac{1}{2}} : (\tilde{A} + Z)x \in \mathcal{H} \right\}$$

regarded as an operator in $\mathcal{H}$ is denoted by $A \dot{+} Z$.

Evidently, $A \dot{+} Z$ is a densely defined closed operator.

LEMMA 2.3. *Let $V \in \mathcal{L}^{(A)}$ be compact. Then $A \dot{+} V$ is defined.*

Assume, in addition, that $\rho(A) \cap \rho(A \dot{+} V) \neq \emptyset$ and $V \in \mathfrak{S}_p$ for some p, $1 \leqslant p \leqslant \infty$. Let $z \in \rho(A) \cap \rho(A \dot{+} V)$. Then

$$R(z) - R(z; A \dot{+} V) \in \mathfrak{S}_p. \tag{2.14}$$

If $V \in \mathcal{L}^{(A)}$ is compact and $[\cdot,\cdot]$-symmetric, then $A \dot{+} V$ is selfadjoint in the Krein space $\mathcal{H}$.

Proof. Let $z \in \rho(A)$. Using the representation (2.12) of V we see that there exists a decomposition $V = Z_1 + Z_2$, $Z_1, Z_2 \in \mathcal{L}^{(A)}$ such that

$$\|Z_1\|_{\mathcal{L}^{(A)}} \|\tilde{R}(z)\|_{\mathcal{L}(\mathcal{H}_{-\frac{1}{2}}, \mathcal{H}^{(*)}_{+\frac{1}{2}})} < 1 \tag{2.15}$$

and Z_2 is of finite rank and fulfils the condition (ii) of Definition 2.2. By (2.15) we have

$$\begin{aligned}((A - zI)^{\sim} + Z_1)^{-1} &= ((I + Z_1\tilde{R}(z))(A - zI)^{\sim})^{-1} = \\ &= \tilde{R}(z) - \tilde{R}(z)\sum_{j=1}^{\infty}(-1)^{j-1}(Z_1\tilde{R}(z))^j.\end{aligned} \tag{2.16}$$

Hence $A\dot{+}V$ is defined.

Assume now that $\rho(A) \cap \rho(A\dot{+}V) \neq \emptyset$ and $z \in \rho(A) \cap \rho(A\dot{+}V)$. Since $(A - zI)^{\sim} + V \in \mathcal{L}^{(A)}$ has a dense range in $\mathcal{H}_{-\frac{1}{2}}$ and, on the other hand, its Fredholm index is zero, it is an isomorphism of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ onto $\mathcal{H}_{-\frac{1}{2}}$. Then, by the relation

$$((A - zI)^{\sim} + V)^{-1} V (A - zI)^{\sim -1} = (A - zI)^{\sim -1} - ((A - zI)^{\sim} + V)^{-1},$$

$V \in \mathfrak{S}_p$ implies (2.14).

To prove the last assertion assume that V is compact and $[\cdot,\cdot]$-symmetric, and $z \in \rho(A)$. Then $\bar{z} \in \rho(A)$ and the norms of $\tilde{R}(z)$ and $\tilde{R}(\bar{z})$ in $\mathcal{L}(\mathcal{H}_{-\frac{1}{2}}, \mathcal{H}^{(*)}_{+\frac{1}{2}})$ coincide. Using (2.13) we find a decomposition of V as above such that Z_1 is $[\cdot,\cdot]$-symmetric. Then (2.15) is also true with z replaced by $\bar{z}$ and we find $z, \bar{z} \in \rho(A\dot{+}Z_1)$. Moreover,

$$[((A - zI)^{\sim} + Z_1)^{-1}x, x] = [x, ((A - \bar{z}I)^{\sim} + Z_1)^{-1}x], \quad x \in \mathcal{H}_{-\frac{1}{2}}.$$

Hence $R(\bar{z}; A\dot{+}Z_1)$ is the Krein space adjoint of $R(z; A\dot{+}Z_1)$. This implies that $A\dot{+}Z_1$ is selfadjoint in $\mathcal{H}$. Then $A\dot{+}V$ is also selfadjoint.

REMARK 2.4. If A is nonnegative and $V \in \mathcal{L}^{(A)}$ is compact and $[\cdot,\cdot]$-nonnegative, then the selfadjoint operator $A\dot{+}V$ is nonnegative. Indeed, by the definition of $A\dot{+}V$ we have $[(A\dot{+}V)x, x] \geqslant 0$ if $x \in \mathcal{D}(A\dot{+}V)$. Then, on account of [14; Folgerung 1.1] the range of $(A\dot{+}V) - zI$ is dense in $\mathcal{H}$ for every nonreal z. On the other hand $(A - zI)^{\sim} + V$ is a Fredholm operator of index 0. Consequently, $\sigma(A\dot{+}V) \subset \mathbb{R}$ and the operator $A\dot{+}V$ is nonnegative.

LEMMA 2.5. *Assume that $V \in \mathcal{L}^{(A)}$ is compact. Let $\mathcal{K}$ be a Hilbert space, $V_1 \in \mathcal{L}(\mathcal{H}^{(*)}_{+\frac{1}{2}}, \mathcal{K})$, $V_2 \in \mathcal{L}(\mathcal{K}, \mathcal{H}_{-\frac{1}{2}})$ such that $V = V_2V_1$, and let at least one of the operators V_1 and V_2 be compact. Assume that $z \in \rho(A)$. Then we have $z \in \rho(A\dot{+}V)$ if and only if*

$$Q(z) := I + V_1\tilde{R}(z)V_2 : \mathcal{K} \to \mathcal{K}$$

is invertibile. If this holds, then

$$((A - zI)^{\sim} + V)^{-1} = \tilde{R}(z) - \tilde{R}(z)V_2Q^{-1}(z)V_1\tilde{R}(z). \tag{2.17}$$

Proof. Assume that $z \in \rho(A)$. If $Q(z)$ is not invertible, then there exists an $x \in \mathcal{K}$, $x \neq 0$, such that $x + V_1\tilde{R}(z)V_2x = 0$. Therefore,

$$V_2x + V_2V_1\tilde{R}(z)V_2x = V_2x + V\tilde{R}(z)V_2x = \{(A - zI)^{\sim} + V\}\tilde{R}(z)V_2x = 0.$$

Hence $(A - zI)^{\sim} + V$ is no isomorphism of $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ onto $\mathcal{H}_{-\frac{1}{2}}$ and, consequently, $z \notin \rho(A\dot{+}V)$. If $Q(z)$ is invertible, then it is easy to see that the right hand side of (2.17) is the inverse of $(A - zI)^{\sim} + V$. Thus $z \in \rho(A\dot{+}V)$.

REMARK 2.6. Lemma 2.5 will be applied below in the following situation: Assume that (2.12) holds. Let, for finite ν, ℓ^2_ν denote the Hilbert space $\mathbb{C}^\nu$ and let ℓ^2_∞ be the Hilbert space ℓ^2. Define

$$V_1 : x \longmapsto (s_1^{\frac{1}{2}}[x, e_1], s_2^{\frac{1}{2}}[x, e_2], \ldots)^t : \mathcal{H}^{(*)}_{+\frac{1}{2}} \longrightarrow \ell^2_\nu,$$

$$V_2 : (a_1, a_2, \ldots)^t \longmapsto a_1 s_1^{\frac{1}{2}} f_1 + a_2 s_2^{\frac{1}{2}} f_2 + \ldots : \ell^2_\nu \longrightarrow \mathcal{H}_{-\frac{1}{2}}.$$

Then V_1 and V_2 are compact and $V = V_2V_1$.

2.7. The following lemma shows that within a certain class of operators A, which will play a role in Section 3, the rigging corresponding to A is preserved under compact perturbations in $\mathcal{L}^{(A)}$.

LEMMA 2.7. *Let A be a selfadjoint operator in $\mathcal{H}$ with $\infty \in \Delta(A)$. Let $V \in \mathcal{L}^{(A)}$ be compact and $[\cdot,\cdot]$-symmetric, and assume that $\infty \in \Delta(A\dot{+}V)$. Assume, further, that there exists a connected open subset Δ of $\overline{\mathbb{R}}$ such that $\infty \in \Delta$, $\overline{\Delta} \subset \Delta(A) \cap \cap\Delta(A\dot{+}V)$, the endpoints of Δ do not belong to $c(A) \cup c(A\dot{+}V)$, and $A \mid E(\Delta; A)\mathcal{H}$ and $A\dot{+}V \mid E(\Delta; A\dot{+}V)\mathcal{H}$ are nonnegative. Then $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A) = \mathcal{H}^{(*)}_{+\frac{1}{2}}(A\dot{+}V)$.*

