J. OPERATOR THEORY
25(1991), 275–282

DILATION THEORY FOR SUBALGEBRAS OF AF ALGEBRAS

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INTRODUCTION

W. B. Arveson has shown in [1] that a (Hilbert space) representation of an algebra has a Stinespring dilation if and only if it is completely contractive. Since then steps forward in dilation theory have been made by P. S. Muhly and B. Solel in [4] and by V. I. Paulsen, S. C. Power and J. D. Ward in [7]. These were positive results about the existence of dilations for contractive representations of a class of structured algebras. V. I. Paulsen, S. C. Power and R. R. Smith in [6] studied representations of unital subalgebras of matrix algebras which are spanned by the matrix units which they contain. These are called finite-dimensional CSL algebras. For a large class of algebras it was shown that all contractive representations are completely contractive and that all representations which are contractive on a set of matrix units are contractive on the whole algebra. Examples of other algebras with contractive but not completely contractive representations were given. In this paper we identify a class of subalgebras of AF algebras for which all contractive representations are completely contractive, based on a criterion on the fundamental relation, a concept introduced by S. C. Power in [9]. It is also shown that all ultraweakly continuous contractive representations of certain atomic CSL algebras are completely contractive. These results are generalizations of the finite-dimensional CSL result in [6].

The definition of a chordal graph is given in Section 2. A subalgebra of an AF algebra is chordal if it has a fundamental relation with chordal underlying graph. The result in [6] that contractive representations of chordal finite-dimensional algebras are completely contractive is extended in Section 2 by showing that all chordal subalgebras of AF algebras are limits of finite-dimensional chordal subalgebras. This has to be proven in several steps due to the complexity of the process of increasing a non-chordal graph into a chordal graph.

This paper is taken from my thesis and I would like to thank my supervisor, Professor S. C. Power for all his help, encouragement and advice.

1. PRELIMINARIES

Let B be an AF C^*-algebra, the closed union of an increasing chain B_1, B_2, ... of finite-dimensional $*$-subalgebras. For a subspace N of B write N_n for $N \cap B_n$ and N_∞ for the union of all these. We say that N is *inductive* if N_∞ is dense in N. Let C be an inductive masa in B such that each C_n is a masa in B_n, called a *cannonical* masa. Let M be a C-bimodule in B. ($CM \subset M$, $MC \subset M$ and M is closed.) It is a key result of S. C. Power [8, Lemma 1.3] that all C-bimodules are inductive.

A partial isometry v in B is said to be *C-normalising* if $vCv^* \subset C$ and $v^*Cv \subset C$. For a functional x in the maximal ideal space $\Phi(C)$ of C and a C-normalising partial isometry v, another functional x_v is defined by $x_v(c) = x(vcv^*)$ for all c in C. The *fundamental relation* of the C-bimodule M is the binary relation defined on $\Phi(C)$ by $x\mathrm{R}(M)y$ if there is a C-normalising partial isometry v in M such that $y = x_v$. This was introduced in [9, Chapter 6] to study subalgebras of AF algebras.

It is always possible to specify a set of matrix units for each B_n such that they are all C-normalising and the matrix units for B_n are all sums of matrix units for B_{n+1}. We shall fix such a system. It is shown in [9] that in the above definition of the fundamental relation, if the partial isometries are replaced with matrix units from any chosen system then the new definition is equivalent to the old. This is used to relate the fundamental relation, which is dependant only on M and C, to finite-dimensional spaces. In fact $x\mathrm{R}(M)y$ if and only if for all large enough n there is a matrix unit of B_n in M implementing this relation. A result in [11], the AF spectral theorem, states that a matrix unit w is in M if and only if whenever x and y are functionals with $y(c) = x(wcw^*)$ for all c in C then $x\mathrm{R}(M)y$.

The binary relation $\mathrm{R}(M)$ also defines a directed graph with vertices $\Phi(C)$ and arrows (x,y) for x and y with $x\mathrm{R}(M)y$. The *underlying* graph of $\mathrm{R}(M)$ is the underlying graph of this directed graph, with the same vertices and arcs $\{x, y\}$ for all x and y with $x\mathrm{R}(M)y$.

