J. OPERATOR THEORY
27(1992), 17–52



# $C^*$-ALGEBRAS GENERATED BY ISOMETRIES AND WIENER-HOPF OPERATORS

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## 1. INTRODUCTION

1.1 AMENABILITY FOR A CLASS OF ORDERED GROUPS. If $G$ is a discrete group and $P$ is a subsemigroup of $G$, the $C^*$-algebra generated by the compression to $\ell^2(P)$ of the left regular representation of $G$ (or, equivalently, of $\ell^1(G)$) is called the $C^*$-algebra of the Wiener-Hopf operators on $(G,P)$ and is denoted by $\mathcal{W}(G,P)$. In this paper we study a class of pairs $(G,P)$ with the property that $\mathcal{W}(G,P)$ is generated by isometries. The $C^*$-algebras generated by one-parameter semigroups of isometries studied by R. Douglas in [5] and the Cuntz algebras are obtained as (or at least related to) particular examples. Moreover, the uniqueness properties which are known to hold for these $C^*$-algebras (see J. Cuntz [3], R. Douglas [5]) are interpreted as being related to the amenability of the corresponding $(G,P)$'s, and may in fact be deduced from amenability phenomena combined with an analysis of the ideals of $\mathcal{W}(G,P)$.

We call the $(G,P)$'s we consider "quasi-lattice ordered groups "(it is not a standard notion; the precise definition and some examples will be given in the next section). This is the context discussed everywhere in the present section, without any other mention.

In order to make things more explicit, let us consider the simple example of $(\mathbf{Z},\mathbf{N})$, and recall a well-known theorem of L. Coburn (see [2]), which asserts that the $C^*$-algebra generated by a non-unitary isometry on a Hilbert space does not depend on the particular choice of the isometry. It is convenient to view this theorem as being implied by the following two facts:

1° For any isometry $V$ on a Hilbert space $H$ there exists a unique $*$-representation $\pi : C^*(S) \to \mathcal{L}(H)$ such that $\pi(S) = V$, where $S$ is the unilateral shift on $\ell^2(\mathbf{N})$ and $C^*(S)$ is the $C^*$-algebra generated by $S$.

2° Every non-zero closed two-sided ideal of $C^*(S)$ contains the compact operators. Indeed, assuming that 1° and 2° are true, consider the non-unitary isometry $V$ on the Hilbert space $H$, and the corresponding representation $\pi$ given by 1°. If $\pi$ were not isometric, then its kernel would contain by 2° the compact operator $I - SS^*$, implying $I - VV^* = \pi(I - SS^*) = 0$, a contradiction.

Now the Wiener-Hopf $C^*$-algebra of $(\mathbb{Z}, \mathbb{N})$ is $\mathcal{W}(\mathbb{Z}, \mathbb{N}) = C^*(S)$, and an isometry on a Hilbert space means a representation by isometries of $\mathbb{N}$ on the corresponding space. Hence assertion 1° above means exactly that any representation by isometries of $\mathbb{N}$ can be uniquely extended to $\mathcal{W}(\mathbb{Z}, \mathbb{N})$. This is one of the possible definitions for the amenability of a "quasi-lattice" ordered group, applied to $(\mathbb{Z}, \mathbb{N})$.

The use of the term "amenability" is justified by several equivalent reformulations, resembling the amenability of (unordered) groups:

1° Besides $\mathcal{W}(G, P)$ one can construct a universal $C^*$-algebra $C^*(G, P)$, which has a canonical $*$-homomorphism onto $\mathcal{W}(G, P)$. $(G, P)$ is amenable if and only if this $*$-homomorphism is one-to-one.

2° The universal $C^*$-algebra $C^*(G, P)$ has a remarkable Abelian $C^*$-subalgebra, and a canonical conditional expectation onto it. $(G, P)$ is amenable if and only if this conditional expectation is faithful.

3° Among the positive forms on $C^*(G, P)$ there are some which are, in a certain natural sense, finitely supported. $(G, P)$ is amenable if and only if these forms are weak* dense in the space of all positive forms on $C^*(G, P)$.

4° Under a suitable natural definition for a positive definite function on $PP^{-1}$, the amenability of $(G, P)$ is implied by the existence of a net of finitely supported positive definite functions on $PP^{-1}$ which converge pointwisely to 1 (but we do not know if the converse is true).

$(G, P)$ is always amenable when $G$ is amenable in the usual sense; this comes out directly from the assertion 4° above. With a simpler proof we can derive the weaker result that $(G, P)$ is amenable if $P$ is Abelian. Quite surprisingly, $(F_n, SF_n)$ is also amenable, where $F_n$ is the free group on $n$ generators, and $SF_n$ the semigroup generated by a free family of generators. The amenability of $(F_n, SF_n)$ is equivalent to the uniqueness property of the Cuntz algebra $O_n$.

1.2. The "Crossed-product type" Structure of $\mathcal{W}(G, P)$. All our results depend on a description of $\mathcal{W}(G, P)$ which is very much alike the one given to the crossed product of a $C^*$-algebra by a discrete group. More precisely, we can define for any $x$ in $PP^{-1}$ a closed subspace $\mathcal{D}_x$ of $\mathcal{W}(G, P)$, called the diagonal subspace of $x$. The spaces $\{\mathcal{D}_x \mid x \in PP^{-1}\}$ establish, in a weak sense, a direct sum decomposition of $\mathcal{W}(G,P)$, and obey the multiplication and involution rules:

$$\mathcal{D}_x\mathcal{D}_y \subseteq \begin{cases} \mathcal{D}_{xy}, & \text{if } xy \text{ is still in } PP^{-1}, \\ \{0\}, & \text{if not,} \end{cases}$$

and $\mathcal{D}_x^* = \mathcal{D}_{x^{-1}}$. In particular $\mathcal{D} = \mathcal{D}_e$ (with $e$ the unit of $G$) is a $C^*$-subalgebra of $\mathcal{W}(G,P)$, called the diagonal subalgebra; it can be shown to be maximal Abelian. For any $x$ in $PP^{-1}$ we have a canonical Banach space isomorphism between $\mathcal{D}$ and $\mathcal{D}_x$, given by a multiplication operator. The form of the isomorphisms $\mathcal{D} \leftrightarrow \mathcal{D}_x$ suggests a set of (not necessarily unital) $*$-endomorphisms $\{\alpha_x \mid x \in PP^{-1}\}$ of $\mathcal{D}$. In addition, there exists a canonical conditional expectation $E : \mathcal{W}(G,P) \to \mathcal{D}$ which can be transported via the canonical isomorphism to give a projection of norm one onto $\mathcal{D}_x$, for any $x$ in $PP^{-1}$. Hence, informally speaking, $\mathcal{W}(G,P)$ is a kind of crossed product of $\mathcal{D}$ by the action $\{\alpha_x | x \in PP^{-1}\}$ of $PP^{-1}$.

Now, $\mathcal{W}(G,P)$ is actually the analogue of the reduced crossed product, the analogue of the full crossed product being the universal $C^*$-algebra $C^*(G,P)$ mentioned above. This is seen by considering covariant representations. One naturally expects such an object to consist of a $*$-representation $\rho$ of $\mathcal{D}$ and a "representation " $U$ of $PP^{-1}$ on the same Hilbert space, tied together by the covariance relation $\rho(\alpha_x(X)) = U(x)\rho(X)U(x)^*$, $\forall x \in PP^{-1}$, $\forall X \in \mathcal{D}$. In our approach, both $\rho$ and $U$ are determined by a representation by isometries of $P$ enjoying a precise property, which will also be called "covariance". $C^*(G,P)$ is the universal object obtained by enveloping all the covariant representations of $P$, while $\mathcal{W}(G,P)$ is associated to a remarkable covariant representation $W : P \to \mathcal{L}(\ell^2(P))$, which will be called the Wiener-Hopf representation.

The remarks on covariance make clear why a necessary and sufficient criterion of amenability is :"Every covariant representation of $P$ can be extended to $\mathcal{W}(G,P)$". In the case of a totally ordered group, and in particular of $(\mathbb{Z},\mathbb{N})$, every representation by isometries of the semigroup is covariant. But for instance in the case of $(F_n, SF_n)$ a representation by isometries of the semigroup is determined by $n$ isometries on the same Hilbert space (the values at the generators) and it is covariant if and only if these $n$ isometries have mutually orthogonal ranges; this explains the relation with $O_n$.

We mention that in his paper [10], G. Murphy describes the Wiener-Hopf $C^*$-algebra of a totally ordered Abelian group as a corner of a crossed product $C^*$-algebra. In this case (which enters our context) his approach and ours have common points, but do not coincide.

1.3. Induced Ideals. As we saw in 1.1, the theorem of L. Coburn can be derived from the amenability of $(\mathbb{Z},\mathbb{N})$ combined with considerations concerning the ideal structure of $\mathcal{W}(\mathbf{Z},\mathbf{N})$. It generally seems useful to have some information about the ideals of $\mathcal{W}(G,P)$. In this paper we put into evidence a class of ideals which are obtained from the invariant ideals of the diagonal subalgebra by an induction process ("invariant" means invariant for the $*$-endomorphisms $\{\alpha_x | x \in PP^{-1}\}$ of 1.2).

There are several possible definitions of $\mathrm{Ind}\,\mathcal{I}$ ($\mathcal{I} \subseteq \mathcal{D}$ closed invariant ideal), which coincide under suitable hypothesis-for instance if $G$ is amenable:

1° We write $\mathcal{I}$ as the kernel of a representation $\rho$ of $\mathcal{D}$, we induce $\rho$ to a representation $\pi$ of $\mathcal{W}(G,P)$ (in the sense of [15]) and define $\mathrm{Ind}\,\mathcal{I} = \mathrm{Ker}\,\pi$.

2° $\mathrm{Ind}\,\mathcal{I} = \{T \in \mathcal{W}(G,P) | E(T^*T) \in \mathcal{I}\}$ with $E : \mathcal{W}(G,P) \to \mathcal{D}$ the conditional expectation (compare with Lemma I 2.2 of [16]).

3° For any $T$ in $\mathcal{W}(G,P)$ we take the projections of $T$ onto the diagonal subspaces $\{\mathcal{D}_x | x \in PP^{-1}\}$ and transport them onto $\mathcal{D}$ with the canonical isomorphisms $\mathcal{D}_x \to \to \mathcal{D}$ ($x \in PP^{-1}$), obtaining thus a family of "coefficients" $\{T_x | x \in PP^{-1}\}$. We define $T$ to be in $\mathrm{Ind}\,\mathcal{I}$ if and only if all the $T_x$'s are in $\mathcal{I}$ (compare with Definition 4.15 of [19]).

4° $\mathrm{Ind}\,\mathcal{I} =$ the closed two-sided ideal of $\mathcal{W}(G,P)$ generated by $\mathcal{I}$ (compare with Corollary I 2.6 of [16], Proposition 5.10 of [19]).

The map $\mathcal{I} \to \mathrm{Ind}\,\mathcal{I}$ is one-to-one, with inverse $\mathcal{J} \to \mathcal{J} \cap \mathcal{D}$, and its range consists of the closed two-sided ideals of $\mathcal{W}(G,P)$ which are invariant to the conditional expectation. Generally, these are not all the closed two-sided ideals of $\mathcal{W}(G,P)$.

1.4. Applications. The spectrum of the diagonal subalgebra $\mathcal{D}$ can be canonically identified to an explicitly described space $\Omega$ having as elements a class of subsets of $P$. Taking into account the induction process, $\Omega$ can be used as an intermediate link in finding connections between $\mathcal{W}(G,P)$ and the order relation determined by $P$ on $G$.

As an application we can prove that $\mathcal{W}(G,P)$ contains the compact operators if and only if there exists a finite subset of $P \setminus \{e\}$ which contains a lower bound for every element of $P\setminus\{e\}$ (the last condition always holds when $P$ is finitely generated). The proof is done by passing through the equivalent statement: "For any $t$ in $P$, the interval $[e,t] = \{a \in P | a \leqslant t\}$ is an open point of $\Omega$". The implication "$[e,t]$ open, $\forall t \in P \Rightarrow \mathcal{W}(G,P) \supseteq \mathcal{K}$" was proved by P. Muhly and J. Renault in a more general case (see Corollary 3.7.2 of [9]), and they conjecture that its converse also holds in general (see [9], 3.7.3). We note that $\mathcal{W}(G,P) \supseteq \mathcal{K}$ is a necessary condition for $\mathcal{W}(G,P)$ to be type I, because $\mathcal{W}(G,P)$ is irreducible; this condition is not sufficient ($\mathcal{W}(F_n, SF_n) \supseteq \mathcal{K}$, but it is not type I because $\mathcal{W}(F_n, SF_n)/\mathcal{K} = O_n$).