Proof. The norms $\|\cdot\|^{(*)}_{+\frac{1}{2}}$ and $x \longmapsto ([AE(\Delta; A)x, x] + \|x\|^2)^{\frac{1}{2}}$ are equivalent on $\mathcal{D}(A)$. Then there exists an $\alpha_0 > 0$ such that for every $\alpha \geqslant \alpha_0$

$$\mathcal{H}^{(*)}_{+\frac{1}{2}}(A) \ni x \longmapsto ([\tilde{A}x, x] + \alpha\|x\|^2)^{\frac{1}{2}}$$

is a norm equivalent to $\|\cdot\|^{(*)}_{+\frac{1}{2}}$ on $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A)$. Now we write V as a sum $V = V_1 + V_2$, $V_1, V_2 \in \mathcal{L}^{(A)}$, where

$$|[V_1x, x]| \leqslant \frac{1}{2}([\tilde{A}x, x] + \alpha_0\|x\|^2), \quad x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}(A),$$

and V_2 is the extension by continuity of a bounded operator in $\mathcal{H}$. Then the norms $x \longmapsto ([(\tilde{A}+V_1)x,x]+\alpha_0\|x\|^2)^{\frac{1}{2}}$ and $\|\cdot\|_{+\frac{1}{2}}^{(*)}$ are equivalent on $\mathcal{H}_{+\frac{1}{2}}^{(*)}$ and there exists an $\alpha_1 \geqslant \alpha_0$ such that for $\alpha \geqslant \alpha_1$

$$\mathcal{H}_{+\frac{1}{2}}^{(*)}(A) \ni x \longmapsto ([(\tilde{A}+V)x,x]+\alpha\|x\|^2)^{\frac{1}{2}} \tag{2.18}$$

is a norm equivalent to $\|\cdot\|_{+\frac{1}{2}}^{(*)}$ on $\mathcal{H}_{+\frac{1}{2}}^{(*)}(A)$. By definition $\mathcal{D}(A\underset{\smile}{+}V)$ is dense in $\mathcal{H}_{+\frac{1}{2}}^{(*)}(A)$. Hence the closure of $\mathcal{D}(A\underset{\smile}{+}V)$ with respect to (2.18) is equal to $\mathcal{H}_{+\frac{1}{2}}^{(*)}(A)$. Now one proves as above that for sufficiently large α the norm of $\mathcal{H}_{+\frac{1}{2}}^{(*)}(A\underset{\smile}{+}V)$ is equivalent to (2.18) on $\mathcal{D}(A\underset{\smile}{+}V)$.

2.8. Given A, t_0 and Δ as in Section 2.4. Let A' be the operator defined in Section 2.4. Then, evidently, $\mathcal{L}^{(A)} = \mathcal{L}^{(A')}$. Let the $[\cdot,\cdot]$-symmetric operator $V \in \mathcal{L}^{(A)}$ be compact. Then $A\underset{\smile}{+}V$ and $A'\underset{\smile}{+}V$ are selfadjoint operators in $\mathcal{H}$ and $\mathcal{H}_A$, respectively. In the sequel we shall need the following connections between $A\underset{\smile}{+}V$ and $A'\underset{\smile}{+}V$.

LEMMA 2.8. (a) *If G is an open subset of $\rho(A)$, then*

$$G \cap \sigma(A\underset{\smile}{+}V) = G \cap \sigma(A'\underset{\smile}{+}V). \tag{2.19}$$

If $\lambda \in G$ is an isolated point of $\sigma(A\underset{\smile}{+}V)$ and $\mathcal{L}_\lambda$ ($\mathcal{L}'_\lambda$) is the root space of $A\underset{\smile}{+}V$ (resp. $A'\underset{\smile}{+}V$) corresponding to α, then $\mathcal{L}_\lambda = \mathcal{L}'_\lambda \subset \mathcal{H}_{+\frac{1}{2}}^{(*)}$ and $A\underset{\smile}{+}V$ and $A'\underset{\smile}{+}V$ coincide on $\mathcal{L}_\lambda$.

(b) *If $A\underset{\smile}{+}V$ and $A'\underset{\smile}{+}V$ are definitizable over $(-\infty,-t_0)\cup(t_0,\infty)$ and ∞ is no accumulation point of $\sigma_0(A\underset{\smile}{+}V)$ (or, equivalently, of $\sigma_0(A'\underset{\smile}{+}V)$), then*

$$(I - E(\overline{\mathbb{R}}\setminus(-s,s); A\underset{\smile}{+}V))\mathcal{H} = (I - E(\overline{\mathbb{R}}\setminus(-s,s); A'\underset{\smile}{+}V))\mathcal{H}_A \tag{2.20}$$

for every real $s > t_0$. The operators $A\underset{\smile}{+}V$ and $A'\underset{\smile}{+}V$ coincide on the subspaces (2.20). In particular, we have $\sigma(A\underset{\smile}{+}V) = \sigma(A'\underset{\smile}{+}V)$.

(c) *If, in addition to the assumptions of* (b), *one of the operators $A\underset{\smile}{+}V$ or $A'\underset{\smile}{+}V$ is definitizable over $\overline{\mathbb{R}}\setminus[-t_0,t_0]$ then the other also has this property.*

Proof. For every $z \in G$ the operator $(A - zI)^{\sim}$ is an isomorphism of $\mathcal{H}_{+\frac{1}{2}}^{(*)}$ onto $\mathcal{H}_{-\frac{1}{2}}$. The operators $(A-zI)^{\sim}+V$, $z \in G$, are Fredholm operators of index 0. A point $z \in G$ belongs to $\sigma(A\underset{\smile}{+}V)$ if and only if $(A - zI)^{\sim}+V$ is not one-to-one. The same holds for $\sigma(A\underset{\smile}{+}V)$ replaced by $\sigma(A'\underset{\smile}{+}V)$. Hence (2.19) is true.

The proofs of the rest of assertion (a) and of assertion (b) are similar to the proof of assertion (iii) of Lemma 2.1.

Let the assumptions of (b) be fulfilled. Assume that $A'\underset{\smile}{+}V$ is definitizable over $\overline{\mathbb{R}}\setminus[-t_0,t_0]$. Then, in view of (a), ∞ is no accumulation point of $\sigma_0(A\underset{\smile}{+}V)$. According to (b) there exists a $t' > t_0$, t', $-t' \notin c(A\dot{+}V)$, such that $(-\infty, -t')$ is of negative type and (t', ∞) is of positive type with respect to $A\dot{+}V$. We set $\Delta' := \overline{\mathbb{R}} \setminus (-t', t')$. To prove that $A\dot{+}V$ is definitizable over $\overline{\mathbb{R}} \setminus [-t_0, t_0]$ it remains to show that

$$E(\Delta'; A\dot{+}V)\mathcal{H} = \bigvee_{n>t'} E((-n, -t') \cup (t', n); A\dot{+}V)\mathcal{H}. \tag{2.21}$$

Since $E(\Delta'; A\dot{+}V)\mathcal{D}(A\dot{+}V)$ is dense in $E(\Delta'; A\dot{+}V)\mathcal{H}$ and by (b)

$$\bigcup_{n>t'} E((-n, -t') \cup (t', n); A\dot{+}V)\mathcal{H} = \bigcup_{n>t'} E((-n, -t') \cup (t', n); A'\dot{+}V)\mathcal{H}_A =: \mathcal{M},$$

it is sufficient to prove that an arbitrary $x \in E(\Delta'; A\dot{+}V)\mathcal{D}(A\dot{+}V)$ can be approximated in $\mathcal{H}$ by elements of $\mathcal{M}$. We have

$$E(\Delta'; A\dot{+}V)\mathcal{D}(A\dot{+}V) = \left\{ y \in \mathcal{D}(A\dot{+}V) \ : \ [y, (I - E(\Delta'; A\dot{+}V))\mathcal{H}] = \{0\} \right\} \subset$$

$$\subset \left\{ y \in \mathcal{H}^{(*)}_{+\frac{1}{2}} \ : \ [y, (I - E(\Delta'; A'\dot{+}V))\mathcal{H}] = \{0\} \right\} = E(\Delta'; A'\dot{+}V)\mathcal{H}^{(*)}_{+\frac{1}{2}}.$$

By assumption every element of $E(\Delta'; A'\dot{+}V)\mathcal{H}^{(*)}_{+\frac{1}{2}}$ can be approximated in $\mathcal{H}$ by elements of $\mathcal{M}$. This implies (2.21). If $A\dot{+}V$ is definitizable over $\overline{\mathbb{R}} \setminus [-t_0, t_0]$ an analogous reasoning applies.

3. PRESERVATION OF THE DEFINITIZABILITY OVER A NEIGHBOURHOOD OF ∞ UNDER PERTURBATIONS

3.1. Let A be a selfadjoint operator definitizable over a neighbourhood of ∞ in the Krein space $\mathcal{H}$. Again we set $R(\cdot) := R(\cdot\,; A)$ and $E(\cdot) := E(\cdot\,; A)$. In what follows we consider operators of the form $A\dot{+}V$ with compact $V \in \mathcal{L}^{(A)}$. The following proposition shows that every operator of this form has a nonempty resolvent set.