All of the above holds for finite-dimensional CSL algebras since they are bimodules for a masa in the containing matrix algebra.

A *representation* ρ of a subalgebra A of a C^*-algebra is an algebra homomorphism taking elements of A to operators on a Hilbert space, H say. It is *completely* contractive if the induced maps $\rho_{(n)} : A \otimes M(n) \to L(H) \otimes M(n) : (a_{ij}) \to (\rho(a_{ij}))$ are contractive for all positive integers n, where the tensor product algebras are treated as algebras of $n \times n$ operator matrices on the direct sum of n copies of H.

2. CHORDAL BIMODULES

In [6] it was shown that any Hilbert space representation of the chordal finite-dimensional CSL algebra which is contractive on the matrix units is completely contractive and therefore has a Stinespring dilation. We shall see that this result extends to bimodule algebras which have a fundamental relation, with respect to some canonical masa, which has a chordal underlying graph.

A *cycle (of length r)* in a graph G is a finite sequence $x_0, \ldots, x_r$ of vertices with the first equal to the last, $x_1, \ldots, x_r$ all distinct and consecutive vertices adjacent in G. A *chord* for the cycle is an arc between two non-consecutive vertices. The graph G is a *t-cycle* graph if every cycle in G of length at least t has a chord in G. Similarly a binary relation is a t-cycle relation if its underlying graph is a t-cycle graph. G is *chordal* if it is a 4-*cycle* graph.

In the finite-dimensional case, if N is a C_n-bimodule in B_n then it is spanned by the matrix units of B_n which it contains, and the fundamental relation of N over C_n corresponds to exactly these matrix units.

LEMMA 2.1. *Let N be a C_n-bimodule contained in M_n. If $\mathrm{R}(M)$ is a t-cycle relation then for every k-cycle in N with $k \geqslant t$ there is an integer $m \geqslant n$ such that M_m contains a chord for all copies of this cycle in the C_m-bimodule generated by N.*

Proof. Suppose that $\mathrm{R}(M)$ is a t-*cycle* relation and k is the length of the longest chordless cycle in N with $k \geqslant t$. The arcs in the graph of N correspond to matrix units. Fix a cycle of length k in the graph of N, this corresponds to a sequence $v_1, \ldots, v_k$ of distinct matrix units of B_n with $v_2 \cdots v_k = v_1^*$ and for each i, one of v_i or v_i^* is in N.

Let x be a functional in $\Phi(C)$ with $x(v_1 v_1^*) = 1$. Set $x = x_1$. Define functionals $x_1, \ldots, x_k$ inductively as follows. Let $x_{i+1}(c) = x_i(v_i c v_i^*)$ for all c in C, and for $i = 1, \ldots, k$. Now $x_i \mathrm{R}(M) x_{i+1}$ or $x_{i+1} \mathrm{R}(M) x_i$ for $i = 1, \ldots, k-1$ and since $x_{k+1} = x_1$

it is also true that $x_1 \mathrm{R}(M) x_k$ or $x_k \mathrm{R}(M) x_1$.

The k-cycle $x_1, \ldots, x_k$ has a chord in $\mathrm{R}(M)$ so for some integers i and j with $|i-j| \geqslant 2$ and $\{i,j\} \neq \{1,k\}$ there is a matrix unit $v(x)$ in M so that $x_i = (x_j)_{v(x)}$. Assuming that $v(x)v(x)^*$ is a subprojection of $v_j v_j^*$ let $e(x) = v_1 v_2 \cdots \cdots v_{j-1} v(x) v(x)^* v_{j-1}^* \cdots v_2^* v_1^*$ so that $x(e(x)) = 1$. If y is another functional in $\Phi(C)$ with $y(e(x)) = 1$ then the cycle $y_1, \ldots, y_k$ defined as in the above paragraph has a chord in $\mathrm{R}(M)$ between y_i and y_j since $y_i = (y_j)_{v(x)}$. The space $v_1 v_1^* C$ is unital so its maximal ideal space is compact, hence by an identification with the subspace $\{x : x(v_1 v_1^*) = 1\}$ of $\Phi(C)$ this is also compact. With $E(x) = \{y \text{ in } \Phi(C) : y(e(x)) = 1\}$ the set $\{E(x) : x(v_1 v_1^*) = 1\}$ is an open cover of the above compact space. Let $E(x_1), \ldots, E(x_s)$ be a finite subcover. Now $e(x_1) + \cdots + e(x_s) \geqslant v_1 v_1^*$, and the multiplicities can be eliminated by finding matrix units $e_1, \ldots, e_s$ which sum to $v_1 v_1^*$ with $e_i \leqslant e(x_i)$ for all i. Reversing the process used to obtain $e(x_i)$ from a matrix unit, a partial isometry finite sum of matrix units v_i in M can be obtained from e_i for all i. For every functional y with $y(v_1 v_1^*) = 1$ if the cycle $y_1, \ldots, y_k$ is produced as in the second paragraph then there is exactly one of $v_1, \ldots, v_s$ which provides a chord for it. The number m greather than n can be chosen large enough so that all of $v_1, \ldots, v_s$ are in M_m. In the C_m-bimodule generated by N and these matrix units the original k-cycle is split up into many k-cycles but each one has exactly one chord.