For another application, let us consider the Theorem 1 of [5], which can be stated as follows: "If the totally ordered Abelian group $(G,P)$ is Archimedean, then any two non-unitary representations by isometries of $P$ generate canonically isomorphic $C^*$--algebras". This is equivalent to the fact that, in the considered setting, any non--unitary representation by isometries of $P$ extends to a faithful representation of $\mathcal{W}(G, P)$, and a proof may be given along these lines. It is interesting that we can also prove "the converse"; more precisely, for a totally ordered Abelian group $(G, P)$ the following are equivalent:

1° $P$ is Archimedean;

2° any two non-unitary representations by isometries of $P$ generate canonically isomorphic $C^*$-algebras;

3° the commutator ideal of $\mathcal{W}(G, P)$ is simple.

(Remark: 2° $\Rightarrow$ 3° is proved in [5], too.)

1.5. THE WIENER-HOPF GROUPOID. It is known that, in a more general case than the one studied here, the $C^*$-algebra of the Wiener-Hopf operators can be presented as $C^*_{\text{red}}(\mathcal{G})$, with $\mathcal{G}$ a locally compact groupoid ([9], see also [11]). In the present context, $\mathcal{G}$ can be got by transposing the action $\{\alpha_x | x \in PP^{-1}\}$ considered at 1.2 on the spectrum $\Omega$ of $\mathcal{D}$; the action of $PP^{-1}$ on $\Omega$ obtained in this manner is only partially defined, and gives exactly a groupoid structure (this is $\mathcal{G}$).

The groupoid interpretation has turned out to be extremely useful during the preparation of this work. Nevertheless we have decided, for lack of space, to omit it in the final presentation, and discuss it separately in a future paper.

1.6 Finally let us give a brief review of the sections into which the paper is subdivided. In Section 2 we present the "quasi-lattice" ordered groups, and show that their Wiener-Hopf $C^*$-algebras are generated by isometries. In Section 3 we put into evidence the "crossed-product type" stucture of $\mathcal{W}(G, P)$. Amenability is discussed in Section 4. In Section 5 we consider two important particular cases: in 5.1 we show how the uniqueness property of the Cuntz algebra implies the amenability of the free partially ordered group; in 5.2 we consider the totally ordered Abelian case and we give a simple proof to a generalization of the Theorem 1 of [5], due to G. Murphy (Theorem 2.9 of [10]). The sixth section, which is the last one, deals with the induced ideals; it also contains the two applications announced at 1.4.

## 2. QUASI-LATTICE ORDERED GROUPS

2.1. DEFINITIONS. By a (partially) ordered group we shall understand a pair $(G, P)$, with $G$ a (not necessarily Abelian) discrete group and $P$ a subsemigroup of $G$. We shall always assume that $P \cap Q$ is the unit $e$ of $G$, where $Q = P^{-1}$; this implies that $x \leqslant y \overset{def}{\Longleftrightarrow} x^{-1}y \in P$ is a partial order relation on $G$ (called the left invariant order relation induced by $P$).

It is clear that $P = \{x \in G | x \geqslant e\}$ and $Q = \{x \in G | x \leqslant e\}$. An important role will be played in what follows by the set $PQ = \{pq | p \in P,\ q \in Q\}$ which can be also described in terms of "$\leqslant$" as $\{x \in G | x$ has upper bounds in $P\}$.

The ordered group $(G, P)$ is said to be *quasi-lattice ordered* if the following condition is satisfied:

(QL) For any $n \geqslant 1$, any $x_1, \dots, x_n$ in $G$ which have common upper bounds (c.u.b.) in $P$, also have a least c.u.b. in $P$.

This condition can also be expressed in a weaker form, i.e. (QL)$\Leftrightarrow$ (QL1)+(QL2), with:

(QL1) Any $x$ in $PQ$ has a least upper bound in $P$;

(QL2) Any $s, t$ in $P$ with c.u.b. have a least c.u.b.;

(the proof of "$\Leftarrow$" is easily done using induction).

If $(G, P)$ is a quasi-lattice ordered group and $x_1, \dots, x_n$ in $G$ have c.u.b. in $P$, then their least c.u.b. in $P$ will be denoted by $\sigma(x_1, \dots, x_n)$. In particular, the least upper bound in $P$ of an arbitrary element $x$ of $PQ$ will be denoted by $\sigma(x)$. We shall also use the notation $\tau(x) = x^{-1}\sigma(x)$, $\forall x \in PQ$; $\tau(x)$ is in $P$, because $x \leqslant \sigma(x)$. It can be seen without difficulty that $\sigma(x^{-1}) = \tau(x)$ and $\tau(x^{-1}) = \sigma(x)$, $\forall x \in PQ$.

The name "quasi-lattice ordered" is justified by the fact that lattice ordered Abelian groups are quasi-lattice ordered. To be more precise, an ordered Abelian group $(G, P)$ with the property that any two elements of $P$ have a least c.u.b. is quasi-lattice ordered. Indeed, this hypothesis is exactly (QL2) (we take into account that in the Abelian case, any two elements of $P$ have c.u.b.—their product for instance). In what concerns (QL1), it suffices to note that for any $x$ in $PQ$ and for an arbitrary writing $x = st^{-1}$ with $s, t$ in $P$, $\sigma(s, t)t^{-1}$ is the least upper bound of $x$ in $P$.

2.2. ELEMENTARY REMARKS. Let $(G, P)$ be a quasi-lattice ordered group.

1° For any $x$ in $PQ$, the set:

$$\Delta_x = \{(s, t) \in P \times P \,|\, st^{-1} = x\} \tag{1}$$

will be called the diagonal of $x$. Clearly, $\{\Delta_x | x \in PQ\}$ is a partition of $P \times P$; note that each $\Delta_x$ is canonically put into bijection with $P$ by the map: $P \ni p \to (\sigma(x)p, \tau(x)p) \in \Delta_x$ (we leave the simple proof to the reader). So an $(s, t) \in P \times P$ is determined by $x = st^{-1} \in PQ$ and an element $p \in P$ such that $s = \sigma(x)p$, $t = \tau(x)p$; $x$ and $p$ can be thought as "diagonal coordinates" of $(s, t)$.

2° $s, t \in P$ have c.u.b. in $P$ if and only if $s^{-1}t \in PQ$. If this happens, then $\sigma(s^{-1}t) = s^{-1}\sigma(s, t)$.

*Proof* "$\Rightarrow$" $s^{-1}t = (s^{-1}\sigma(s,t))(t^{-1}\sigma(s,t))^{-1} \in PQ$; at the same time we get $\sigma(s^{-1}t) \leqslant s^{-1}\sigma(s,t)$.

"$\Leftarrow$" $t = s(s^{-1}t) \leqslant s\sigma(s^{-1}t)$ and obviously $s \leqslant s\sigma(s^{-1}t)$, hence $s$ and $t$ have c.u.b. and $\sigma(s,t) \leqslant s\sigma(s^{-1}t)(\Longleftrightarrow s^{-1}\sigma(s,t) \leqslant \sigma(s^{-1}t))$. ∎

3° Let $s, t_1, t_2$ be in $P$. $t_1$ and $t_2$ have c.u.b. if and only if $st_1$ and $st_2$ have. If this happens, then $\sigma(st_1, st_2) = s\sigma(t_1, t_2)$. (This is immediate from 2°.)

2.3. EXAMPLES. 1° *Any totally ordered group* is clearly quasi-lattice ordered. In particular, if $G$ is a subgroup of $\mathbb{R}$ and $P = G \cap [0,\infty)$, then $(G,P)$ is quasi-lattice ordered. By a theorem of Hölder (see [7], Chapter IV, Section 1, Theorem 1) any totally ordered Archimedean group can be put into this form.

2° *Direct products.* If $\{(G_j,P_j)|1 \leqslant j \leqslant n\}$ are quasi-lattice ordered groups, then so is $(G_1 \times \cdots \times G_n,\ P_1 \times \cdots \times P_n)$, too, because the order on it is the "product order". In particular, if $\{G_j|1 \leqslant j \leqslant n\}$ are subgroups of $\mathbb{R}$ and if we put $G = G_1 \times \cdots \times G_n$, $P = G \cap [0,\infty)^n$, then $(G,P)$ is quasi-lattice ordered.

3° *Semi-direct products.* Let $(G,P)$ and $(H,R)$ be quasi-lattice ordered groups such that $G$ has an action $\Phi$ by automorphisms on $H$, and such that $R$ is $\Phi$-invariant. It is then easy to see that $P \times R$ gives on $G \rtimes_\Phi H$ the product order; this entails that $(G \rtimes_\Phi H, P \times R)$ is quasi-lattice ordered.

In particular, we may consider the action of $((0,\infty),\cdot)$ on $(\mathbb{R},+)$ by multiplication, which leaves fixed the semigroup $[0,\infty) \subseteq \mathbb{R}$, and obtain a natural quasi-lattice order on the "$ax+b$" group (with positive semigroup $[1,\infty) \times [0,\infty)$).

4° *Free groups.* Let $F_n$ be the free group on $n$ generators $a_1, \dots, a_n$, and denote by $SF_n$ the semigroup generated by $a_1, \dots, a_n$. We claim that $(F_n, SF_n)$ is quasi-lattice ordered. To prove this, we first remark that for any $t$ in $SF_n$ the set $\{s \in SF_n | s \leqslant t\}$ can also be written as $\{e \lneqq a_{j_1} \lneqq a_{j_1}a_{j_2} \lneqq \cdots \lneqq a_{j_1}a_{j_2}\cdots a_{j_m} = t\}$, with $t = a_{j_1}a_{j_2}\cdots a_{j_m}$ the "spelling" of $t$, and is hence totally ordered. This implies that any $s,t$ in $SF_n$ which have c.u.b. are comparable; consequently, the greater one of the two elements is also their least c.u.b., and (QL2) is satisfied. In what concerns (QL1), it is easy to see that any $x$ in $(SF_n)(SF_n)^{-1}$ has a unique reduced writing $x = st^{-1}$ with $s,t \in SF_n$, and that $\sigma(x) = s$.

Using a slight adaptation of this argument, one can prove without difficulty the more general fact that if $\{(G_i,P_i)|i \in I\}$ is a family of quasi-lattice ordered groups, then the free product $(\mathop{\ast}_{i\in I} G_i, \mathop{\ast}_{i\in I} P_i)$ is also quasi-lattice ordered.

5° If $G = ((0,\infty),\ \cdot)$ and $P = \mathbb{N} \setminus \{0\}$, then any two elements of $P$ have a least c.u.b., which is their least common multiple, and any element of $G = PQ$ has a least upper bound in $P$, which is the numerator of its unique representation as an irreducible fraction. Hence $(G,P)$ is a quasi-lattice ordered group.

2.4. We now prove that *the Wiener-Hopf $C^*$-algebra* of a quasi-lattice ordered group *is generated by isometries.*

For any ordered group $(G,P)$, the Wiener-Hopf $C^*$-algebra $\mathcal{W}(G,P)$ can be defined as the $C^*$-subalgebra of $\mathcal{L}(\ell^2(P))$ generated by $\{J^*\Lambda(x)J \mid x \in G\}$ with $\Lambda : G \to \mathcal{L}(\ell^2(G))$ the left regular representation and $J : \ell^2(P) \to \ell^2(G)$ the inclusion operator. For any $t$ in $P$, the operator $J^*\Lambda(t)J$, which will be from now on denoted by $W(t)$, is an isometry; it is determined by the fact that $W(t)\delta_a = \delta_{ta}$, $\forall a \in P$, where $(\delta_a)_{a\in P}$ is the canonical basis of $\ell^2(P)$. We note that $W(s)W(t) = W(st)$, $\forall s,t \in P$, and that $W(e) = I_{\ell^2(P)}$, i.e. $W : P \to \mathcal{L}(\ell^2(P))$ is a representation by isometries. $W$ will be called the Wiener-Hopf representation.

Let us remark that (in the case of any ordered group) $J^*\Lambda(x)J = 0$ if $x \notin PQ$; indeed, it is immediate that:

$$(J^*\Lambda(x)J)\delta_a = \begin{cases} \delta_{xa}, & \text{if } xa \in P, \\ 0, & \text{if } xa \notin P, \end{cases} \tag{2}$$

so that: $J^*\Lambda(x)J \neq 0 \Longrightarrow \exists a \in P$ such that $xa \in P \Longrightarrow x \in PQ$. Hence we can also write $\mathcal{W}(G,P) = C^*\ (\{J^*\Lambda(x)J \mid x \in PQ\})$.

Assume now that $(G,P)$ is quasi-lattice ordered. An easy computation shows that for any $x$ in $PQ$ and $a$ in $P$:

$$W(\sigma(x))W(\tau(x))^*\delta_a = \begin{cases} \delta_{xa}, & \text{if } \tau(x) \leqslant a, \\ 0, & \text{if } \tau(x) \not\leqslant a; \end{cases} \tag{3}$$

but $\tau(x) \leqslant a \iff \sigma(x^{-1}) \leqslant a \iff x^{-1} \leqslant a \iff xa \in P$. So (2) and (3) imply $J^*\Lambda(x)J = W(\sigma(x))W(\tau(x))^*$, $\forall x \in PQ$, and this makes clear that $\mathcal{W}(G,P) = C^*\ (\{W(t) \mid t \in P\})$. Hence $\mathcal{W}(G,P)$ is indeed generated by isometries.