PROPOSITION 3.1. *Let A be a selfadjoint operator in $\mathcal{H}$ definitizable over neighbourhood of ∞. Let $V \in \mathcal{L}^{(A)}$ be compact. Then there exists an $\eta_0 > 0$ such that $i\eta \in \rho(A\dot{+}V)$ for $|\eta| \geqslant \eta_0$.*

Proof. We use the notations introduced in (2.12) and Remark 2.6. On account of Lemma 2.5 it is sufficient to prove that

$$\lim_{|\eta|\uparrow\infty, \eta\in\mathbb{R}} V_1 \tilde{R}(i\eta) V_2 = 0$$

with respect to the operator norm in ℓ^2_ν. For $\mathrm{i}\eta \in \rho(A)$ we have

$$V_1 \tilde{R}(\mathrm{i}\eta) V_2 = \begin{pmatrix} s_1^{\frac{1}{2}} & 0 & & \\ 0 & s_2^{\frac{1}{2}} & & \\ & & \ddots & 0 \\ & & & \ddots \\ & 0 & & \ddots \end{pmatrix} \begin{pmatrix} [\tilde{R}(\mathrm{i}\eta)f_1, e_1] & [\tilde{R}(\mathrm{i}\eta)f_2, e_1] & \dots \\ [\tilde{R}(\mathrm{i}\eta)f_1, e_2] & [\tilde{R}(\mathrm{i}\eta)f_2, e_2] & \dots \\ \vdots & \vdots & \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} s_1^{\frac{1}{2}} & 0 & & \\ 0 & s_2^{\frac{1}{2}} & & \\ & & \ddots & 0 \\ & & & \ddots \\ & 0 & & \ddots \end{pmatrix}.$$

Since the two diagonal operators in this relation are compact, it is sufficient to prove that the operator in the middle converges to 0 in the weak sense if $|\eta| \uparrow \infty$, $\eta \in \mathbb{R}$, or, equivalently,

$$\lim_{\eta \uparrow \infty} [\tilde{R}(\mathrm{i}\eta) f, f] = 0 \tag{3.1}$$

for every $f \in \mathcal{H}_{-\frac{1}{2}}$. This holds if and only if

$$\lim_{\eta \uparrow \infty} [\tilde{R}(\mathrm{i}\eta) \tilde{E}(\Delta_\infty) f, \tilde{E}(\Delta_\infty) f] = 0 \tag{3.2}$$

for some open neighbourhood Δ_∞ of ∞ in $\overline{\mathbb{R}}$ and every $f \in \mathcal{H}_{-\frac{1}{2}}$.

If $(E(\Delta_\infty)\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space and $f \in \mathcal{H}_{-\frac{1}{2}}$, then

$$[\tilde{R}(\mathrm{i}\eta)\tilde{E}(\Delta_\infty) f, \tilde{E}(\Delta_\infty) f] = \int_{\Delta_\infty} (|t|^2 + 1)^{\frac{1}{2}} (t - \mathrm{i}\eta)^{-1} \mathrm{d}[E(t)x, x]$$

for some $x \in E(\Delta_\infty)\mathcal{H}$ (see the remark at the end of Section 2.3). Evidently, the right hand side of this relation converges to 0 if $|\eta| \uparrow \infty$.

Assume now that $(E(\Delta_\infty)\mathcal{H}, [\cdot,\cdot])$ is a Krein space such that the restriction of A to $E(\Delta_\infty)\mathcal{H}$ is nonnegative. Let $f \in \mathcal{H}_{-\frac{1}{2}}$ and $g := \tilde{E}(\Delta_\infty) f$. We have $\mathrm{Im}\{z[\tilde{R}(z)g, g]\} = (\mathrm{Im}\, z)[A\tilde{R}(z)g, \tilde{R}(z)g] \geqslant 0$ if $\mathrm{Im}\, z > 0$. If we set $r(z) := [\tilde{R}(z)g, g]$, $z \in \rho(A)$, then according to [11; §1.1]

$$\lim_{\eta \uparrow \infty} r(\mathrm{i}\eta) = \lim_{\eta \uparrow \infty} (\mathrm{i}\eta)^{-1} \mathrm{i}\eta r(\mathrm{i}\eta) = \lim_{\eta \uparrow \infty} \eta^{-1} \mathrm{Im}(\mathrm{i}\eta r(\mathrm{i}\eta)) = \lim_{\eta \uparrow \infty} \mathrm{Re}\, r(\mathrm{i}\eta).$$

A simple computation shows that

$$\mathrm{Re}\, r(\mathrm{i}\eta) = [\tilde{A}\tilde{R}(\mathrm{i})g, \tilde{R}(\mathrm{i})g] + (1 - \eta^2)[AR(\mathrm{i}\eta)\tilde{R}(\mathrm{i})g, R(\mathrm{i}\eta)\tilde{R}(\mathrm{i})g].$$

By [10; Proposition 3.1] the right hand side of this relation converges to 0 if $\eta \uparrow \infty$, and the Proposition 3.1 is proved.

REMARK 3.2. Let A, V and η_0 be as in Proposition 3.1. Then according to (2.17) and to the fact that the operators $\tilde{R}(i\eta)$, $|\eta| \geqslant \eta_0$, are uniformly bounded there exists an $\varepsilon_0 > 0$ such that for every compact $W \in \mathcal{L}^{(A)}$ with $\|W\|_{\mathcal{L}^{(A)}} \leqslant \varepsilon_0$ we have $i\eta \in \rho(A\dot{+}(V+W))$ for $|\eta| \geqslant \eta_0$.

The following proposition is a consequence of Proposition 3.1 and a result from [9].

PROPOSITION 3.3. *Let A and V be as in Proposition 3.1. Assume, in addition, that the quadratic form $x \longmapsto [(\tilde{A}+V)x, x]$ on $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ is symmetric and has a finite number of negative squares. Then $A\dot{+}V$ is definitizable and $(0,\infty)$ ($(-\infty,0)$) is of type π_+ (resp. π_-) with respect to $A\dot{+}V$.*

If A is nonnegative, $0 \in \rho(A)$, and $V \in \mathcal{L}^{(A)}$ is compact and $[\cdot,\cdot]$-symmetric, the assumptions are fulfilled.

Proof. It is easy to see that there exists a $[\cdot,\cdot]$-symmetric $V_0 \in \mathcal{L}^{(A)}$ of finite rank such that $\tilde{A}+V+V_0$ is $[\cdot,\cdot]$-nonnegative. Then, by Proposition 3.1, $\rho_0 := \rho(A\dot{+}V) \cap \rho(A\dot{+}(V+V_0)) \neq \emptyset$. For $z \in \rho_0$ the operator

$$((A-zI)^{\sim}+V)^{-1} - ((A-zI)^{\sim}+V+V_0)^{-1}$$

is of finite rank. Hence the difference of the resolvents of $A\dot{+}V$ and $A\dot{+}(V+V_0)$ is of finite rank. Since $A\dot{+}(V+V_0)$ is nonnegative, it follows by [9; Theorem 1] that $A\dot{+}V$ is definitizable and that the above statements on the type of $(0,\infty)$ and $(-\infty,0)$ are true.

REMARK 3.4. Making use of this proposition the results of [5] concerning the preservation of definitizability can be extended to the case when the unperturbed operator is not fundamentally reducible.

3.2. The following lemma will be needed in the proofs of our main results in Section 3.3.

For a closed operator B in $\mathcal{H}$ such that $it \in \rho(B)$ for every real t with $\eta_0 \leqslant |t| < \infty$, where η_0 is some positive number, we set

$$T(\eta_0,\eta;B) := -(2\pi i)^{-1}\Big(\int_{-i\eta}^{-i\eta_0} R(z;B)dz + \int_{i\eta_0}^{i\eta} R(z;B)dz,$$

$\eta \in (\eta_0,\infty)$. If the strong limit $\operatorname{s-lim}_{\eta\uparrow\infty} T(\eta_0,\eta;B)$ exists, it is denoted by $T(\eta_0;B)$. If B is selfadjoint in $\mathcal{H}$, then the operators $T(\eta_0,\eta;B)$ and $T(\eta_0;B)$ are also selfadjoint.

The operators $T(\eta_0; B)$ were first used by K. Veselić in the study of nonnegative operators in Krein spaces ([21], see also [6], [5]).

LEMMA 3.5. *Let A be a selfadjoint operator in $\mathcal{H}$ definitizable over a neighbourhood of ∞ such that $\infty \notin c_s(A)$. Let $V \in \mathcal{L}^{(A)}$ be compact. Then for sufficiently large $\eta_0 > 0$ the strong limit*

$$\text{s-}\lim_{\eta\uparrow\infty} T(\eta_0, \eta; A \dot{+} V)$$

exists. $T(\eta_0; A\dot{+}V)$ depends continuously on V with respect to the norm topologies and we have

$$T(\eta_0; A\dot{+}V) - T(\eta_0; A) \in \mathfrak{S}_\infty. \tag{3.3}$$

If, in addition, V belongs to $\mathfrak{S}_p$, $p \in [1, \infty)$, then

$$T(\eta_0; A\dot{+}V) - T(\eta_0; A) \in \mathfrak{S}_p. \tag{3.4}$$

Proof. 1. Let t_0 be a real number, $t_0 > 1$, such that A is definitizable over a neighbourhood (in $\overline{\mathbb{R}}$) of $\Delta := \overline{\mathbb{R}} \setminus (-t_0, t_0)$, and $t_0, -t_0 \notin c(A)$. We assume that $E([t_0, \infty))$ is nonnegative and $E((-\infty, -t_0])$ is either nonnegative or nonpositive. This is no restriction. We choose the Hilbert scalar product $(\cdot, \cdot)$ on $\mathcal{H}$ as in (2.3) and define $A_\Delta := A \mid E(\Delta)\mathcal{H}$. By Proposition 3.1 there exists an $\eta_0 > 0$ such that

$$\{\mathrm{i}t \ : \ t \in \mathbb{R}, \ |t| \geqslant \eta_0\} \subset \rho(A) \cap \rho(A\dot{+}V). \tag{3.5}$$