■

The above lemma is not enough to ensure that a bimodule with a t-cycle fundamental relation is the limit of finite-dimensional t-cycle bimodules, as a chord for a k-cycle might generate a longer chordless cycle.

Suppose that N is a C_n-bimodule in B_n. As a finite dimensional C^*-algebra, B_n is isomorphic to a direct sum of matrix algebras, say for some r the sum $M(i_1) \oplus \cdots \oplus M(i_r)$, with the isomorphism carrying C_n onto the diagonal. With respect to this we can also write N as the direct sum, $N = \bigoplus_{i=1}^{r} N(i)$ where $N(i)$ is the part of N in $M(i)$. Considering the minimal projections in C_m for some $m \geqslant n$ the C_m-bimodule generated by N can be written as $\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s(i)} N(i,j)$ where for all j the bimodule $N(i,j)$ is isomorphic to $N(i)$, which is a subspace of $\bigoplus_{j=1}^{s(i)} N(i,j)$.

LEMMA 2.2. *Let n be a positive integer. If $\mathrm{R}(M)$ is a t-cycle relation then there is an integer m and a $*$-algebra D which contains $D \cap C$ as a masa, with $B_n \subset D \subset B_m$ and $M \cap D$ has a t-cycle graph.*

Proof. We define inductively an increasing sequence of finite-dimensional C^*-algebras, at each stage splitting the algebra up into factors. If the part of M in a factor has a t-cycle relation then we leave this factor unchanged for the next stage. If there is a cycle of length at least t then we shall use Lemma 2.1 to replace this factor with a direct sum of factors of the same size in such a way that the part of M in each one has an extra chord for a cycle in its relation. The notation is complicated. Let $B_n = D(1) \oplus \ldots \oplus D(s)$ be the decomposition of B_n into finite dimensional factors. Let $S(1) = \{(1), \ldots, (s)\}$, let $D(S(1)) = B_n$, let $D((i)) = D(i)$ and let $m(1) = n$. Define a sequence of $*$-subalgebras of B inductively as follows.

Suppose that $D(S(r)) = \oplus D(\xi)$ is a direct sum of finite-dimensional factors in $B_{m(r)}$ indexed by a set $S(r)$ of finite sequences ξ of length r, with masa $D(S(r)) \cap C$. If $D(\xi)$ is in this direct sum then $N = M \cap D(\xi)$ is a $D(S(r)) \cap C$-bimodule. If it does not have a t-cycle fundamental relation then let $m(\xi)$ be the integer obtained from N by Lemma 2.1. Since $S(r)$ is finite we can let $m(r+1)$ be the largest of all the $m(\xi)$ for ξ in $S(r)$.