## 3. THE "CROSSED-PRODUCT TYPE" STRUCTURE OF $\mathcal{W}(G,P)$

In this section we fix a quasi-lattice ordered group $(G,P)$.

3.1. THE DIAGONAL SUBALGEBRA. We introduce the notation $M(t) = W(t)W(t)^*$ – the multiplication operator with the characteristic function of $\{a \in P \mid a \geqslant t\}$, $\forall t \in P$, and define $\mathcal{D} = \operatorname{clos\,sp}\{M(t) \mid t \in P\}$. It is immediate that:

$$M(s)M(t) = \begin{cases} M(\sigma(s,t)), & \text{if } s \text{ and } t \text{ have c.u.b.}, \\ 0, & \text{if they haven't.} \end{cases} \tag{4}$$

This implies that $\{M(t)|t \in P\} \cup \{0\}$ is a commuting and closed under multiplication family of selfadjoint projections of $\mathcal{W}(G, P)$, and makes clear that $\mathcal{D}$ is an Abelian $C^*$-subalgebra of $\mathcal{W}(G, P)$. $\mathcal{D}$ is also unital, because $M(e) = I_{\ell^2(P)}$; it will be called the diagonal subalgebra of $\mathcal{W}(G, P)$. (Remark: it can be shown that $\mathcal{D}$ is maximal Abelian.)

3.2. THE "PICTURE" OF $\mathcal{W}(G, P)$. The name of $\mathcal{D}$ is justified by the following

PROPOSITION. *The operators $\{W(s)W(t)^* | s, t \in P\}$ are linearly independent, and their linear span is a dense unital $*$-subalgebra of $\mathcal{W}(G, P)$.*

Indeed, the proposition asserts that, in an informal sense, $\mathcal{W}(G, P)$ is "the closed linear span of $P \times P$"; in the same informal sense, $\mathcal{D}$ is the closed linear span of $\Delta_e$, the principal diagonal of $P \times P$ (see relation (1) of 2.2).

*Proof.* Let us suppose that $\{W(s)W(t)^* | s, t \in P\}$ are linearly dependent. We can then find $\{\lambda_j | 1 \leqslant j \leqslant n\}$ in $\mathbb{C} \setminus \{0\}$ and $\{(s_j, t_j) | 1 \leqslant j \leqslant n\}$ with $(s_j, t_j) \neq (s_k, t_k)$ for $j \neq k$, such that $\sum_{j=1}^{n} \lambda_j W(s_j)W(t_j)^* = 0$. We have of course $n \geqslant 2$, because $W(s)W(t)^*$ can never be zero ($W(s)W(t)^*\delta_t = \delta_s$). The finite set $\{t_1, \ldots, t_n\}$ must contain an element $t_k$ which is minimal, in the sense that for any $1 \leqslant j \leqslant n$, $t_j \leqslant t_k$ implies $t_j = t_k$. But:

$$0 = \left( \sum_{j=1}^{n} \lambda_j W(s_j)W(t_j)^* \right) \delta_{t_k} = \lambda_k \delta_{s_k} + \sum_{\substack{j \neq k \\ t_j \leqslant t_k}} \lambda_j \delta_{s_j t_j^{-1} t_k}$$

implies that there exists at least one $j \neq k$ such that $t_j \leqslant t_k$ and $s_j t_j^{-1} t_k = s_k$; for such a $j$ we clearly obtain $t_j = t_k$, $s_j = s_k$ — contradiction.

For the rest of the proof it clearly suffices to show that $\{W(s)W(t)^* | s, t \in P\} \cup \{0\}$ is closed under multiplication and $*$-operation. The latter fact is clear. To prove the first one, we remark that if for arbitrary $t$ and $u$ in $P$ we multiply the relation (4) of 3.1, written for $t$ and $u$, with $W(t)^*$ on the left and with $W(u)$ on the right, we obtain:

$$W(t)^* W(u) = \begin{cases} W(t)^* W(\sigma(t, u)) W(\sigma(t, u))^* W(u), & \text{if } t \text{ and } u \text{ have c.u.b.,} \\ 0, & \text{if they haven't.} \end{cases}$$

But:

$$W(t)^* W(\sigma(t, u)) = W(t)^* (W(t) W(t^{-1}\sigma(t, u))) = W(t^{-1}\sigma(t, u)),$$

and similarly $W(\sigma(t, u))^* W(u) = W(u^{-1}\sigma(t, u))^*$. Hence for any $t, u$ in $P$:

$$W(t)^* W(u) = \begin{cases} W(t^{-1}\sigma(t, u)) W(u^{-1}\sigma(t, u))^*, & \text{if } t \text{ and } u \text{ have c.u.b.,} \\ 0, & \text{if they haven't,} \end{cases}$$

and it becomes clear that for any $s,t,u,v$ in $P$:

$$(5)\quad (W(s)W(t)^*)(W(u)W(v)^*) = \begin{cases} W(st^{-1}\sigma(t,u))W(vu^{-1}\sigma(t,u))^*, & \text{if } t \text{ and } u \text{ have c.u.b.,} \\ 0, & \text{if they haven't.} \end{cases}$$

■

3.3. An other Justification for the Name of $\mathcal{D}$ is given by the equivalent characterization:

$$\mathcal{D} = \{T \in \mathcal{W}(G,P) | T \text{ has diagonal matrix relatively to the canonical basis of } \ell^2(P)\}$$

*Proof.* Every $M(t)$ $(t \in P)$ is diagonal relatively to the canonical basis, because:

$$(6)\qquad M(t)\delta_a = \begin{cases} \delta_a, & \text{if } a \geqslant t, \\ 0, & \text{otherwise.} \end{cases}$$

Since the property of being diagonal can be passed through the closed linear span, we get "$\subseteq$ ".

In order to prove "$\supseteq$ ", we denote by $\mathcal{D}_0$ the $C^*$-subalgebra of $\mathcal{L}(\ell^2(P))$ consisting of all the operators having diagonal matrix relatively to the canonical basis. It is well-known that there exists a linear and contractive map $E_0 : \mathcal{L}(\ell^2(P)) \to \mathcal{D}_0$ determined by the following rule: the matrix of $E_0(T)$ (relatively to the canonical basis) is obtained from the one of $T$ by replacing with zero all the entries which are not situated on the principal diagonal. If $s \neq t$, then $E_0(W(s)W(t)^*) = 0$, because for every $a$ in $P$:

$$\langle W(s)W(t)^*\delta_a|\delta_a\rangle = \langle W(t)^*\delta_a|W(s)^*\delta_a\rangle =$$

$$\left\{\begin{matrix} \langle \delta_{t^{-1}a}|\delta_{s^{-1}a}\rangle, & \text{if } s,t \leqslant a, \\ 0, & \text{otherwise} \end{matrix}\right\} = 0.$$

On the other hand $E_0(W(t)W(t)^*) = W(t)W(t)^*$, $\forall t \in P$, because $W(t)W(t)^* = M(t)$ is diagonal. Hence the closed linear subspace $\{T \in \mathcal{W}(G,P)|E_0(T) \in \mathcal{D}\}$ of $\mathcal{W}(G,P)$ contains $W(s)W(t)^*$ for any $s,t$ in $P$, and therefore must be $\mathcal{W}(G,P)$ itself, by Proposition 3.2. Finally, if $T$ in $\mathcal{W}(G,P)$ has diagonal matrix relatively to the canonical basis, then $T = E_0(T) \in \mathcal{D}$. ■

3.4. A first Evidence that $\mathcal{W}(G,P)$ has a "crossed-product type" structure is obtained by remarking that $\mathcal{D}$ is only one of the diagonal subspaces of $\mathcal{W}(G,P)$ that can be considered. More precisely, for any $x$ in $PQ$ we can define the diagonal subspace of $x$, $\mathcal{D}_x = \operatorname{clos sp}\{W(s)W(t)^* | (s,t) \in \Delta_x\}$ ($\Delta_x$ is defined in 2.2, relation (1). Clearly $\mathcal{D} = \mathcal{D}_e$.) The subspaces $\{\mathcal{D}_x | x \in PQ\}$ obey the following rules of multiplication and involution :

$$(7) \qquad \mathcal{D}_x\mathcal{D}_y \subseteq \begin{cases} \mathcal{D}_{xy}, & \text{if } xy \text{ is still in } PQ, \\ \{0\}, & \text{otherwise,} \end{cases} \qquad (\forall x, y \in PQ),$$

$$(8) \qquad \mathcal{D}_x^* = \mathcal{D}_{x^{-1}} \quad (\forall x \in PQ)$$

The formula (8) is clear. To prove (7), let $x, y$ be in $PQ$ such that $\mathcal{D}_x\mathcal{D}_y \neq \{0\}$; then there exists $(s,t) \in \Delta_x$ and $(u,v) \in \Delta_y$ such that $(W(s)W(t)^*)(W(u)W(v)^*) \neq$ $\neq 0$. According to the relation (5) of 3.2, $t$ and $u$ must have c.u.b., and the product is $W(st^{-1}\sigma(t,u))W(vu^{-1}\sigma(t,u))^*$. But $(st^{-1}\sigma(t,u))(vu^{-1}\sigma(t,u))^{-1} = xy$, and this makes clear that $xy \in PQ$ and $\mathcal{D}_x\mathcal{D}_y \subseteq \mathcal{D}_{xy}$.

Moreover, Proposition 3.2 implies that the set of finite sums of elements of the $\mathcal{D}_x$'s is a dense linear subspace of $\mathcal{W}(G,P)$. It can be shown that every element $T$ of this subspace has a unique writing $T = \sum_{x \in PQ} T_x$, with $T_x \in \mathcal{D}_x$, $\forall x \in PQ (T_x \neq$ $\neq 0$ only for a finite number of $x$'s). Hence, in a weak sense, the diagonal subspaces establish a direct sum decomposition of $\mathcal{W}(G,P)$.

Using the method of 3.3, one can describe the operators belonging to a given $\mathcal{D}_x$ in terms of their matrices relatively to the canonical basis. This description offers a better understanding of the picture of $\mathcal{W}(G,P)$; but however, since we shall not be using it anywhere in the paper, we leave its details to the reader.

3.5. A SECOND EVIDENCE for the "crossed-product type" structure of $\mathcal{W}(G,P)$ is given by the fact that for any $x$ in $PQ$, the map $X \to W(\sigma(x))XW(\tau(x))^*$ is an isometric isomorphism (of Banach spaces) between $\mathcal{D} = \mathcal{D}_e$ and $\mathcal{D}_x$; its inverse is $\mathcal{D}_x \ni Y \to W(\sigma(x))^*YW(\tau(x)) \in \mathcal{D}$. The only non-trivial point in the proof of this fact is that for any $(s,t) \in \Delta_x$ we have, putting $Y = W(s)W(t)^*$, that $W(\sigma(x))^*YW(\tau(x)) \in \mathcal{D}$ and that $W(\sigma(x))(W(\sigma(x))^*YW(\tau(x)))W(\tau(x))^* = Y$. In order to show this, we take the diagonal coordinates of $(s,t)$, i.e. we write $s =$ $= \sigma(x)p$, $t = \tau(x)p$ for a uniquely determined $p$ in $P$ (see 2.2.1°), and we easily get: $W(\sigma(x))^*YW(\tau(x)) = M(p) \in \mathcal{D}$, $W(\sigma(x))M(p)W(\tau(x))^* = W(s)W(t)^* = Y$.

3.6. A THIRD EVIDENCE for the "crossed-product type" structure of $\mathcal{W}(G,P)$ is the existence of a canonical conditional expectation of $\mathcal{W}(G,P)$ onto $\mathcal{D}$. More precisely, we have the following

PROPOSITION. *There exists a unique bounded linear map $E : \mathcal{W}(G,P) \to \mathcal{D}$ such that for any $s, t$ in $P$:*

$$E(W(s)W(t)^*) = \begin{cases} W(s)W(t)^*, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

*E is a conditional expectation.*

*Proof.* $E_0 : \mathcal{L}(\ell^2(P)) \to \mathcal{D}_0$ considered in the prof of 3.3 gives by restriction and corestriction a contractive linear map acting as in the statement of the proposition. This proves the existence of $E$. Its uniqueness is clear (from 3.2), and the fact that it is a conditional expectation is immediately implied by the theorem of Tomiyama. ∎

COROLLARY. *For any $x$ in $PQ$ there exists a canonical projection of norm one $E_x$ of $\mathcal{W}(G,P)$ onto $\mathcal{D}_x$, determined by:*

$$E_x(W(s)W(t)^*) = \begin{cases} W(s)W(t)^*, & \text{if } (s,t) \in \Delta_x, \\ 0, & \text{otherwise.} \end{cases}$$

*(Of course, $E_e = E$.)*

*Proof.* The map $T \to W(\sigma(x))E(W(\sigma(x))^* T W(\tau(x)))W(\tau(x))^*$ is easily seen to have the properties requested in the statement of the corollary. On the other hand, the uniqueness of $E_x$ (with these properties) is obvious from 3.2. ∎

REMARK. *$E$ is faithful.* Indeed, taking into account that $E$ is obtained from $E_0 : \mathcal{L}(\ell^2(P)) \to \mathcal{D}_0$ defined in the proof of 3.3, we easily obtain the formula:

$$\langle E(T)\delta_a | \delta_a \rangle = \langle T\delta_a | \delta_a \rangle, \quad \forall T \in \mathcal{W}(G,P),\ a \in P. \tag{9}$$

So, if a positive $T \in \mathcal{W}(G,P)$ has $E(T) = 0$, then: $\|\sqrt{T}\delta_a\|^2 = \langle T\delta_a|\delta_a\rangle = \langle E(T)\delta_a|\delta_a\rangle = 0$, $\forall a \in P$, hence $\sqrt{T} = 0$ and $T = 0$.