We use the notations introduced in (2.4), (2.12), Lemma 2.5 and Remark 2.6. By Lemma 2.5 and Proposition 3.1 the function Q^{-1} is bounded on $\{\mathrm{i}t \ : \ t \in \mathbb{R}, |t| \geqslant \eta_0\}$. Let $\eta_0 \leqslant \eta \leqslant \eta'$ and let x and y be arbitrary elements of $\mathcal{H}$. Then according to (2.17)

$$\begin{aligned}
&\Big|\Big[\Big\{T(\eta_0, \eta'; A\dot{+}V) - \\
&-T(\eta_0, \eta'; A) - \Big(T(\eta_0, \eta; A\dot{+}V) - T(\eta_0, \eta; A)\Big)\Big\}x, y\Big]\Big| = \\
&= (2\pi)^{-1}\left| \int_{(-\eta', -\eta)\cup(\eta, \eta')} \Big[\Big\{R(\mathrm{i}\mu; A\dot{+}V) - R(\mathrm{i}\mu)\Big\}x, y\Big]\,\mathrm{d}\mu\right| = \\
&= (2\pi)^{-1}\left| \int_{(-\eta', -\eta)\cup(\eta, \eta')} \Big[V_2 Q^{-1}(\mathrm{i}\mu) V_1 \tilde{R}(\mathrm{i}\mu)x, \tilde{R}(-\mathrm{i}\mu)y\Big]\,\mathrm{d}\mu\right| \leqslant \\
&\leqslant M \int_{(-\eta', -\eta)\cup(\eta, \eta')} \|R(\mathrm{i}\mu)x\|^{(*)}_{+\frac{1}{2}} \|R(-\mathrm{i}\mu)y\|^{(*)}_{+\frac{1}{2}}\,\mathrm{d}\mu \leqslant \\
&\leqslant M\left(\int_{(-\eta', -\eta)\cup(\eta, \eta')} \|R(\mathrm{i}\mu)x\|^{(*)2}_{+\frac{1}{2}}\,\mathrm{d}\mu\right)^{\frac{1}{2}} \cdot \\
&\cdot\left(\int_{(-\eta', -\eta)\cup(\eta, \eta')} \|R(-\mathrm{i}\mu)y\|^{(*)2}_{+\frac{1}{2}}\,\mathrm{d}\mu\right)^{\frac{1}{2}},
\end{aligned} \tag{3.6}$$

where M is a constant. Further, we have

$$\begin{aligned}\|R(\mathrm{i}\mu)x\|_{+\frac{1}{2}}^{(*)2} = \int_{\Delta} \left(|t|^2+1\right)^{\frac{1}{2}} \left(|t|^2+\mu^2\right)^{-1} \mathrm{d}\left(E(t)x,x\right) + \\ +\left\| \left(I-E(\Delta)\right) R(\mathrm{i}\mu)x\right\|_{0,+}^2 \leqslant \\ \leqslant 2\int_{\Delta} |t|\left(|t|^2+\mu^2\right)^{-1} \mathrm{d}\left(E(t)x,x\right) + \left\|\left(I-E(\Delta)\right)R(\mathrm{i}\mu)x\right\|_{0,+}^2.\end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}\int_{(-\eta',-\eta)\cup(\eta,\eta')} \|R(\mathrm{i}\mu)x\|_{+\frac{1}{2}}^{(*)2}\mathrm{d}\mu \leqslant \\ \leqslant 4\int_{\Delta}\left(\arctan\eta'|t|^{-1} - \arctan\eta|t|^{-1}\right)\mathrm{d}\left(E(t)x,x\right) + M'\int_{(\eta,\eta')}|\mu|^{-2}\mathrm{d}\mu,\end{aligned} \tag{3.8}$$

where the constant M' does not depend on η, η' and x, if x belongs to a fixed bounded set. It follows that for fixed x the left hand side of (3.8) converges to 0 if $\eta\uparrow\infty$ and $\eta'\uparrow\infty$.

A relation similar to (3.8) shows that there exits an $M''<\infty$ such that

$$\int_{(-\eta',-\eta)\cup(\eta,\eta')} \|R(-\mathrm{i}\mu)y\|_{+\frac{1}{2}}^{(*)}\mathrm{d}\mu \leqslant M''$$

for all $y\in\mathcal{H}$ with $|y|\leqslant 1$ and all real numbers η, η' with $\eta_0\leqslant\eta\leqslant\eta'$. Therefore, on account of (3.6), $T(\eta_0,\eta;A\dot{+}V)-T(\eta_0,\eta;A)$ converges strongly if $\eta\uparrow\infty$. Since by the assumptions on A, $T(\eta_0,\eta;A)$ converges in the strong sense, the strong limit $\operatorname{s-lim}_{\eta\uparrow\infty} T(\eta_0,\eta;A\dot{+}V)$ exists.

2. Now we are going to prove the continuity statement. Let (V_n) be a sequence of compact operators in $\mathcal{L}^{(A)}$ converging to $V_\infty := V$ with respect to the operator norm. We may assume that

$$\{\mathrm{i}t\ :\ t\in\mathbb{R},\ \ |t|\geqslant\eta_0\}\subset\rho(A\dot{+}V_n),\qquad n=1,2,\ldots,\infty$$

(see Remark 3.2). Let d be a positive number such that

$$\sup\left\{\|R(\mathrm{i}t)x\|_{+\frac{1}{2}}^{(*)} : x\in\mathcal{H},\ \|x\|_{-\frac{1}{2}}\leqslant 1,\ \ |t|\geqslant\eta_0\right\}\leqslant d. \tag{3.9}$$

By the compactness of V_∞ there exists a compact $V_0\in\mathcal{L}^{(A)}$ with

$$\|V_0\|_{\mathcal{L}^{(A)}} < d^{-1}$$

such that $V_\infty = V_0 + V_0'$, where V_0' is an operator of finite rank in $\mathcal{H}$. Then the sequence of the operators $V_{0,n} := V_n - V_0'$, $n = 1, 2, \ldots$, converges to $V_{0,\infty} := V_0$. We may assume that

$$\sup\left\{ \|V_{0,n}\|_{\mathcal{L}^{(A)}} : n = 1, 2, \ldots \right\} < d^{-1}.$$

Then according to (3.9) it follows that

$$\{\mathrm{i}t \ : \ t \in \mathbb{R}, |t| \in [\eta_0, \infty)\} \subset \rho(A \dot{+} V_{0,n}), \quad n = 1, 2, \ldots, \infty.$$

Moreover, we have

$$\left| \left[\left\{ T(\eta_0, \eta; A \dot{+} V_n) - T(\eta_0, \eta; A \dot{+} V_\infty) \right\} x, y \right] \right| = (2\pi)^{-1} \cdot$$
$$\cdot \left| \int\limits_{(-\eta, -\eta_0) \cup (\eta_0, \eta)} \left[\sum_{j=1}^{\infty} (-1)^j \left\{ (V_{0,n} \tilde{R}(\mathrm{i}\mu))^j - (V_{0,\infty} \tilde{R}(\mathrm{i}\mu))^j \right\} x, \tilde{R}(-\mathrm{i}\mu) y \right] \mathrm{d}\mu \right|,$$
$$n = 1, 2, \ldots .$$

Then analogously to part 1 of the proof ((3.7) and the following considerations) it follows that

$$\lim_{n \to \infty} \|TA \dot{+} V_{0,n} - TA \dot{+} V_{0,\infty}\| = 0 \tag{3.10}$$

uniformly for $\eta \in [\eta_0, \infty)$. Similarly we obtain

$$\lim_{n \to \infty} \|R(\mathrm{i}\mu; A \dot{+} V_{0,n}) - R(\mathrm{i}\mu; A \dot{+} V_{0,\infty})\| = 0 \tag{3.11}$$

uniformly for $\mu \in [\eta_0, \infty)$ and

$$R(\mathrm{i}\mu; A \dot{+} V_{0,n})\| = O(|\mu|^{-1}), \quad |\mu| \to \infty, n = 1, 2, \ldots, \infty, \tag{3.12}$$

uniformly with respect to $n = 1, 2, \ldots, \infty$.

It easy to see that for every $\mu \in \mathbb{R}$ with $\|\mu\| \geqslant \eta_0$ we have

$$\begin{aligned} R(\mathrm{i}\mu; A \dot{+} V_n) &= R(\mathrm{i}\mu; A \dot{+} V_{0,n}) - R(\mathrm{i}\mu; A \dot{+} V_{0,n}) V_0' \times \\ &\times (I + R(\mathrm{i}\mu; A \dot{+} V_{0,n}) V_0')^{-1} R(\mathrm{i}\mu; A \dot{+} V_{0,n}), \quad n = 1, 2, \ldots, \infty. \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} & T(\eta_0, \eta; A \dot{+} V_\infty) - T(\eta_0, \eta; A \dot{+} V_n) = \\ & = T(\eta_0, \eta; A \dot{+} V_{0,\infty}) - T(\eta_0, \eta; A \dot{+} V_{0,n}) + \\ & + (2\pi \mathrm{i})^{-1} \int\limits_{(-\eta, -\eta_0) \cup (\eta_0, \eta)} R(\mathrm{i}\mu; A \dot{+} V_{0,\infty}) \cdot \\ & \cdot V_0' (I + R(\mathrm{i}\mu; A \dot{+} V_{0,\infty}) V_0')^{-1} R(\mathrm{i}\mu; A \dot{+} V_{0,\infty}) \mathrm{d}\mu - \\ & - (2\pi \mathrm{i})^{-1} \int\limits_{(-\eta', -\eta) \cup (\eta, \eta')} R(\mathrm{i}\mu; A \dot{+} V_{0,n}) \cdot \\ & \cdot V_0' (I + R(\mathrm{i}\mu; A \dot{+} V_{0,n}) V_0')^{-1} R(\mathrm{i}\mu; A \dot{+} V_{0,n}) \mathrm{d}\mu. \end{aligned} \tag{3.14}$$

Then (3.10), (3.11) and (3.12) yield

$$\lim_{n\to\infty} \|T(\eta_0,\eta;A\dot{+}V_\infty) - T(\eta_0,\eta;A\dot{+}V_n)\| = 0$$

uniformly for $\eta \in [\eta_0,\infty)$. Therefore $T(\eta_0;A\dot{+}V)$ depends continuously on V with respect to the norm topologies.