Suppose that $\xi = (t_1, \ldots, t_r)$ is a sequence in $S(r)$. If $D(\xi) \cap M$ is a t-cycle bimodule then let $D((t_1, \ldots, t_r, 0)) = D(\xi)$ otherwise, with k the multiplicity of the bimodule generated by $D(\xi)$ in $B_{m(r+1)}$ let $D((t_1, \ldots, t_r, 1)), \ldots, D((t_1, \ldots, t_r, k))$ be defined to be the k full factors in that bimodule. Notice that if $i \geqslant 1$, $M \cap D((t_1, \ldots, \ldots, t_r, i))$ contains at least one more matrix unit of $B_{m(r+1)}$ than $M \cap D((t_1, \ldots, t_r))$ does of $B_{m(r)}$. Let $S(r+1)$ be the set of all sequences ξ of length $n+1$ for which $D(\xi)$ has been defined, and $D(S(r+1))$ the direct sum of all of these. The sequence $D(S(1)), D(S(2)), \ldots$ is now well defined.

Suppose that $\xi = (i, t_1, \ldots, t_r, 1)$ is in $S(r+2)$. By construction $D(\xi)$ is isomorphic to $D(i)$ and since the sequence ξ does not end in 0, the bimodule $M \cap \cap D((i, t_1, \ldots, t_r))$ does not have a t-cycle relation. In fact none of $M \cap D((i)), \ldots, M \cap \cap D((i, t_1, \ldots, t_r))$ have a t-cycle relation, so by construction $M \cap D(\xi)$ must contain at least $r+1$ more matrix units of $B_{m(r+2)}$ than $M \cap D(i)$ does of B_n. However, $M \cap D(\xi)$ contains at most $\dim D(i)$ matrix units of $B_{m(r+2)}$ so $r+1 < \dim D(i)$. Let $m = m(\max\{\dim D(1), \ldots, \dim D(s)\})$ and let $D = D(S(m))$. Since the sequence is constant at $S(m)$, $M \cap D$ must have a t-cycle fundamental relation. ∎

THEOREM 2.3. *If $\mathbb{R}(M)$ is a t-cycle relation then there is an increasing sequence of positive integers $m(1)$, $m(2)$,... and a sequence $D_1, D_2, \ldots$ of $*$-subalgebras of B with $B_n \subset D_n \subset B_{m(n)}$ for $n = 1, 2, \ldots$ and $M \cap D_n$ has a t-cycle graph.*

Proof. Apply the lemma to get D_1 and $m(1)$ with $B_1 \subset D_1 \subset B_{m(1)}$ and $M \cap D_1$

has a t-cycle graph.

If $D_1, \ldots, D_n$ have been defined then apply the lemma to $B_{m(n)+1}$ to get D_{n+1} and $m(n+1)$ with the required properties. ∎

If M is an algebra containing C and $\mathrm{R}(M)$ is a chordal relation then by the theorem it is the limit of an increasing sequence of finite-dimensional chordal CSL algebras. If $\rho : M \rightarrow L(K)$ is an algebra representation which is contractive on the matrix units then ρ is completely contractive, since it is completely contractive on the finite-dimensional chordal subalgebras from [6].

We will construct an example of a subalgebra of an AF algebra with a chordal fundamental relation, writing it as the limit of non-chordal subalgebras with respect to some increasing sequence of finite-dimensional $*$-algebras with closed union the whole AF algebra. Thus the fundamental relation can be used to see through an apparantly non-chordal situation.

Let D_n be the finite-dimensional $*$-algebra which is the direct sum of $D(n,1), \ldots$ $\ldots, D(n,2n)$ where each of these subalgebras is a copy of the 4×4 matrix algebra $\mathrm{M}(4)$. Embed D_n in D_{n+1} as follows. Let $D(n+1,i) = D(n,i)$ for $i = 1, \ldots,$ $\ldots, 2n-2$. Let $D(n, 2n-1)$ be embedded in $D(n+1, 2n-1) \oplus D(n+1, 2n)$ and $D(n,2n)$ in $D(n+1, 2n+1) \oplus D(n+1, 2n+2)$ both using standard embeddings. Let D be the limit algebra of this sequence.

Define the following subalgebras of $\mathrm{M}(4)$ as the span of the matrix units which they contain: $T = \langle e_{13}, e_{23}, e_{24} \rangle, U = \langle T, e_{14} \rangle, W = \langle U, e_{12} \rangle$, $Z = \langle U, e_{21} \rangle$. These algebras will be used to define a chordal bimodule. The graph of U is not chordal but the graph of all of the other algebras is chordal.