$E$ is always faithful because actually $\mathcal{W}(G,P)$ is the analogue of a reduced crossed product ("of $\mathcal{D}$ by $PQ$"). An analogue of the full crossed product will be considered in the next section, in connection with amenability phenomena.

3.7. THE ACTION OF $PQ$ ON $\mathcal{D}$ The last three subsections indicate that, in an informal way, $\mathcal{W}(G,P)$ must be the analogue of a (reduced) crossed product $C^*$-algebra "of $\mathcal{D}$ by $PQ$". It is believable that there exists a natural "action" of $PQ$ on $\mathcal{D}$ connected to this crossed product structure. Recalling the way things look like in the theory of crossed products by discrete groups, it is also believable that the map $\mathcal{D} \to \mathcal{D}$ given by an $x \in PQ$ must be tied to the canonical isomorphism $\mathcal{D} \to \mathcal{D}_x$ of 3.5. A natural candidate is :

$$\alpha_x(X) = (W(\sigma(x))W(\tau(x))^*)X(W(\sigma(x))W(\tau(x))^*)^*, \quad \forall x \in PQ,\ \forall X \in \mathcal{D}, \tag{10}$$

which makes sense and is a $*$-endomorphism of $\mathcal{D}$, by the following

PROPOSITION AND DEFINITION. *For any $s,t$ in $P$, the map $X \to (W(s)W(t)^*) \cdot X(W(s)W(t)^*)^*$ is a (not necessarily unital) $*$-endomorphism of $\mathcal{D}$, denoted by $\alpha_{s,t}$. We briefly write $\alpha_x$ instead of $\alpha_{\sigma(x),\tau(x)}$ ($\forall x \in PQ$).*

*Proof.* It suffices to show that for any $t$ in $P$, $X \to W(t)XW(t)^*$ and $X \to W(t)^*XW(t)$ are $*$-endomorphisms of $\mathcal{D}$. It is obvious that these maps are $*$-morphisms of $\mathcal{D}$ into $\mathcal{W}(G,P)$. (Let us check for instance the multiplicativity of the second. For $t$ in $P$ and $X_1, X_2$ in $\mathcal{D}$ we have:

$$(W(t)^*X_1W(t))(W(t)^*X_2W(t)) = W(t)^*X_1M(t)X_2W(t) = \\ = W(t)^*M(t)X_1X_2W(t) = W(t)^*X_1X_2W(t),$$

where we used the commutativity of $\mathcal{D}$ and the fact that $W(t)^*M(t) = W(t)^*$.) So all we need to verify is that the two considered maps take values in $\mathcal{D}$. It clearly suffices to make the verifications on the generators $\{M(s)|s \in P\}$ of $\mathcal{D}$. But simple computations show that for any $s,t$ in $P$:

$$W(t)M(s)W(t)^* = M(ts), \tag{11}$$

$$W(t)^*M(s)W(t) = \begin{cases} M(t^{-1}\sigma(s,t)), & \text{if } s \text{ and } t \text{ have a c.u.b.,} \\ 0, & \text{if they haven't.} \end{cases} \tag{12}$$

■

3.8. Remark. We can sum up the results of this section into the formula $\mathcal{W}(G,P) =$"$\mathcal{D} \rtimes_\alpha PQ$" (with $\alpha$ defined in 3.7). Now, of course, $PQ$ is generally not a group (see for instance the Example 4° of 2.3). Even if it is, it does not "act" on $\mathcal{D}$ by automorphisms, but only by a class of not necessarily unital $*$-endomorphisms, which is not closed under composition. (Indeed, the semigroup generated by $\{\alpha_x|x \in PQ\}$ is $\{\alpha_{s,t}|s,t \in P\}$ when any two elements of $P$ have c.u.b., and $\{\alpha_{s,t}|s,t \in P\} \cup \{0\}$ in the opposite case. The only observation needed to prove this, besides a trivial use of relation (5) of 3.2, is that $\alpha_{s,t} = \alpha_{s,e} \circ \alpha_{e,t} = \alpha_s \circ \alpha_{t^{-1}}$, $\forall s,t \in P$.) Hence even in the simplest cases, we do not have a crossed product structure in the proper sense.

3.9. Covariant Representations. One naturally expects such an object ( a "covariant representation of ($\mathcal{D}$, $PQ$, $\alpha$)") to consist of a unital $*$-representation $\rho$ of $\mathcal{D}$ and a "representation" $U$ of $PQ$ on the same Hilbert space $H$, tied toghether by the covariance relation:

$$\rho(\alpha_x(X)) = U(x)\rho(X)U(x)^*, \quad \forall x \in PQ,\ X \in \mathcal{D}. \tag{13}$$

But let us remark that if we put in (13) $x = t \in P$ and $X = I$, we get:

$$\rho(M(t)) = U(t)U(t)^*, \quad \forall t \in P;$$

this clearly implies that $\rho$ is determined by $U$. So $U$ alone gives the covariant representation, provided it satisfies a certain condition extracted from the following lemma:

LEMMA. *Let $\{L(t)|t \in P\}$ be a family of selfadjoint projections of the unital $C^*$-algebra $\mathfrak{A}$. There exists a $*$-homomorphism $\rho : \mathcal{D} \to \mathfrak{A}$ such that $\rho(M(t)) = L(t)$, $\forall t \in P$, if and only if for any $s,t$ in $P$:*

$$L(s)L(t) = \begin{cases} L(\sigma(s,t)), & \text{if } s \text{ and } t \text{ have c.u.b.,} \\ 0, & \text{if they haven't.} \end{cases} \tag{14}$$

*Proof.* "$\Longrightarrow$" clearly follows from (4) of 3.1. To prove "$\Longleftarrow$" it is suffices to show that :

$$\left\| \sum_{j=1}^{n} \lambda_j L(t_j) \right\| \leqslant \left\| \sum_{j=1}^{n} \lambda_j M(t_j) \right\|, \quad \forall t_1, \dots, t_n \in P,\ \lambda_1, \dots, \lambda_n \in \mathbb{C}.$$

Because the operator $T = \sum_{j=1}^{n} \lambda_j M(t_j)$ is diagonal relatively to the canonical basis of $\ell^2(P)$ (see 3.3), its norm equals $\sup_{a \in P} |\langle T\delta_a | \delta_a \rangle|$, and so we get :

$$\left\| \sum_{j=1}^{n} \lambda_j M(t_j) \right\| = \sup_{a \in P} \left| \sum_{\substack{1 \leqslant j \leqslant n \\ t_j \leqslant a}} \lambda_j \right|. \tag{15}$$

On the other hand we have:

$$\left\| \sum_{j=1}^{n} \lambda_j L(t_j) \right\| = \sup \left\{ \left| \sum_{j \in A} \lambda_j \right| \,\middle|\, \emptyset \neq A \subseteq \{1, \dots, n\} \right. \\ \left. \prod_{j \in A} L(t_j) \prod_{k \notin A} (I - L(t_k)) \neq 0 \right\} \tag{16}$$

The proof of (16) is done by writing for every $j$ : $L(t_j) = L(t_j) \prod_{k \neq j} (L(t_k) + (I - L(t_k)))$, and effectively computing the product, which is thereafter substituted in $\sum_{j=1}^{n} \lambda_j L(t_j)$. We leave the details to the reader.

Now, comparing (15) and (16) we see that it suffices, in order to end the proof, to take $\emptyset \neq A \subseteq \{1, \dots, n\}$ as in (16) and find an element $t$ in $P$ such that $A = \{j | 1 \leqslant j \leqslant n,\ t_j \leqslant t\}$. For such an $A$ we have in particular $\prod_{j \in A} L(t_j) \neq 0$; taking (14) into account, we deduce that $\{t_j | j \in A\}$ have c.u.b. and in fact $\prod_{j\in A} L(t_j) = L(t)$, with $t = \sigma(\{t_j | j \in A\})$. We finally remark that for any $k$ in the complement of $A$ we have $L(t)(I - L(t_k)) \neq 0 \Longrightarrow L(t) \not\leqslant L(t_k) \Longrightarrow t_k \not\leqslant t$ (the last implication holds because, as an immediate consequence of (14), $s \to L(s)$ is decreasing). This makes clear that $A = \{j | 1 \leqslant j \leqslant n,\ t_j \leqslant t\}$, which completes the proof. ■

Now, it is not very clear to us what is the correct definition for the notion of "representation of $PQ$" (a hint is given by the fact that $\alpha$ defined at 3.7 must be an "action" of $PQ$). We have made the simplifying assumption that such a representation should be determined by its restriction to $P$, arriving thus to the following

DEFINITION. Let $V$ be a representation by isometries of $P$ on the Hilbert space $H$ ($V(t)^*V(t) = I$, $\forall t$, $V(s)V(t) = V(st)$, $\forall s,t$, $V(e) = I$). $V$ is said to be *covariant* if (14) holds with $L(t) = V(t)V(t)^*$, $\forall t \in P$.

By the previous lemma, a covariant representation $V : P \to \mathcal{L}(H)$ gives a unital $*$-representation $\rho : \mathcal{D} \to \mathcal{L}(H)$. It is not difficult to see that $\rho$ and $U$ satisfy (13), where $U(x) = V(\sigma(x))V(\tau(x))^*$, $\forall x \in PQ$.

## 4. AMENABILITY FOR QUASI-LATTICE ORDERED GROUPS

4.1 THE UNIVERSAL $C^*$-ALGEBRA $C^*(G,P)$. From the $C^*$-algebraic point of view, amenability means the canonical coincidence of two $C^*$-algebras, one of them being universal, obtained by enveloping a certain class of representations, and the other one being associated to a remarkable representation of the class. Now let $(G,P)$ be a quasi-lattice ordered group. We have the class of covariant representations of $P$, and one remarkable covariant representation, namely the Wiener-Hopf one (defined in 2.4; $W : P \to \mathcal{L}(\ell^2(P))$ is covariant because of the formula (4) of 3.1). The $C^*$-algebra generated by $W$ is $\mathcal{W}(G,P)$ (see 2.4), so that $\mathcal{W}(G,P)$ naturally plays the role of "reduced $C^*$-algebra of $(G,P)$". Our next task is to construct the envelope of the covariant representations of $P$, i.e. the "full $C^*$-algebra of $(G,P)$".

Let us first remark that any covariant representation $V : P \to \mathcal{L}(H)$ can be extended to the dense $*$-subalgebra $\operatorname{sp}\{W(s)W(t)^* | s,t \in P\}$ put into evidence in Proposition 3.2. Indeed, since by the same proposition $\{W(s)W(t)^* | s,t \in P\}$ are linearly independent, there exists a unique linear map $\pi_V : \operatorname{sp}\{W(s)W(t)^* | s,t \in P\} \to \mathcal{L}(H)$ such that $\pi_V(W(s)W(t)^*) = V(s)V(t)^*$, $\forall s,t \in P$. Some simple algebraic computations, similar to those made in the proof of the Proposition 3.2, show that $\pi_V$ is a $*$-representation.

The next fact to be observed is that $\operatorname{sp}\{W(s)W(t)^* | s,t \in P\}$ has an obvious identification with $C_c(P \times P)$, the space of finitely supported complex functions on $P \times P$, such that, for any $s, t$ in $P$, $W(s)W(t)^*$ becomes $\chi_{s,t}$ = the characteristic function of $\{(s,t)\}$. Carrying multiplication and involution through this identification, we get a $*$-algebraic structure on $C_c(P \times P)$, determined by the relations:

$$\chi_{s,t}\chi_{u,v} = \begin{cases} \chi_{st^{-1}\sigma(t,u),\, vu^{-1}\sigma(t,u)}, & \text{if } t \text{ and } u \text{ have c.u.b.,} \\ 0, & \text{if they haven't,} \end{cases}$$

$$\chi_{s,t}^* = \chi_{t,s}.$$

$\chi_{e,e}$ is the unit of $C_c(P \times P)$. The preceding remark and the clear fact that for any $s, t$ in $P$:

$$\chi_{s,e}\chi_{t,e} = \chi_{st,e},$$

$$\chi_{s,s}\chi_{t,t} = \begin{cases} \chi_{\sigma(s,t),\sigma(s,t)}, & \text{if } s \text{ and } t \text{ have c.u.b.,} \\ 0, & \text{if they haven't,} \end{cases}$$

show together that there exists a canonical bijection between the unital $*$-representations of $C_c(P \times P)$ on a given Hilbert space and the covariant representations of $P$ on the same space.