Let (W_n) be a sequence of operators of finite rank in $\mathcal{H}$ which converges to V in $\mathcal{L}^{(A)}$. Then

$$\lim_{n\to\infty} \|T(\eta_0;A+W_n) - T(\eta_0;A) - (T(\eta_0;A\dot{+}V) - T(\eta_0;A))\| = 0.$$

Since by (3.13) and (3.12) (with $V_{0,n} = 0$, $V_0' = W_n$) we have $T(\eta_0;A+W_n) - -T(\eta_0;A) \in \mathfrak{S}_\infty$ for every n, it follows that (3.3) holds.

3. Assume now that $V \in \mathfrak{S}_1$. We shall prove that $T(\eta_0;A\dot{+}V) - T(\eta_0;A) \in \mathfrak{S}_1$. For every $\eta \geqslant \eta_0$ we have

$$\begin{aligned}(3.15)\qquad &\|T(\eta_0,\eta;A\dot{+}V) - T(\eta_0,\eta;A)\|_{\mathfrak{S}_1} \leqslant \\ &\leqslant M_0 \int\limits_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|\tilde{R}(\mathrm{i}\mu)V_2\|_{\mathfrak{S}_2}\|V_1\tilde{R}(\mathrm{i}\mu)\|_{\mathfrak{S}_2}\mathrm{d}\mu \leqslant M_0 \cdot \\ &\cdot\Bigg(\int\limits_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|\tilde{R}(\mathrm{i}\mu)V_2\|^2_{\mathfrak{S}_2}\mathrm{d}\mu\Bigg)^{\frac12}\Bigg(\int\limits_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|V_1R(\mathrm{i}\mu)\|^2_{\mathfrak{S}_2}\mathrm{d}\mu\Bigg)^{\frac12},\end{aligned}$$

where M_0 does not depend on η. Here $\|\cdot\|_{\mathfrak{S}_2}$ denotes the Hilbert-Schmidt norm operators in $\mathfrak{S}_2(\ell^2_\nu,\mathcal{H})$ or $\mathfrak{S}_2(\mathcal{H},\ell^2_\nu)$. To prove that the right hand side of (3.15) is uniformly bounded for $\eta \in [\eta_0,\infty)$ it is sufficient to prove that the integrals

$$(3.16)\qquad \int\limits_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|E(\Delta)\tilde{R}(\mathrm{i}\mu)V_2\|^2_{\mathfrak{S}_2}\mathrm{d}\mu \quad\text{and}\quad \int\limits_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|V_1R(\mathrm{i}\mu)E(\Delta)\|^2_{\mathfrak{S}_2}\mathrm{d}\mu$$

are uniformly bounded for $\eta \in [\eta_0,\infty)$. We have

$$\begin{aligned}\|E(\Delta)\tilde{R}(\mathrm{i}\mu)V_2\|^2_{\mathfrak{S}_2} &= \|E(\Delta)\tilde{R}(\mathrm{i}\mu)V_2(E(\Delta)\tilde{R}(\mathrm{i}\mu)V_2)^*\|_{\mathfrak{S}_1} \leqslant \\ &\leqslant \sum_{j=1}^{\nu} s_j\|E(\Delta)\tilde{R}(\mathrm{i}\mu)f_j\|^2\end{aligned}$$

and

$$\begin{aligned}\|V_1R(\mathrm{i}\mu)E(\Delta)\|^2_{\mathfrak{S}_2} &= \|(V_1R(\mathrm{i}\mu)E(\Delta))^*V_1R(\mathrm{i}\mu)E(\Delta)\|_{\mathfrak{S}_1} \leqslant \\ &\leqslant \sum_{j=1}^{\nu} s_j\|E(\Delta)\tilde{R}(-\mathrm{i}\mu)e_j\|^2.\end{aligned}$$

We define two bounded subsets $\{x_j\}$ and $\{y_j\}$ of $E(\Delta)\mathcal{H}$ of cardinality ν by $\tilde{E}(\Delta)e_j = \vec{r}_\Delta x_j$ and $\tilde{E}(\Delta)f_j = \vec{r}_\Delta y_j$ (see (2.5)). In view of (2.4) we have

$$\|E(\Delta)\tilde{R}(\mathrm{i}\mu)f_j\|^2 = \|R(\mathrm{i}\mu)y_j\|_{+\frac{1}{2}}^{(*)2}$$

and

$$\|E(\Delta)\tilde{R}(-\mathrm{i}\mu)e_j\|^2 = \|R(-\mathrm{i}\mu)x_j\|_{+\frac{1}{2}}^{(*)2}.$$

As in part 1 of the proof we see that the integrals

$$\int_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|R(\mathrm{i}\mu)y_j\|_{+\frac{1}{2}}^{(*)2}\mathrm{d}\mu \quad \text{and} \quad \int_{(-\eta,-\eta_0)\cup(\eta_0,\eta)} \|R(-\mathrm{i}\mu)x_j\|_{+\frac{1}{2}}^{(*)2}d\mu$$

are uniformly bounded for $\eta \in [\eta_0, \infty)$. Since by assumption $\sum_{j=1}^{\nu} s_j < \infty$, the integrals (3.16) are uniformly bounded for $\eta \in [\eta_0, \infty)$. Then it follows from part 1 of the proof and [20; Theorem 2.7] that

$$T(\eta_0; A\pm V) - T(\eta_0; A) \in \mathfrak{S}_1.$$

4. Assume now that $V \in \mathfrak{S}_p$ for some $p \in [1,\infty)$. We write V in the form

$$V = \sum_{j=1}^{\nu} s_j[\,\cdot\,,e_j]f_j,$$

where s_j, e_j, f_j are as in (2.12). We have $\sum_{j=1}^{\nu} s_j^p < \infty$. Choose now $\delta_j \in (0,1)$, $j = 1,\dots,\nu$, such that $\lim_{j\to\infty} \delta_j = 0$ if $\nu = \infty$, and $\sum_{j=1}^{\nu} s_j^p\delta_j^{-p} < \infty$ (see [19; Lemma 1.7.4]). We define a continuous function $\hat{V}$ on the closure $\overline{G}$ (in $\mathbf{C}$) of $G := \{z : 0 < \operatorname{Re} z < p\}$ with values in $\mathcal{L}^{(A)}$ by

$$\hat{V}(z) := \sum_{j=1}^{\nu}(s_j\delta_j^{-1})^{p-z}\delta_j[\,\cdot\,,e_j]f_j, \quad z \in \overline{G},$$

where $(s_j\delta_j^{-1})^{p-z} := \exp\{(p-z)\log_0(s_j\delta_j^{-1})\}$ and $\log_0$ denotes the principal value of the logarithm. The values of $\hat{V}$ are compact. Set

$$s_j' := \max\{(s_j\delta_j^{-1})^t\delta_j : t \in [0,p]\}.$$

In this part of the proof we additionally assume on η_0 that

$$\sup\left\{\|V_1'\tilde{R}(\mathrm{i}t)V_2'\|_{\ell^2_\nu\to\ell^2_\nu} : t \in \mathbf{R},\, |t| \geqslant \eta_0\right\} \leqslant \frac{1}{2}, \tag{3.17}$$

where V_1' and V_2' are defined as V_1 and V_2 in Remark 2.6 with s_j replaced by s_j'. Then it follows by Lemma 2.5 that

$$\{it \; : \; t \in \mathbb{R}, |t| \geqslant \eta_0\} \subset \rho(A) \cap \rho(A \dot{+} \hat{V}(z))$$

for all $z \in \overline{G}$.

We consider the operator functions F_η, $\eta \in [\eta_0, \infty)$, on $\overline{G}$ defined by

$$F_\eta(z) := T(\eta_0, \eta; A \dot{+} \hat{V}(z)) - T(\eta_0, \eta; A), \quad z \in \overline{G}.$$

According to part 2 of this proof the functions F_η are continuous on $\overline{G}$. The function $\hat{V}$ is holomorphic on G. Then one verifies without difficulty, making use of Lemma 2.5, that $z \mapsto R(it; A \dot{+} \hat{V}(z))$ is holomorphic in G for every fixed $t \in \mathbb{R}$ with $|t| \geqslant \eta_0$. As a consequence the functions F_η, $\eta \in [\eta_0, \infty)$, are holomorphic in G.