Define a chordal algebra N by specifying that

$N \cap D(n,i)$ is a copy of W if $i \leqslant 2n-2$ and i is odd,

$N \cap D(n,i)$ is a copy of Z if $i \leqslant 2n-2$ and i is even,

$N \cap D(n,i)$ is a copy of U if $i = 2n-1$ or,

$N \cap D(n,i)$ is a copy of T if $i = 2n$.

Each N_m is isomorphic to the direct sum of $2n-2$ copies of W and Z with a copy each of U and T so it is not chordal for any m.

Let $G_n = D(n,1) \oplus \ldots \oplus D(n, 2n-2) \oplus D(n+1, 2n-1) \oplus D(n+1, 2n) \oplus D(n, 2n)$. The algebra $G_n \cap N$ is always chordal and the increasing sequence $G_1, G_2, \ldots$ has closed union D. This shows that N is chordal, and in fact this sequence is the one produced by the proof of Theorem 2.3.

3. CHORDAL ATOMIC CSL ALGEBRAS

Let L be an atomic CSL on a Hilbert space H. (A lattice of commuting projections contained in an atomic von Neumann algebra.) Let $\{q_n\}$ be a sequence of mutually orthogonal rank one projections which are atoms of a von Neumann masa containing L. Let Q be the binary relation defined on $\mathbf{N}$ by iQj if there is an operator a in $\operatorname{alg} L$ such that $q_i a q_j$ is non-zero. Let A be the set of all operators in $L(H)$ with $q_i a q_j = 0$ if iQj is false. Since an atomic CSL is synthetic (see [3] for this and general CSL theory) and the algebra A is ultraweakly closed containing a von Neumann masa with $\operatorname{lat} A = L$ it follows by definition that $A = \operatorname{alg} L$.

We say that the atomic CSL algebra A is *chordal* if the underlying graph of Q is chordal. In fact this is a property of L not dependant on the particular choice of $\{q_n\}$.

PROPOSITION 3.1. *If A is chordal and ρ is a contractive ultraweakly continuous representation of A then it is completely contractive.*

Proof. The space $(q_1 + \dots + q_n)A(q_1 + \dots + q_n)$ is chordal if A is and then ρ is completely contractive on it. For any operator a in A, the sequence $(q_1 + \dots + q_n)a(q_1 + \dots + q_n)$ converges ultraweakly to a. Now for any operator matrix (a_{ij}) in $A \otimes M(k)$ the sequence of operator matrices $((q_1 + \dots + q_n)a_{ij}(q_1 + \dots + q_n))$ converges ultraweakly to (a_{ij}). This means that ρ must be completely contractive. ■

Supported at Lancaster University by the Science and Engineering Research Council.

REFERENCES

1. ARVESON, W. B., Subalgebras of C^*-algebras, *Acta Math.*, **123**(1969), 271-308.
2. BRATTELI, O., Inductive limits of finite dimensional C^*-algebras, *Trans. Amer. Math. Soc.*, **171** (1972), 195–234.
3. DAVIDSON, K. R., *Nest algebras*, Pitman Research Notes, vol. **191**, 1988.
4. MUHLY, P. S.; SOLEL, B., Dilations for representations of triangular algebras, Preprint, 1989.
5. PAULSEN, V. I., *Completely bounded maps and dilations*, Pitman Research Notes, vol. **146**, New York, 1986.
6. PAULSEN, V. I. ; POWER, S. C. ; SMITH, R. R., Schur products and matrix completions, Preprint, 1986.
7. PAULSEN, V. I. ; POWER, S. C. ; WARD, J. D., Semi-discreteness and dilations theory for nest algebras, *J. Funct. Anal.*, **80** (1988), 76–87.
8. POWER, S.C., On ideals of nest subalgebras of C^*-algebras, *Proc. London Math. Soc.* (3), **50**(1985), 314–332.

9. POWER, S. C., Classification of tensor products of triangular operator algebras, *Proc. London Math. Soc., to appear.*
10. THELWALL, M. A., Bimodule theory in the study of non-self-adjoint operator algebras, Thesis, University of Lancaster, 1989.
11. THELWALL, M. A., Maximal triangular subalgebras of AF Algebras, *J. Operator Theory*, **25**(1991), to appear.

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Received September 4, 1989; revised February 10, 1990.