The only thing left to be done is the enveloping of $C_c(P \times P)$. We define for any $f$ in $C_c(P \times P)$:

$$\|f\| = \sup\{\|\pi(f)\| \mid \pi \text{ unital } *\text{-representation of } C_c(P \times P)\}.$$

$\|f\|$ is finite and actually not greater than $\sum_{s,t \in P} |f(s,t)|$, because $f = \sum_{s,t \in P} f(s,t)\chi_{s,t}$ and each $\chi_{s,t}$ is a partial isometry. On the other hand, the canonical identification of $C_c(P \times P)$ with $\mathrm{sp}\{W(s)W(t)^* | s, t \in P\}$ gives an injective unital $*$-representation, hence $\|f\| > 0$ for $f \neq 0$. It follows then immediately that $\|\cdot\|$ is a $C^*$-norm on $C_c(P \times P)$.

DEFINITION. The completion of $C_c(P \times P)$ with respect to $\|\cdot\|$ will be denoted by $C^*(G, P)$ and will be called *the universal $C^*$-algebra of* $(G, P)$.

REMARK. In his paper [10], G. Murphy constructs, for an ordered group, a $C^*$-algebra which envelops all the representations by isometries of the positive semigroup. This $C^*$-algebra is not fit for studying amenability, because it is too large (they generally exist non-covariant representations of the positive semigroup, and the relation (4) of 3.1 clearly shows that these representations can not be factored through the Wiener-Hopf operators).

4.2. THE DEFINITION OF AMENABILITY. It is clear that covariant representations of $P$ extend to unital $*$-representations of $C^*(G, P)$. In particular, the Wiener-Hopf representation $W : P \to \mathcal{L}(\ell^2(P))$ extends to $\pi_W : C^*(G, P) \to \mathcal{L}(\ell^2(P))$.

DEFINITION. The quasi-lattice ordered group $(G, P)$ is said to be *amenable* if (and only if) $\pi_W$ is one-to-one.

It is obvious that the range of $\pi_W$ is $\mathcal{W}(G,P)$. So, if $(G, P)$ is amenable, $\pi_W$ establishes a canonical isomorphism between $C^*(G, P)$ and $\mathcal{W}(G, P)$. It is also obvious that we have the following equivalent reformulation:

PROPOSITION. *$(G, P)$ is amenable if and only if every covariant representation $V : P \to \mathcal{L}(H)$ can be factored through the Wiener-Hopf representation $W$, in the sense that there exists a $*$-representation $\pi : \mathcal{W}(G,P) \to \mathcal{L}(H)$ such that $\pi(W(t)) = V(t)$, $\forall t \in P$.*

Other less trivial reformulations of the amenability concept will be discussed in the next two subsections.

4.3 AMENABILITY IN TERMS OF THE CONDITIONAL EXPECTATION $C^*(E)$. In the same notation as above, let us define

$$C^*(\mathcal{D}) = \text{ clos sp}\,\{\chi_{t,t} | t \in P\} \subseteq C^*(G,P).$$

Exactly as in 3.1 we see that $C^*(\mathcal{D})$ is a unital Abelian $C^*$-subalgebra of $C^*(G, P)$. Moreover, since $\pi_W(\chi_{t,t}) = M(t)$, $\forall t \in P$, it is clear that $\pi_W(C^*(\mathcal{D})) = \mathcal{D}$.

LEMMA 1. *$\pi_W | C^*(\mathcal{D})$ is isometric (hence it is an isomorphism between $C^*(\mathcal{D})$ and $\mathcal{D}$).*

*Proof.* By Lemma 3.9, there exists a unital $*$-homomorphism $\rho : \mathcal{D} \to C^*(\mathcal{D})$ such that $\rho(M(t)) = \chi_{t,t}$, $\forall t \in P$, which is clearly an inverse for $\pi_W|C^*(\mathcal{D})$. ■

LEMMA 2 AND DEFINITION. *There exists a unique linear bounded map $C^*(E) : C^*(G,P) \to C^*(\mathcal{D})$ such that:*

$$C^*(E)\chi_{s,t} = \begin{cases} \chi_{s,t}, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

*$C^*(E)$ is a conditional expectation.*

*Proof.* The uniqueness of $C^*(E)$ is clear, and its existence follows from the fact that in the diagram:

$$\begin{array}{ccc} C^*(G,P) & \xrightarrow{\quad \pi_W \quad} & \mathcal{W}(G,P) \\ {\scriptstyle C^*(E)}\big\downarrow & & \big\downarrow{\scriptstyle E} \\ C^*(\mathcal{D}) & \xrightarrow[\pi_W|C^*(\mathcal{D})]{} & \mathcal{D} \end{array}$$

we can reverse the horizontal arrow of the bottom, due to the previous lemma. From this diagram it also results that $C^*(E)$ is a conditional expectation. ■

PROPOSITION. *$(G,P)$ is amenable if and only if the conditional expectation $C^*(E)$ is faithful.*

*Proof.* In the commutative diagram of Lemma 2, all the arrows are positive maps between $C^*$-algebras, and $E$ and $\pi_W|C^*(\mathcal{D})$ are faithful. It is then easy to see that $\pi_W$ is faithful if and only if $C^*(E)$ is so. ■

REMARK. No matter if $(G,P)$ is amenable or not, the above argument shows that $(\operatorname{Ker}\pi_W)\cap C^*(G,P)_+ = (\operatorname{Ker}C^*(E))\cap C^*(G,P)_+$, where $C^*(G,P)_+$ is the set of positive elements of $C^*(G,P)$.

As an application, one can use an idea of R. Douglas [5] to obtain the following

COROLLARY. *If $P$ is Abelian, then $(G,P)$ is amenable.*

*Proof.* It can be seen without difficulty that the compact group $\hat{P}$ of the characters of $P$ has a continuous action by automorphisms $\beta$ on $C^*(G,P)$, determined by:

$$\beta_c(\chi_{s,t}) = c(s)\overline{c(t)}\chi_{s,t}, \quad \forall c\in\hat{P},\ s,t\in P.$$

Moreover, we have:

$$C^*(E)\chi_{s,t} = \int_{\hat{P}} \beta_c(\chi_{s,t})\mathrm{d}c, \quad \forall s,t\in P \tag{17}$$

(if $s=t$, this equality is clear; if not, it amounts to $\int_{\hat{P}} c(s)\overline{c(t)}\mathrm{d}c = 0$, and it holds because $c\to c(s)$ and $c\to c(t)$ are two characters of $\hat{P}$, which are different by a theorem of Hewitt and Zuckermann — see Chapter V of [1]).

Using relation (17) and the fact that $\operatorname{clos}\operatorname{sp}\{\chi_{s,t}|s,t\in P\} = C^*(G,P)$, we easily infer that:

$$C^*(E)f = \int_{\hat{P}} \beta_c(f)\mathrm{d}c, \quad \forall f\in C^*(G,P).$$

Finally, for a positive $f\in C^*(G,P)$ every $\beta_c(f)$ is also positive, hence $C^*(E)f = 0 \Rightarrow \int_{\hat{P}} \beta_c(f)\,\mathrm{d}c = 0 \Rightarrow \beta_c(f)=0,\ \forall c\in\hat{P} \Rightarrow f=\beta_{\mathbf{1}}(f)=0$, with $\mathbf{1}$ the unit of $\hat{P}$. ■

4.4. AMENABILITY IN TERMS OF POSITIVE FORMS. Let us consider a quasi-lattice ordered group $(G,P)$ and make the following

DEFINITION. A positive form $\varphi$ on $C^*(G,P)$ is said to be finitely d-supported if the set $\text{d-supp}\,\varphi = \{x \in PQ | \exists (s,t) \in \Delta_x \text{ such that } \varphi(\chi_{s,t}) \neq 0\}$ is finite ($\Delta_x$=the diagonal of $x$, defined in 2.2.1°.)

The significance of this definition is clear if we recall that $C_c(P \times P)$ is a dense $*$-subalgebra of $C^*(G,P)$, hence that, exactly as in the case of $\mathcal{W}(G,P)$, we can imagine $C^*(G,P)$ as "the closed linear span of $P \times P$". The positive form $\varphi$ on $C^*(G,P)$ is determined by the map $(s,t) \to \varphi(\chi_{s,t})$, and it is finitely d-supported if and only if this map vanishes outside a finite set of diagonals.

PROPOSITION. *$(G,P)$ is amenable if and only if the set of finitely d-supported positive forms on $C^*(G,P)$ is weak* dense in the space of all positive forms.*

*Proof* "$\Rightarrow$" Since $\pi_W$ is isometric, the sums of positive forms of the type $\langle \pi_W(\cdot)\xi|\xi\rangle$ with $\xi$ in $\ell^2(P)$ are weak* dense in the space of all positive forms on $C^*(G,P)$. A simple approximation argument shows that this is still true if we assume only $\xi \in C_c(P)$. But for any $\xi$ in $C_c(P)$, $\langle \pi_W(\cdot)\xi|\xi\rangle$ is finitely d-supported. Indeed, if $\xi = \sum_{j=1}^{n} \lambda_j \delta_{a_j}$ $(\lambda_1,\dots,\lambda_n \in \mathbb{C},\ a_1,\dots,a_n \in P)$, a simple computation shows that for any $s,t$ in $P$:

$$\langle \pi_W(\chi_{s,t})\xi|\xi\rangle = \sum_{\substack{1 \leqslant j,\ k \leqslant n \\ t \leqslant a_j,\ s \leqslant a_k \\ t^{-1}a_j = s^{-1}a_k}} \lambda_j \overline{\lambda_k},$$

and this immediately implies that $\text{d-supp}\,\langle \pi_W(\cdot)\xi|\xi\rangle \subseteq \{a_j a_k^{-1} | 1 \leqslant j,\ k \leqslant n\}$. Finally, it is clear that a finite sum of finitely d-supported positive forms is still finitely d-supported.

"$\Leftarrow$" We need a

LEMMA. *Let $\varphi$ be a finitely d-supported positive form on $C^*(G,P)$. Define $\varphi_1(f) = \varphi(C^*(E)f)$, $\forall f \in C^*(G,P)$, obtaining another positive form $\varphi_1$ on $C^*(G,P)$, and consider the GNS representation of $\varphi_1$, $\pi : C^*(G,P) \to \mathcal{L}(H)$, with canonical cyclic vector $\xi \in H$. Then there exists a vector $\eta \in H$ such that $\varphi(f) = \langle \pi(f)\xi|\eta\rangle$, $\forall f \in C^*(G,P)$.*

*Proof of the Lemma.* Let $\text{d-supp}\,\varphi = \{x_1,\dots,x_n\}$. We shall prove that:

$$|\varphi(f)|^2 \leqslant n\|\varphi\|\varphi_1(f^*f), \quad \forall f \in C^*(G,P). \tag{18}$$

This inequality entails the statement of the Lemma. Indeed, it can be also written $|\varphi(f)| \leqslant \sqrt{n\|\varphi\|}\,\|\pi(f)\xi\|$, $\forall f \in C^*(G,P)$, and (since $\xi$ is cyclic for $\pi$) it implies the existence of a linear bounded functional on $H$ such that $\pi(f)\xi \to \varphi(f)$,

$\forall f \in C^*(G,P)$. By the theorem of Riesz, this functional must be the inner product with a certain $\eta \in H$.

It clearly suffices to prove (18) for $f$ in $C_c(P \times P)$. We fix such an $f$ and write it as a sum, $f = \sum_{x \in PQ} f_x$, with $f_x = \sum_{(s,t) \in \Delta_x} f(s,t)\chi_{s,t}$, $\forall x \in PQ$ (this holds because $\{\Delta_x | x \in PQ\}$ is a partition of $P \times P$). Only a finite number of the $f_x$'s are in fact non-zero, hence we can find a finite subset $F$ of $PQ$, about which we may assume that it contains d-supp $\varphi$, such that $f = \sum_{x \in F} f_x$. Since for any $x$, $f_x$ belongs to sp $\{\chi_{s,t} | (s,t) \in \Delta_x\}$, it is clear that $\varphi(f_x) = 0$ for $x$ in $F \setminus$(d-supp $\varphi$), hence $\varphi(f) = \sum_{x \in F} \varphi(f_x) = \sum_{j=1}^{n} \varphi(f_{x_j})$. We majorize:

$$|\varphi(f)|^2 = \left| \sum_{j=1}^{n} \varphi(f_{x_j}) \right|^2 \overset{\text{CS}}{\leqslant} n \sum_{j=1}^{n} |\varphi(f_{x_j})|^2 \overset{\text{CS for } \varphi}{\leqslant}$$

$$\overset{\text{CS for} \varphi}{\leqslant} n \sum_{j=1}^{n} \|\varphi\| \varphi(f_{x_j}^* f_{x_j}) = n\|\varphi\| \varphi \left( \sum_{j=1}^{n} f_{x_j}^* f_{x_j} \right)$$

(where we used the Cauchy-Schwartz inequality, first for $n$-tuples of complex numbers, then for the form $\varphi$). We are only left to prove that:

$$\sum_{j=1}^{n} f_{x_j}^* f_{x_j} \leqslant C^*(E)(f^* f), \tag{19}$$

because, assuming this true, we can continue our majorization with

$$n\|\varphi\| \varphi(C^*(E)(f^* f)) = n\|\varphi\| \varphi_1(f^* f),$$

obtaining thus (18).