By part 2 of this proof the values of F_η are compact operators. We have

$$F_\eta(p-1) = T(\eta_0, \eta; A \dot{+} V) - T(\eta_0, \eta; A), \quad \eta \in [\eta_0, \infty).$$

From (3.17), Lemma 2.5 and part 1 of this proof we conclude that

$$\sup\Big\{ \|F_\eta(p + iy)\| \; : \; y \in \mathbb{R}, \eta \in [\eta_0, \infty)\Big\} < \infty.$$

According to part 3 of the proof we have

$$\sup\{\|F_\eta(iy)\|_{\mathfrak{S}_1} \; : \; y \in \mathbb{R}, \eta \in [\eta_0, \infty)\} < \infty.$$

Then on account of [4; Theorem 13.1] it follows that the operators

$$T(\eta_0, \eta; A \dot{+} V) - T(\eta_0, \eta; A)$$

are uniformly bounded in $\mathfrak{S}_p$. Then [20; Theorem 2.7] gives

$$T(\eta_0; A \dot{+} V) - T(\eta_0; A) \in \mathfrak{S}_p,$$

which proves the lemma.

3.3. Now we present our main results. Let $\mathcal{H}' = (\mathcal{H}, [\cdot,\cdot]')$ and $\mathcal{H}'' = (\mathcal{H}, [\cdot,\cdot]'')$ be Krein spaces with the same underlying linear space $\mathcal{H}$. Assume that there exists a scalar product $(\cdot,\cdot)$ on $\mathcal{H}$ such that $(\mathcal{H}, (\cdot,\cdot))$ is a separable Hilbert space and $[\cdot,\cdot]'$ and $[\cdot,\cdot]''$ are continuous with respect to $(\cdot,\cdot)$. We define bounded selfadjoint operators G' and G'' in $(\mathcal{H}, (\cdot,\cdot))$ by

$$(G'x, y) = [x,y]', \quad (G''x, y) = [x,y]'', \quad x, y \in \mathcal{H}.$$

THEOREM 3.6. *Let A be a selfadjoint operator in $\mathcal{H}'$ such that $\sigma_0(A)\setminus\sigma_{\mathrm{p,norm}}(A)$ is empty or finite, A is definitizable over $\Delta := (t_2,\infty)\cup\{\infty\}\cup(-\infty,t_1)$, where t_1 and t_2 are two real points belonging to $\rho(A)$, $t_1 < t_2$, and (t_2,∞) $((-\infty,t_1))$ is of type π_+ (resp. π_-). Let $V \in \mathcal{L}^{(A)}$ be compact and let $A\dot{+}V$ be symmetric in $\mathcal{H}''$.*

Assume that one of the following conditions is fulfilled:

(i) *$G' - G''$ is compact and $\infty \notin \mathrm{c_s}(A)$.*

(ii) *$G' = G''$ and V is $[\cdot,\cdot]'$-symmetric.*

Then $A\dot{+}V$ is a selfadjoint operator in $\mathcal{H}''$ definitizable over Δ and (t_2,∞) $((-\infty,t_1))$ is of type π_+ (resp. π_-) with respect to $A\dot{+}V$. Moreover, if (i) holds, then $\infty \notin \mathrm{c_s}(A\dot{+}V)$. If (ii) holds, then $\infty \in \mathrm{c_s}(A)$ if and only if $\infty \in \mathrm{c_s}(A\dot{+}V)$.

Proof. Let first condition (i) be fulfilled. We assume that $t_1 < 0 < t_2$ and $t_1, t_2 \in \rho(A)\cap\rho(A\dot{+}V)$. This is no restriction. Let $\mathcal{O}$ be a bounded simply connected C^∞ domain of $\mathbf{C}$ symmetric with respect to $\mathbf{R}$ such that the following holds:

(α) $\mathcal{O}\cap\mathbf{R} = (t_1,t_2)$. $\mathcal{O}\cap \mathrm{i}\mathbf{R} = \{\mathrm{i}t : t \in (-\eta_0,\eta_0)\}$, where η_0 is as in Proposition 3.1.

(β) $\sigma(A)\setminus\Delta \subset \mathcal{O}$.

(γ) $\mathcal{C} := \partial\mathcal{O}$ is contained in $\rho(A\dot{+}V)$.

(δ) The sets $\mathcal{O}_1 := \{z : \operatorname{Re} z < 0\}\setminus\overline{\mathcal{O}}$ and $\mathcal{O}_\mathrm{r} := \{z : \operatorname{Re} z > 0\}\setminus\overline{\mathcal{O}}$ are connected.

We provide the curves $\mathcal{C} := \mathcal{C}\cap\{z : \operatorname{Re} z \leqslant 0\}$ and $\mathcal{C}_\mathrm{r} := \mathcal{C}\cap\{z : \operatorname{Re} z \geqslant 0\}$ with the orientation induced by $\mathcal{C}$. By the functional calculus of A the operators

$$
\begin{aligned}
P_\mathrm{l}(A) &:= T(\eta_0;A) + (2\pi\mathrm{i})^{-1}\int_{\mathcal{C}_\mathrm{l}} R(\lambda;A)\mathrm{d}\lambda + \frac{1}{2}I,\\
P_\mathrm{r}(A) &:= -T(\eta_0;A) + (2\pi\mathrm{i})^{-1}\int_{\mathcal{C}_\mathrm{r}} R(\lambda;A)\mathrm{d}\lambda + \frac{1}{2}I, \qquad (3.18)\\
P_0(A) &:= -(2\pi\mathrm{i})^{-1}\int_{\mathcal{C}} R(\lambda;A)\mathrm{d}\lambda
\end{aligned}
$$

are pairwise commuting projection and we have

$$P_\mathrm{l}(A)P_\mathrm{r}(A) = P_\mathrm{l}(A)P_0(A) = P_\mathrm{r}(A)P_0(A) = 0$$

and

$$P_\mathrm{l}(A) + P_\mathrm{r}(A) + P_0(A) = I.$$

We now choose an $\varepsilon > 0$ such that for any compact $V' \in \mathcal{L}^{(A)}$ with $\|V' - -V\|_{\mathcal{L}^{(A)}} \leqslant \varepsilon_0$,

$$\{\mathrm{i}t : |t| \geqslant \eta_0\} \cup \mathcal{C} \subset \rho(A\dot{+}V')$$

(see Remark 3.2). For every such operator V' (including $V' = V$) we define the operators $P_\mathrm{l}(A\dot{+}V')$, $P_\mathrm{r}(A\dot{+}V')$, $P_0(A\dot{+}V')$ as in (3.18). Let n_0 be an integer, $n_0 >$

$> \max(|t_1|, |t_2|)$, such that the intervals $(-\infty, -n_0)$ and (n_0, ∞) are of definite type with respect to A.

Assume now that some operator V' with the properties mentioned above has the form

$$V' = \sum_{j=1}^{k} s_j'[\cdot, e_j']f_j', \quad \text{where} \tag{3.19}$$
$$e_j', f_j' \in \cup\{(I - E(\overline{\mathbf{R}} \setminus (-n, n); A))\mathcal{H} \ : \ n \in \mathbf{N}, n > n_0\}, \quad j = 1, 2, \ldots, k.$$

Then there exists an $n_1 \in \mathbf{N}$, $n_1 > n_0$, such that the decomposition

$$\mathcal{H} = E_{n_1}\mathcal{H} + (I - E_{n_1})\mathcal{H}, \quad E_{n_1} := E(\overline{\mathbf{R}} \setminus (-n_1, n_1); A),$$

reduces $A+V'$ and the operator A and $A+V'$ coincide on $E_{n_1}\mathcal{H}$. $A+V' \,|\, (I - E_{n_1})\mathcal{H}$ is bounded. This implies that $P_l(A+V')$, $P_r(A+V')$ and $P_0(A+V')$ have the properties of the operators $P_l(A)$, $P_r(A)$ and $P_0(A)$ mentioned above.

By the compactness of V there exists a sequence (V_n') of operators of the form (3.19) which converges to V in $\mathcal{L}^{(A)}$. Let $\|V_n' - V\|_{\mathcal{L}^{(A)}} \leqslant \varepsilon_0$ for all n. For brevity we set $B := A \dot{+} V$. Then according to Lemma 3.5 the operators $P_l(A + V_n')$, $P_r(A + V_n')$, $P_0(A + V_n')$ converge to $P_l(B)$, $P_r(B)$, $P_0(B)$, respectively, with respect to the operator norm. Hence the latter operators are also pairwise commuting projections satisfying the relations

$$P_l(B)P_r(B) = P_l(B)P_0(B) = P_r(B)P_0(B) = 0$$

and

$$P_l(B) + P_r(B) + P_0(B) = I.$$

The operators (3.18) are selfadjoint in $\mathcal{H}'$. Since B is symmetric in $\mathcal{H}'$ and $\rho(B) \cap \mathbf{R} \neq \emptyset$ it follows that B and the projections $P_l(B)$, $P_r(B)$ and $P_0(B)$ are selfadjoint in $\mathcal{H}''$. By Lemma 3.5

$$P_l(B) - P_l(A) \in \mathfrak{S}_\infty \tag{3.20}$$

and

$$P_r(B) - P_r(A) \in \mathfrak{S}_\infty. \tag{3.21}$$

According to our assumptions we have

$$\kappa_+(P_l(A)\mathcal{H}') < \infty \tag{3.22}$$

and

$$\kappa_-(P_r(A)\mathcal{H}') < \infty. \tag{3.23}$$

Then, on account of [7; Theorem 3.1], (3.20) and (3.22) imply

$$\kappa_+(P_l(B)\mathcal{H}'') < \infty, \tag{3.24}$$

and (3.21) and (3.23) imply

$$\kappa_-(P_r(B)\mathcal{H}'') < \infty. \tag{3.25}$$

We set $B_l := B \mid P_l(B)\mathcal{H}''$, $B_r := B \mid P_r(B)\mathcal{H}''$ and $B_0 := B \mid P_0(B)\mathcal{H}''$. According to (3.20), (2.14) and to the relation $\sigma(A \mid P_l(A)\mathcal{H}') \subset (-\infty, t_1)$ we have

$$\sigma(B_l) \setminus \sigma_{\mathrm{p,norm}}(B_l) \subset (-\infty, t_1). \tag{3.26}$$

We claim that

$$\sigma(B_l) \subset \mathcal{O}_l. \tag{3.27}$$

Indeed, suppose that $\lambda \in \sigma(B_l) \cap (\mathcal{O} \cup \mathcal{O}_r)$. Then by (3.26) there exists an $x \in \in P_l(B)\mathcal{H}''$, $x \neq 0$, such that $Bx = \lambda x$. From this relation we find $P_l(B)x = 0$, a contradiction. Hence (3.27) is true. Similarly,

$$\sigma(B_r) \subset \mathcal{O}_r. \tag{3.28}$$

Furthermore, we have

$$\sigma(B_0) \subset \mathcal{O}. \tag{3.29}$$

B_l and B_r are selfadjoint operators in the Pontrjagin spaces $P_l(B)\mathcal{H}''$ and $P_r(B)\mathcal{H}''$, respectively. Let E_l and E_r denote the spectral functions of B_l and B_r, respectively, and let $\mathfrak{B}'$ be the Boolean algebra of Borel sets $b \subset \overline{\mathbb{R}}$ with the property

$$\partial b \cap (\overline{\mathcal{O}} \cup \mathrm{c}(B_l) \cup \mathrm{c}(B_r) \cup \sigma_0(B_l) \cup \sigma_0(B_r) \cup \{\infty\}) = \emptyset.$$

Then the mapping E' defined by

$$E'(b) := E_l(b)P_l(B) + E_r(b)P_r(B) + \varepsilon P_0(B),$$

where $\varepsilon = 0$ if $\overline{\mathcal{O}} \cap b = \emptyset$ and $\varepsilon = 1$ if $\overline{\mathcal{O}} \subset b$, which is a homomorphism of $\mathfrak{B}'$ into a Boolean algebra of projections of $\mathcal{H}$, fulfils the conditions of [7; Theorem 2.6].

As a consequence E' coincides with the spectral function $E(\,\cdot\,; B)$ of B restricted to $\mathfrak{B}'$. Then the assertions of the theorem in the case when (i) holds follow easily from (3.24), (3.25), (3.27) – (3.29)) and well-known properties of E_l and E_r.

Assume now that (ii) holds and that $\infty \in c_s(A)$. Set $\mathcal{H}' = \mathcal{H}'' =: \mathcal{H}$. Then by Section 2.4 and Lemma 2.1 the operator A' in $\mathcal{H}_A$ (see (2.7)) and the operator $V \in \mathcal{L}^{(A)}$ fulfil the assumptions of Theorem 3.6 with condition (i). Hence the statements of Theorem 3.6 are true for $A \dotplus V$ replaced by $A' \dotplus V$. Then, on account of Lemma 2.8, they are also true for $A \dotplus V$.

Suppose that $\infty \notin c_s(A \dotplus V)$. Then Lemma 2.7 and this proof yield $\infty \notin c_s(A)$, a contradiction. Theorem 3.6 is proved.

REMARK 3.7. Under the conditions (ii) Theorem 3.6 can also be proved similarly to Theorem 3 in [5].

REMARK 3.8. Let the assumptions of Theorem 3.6 be fulfilled with $\infty \notin c_s(A)$ and let $V \in \mathfrak{S}_p$ for some $p \in [1,\infty) \cup \{\infty\}$. If $t_1' \in (-\infty, t_1)$, $t_2' \in (t_2, \infty)$, t_1', $t_2' \in \rho(A) \cap \rho(A \dotplus V)$, then

$$E((-\infty, t_1'); A \dotplus V) - E((-\infty, t_1'); A) \in \mathfrak{S}_p,$$

$$E((t_2', \infty); A \dotplus V) - E((t_2', \infty); A) \in \mathfrak{S}_p.$$

This is a consequence of Lemma 3.5.

REMARK 3.9. If A and V are as in Theorem 3.6 and ∞ does not belong to $c_s(A)$, then the operator $i(A \dotplus V)$ is the infinitesimal generator of a strongly continuous group of unitary operators in the Krein space $\mathcal{H}''$.

Now we replace the requirement $t_1, t_2 \in \rho(A)$ by the weaker condition $t_1, t_2 \notin \sigma_0(A \dotplus V)$. We remark that in the perturbation theory of differential operators this condition is a natural one. At the same time the compactness of V is replaced by the stronger condition $V \in \mathfrak{S}_p$, $1 \leqslant p < \infty$.

THEOREM 3.10. *Let A be a selfadjoint operator in $\mathcal{H}'$ such that $\sigma_0(A) \backslash \sigma_{p,\text{norm}}(A)$ is empty or finite, A is definitizable over $\Delta := (t_2, \infty) \cup \{\infty\} \cup (-\infty, t_1)$, where t_1 and t_2 are two real points, $t_1 < t_2$, and the interval (t_2, ∞) ($(-\infty, t_1)$) is of type π_+ (resp. π_-). Let $V \in \mathcal{L}^{(A)}$ belong to $\mathfrak{S}_p$ for some $p \in [1, \infty)$ and let $A \dotplus V$ be symmetric in $\mathcal{H}''$. Assume that t_1 and t_2 are no accumulation points of $\sigma_0(A \dotplus V)$ and that one of the following conditions is fulfilled:*

(i) *$G' - G'' \in \mathfrak{S}_p$ and $\infty \notin c_s(A)$.*

(ii) *$G' = G''$ and V is $[\,\cdot\,,\cdot\,]'$-symmetric.*

Then $A \dotplus V$ is a selfadjoint operator in $\mathcal{H}''$ definitizable over Δ and the interval (t_2, ∞) ($(-\infty, t_1)$) is of type π_+ (resp. π_-) with respect to $A \dotplus V$.

Moreover, if (i) *holds, then* $\infty \notin c_s(A \dot{+} V)$. *If* (ii) *holds, then we have* $\infty \in c_s(A)$ *if and only if* $\infty \in c_s(A \dot{+} V)$.

Proof. 1. Suppose that Theorem 3.10 is already proved under the condition (i). Then a reasoning similar to that at the end of the proof of Theorem 3.6 shows that Theorem 3.10 is also true under the condition (ii).

We now assume that the condition (i) is fulfilled. We set again $B := A \dot{+} V$. By Proposition 3.1 there exists an $\eta_0 > 0$ such that

$$\{it \ : \ |t| \in [\eta_0, \infty)\} \subset \rho(A) \cap \rho(B).$$

Then the symmetry of B in $\mathcal{H}''$ implies the selfadjointness of B in $\mathcal{H}''$. According to the assumptions on A and Lemma 3.5 the operators $T_A := T(\eta_0; A)$ and $T_B :=$ $:= T(\eta_0; B)$ exist. They are selfadjoint in $\mathcal{H}'$ and $\mathcal{H}''$, respectively. In what follows we assume that

$$\mathrm{i}, \ -\mathrm{i} \in \rho(A) \cap \rho(B).$$

This is no restriction.

Let Arc tan denote the principal branch of the arc tangent function, which maps $\mathbb{C} \setminus \{\mathrm{i}y \ : \ y \in \mathbb{R}, |y| \geqslant 1\}$ conformally onto $\{z \ : \ |\mathrm{Re}\, z| < \pi/2\}$ (see e.g. [17; 2, §5]). We denote the mapping $z \mapsto -(1/\pi)\mathrm{Arc}\ \tan(\eta_0^{-1} z)$ defined on $\mathbb{C} \setminus \{\mathrm{i}y \ : \ y \in$ $\in \mathbb{R}, |y| \geqslant \eta_0\}$ by φ. Then making use of a functional calculus for selfadjoint operators in Krein spaces which are definitizable over open subsets of $\overline{\mathbb{R}}$ (see [7; Section 2.2]) we may write

$$\varphi(A) = T_A. \tag{3.30}$$

Let K_0 denote a compact subset of $\mathbb{C}$ which is symmetric with respect to $\mathbb{R}$ such that the following holds:

(a) $\{z \in \mathbb{C} \ : \ \mathrm{Re}\, z > 0\} \setminus K_0$ and $\{z \in \mathbb{C} \ : \ \mathrm{Re}\, z < 0\} \setminus K_0$ are simply connected domains, $\{it \ : \ |t| \in [\eta_0, \infty)\} \cap K_0 = \emptyset$, $-\mathrm{i}$, $\mathrm{i} \notin K_0$.

(b) $\overline{\mathbb{R}} \setminus K_0 = \Delta$.

(c) $\sigma(A) \subset \mathbb{R} \cup K_0$.

Then by Lemma 2.3, $\sigma(B) \setminus (\mathbb{R} \cup K_0) \subset \sigma_{\mathrm{p,norm}}(B)$. By (3.30) we have $\sigma(T_A) \subset$ $\subset \overline{\varphi(\mathbb{R} \cup K_0)}$. Since according to Lemma 3.5, $T_B - T_A \in \mathfrak{S}_\infty$, it follows that $\sigma(T_B) \setminus$ $\setminus \overline{\varphi(\mathbb{R} \cup K_0)} \subset \sigma_{\mathrm{p,norm}}(T_B)$.