Finally, in order to get (19) we write:

$$C^*(E)(f^* f) = C^*(E) \left( \left( \sum_{x \in F} f_x \right)^* \left( \sum_{x \in F} f_x \right) \right) =$$

$$= \sum_{x,y \in F} C^*(E)(f_x^* f_y).$$

An argument similar to the one which proved the relation (7) of 3.4 shows that:

$$f_x^* f_y \in \begin{cases} \text{sp}\, \{\chi_{s,t} | (s,t) \in \Delta_{x^{-1}y}\}, & \text{if } x^{-1}y \in PQ, \\ \{0\}, & \text{if } x^{-1}y \notin PQ, \end{cases}$$

and this yields:

$$C^*(E)(f_x^* f_y) = \begin{cases} f_x^* f_y, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

Hence $C^*(E)(f^*f) = \sum_{x \in F} f_x^* f_x \geqslant \sum_{j=1}^{n} f_{x_j}^* f_{x_j}$, and the proof of the Lemma is complete.

Now let us fix a positive $f$ in $C^*(G,P)$ such that $\pi_W(f) = 0$. We shall prove that $f = 0$ (obtaining thus the faithfulness of $\pi_W$). Taking into account the hypothesis it suffices to prove that $\varphi(f) = 0$ for every finitely d-supported positive form $\varphi$ on $C^*(G,P)$. We also fix such a $\varphi$, and define $\varphi_1$ and $\pi$ as in the statement of the Lemma. We have: $\pi_W(g^*fg) = 0$, $\forall g \in C^*(G,P) \Rightarrow C^*(E)(g^*fg) = 0$, $\forall g \in C^*(G,P)$ (by the remark following Proposition 4.3) $\Rightarrow \varphi_1(g^*fg) = 0$, $\forall g \in C^*(G,P)$ (by the definition of $\varphi_1$)$\Rightarrow \pi(f) = 0$ (by the definition of the GNS representation)$\Rightarrow \varphi(f) = 0$ (by the Lemma). ∎

4.5. Amenability in Terms of Positive Definite Functions. Recalling the development of the theory for unordered groups, it is natural to try to find, at this moment, a fourth description of the amenability concept, made in terms of positive definite functions. The notion of a positive definite function on a group (see for instance [13], 7.1.9) can be adapted to work in our situation, with the following remark: it is not generally true that $x, y \in PQ \Rightarrow x^{-1}y \in PQ$, but this is the case if $x$ and $y$ have c.u.b. in $P$, because for any common upper bound $t$ we can write $x^{-1}y = (x^{-1}t)(y^{-1}t)^{-1}$. We can thus make the following definitions:

Definition 1. Let $(G,P)$ be a quasi-lattice ordered group. A function $\theta : PQ \rightarrow \mathbf{C}$ is said to be *positive definite* if for any $x_1, \ldots, x_n$ in $PQ$ there exists a positive definite matrix $(\theta_{j,k})_{1 \leqslant j,\, k \leqslant n}$ such that $\theta_{j,k} = \theta(x_j^{-1}x_k)$ whenever $x_j$ and $x_k$ have c.u.b. in $P$.

Definition 2. The quasi-lattice ordered group $(G,P)$ is said to have the *approximation property for positive definite functions* if there exists a net $(\theta_i)_i$ of positive definite functions with finite support on $PQ$ such that $\theta_i(x) \xrightarrow[i]{} 1$, $\forall x \in PQ$.

It is known that a necessary condition for the amenability of $G$ (discrete group) is the existence of a net of positive definite functions on $G$, with finite support, which converge to 1 pointwisely (see 7.3.8 of [13]). On the other hand, it is clear that if $(G,P)$ is a quasi-lattice ordered group, then the restriction to $PQ$ of a positive definite function on $G$ is positive definite in the sense of Definition 1. Hence we obviously have:

Proposition 1. *If $G$ is amenable, then $(G,P)$ has the approximation property for positive definite functions.*

On the other hand we have:

PROPOSITION 2. *A quasi-lattice ordered group with the approximation property for positive definite functions is amenable.*

The proof of the latter result leans upon the fact that positive definite functions on $PQ$ naturally "perturbate" the positive forms on $C^*(G,P)$:

PROPOSITION 3. *Let $(G,P)$ be a quasi-lattice ordered group, $\theta : PQ \to \mathbb{C}$ a positive definite function and $\varphi$ a positive form on $C^*(G,P)$. There exists a unique positive form $\psi$ on $C^*(G,P)$ such that $\psi(\chi_{s,t}) = \theta(st^{-1})\varphi(\chi_{s,t})$ for any $s,t$ in $P$.*

*Proof of Proposition 2 (using the Proposition 3).* By Proposition 4.4 it suffices to show that an arbitrary positive form $\varphi$ on $C^*(G,P)$ is the weak* limit of a net of finitely d-supported positive forms. In order to do this, we just have to consider a net $(\theta_i)_i$ as in the Definition 2, and take for any $i$ the positive form $\varphi_i$ on $C^*(G,P)$ determined by $\varphi_i(\chi_{s,t}) = \theta_i(st^{-1})\varphi(\chi_{s,t})$, $\forall s,t \in P$. (Then d-supp $\varphi_i \subseteq$ supp $\theta_i$, $\forall i$, and $\varphi_i \xrightarrow[i]{w^*} \varphi$, since clearly $\varphi_i(\chi_{s,t}) \xrightarrow[i]{} \varphi(\chi_{s,t})$, $\forall s,t \in P$, and since $\|\varphi_i\| = \varphi_i(\chi_{e,e}) \xrightarrow[i]{} \varphi(\chi_{e,e}) = \|\varphi\|$, so that $\|\varphi_i\|$ is uniformly bounded for sufficiently large $i$.) ■

*Proof of Propositon 3.* We break the argument in two steps.

*Step 1.* There clearly exists a unique linear map $\psi_0 : C_c(P \times P) \to \mathbb{C}$ such that $\psi_0(\chi_{s,t}) = \theta(st^{-1})\varphi(\chi_{s,t})$, $\forall s,t \in P$. We shall prove that $\psi_0$ is positive on the $*$-algebra $C_c(P \times P)$.

In order to do this, let us fix an $f$ in $C_c(P \times P)$. We can write $f = \sum_{j=1}^{n} \lambda_j \chi_{s_j,t_j}$ for some $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n$ in $P$, and we clearly have:

$$f^* f = \sum_{j,k=1}^{n} \overline{\lambda_j} \lambda_k \chi_{t_j,s_j} \chi_{s_k,t_k},$$

hence

$$\psi_0(f^* f) = \sum_{j,k=1}^{n} \overline{\lambda_j} \lambda_k \psi_0(\chi_{t_j,s_j} \chi_{s_k,t_k}).$$

Now let us put $x_j = s_j t_j^{-1}$, $\forall 1 \leqslant j \leqslant n$, and let us consider a positive definite matrix $(\theta_{j,k})_{1 \leqslant j,\, k \leqslant n}$ such that $\theta_{j,k} = \theta(x_j^{-1} x_k)$ whenever $x_j$ and $x_k$ have c.u.b. in $P$. We claim that for any $j$ and $k$ we have $\psi_0(\chi_{t_j,s_j}\chi_{s_k,t_k}) = \theta_{j,k}\varphi(\chi_{t_j,s_j}\chi_{s_k,t_k})$. Indeed, if $s_j$ and $s_k$ have no c.u.b., then both sides of this equality are zero, because $\chi_{t_j,s_j}\chi_{s_k,t_k}$ is so. If $s_j$ and $s_k$ have c.u.b., then $x_j$ and $x_k$ have c.u.b. in $P$, because $x_j \leqslant s_j$ and $x_k \leqslant s_k$; in this case the multiplication rule of $C_c(P \times P)$ and the definition of $\psi_0$ immediately yield: $\psi_0(\chi_{t_j,s_j}\chi_{s_k,t_k}) = \theta(x_j^{-1} x_k)\varphi(\chi_{t_j,s_j}\chi_{s_k,t_k})$.

Hence we can write:

$$\psi_0(f^*f) = \sum_{j,k=1}^{n} \overline{\lambda_j}\lambda_k\theta_{j,k}\varphi(\chi^*_{s_j,t_j}\chi_{s_k,t_k}).$$

But the matrix $(\theta_{j,k}\varphi(\chi^*_{s_j,t_j}\chi_{s_k,t_k}))_{1\leqslant j,\ k\leqslant n}$ is positive definite, because it is the pointwise product of the positive definite matrices $(\theta_{j,k})_{1\leqslant j,\ k\leqslant n}$ and $(\varphi(\chi^*_{s_j,t_j}\chi_{s_k,t_k}))_{1\leqslant j,\ k\leqslant n}$ (see for instance [13], the proof of 7.1.10), and this makes clear that $\psi_0(f^*f) \geqslant 0$.

*Step 2.* The positive form $\psi_0$ on the $*$-algebra $C_c(P\times P)$ can be uniquely extended to a positive form $\psi$ on $C^*(P \times P)$.

Taking into account the definition of $C^*(G,P)$ it suffices to show that there exists a unital $*$-representation $\pi : C_c(P \times P) \to \mathcal{L}(H)$ and a vector $\xi$ in $H$ such that $\psi_0(f) = \langle\pi(f)\xi|\xi\rangle$, $\forall f \in C_c(P \times P)$ (then we can extend $\pi$ to $C^*(G,P)$, and define $\psi(f) = \langle\pi(f)\xi|\xi\rangle$, $\forall f \in C^*(G,P)$; the uniqueness of $\psi$ is clear, since any positive form on a $C^*$-algebra is bounded).

Thus we only need to prove that the GNS construction can be performed on $C_c(P\times P)$. It is known that a sufficient condition for this to happen is the fulfillment of the Combes' axiom: for any $f$ in $C_c(P \times P)$ there exists a constant $k(f) \geqslant 0$ such that $f^*f \leqslant k(f)\chi_{e,e}$. But, as one can immediately check, the set of those $f$ enjoying the latter property is a linear subspace of $C_c(P \times P)$; this set contains every $\chi_{s,t}$, because $\chi^*_{s,t}\chi_{s,t} = \chi_{t,t} \leqslant \chi_{e,e}$ ($\chi_{e,e} - \chi_{t,t}$ being a selfadjoint projection), and the proof is over.

∎

## 5. TWO PARTICULAR CASES

5.1. THE CASE OF THE FREE GROUP. With the exception of the partially ordered free group of 2.3.4°, all the examples of quasi-lattice ordered groups given at 2.3 are amenable, by the Proposition 1 of 4.5. Quite surprisingly, $(F_n, SF_n)$ is amenable too; this comes out from the uniqueness property of the extended Cuntz algebra, to which both $\mathcal{W}(F_n, SF_n)$ and $C^*(F_n, SF_n)$ are naturally isomorphic. (Remark also that one has $C^*(\mathop{\ast}_{1\leqslant i\leqslant n} \mathbb{Z}, \mathop{\ast}_{1\leqslant i\leqslant n} \mathbb{N}) = \mathop{\ast}_{1\leqslant i\leqslant n} C^*(\mathbb{Z},\mathbb{N})$, where the free product of $C^*$-algebras is considered in the sense of D. Voiculescu [18].) Thus:

PROPOSITION. *$(F_n, SF_n)$ is amenable.*

*Proof.* We denote by $a_1, \ldots, a_n$ the free generators of $F_n$ and $SF_n$.

LEMMA 1. *$\mathcal{W}(F_n, SF_n) \supseteq \mathcal{K}(\ell^2(SF_n))$, and their quotient is $O_n$.*

*Proof of Lemma 1.* One can immediately see that $I - \sum_{j=1}^{n} M(a_j) = \langle \cdot | \delta_e \rangle \delta_e$ (because any $t \neq e$ in $SF_n$ is greater then exactly one of the $a_j$'s), and hence that $\langle \cdot | \delta_s \rangle \delta_t = W(t)(I - \sum_{j=1}^{n} M(a_j))W(s)^* \in \mathcal{W}(F_n, SF_n)$, $\forall s, t \in P$. Consequently, $\mathcal{K}(\ell^2(SF_n)) = \operatorname{clos\,sp}\{\langle \cdot | \delta_s \rangle \delta_t | s, t \in SF_n\} \subseteq \mathcal{W}(F_n, SF_n)$.