We claim that

$$\varphi(\sigma(B) \setminus (\mathbb{R} \cup K_0)) = \sigma(T_B) \setminus \overline{\varphi(\mathbb{R} \cup K_0)}. \tag{3.31}$$

Evidently, by the spectral mapping theorem we have

$$\varphi(\sigma(B) \setminus (\mathbb{R} \cup K_0)) = \sigma(T_B) \setminus \overline{\varphi(\mathbb{R} \cup K_0)}.$$

To prove the opposite inclusion we first observe that the operator V can be approximated in $\mathcal{L}^{(A)}$ by operators V' of finite rank of the form (3.19). By the continuity of the mapping $W \mapsto T(\eta_0; A \dot{+} W)$ on some neighbourhood of V (see Lemma 3.5) and the upper semicontinuity of the spectrum with respect to the operator norm it is sufficient to prove (3.31) for V replaced by an operator V' of the form (3.19) such that $\{it : |t| \in [\eta_0, \infty)\} \subset \rho(A + V')$. In this case there exists an $n_1 \in \mathbb{N}$ such that the decomposition

$$\mathcal{H} = E_{n_1}\mathcal{H} + (I - E_{n_1})\mathcal{H}, \quad E_{n_1} := E(\overline{\mathbb{R}} \setminus (-n_1, n_1); A),$$

reduces $A + V'$ and the operators A and $A + V'$ coincide on $E_{n_1}\mathcal{H}$. $A + V' \mid (I - E_{n_1})\mathcal{H}$ is bounded. Hence

$$\varphi(A + V' \mid (I - E_{n_1})\mathcal{H}) = T(\eta_0; A + V') \mid (I - E_{n_1})\mathcal{H}$$

in the sense of the Riesz-Dunford functional calculus. Then the spectral mapping theorem implies (3.31) for V replaced by V', and (3.31) is proved.

Assign $\varepsilon > 0$. Then by assumptions and the relation (3.31) there exist points $t_1' \in (t_1 - \varepsilon, t_1)$ and $t_2' \in (t_2, t_2 + \varepsilon)$ such that $\varphi(t_1')$ and $\varphi(t_2')$ are no accumulation points of $\sigma_0(T_B)$. Then it follows from [8] that the bounded operator T_B is definitizable over $(\varphi(t_1'), \infty) \cup (-\infty, \varphi(t_2'))$. As a consequence of (3.31) and since ε was arbitrary, Δ contains no accumulation points of $\sigma_0(B)$. Hence, again by [8], B is definitizable over $(-\infty, t_1) \cup (t_2, \infty)$, and the interval (t_2, ∞) ($(-\infty, t_1)$) is of type π_+ (resp. π_-). Moreover, there exists an $M_0 > 0$ with $-M_0 < t_1, t_2 < M_0$ such that $\varphi((-\infty, -M_0))$ is of negative type and $\varphi((M_0, \infty))$ is of positive type with respect to T_B. Let $\delta_0 \subset (M_0, \infty)$ be a compact interval whose endpoints are no critical points of B. Then $T_B E(\delta_0; B)\mathcal{H}'' \subset E(\delta_0; B)\mathcal{H}''$ and $\sigma(T_B \mid E(\delta_0; B)\mathcal{H}'') \subset \varphi(\delta_0)$. On the other hand, from the general properties of the spectral functions of self-adjoint operators definitizable over open subsets of $\overline{\mathbb{R}}$ ([7; Theorem 2.6]) it follows that $E(\varphi(\delta_0); T_B)\mathcal{H}''$ is a spectral maximal space (see [2; Definition 1.3.1]) with respect to T_B. Thus $E(\delta_0; B)\mathcal{H}'' \subset E(\varphi(\delta_0); T_B)\mathcal{H}''$ and $E(\delta_0; B)$ is nonnegative. If $\delta_0 \subset (-\infty, -M_0)$, then a similar reasoning gives that $E(\delta_0; B)$ is nonpositive.

2. Set $\Delta' := \overline{\mathbb{R}} \setminus [-M_0 - 1, m_0 + 1]$ and suppose that

$$E(\Delta'; B)\mathcal{H}'' = \bigvee \{E((-m, -M_0 - 1) \cup (M_0 + 1, m); B)\mathcal{H}'' : \; : m \in \mathbb{N}, |m| > M_0 + 1\}. \tag{3.32}$$

This relation implies that $B \mid E(\Delta'; B)\mathcal{H}''$ is nonnegative. Then $T_B \mid E(\Delta'; B)\mathcal{H}''$ is also nonnegative and it follows as in [6; proof of Theorem 1.1] that the projections $E(\delta; B)$, where δ runs through all compact intervals contained in Δ', are uniformly bounded, i.e. $\infty \notin c_s(B)$. Therefore, to complete the proof it is sufficient to verify the relation (3.32).

Suppose that (3.32) does not hold. Then the $[\cdot,\cdot]''$-orthogonal complement of the right hand side of (3.32) in $E(\Delta'; B)\mathcal{H}''$ is a nontrivial closed subspace $\mathcal{H}_\infty$ of $E(\Delta'; B)\mathcal{H}''$ which is invariant with respect to the resolvent and the Cayley transform $\psi(B), \psi(z) := -(z-\mathrm{i})(z+\mathrm{i})^{-1}$, such that

$$\sigma(\psi(B)\,|\,\mathcal{H}_\infty) \subset \{-1\}. \tag{3.33}$$

We claim that for every $x \in \mathcal{H}_\infty$ the function $z \mapsto (R(z;B)x, x)$ is an entire function of finite order, i.e. there exist positive numbers m and M such that

$$\max\{\,|(R(z;B)x,x)\ :\ |z| = r\} \leqslant Me^{r^m}, \quad r \geqslant 0.$$

Since there exists an $N > 0$ such that

$$|\,R(\zeta;\psi(A))\,| \leqslant N\Big|1 - |\zeta|\Big|^{-1}$$

for all $\zeta \notin \mathbb{T}$ in a neighbourhood of -1, and the difference $\psi(B) - \psi(A)$ belongs to $\mathfrak{S}_p$, there exist $N_1 > 0$ and $N_2 > 0$ such that

$$|\,R(\zeta;\psi(B))\,| \leqslant N_1 \exp\left\{N_2\Big|1 - |\zeta|\Big|^{-p-1}\right\} \tag{3.34}$$

for all $\zeta \notin \mathbb{T}$ in a neighbourhood of -1 (see [2; proof of Theorem 5.5.2], [7; proof of Theorem 3.6]).

For an arbitrary $x \in \mathcal{H}_\infty$ we consider now the linear functional T,

$$T(f) := (f(\mathrm{i}I + 2\mathrm{i}R(1;\psi(B))\,|\,\mathcal{H}_\infty)x, x),$$

which on account of (3.33) is defined for all functions f locally holomorphic at 0. Making use of (3.34) one verifies without difficulty that there exist positive numbers N_1' and N_2' such that

$$|T(f_z)| \leqslant N_1' \exp\{N_2'\,|\operatorname{Im} z\,|^{-p-1}\}, \qquad |\operatorname{Im} z\,| \neq 0, \tag{3.35}$$

where $f_z(\lambda) := (\lambda - z)^{-1}$. On account of [1; Theorems 1.4, 1.2, 1.1] it follows from (3.35) that there exist complex numbers α_n, $n = 0, 1, \dots$, satisfying the relations

$$|\,\alpha_n\,| \leqslant cL^n(n!)^{-1-(2p+2)^{-1}}, \qquad n = 0, 1, \dots, \tag{3.36}$$

with some $c > 0$ and $L > 0$ independent of n, such that T can be written in the form

$$T(f) = \sum_{n=0}^{\infty} \alpha_n f^{(n)}(0)$$

for any function f locally holomorphic at 0. Set $g_z(\lambda) := (\lambda^{-1} - z)^{-1}$, $z \neq \infty$. Then using the definition of T we find

$$(R(z;B)x,x) = T(g_z) = z^{-1}\sum_{n=1}^{\infty}\alpha_n n! z^n. \tag{3.37}$$

Now by a well-known result on the order of entire functions (see [16; I, §2, Lehrsatz 2]) it follows from (3.36) and (3.37) that $z \mapsto (R(z;B)x,x)$ is an entire function of finite order.

For arbitrary $x \in \mathcal{H}_\infty$ the function $z \mapsto (R(z;B)x,x)$ is bounded on any set of the form

$$M_\varepsilon := \{z \ : \ \arg z \in [\varepsilon, \pi - \varepsilon] \cup [-\pi + \varepsilon, -\varepsilon]\}, \quad \varepsilon \in (0, \frac{\pi}{2}).$$

This follows from (2.17) and the fact that $V_1\tilde{R}(z)V_2$, $z \in M_\varepsilon$, converges to 0 if $|\mathrm{Im}\, z| \uparrow \infty$, which can be shown as in the proof of Proposition 3.1. On account of the Phragmen-Lindelöf theorem the functions $z \mapsto (R(z;B)x,x)$, $x \in \mathcal{H}_\infty$, are bounded and, hence, equal to constants. It follows that $\mathcal{H}_\infty = \{0\}$, which contradicts our assumption. This proves Theorem 3.10.

REMARK 3.11. If A and V are as in Theorem 3.10 and $\infty \notin c_s(A)$, then the operator $\mathrm{i}(A \dot{+} V)$ is the infinitesimal generator of a strongly continuous group of unitary operators in the Krein space $\mathcal{H}''$.

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