Since $\{a_j | 1 \leqslant j \leqslant n\}$ generates $SF_n$, the isometries $\{W(a_j) | 1 \leqslant j \leqslant n\}$ generate the same $C^*$-algebra as $\{W(t) | t \in SF_n\}$, which is $\mathcal{W}(F_n, SF_n)$ (by 2.4). It follows that $\mathcal{W}(F_n, SF_n)/\mathcal{K}$ is generated by the isometries $\{W(a_j) + \mathcal{K} | 1 \leqslant j \leqslant n\}$. But $(I + \mathcal{K}) - \sum_{j=1}^{n} (W(a_j) + \mathcal{K})(W(a_j) + \mathcal{K})^* = \langle \cdot | \delta_e \rangle \delta_e + \mathcal{K} = \mathcal{K}$, and it is clear that $\mathcal{W}(F_n, SF_n)/\mathcal{K} = \mathcal{O}_n$.

LEMMA 2. *A representation by isometries $V : SF_n \to \mathcal{L}(H)$ is covariant if and only if the subspaces $\{\operatorname{Ran} V(a_j) | 1 \leqslant j \leqslant n\}$ of $H$ are mutually orthogonal.*

*Proof of Lemma 2.* We put $L(t) = V(t)V(t)^*$, $\forall t \in SF_n$. If $V$ is covariant, then $L(a_j)L(a_k) = 0$, $\forall j \neq k$, by the relation (14) of 3.9 and the fact that $a_j$ and $a_k$ have no c.u.b.; hence $\operatorname{Ran} V(a_j) \perp \operatorname{Ran} V(a_k)$ for $j \neq k$.

Conversely, let us assume that $\{\operatorname{Ran} V(a_j) | 1 \leqslant j \leqslant n\}$ are mutually orthogonal, and take two arbitrary elements $s$ and $t$ of $SF_n$. As we saw in 2.3, Example 4°, only three possibilities can occur: (a) $s \leqslant t$; (b) $s \geqslant t$; (c) $s$ and $t$ have no c.u.b. If it is (a), then $\operatorname{Ran} V(t) = \operatorname{Ran} V(s)V(s^{-1}t) \subseteq \operatorname{Ran} V(s)$, and we obviously have $L(s)L(t) = L(t) = L(\sigma(s,t))$. Situation (b) is treated in the same manner. Finally, if (c) takes place, we easily infer that there exist $p, s', t'$ in $SF_n$ and $j \neq k$ such that $s = pa_js'$, $t = pa_kt'$. We have $\operatorname{Ran} V(s) \subseteq \operatorname{Ran} V(pa_j) = V(p)(\operatorname{Ran} V(a_j))$, and similarly $\operatorname{Ran} V(t) \subseteq V(p)(\operatorname{Ran} V(a_k))$. Since $\operatorname{Ran} V(a_j) \perp \operatorname{Ran} V(a_k)$ and $V(p)$ is an isometry, it follows that $\operatorname{Ran} V(s) \perp \operatorname{Ran} V(t)$, i.e. $L(s)L(t) = 0$. In conclusion (14) of 3.9 takes place, and $V$ is covariant.

LEMMA 3. *Any covariant representation $V : SF_n \to \mathcal{L}(H)$ can be written as a direct sum such that one of the summands, say $V_0$, has $\sum_{j=1}^{n} V_0(a_j)V_0(a_j)^* = I$, and any other one is unitarily equivalent to the Wiener-Hopf representation $W$. (Remark: $V_0$ can be missing, or it can be the only term of the direct sum.)*

*Proof of Lemma 3.* The proof is carried over in the spirit of the Wold decomposition for semigroups of isometries (see Chapter IX of [17]). We only indicate the main idea. If $\sum_{j=1}^{n} V(a_j)V(a_j)^* = I$, then we take $V_0 = V$, the only term of the direct sum. If not, we consider an orthonormal basis $(\xi_i)_i$ of the space $H \ominus (\bigoplus_{j=1}^{n} \operatorname{Ran} V(a_j))$, and define $H_i = \operatorname{clos}\operatorname{sp}\{V(t)\xi_i | t \in SF_n\}$, $\forall i$. Then $(H_i)_i$ are mutually orthogonal reducing spaces for $V$, and for each $i$ the restriction of $V$ to $H_i$ is unitarily equivalent to $W$. $V_0$ is taken to be the restriction of $V$ to $H \ominus (\bigoplus_i H_i)$, if the latter space isn't zero.

Let us finally prove the statement of the proposition. We use the Proposition 4.2, i.e. we consider a covariant representation $V : SF_n \to \mathcal{L}(H)$ and show that it can be extended to a representation of $\mathcal{W}(F_n, SF_n)$. Using Lemma 3 and a direct sum decomposition argument, we reduce ourselves to the situation when either $V$ is unitarily equivalent to $W$, or $\sum_{j=1}^{n} V(a_j)V(a_j)^* = I$. The first alternative is trivial. For the second, we use the uniqueness of $O_n$: there exists a $*$-representation $\pi :$ $: \mathcal{W}(F_n, SF_n)/\mathcal{K} \to \mathcal{L}(H)$ such that $\pi(W(a_j) + \mathcal{K}) = V(a_j)$, $\forall 1 \leqslant j \leqslant n$, and we only have to compose $\pi$ with the canonical surjection $\mathcal{W}(F_n, SF_n) \to \mathcal{W}(F_n, SF_n)/\mathcal{K}$. ∎

REMARK 1. We do not know whether $(F_n, SF_n)$ has the approximation property of 4.5, Definition 2.

REMARK 2. At this moment one can naturally ask if there do indeed exist any non-amenable quasi-lattice ordered groups. Related to this, let us make the following remark:

Let $(G, P)$ be a quasi-lattice ordered group such that any two elements of $P$ have c.u.b. If $(G, P)$ is amenable, then $P$ is amenable (in the sense of invariant means—see Chapter I of [8]).

*Proof.* The identically one representation of $P$ on $\mathbb{C}$ is clearly covariant, hence it can be extended to $\mathcal{W}(G, P)$. We obtain the inequality $\left\| \sum_{j=1}^{n} \lambda_j W(s_j)W(t_j)^* \right\| \geqslant$ $\geqslant \left| \sum_{j=1}^{n} \lambda_j \right|$, $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in P$. In particular, for any $t_1, \ldots, t_n$ in $P$ and $\lambda_1, \ldots, \lambda_n$ in $[0, \infty)$ we get $\left\| \sum_{j=1}^{n} \lambda_j W(t_j) \right\| \geqslant \sum_{j=1}^{n} \lambda_j$, and we obviously must have equality. By a criterion of M. Day [4], the semigroup $P$ must be amenable. ∎

Hence a quasi-lattice ordered group $(G, P)$ is amenable if $P$ is non-amenable and any two elements of $P$ have c.u.b.; this is for instance the case for a total order on a free group (as in Section 4.2 of [7]).

5.2. THE CASE OF TOTALLY ORDERED ABELIAN-GROUPS. In this case, every representation by isometries of the semigroup is covariant (trivial verification). Since amenability is ensured by Propositions 1 and 2 of 4.5 (or by the Corollary 4.3), we have that any such representation can be uniquely extended to the $C^*$-algebra of Wiener-Hopf operators.

Let us make at this point the connection with a theorem of R. Douglas ([5], Theorem 1); using the characterization of totally ordered Archimedean groups cited at 2.3, Example 1°, we can state it as follows:

(D) If $(G, P)$ is Abelian, totally ordered and Archimedean, then any two non-unitary representations by isometries of $P$ generate canonically isomorphic $C^*$-algebras.

It is clear that (D) can be restated by saying that for any non-unitary representation by isometries of $P$, the corresponding representation of $\mathcal{W}(G, P)$ is isometric.

Now, it is an easy exercise to see that if the Abelian totally ordered group $(G, P)$ is Archimedean, then for any representation by isometries $V : P \to \mathcal{L}(H)$ either $V(t)$ is unitary for every $t$ in $P$, or $V(t)$ is non-unitary for every $t$ in $P$. Hence the following result of G. Murphy (Theorem 2.9 of [10]) is a generalization of (D):

(M) Let $(G, P)$ be an Abelian totally ordered group and let $V : P \to \mathcal{L}(H)$ be a representation by isometries, such that every $V(t)$ $(t \in P)$ is non-unitary. Then the corresponding representation $\pi_V : \mathcal{W}(G, P) \to \mathcal{L}(H)$ is isometric.

We give here a new proof of (M), which is sensibly simpler than the original one. The proof is obtained by adapting the techniques of R. Douglas [5] to this situation, when a universal object attached to $(G, P)$ has been put into evidence.

*Proof of* (M). The law of $G$ will be written additively.

We first remark that $\pi_V$ is isometric on $\mathcal{D}$. For any $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ and $t_1, \dots, t_n$ in $P$ we have $\pi_V \left( \sum_{j=1}^{n} \lambda_j M(t_j) \right) = \sum_{j=1}^{n} \lambda_j L(t_j)$, where we use the notation $L(t) = V(t)V(t)^*$, $\forall t \in P$. The hypothesis implies that $L(s) \neq L(t)$ for $s \neq t$ (if for instance $L(s) = L(t)$ for $s \lneq t$, then multiplying this equality with $V(s)^*$ on the left and with $V(s)$ on the right, we obtain $L(t - s) = I$, contradiction). But then the formulae (15) and (16) deduced in the proof of Lemma 3.9 imply that (for any $\lambda_1, \dots, \lambda_n, t_1, \dots, t_n$) both $\left\| \sum_{j=1}^{n} \lambda_j M(t_j) \right\|$ and $\left\| \sum_{j=1}^{n} \lambda_j L(t_j) \right\|$ equal $\max(|\lambda_1|, |\lambda_1 + \lambda_2|, \dots, |\lambda_1 + \lambda_2 + \dots + \lambda_n|)$. Hence $\pi_V | \mathcal{D}$ is isometric.

The faithfulness of $\pi_V | \mathcal{D}$ will be lifted to the whole of $\mathcal{W}(G, P)$ by using the following

LEMMA. *There exists a linear bounded map* $E' : \operatorname{clos}\operatorname{sp}\{V(s)V(t)^* | s, t \in P\} \to$ $\to \operatorname{clos}\operatorname{sp}\{V(s)V(s)^* | s \in P\}$ *such that for any* $s, t$ *in* $P$*:*

$$E'(V(s)V(t)^*) = \begin{cases} V(s)V(t)^*, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

Indeed, assuming this lemma true, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{W}(G,P) & \xrightarrow{\ \pi_V\ } & \operatorname{Ran}\pi_V \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E'} \\ \mathcal{D} & \xrightarrow[\ \pi_V|\mathcal{D}\ ]{} & \pi_V(\mathcal{D}) \end{array}$$

and the implications: $\pi_V(T) = 0 \Rightarrow \pi_V(T^*T) = 0 \Rightarrow E'(\pi_V(T^*T)) = 0 \Rightarrow (\pi_V|\mathcal{D}) \cdot$ $\cdot (E(T^*T)) = 0 \Rightarrow E(T^*T) = 0$ (by the Lemma) $\Rightarrow T^*T = 0$ (because $E$ is also faithful—3.6)$\Rightarrow T = 0$.

*Proof of the Lemma.* It suffices to prove that for any $t_1 \lneq t_2 \lneq \ldots \lneq t_n$ in $P$ and $\{\lambda_{j,k} | 1 \leqslant j,\ k \leqslant n\}$ in $\mathbf{C}$, the inequality

$$\left\| \sum_{j=1}^{n} \lambda_{j,j} V(t_j)V(t_j)^* \right\| \leqslant \left\| \sum_{j,k=1}^{n} \lambda_{j,k} V(t_j)V(t_k)^* \right\|$$

holds. As remarked earlier, the left side of this inequality is $\max(|\lambda_1|,\ |\lambda_1 +$ $+ \lambda_2|, \ldots,\ |\lambda_1 + \lambda_2 + \cdots + \lambda_n|)$, hence we only have to prove that in the same notations and for an arbitrary $1 \leqslant m \leqslant n$:

$$\left| \sum_{j=1}^{m} \lambda_{j,j} \right| \leqslant \left\| \sum_{j,k=1}^{n} \lambda_{j,k} V(t_j)V(t_k)^* \right\| \tag{20}$$

This is done as follows: let $t$ be the least of the elements $t_2 - t_1, \ldots, t_n - t_{n-1}$. $V(t)$ is non-unitary because $t \neq e$ and by the hypothesis; hence we can consider a vector of norm one $\xi$ in $H \ominus \operatorname{Ran}V(t) = \operatorname{Ker} V(t)^*$. It is immediate that for any $s \geqslant t$ in $P$ we have $V(s)^*\xi = 0$ and $\langle V(s)\xi | \xi \rangle = 0$. Using these facts, a simple computation shows that:

$$\left\langle \left( \sum_{j,k=1}^{n} \lambda_{j,k} V(t_j)V(t_k)^* \right) \eta_m \middle| \eta_m \right\rangle = \sum_{j=1}^{m} \lambda_{j,j}$$

where $\eta_m = V(t_m)\xi$. Since $\|\eta_m\| = 1$, we obtain (20) and the proof is over. ∎

## 6. INDUCED IDEALS AND APPLICATIONS

In this section $(G,P)$ is a fixed quasi-lattice ordered group. "Ideal" means everywhere "two-sided closed ideal".

6.1. EQUIVALENT DEFINITIONS FOR Ind $\mathcal{I}$. The process of induction is naturally defined as follows: let $\mathcal{I}$ be an ideal of $\mathcal{D}$; we take a unital $*$-representation $\rho$ of $\mathcal{D}$ such that Ker $\rho = \mathcal{I}$, we induce $\rho$ to a representation $\pi$ of $\mathcal{W}(G,P)$ (in the sense of M. Rieffel [15]), and define Ind $\mathcal{I}$ = Ker $\pi$. Ind $\mathcal{I}$ depends only on $\mathcal{I}$, and not on the particular choice of $\rho$, because the process of induction respects weak containment. Considering the details of the construction of M. Rieffel [15], the reader may easily check that we have:

$$\text{(21)} \qquad \operatorname{Ind}\mathcal{I} = \{T \in \mathcal{W}(G,P) | \alpha_{s,t}(E(T^*T)) \in \mathcal{I},\ \forall s,t \in P\},$$

with $E : \mathcal{W}(G,P) \to \mathcal{D}$ the conditional expectation of 3.6 and $\{\alpha_{s,t} | s,t \in P\}$ the set of endomorphisms of $\mathcal{D}$ defined in 3.7.

Let us call the ideal $\mathcal{I}$ of $\mathcal{D}$ "invariant" if $\alpha_x(X) \in \mathcal{I}$, $\forall x \in PQ$, $\forall X \in \mathcal{I}$, with $\{\alpha_x | x \in PQ\}$ the "action" of $PQ$ on $\mathcal{D}$ (see again 3.7). Since, as shown at 3.8, the semigroup generated by $\{\alpha_x | x \in PQ\}$ contains $\{\alpha_{s,t} | s,t \in P\}$, we have in fact for the invariant ideal $\mathcal{I}$ : $s,t \in P,\ X \in \mathcal{I} \Rightarrow \alpha_{s,t}(X) \in \mathcal{I}$. It is clear that for such an ideal (21) becomes:

$$\text{(22)} \qquad \operatorname{Ind}\mathcal{I} = \{T \in \mathcal{W}(G,P) | E(T^*T) \in \mathcal{I}\}.$$

In what follows we shall consider invariant ideals only; the reason is that for an arbitrary ideal $\mathcal{I}$ of $\mathcal{D}$, the set $\mathcal{I}_0 = \bigcap_{s,t \in P} \alpha_{s,t}^{-1}(\mathcal{I})$ is an invariant ideal with Ind $\mathcal{I}_0 =$ $=$ Ind $\mathcal{I}$ (immediate verification); hence any induced ideal of $\mathcal{W}(G,P)$ can be obtained from an invariant ideal of $\mathcal{D}$. Moreover, for any invariant ideal $\mathcal{I} \subseteq \mathcal{D}$ we have (Ind $\mathcal{I}) \cap \mathcal{D} = \mathcal{I}$, because $X \in$ (Ind $\mathcal{I}) \cap \mathcal{D} \Leftrightarrow X \in \mathcal{D}$ and $X^*X \in \mathcal{I} \Leftrightarrow X \in \mathcal{I}$; this implies that the map $\mathcal{I} \to$ Ind $\mathcal{I}$ is one-to-one on the set of invariant ideals of $\mathcal{D}$.

The range of $\mathcal{I} \to$ Ind $\mathcal{I}$ (i.e. the set of induced ideals) can be characterized as $\{\mathcal{J} \in \mathcal{W}(G,P)$, ideal $| T \in \mathcal{J} \Rightarrow E(T) \in \mathcal{J}\}$ (see the Corollary below). Generally, this is not the set of all the ideals of $\mathcal{W}(G,P)$. For instance in the case of $(\mathbf{Z},\mathbf{N})$ there exists exactly one non-trivial invariant ideal of $\mathcal{D}$, which induces the compact operators; in spite of that, $\mathcal{W}(\mathbf{Z},\mathbf{N})$ (=the $C^*$-algebra of the shift) has a rich family of ideals, indexed by the closed subsets of the unit circle (see [2]). A sufficient condition ensuring that any ideal of $\mathcal{W}(G,P)$ is induced from $\mathcal{D}$ can be given by using the groupoid interpretation of $\mathcal{W}(G,P)$ and a result of J. Renault ([14], Chapter III, Proposition 4.6); this condition holds for instance for the partially ordered free group with at least two generators (2.3, Example 4°).

The main result of this subsection is the following:

PROPOSITION. *Assume that $(G, P)$ has the approximation property of 4.5, Definition 2. Then for any invariant ideal $\mathcal{I}$ of $\mathcal{D}$, $\operatorname{Ind}\mathcal{I}$ can also be described as:*

*1° $\{T \in \mathcal{W}(G,P) | W(\sigma(x))^* E_x(T) W(\tau(x)) \in \mathcal{I},\ \forall x \in PQ\}$.*

*2° The ideal of $\mathcal{W}(G,P)$ generated by $\mathcal{I}$.*

Before passing to the proof, let us make some explanatory remarks. In 1° above, $E_x : \mathcal{W}(G,P) \to \mathcal{D}_x$ is the canonical projection onto the diagonal subspace of $x \in PQ$ (see 3.6); so 1° says that $T$ is in $\operatorname{Ind}\mathcal{I}$ if and only if all its projections on the diagonals, when canonically transported on the principal diagonal (see also 3.5) lie in $\mathcal{I}$. This is exactly the analogue of the definition used by G. Zeller-Meier for induced ideals in the theory of crossed products by discrete groups (see 4.15 of [19]). The analogy with the theory of crossed products is not a surprise, if we take into account the results of Section 3. More surprising is the strong resemblance with the theory of induced ideals developed in [16] by Ş. Strătilă and D. Voiculescu, who use exactly the formula (22) to induce an invariant ideal of a maximal Abelian subalgebra of an AF-algebra (see Lemma I 2.2 of [16]). Both Ş. Strătilă and D. Voiculescu [16] and G. Zeller-Meier [19] prove the analogue of the characterization 2°.

*Proof of the Proposition.* We denote by $\mathcal{J}_1$ and $\mathcal{J}_2$ the sets appearing at 1° and 2° respectively. We shall prove that $\operatorname{Ind}\mathcal{I} \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \operatorname{Ind}\mathcal{I}$. The last inclusion is clear because $\operatorname{Ind}\mathcal{I}$ is an ideal of $\mathcal{W}(G,P)$ and contains $\mathcal{I}$.

$\operatorname{Ind}\mathcal{I} \subseteq \mathcal{J}_1$. Let us fix a $T$ in $\operatorname{Ind}\mathcal{I}$ and make the notation $W(\sigma(x))^* E_x(T) \cdot W(\tau(x)) = T_x$, $\forall x \in PQ$. 3.5 implies that every $T_x$ is in $\mathcal{D}$ and that we have $E_x(T) = W(\sigma(x)) T_x W(\tau(x))^*$. It follows that $E_x(T)^* E_x(T) = W(\tau(x))(T_x^* T_x) W(\tau(x))^*$ and hence that $T_x^* T_x = W(\tau(x))^*(E_x(T)^* E_x(T)) W(\tau(x)) = \alpha_{e,\tau(x)}(E_x(T)^* E_x(T))$, $\forall x \in PQ$. Now, our goal is to prove that $T_x \in \mathcal{I}$, $\forall x \in PQ$, which is clearly equivalent to $T_x^* T_x \in \mathcal{I}$, $\forall x \in PQ$. Due to the latter equality and the fact that $\mathcal{I}$ is invariant, it suffices to show that $E_x(T)^* E_x(T) \in \mathcal{I}$, $\forall x \in PQ$. (Remark: $E_x(T)^* E_x(T)$ belongs in any case to $\mathcal{D}$, because of (7) and (8) of 3.4.) An argument similar to the one which proved relation (19) (in the proof of Proposition 4.4) shows that $E_x(T)^* E_x(T) \leqslant E(T^*T)$, $\forall x \in PQ$; but $E(T^*T)$ is in $\mathcal{I}$ (because $T \in \operatorname{Ind}\mathcal{I}$), and $\mathcal{I}$ is hereditary, hence all the $E_x(T)^* E_x(T)$'s are indeed in $\mathcal{I}$.

$\mathcal{J}_1 \subseteq \mathcal{J}_2$. Define $\mathcal{J}_1^\circ = \{T \in \mathcal{J}_1 | \exists F \subseteq PQ \text{ finite such that } T \in \operatorname{clos\,sp}(\bigcup_{x \in F} \mathcal{D}_x)\}$. It is clear that for $T$ in $\mathcal{J}_1^\circ$ and $F$ taken as in the above definition we have $T = \sum_{x \in F} E_x(T)$; then using the same notations and invoking the same arguments as in the proof of "Ind $\mathcal{I} \subseteq \mathcal{J}_1$" we get $T = \sum_{x \in F} W(\sigma(x)) T_x W(\tau(x))^* \in \mathcal{J}_2$. Hence it suffices to show that $\operatorname{clos} \mathcal{J}_1^\circ = \mathcal{J}_1$. The point is in proving the following

LEMMA. *For any positive definite function $\theta : PQ \to \mathbb{C}$ (Definition 1 of 4.5) there exists a bounded linear operator $M_\theta$ on $\mathcal{W}(G,P)$ such that $M_\theta(W(s)W(t)^*) = \theta(st^{-1})W(s)W(t)^*$, $\forall s,t \in P$; we have $\|M_\theta\| \leqslant 2\theta(e)$.*

Assuming this lemma true, we consider a net $(\theta_i)_i$ of finitely supported positive definite functions on $PQ$ having the property that $\theta_i(x) \underset{i}{\to} 1$, $\forall x \in PQ$, and we see that for any $T$ in $\mathcal{J}_1$ we have $M_{\theta_i}(T) \xrightarrow[i]{\|\cdot\|} T$ and $M_{\theta_i}(T) \in \mathcal{J}_1^\circ$, $\forall i$. (Indeed, $(M_{\theta_i})_i$ converges strongly to the identity because $(\|M_{\theta_i}\|)_i$ is bounded and $M_{\theta_i}(S) \xrightarrow[i]{\|\cdot\|} S$ for $S \in \operatorname{sp}\{W(s)W(t)^* \mid s,t \in P\}$ which is dense. On the other hand it is immediate that $E_x(M_{\theta_i}(S)) = \theta_i(x)E_x(S)$, $\forall x \in PQ$, $\forall S \in \mathcal{W}(G,P)$, $\forall i$, and this shows that $\mathcal{J}_1$ is invariant to every $M_{\theta_i}$. It is also immediate that $\operatorname{Ran} M_{\theta_i} \subseteq \operatorname{clos} \operatorname{sp}(\bigcup_{x \in \operatorname{supp}\theta_i} \mathcal{D}_x)$, $\forall i$.)

In the statement of the lemma it implicitly appears that that $\theta(e) \geqslant 0$ for any positive definite function $\theta : PQ \to \mathbb{C}$. In fact, exactly as in the case of positive definite functions on groups, it can be checked that $\theta(e) = \|\theta\|_\infty$. The same holds for the assertion: "$\theta(x^{-1}) = \overline{\theta(x)}$, $\forall x \in PQ$".

*Proof of the Lemma.* Since $(G,P)$ is amenable, we may very well construct $M_\theta$ on $C^*(G,P)$ instead of $\mathcal{W}(G,P)$. There obviously exists a linear map $L_\theta : C_c(P \times P) \to C_c(P \times P)$ such that $L_\theta(\chi_{s,t}) = \theta(st^{-1})\chi_{s,t}$, $\forall s,t \in P$. We will show that $\|L_\theta(f)\| \leqslant \theta(e)\|f\|$ for any selfadjoint $f$ in $C_c(P \times P)$; this will immediately imply that $\|L(f)\| \leqslant 2\theta(e)\|f\|$, $\forall f \in C_c(P \times P)$, and hence the fact that $L_\theta$ can be extended to $M_\theta \in \mathcal{L}(C^*(G,P))$ with $\|M_\theta\| \leqslant 2\theta(e)$.

So let us consider the selfadjoint element $f \in C_c(P \times P)$. $L_\theta(f)$ is selfadjoint, too (it is easy to see that $L_\theta(\chi_{s,t}^*) = (L_\theta(\chi_{s,t}))^*$, $\forall s,t \in P$, and this implies $L_\theta(g^*) = (L_\theta(g))^*$, $\forall g \in C_c(P \times P)$); hence we can write $\|L_\theta(f)\| = \sup_\varphi |\varphi(L_\theta(f))|$, with the supremum taken after all the states of $C^*(G,P)$. Now for any such $\varphi$ there exists a positive form $\psi$ on $C^*(G,P)$ such that $\psi(\chi_{s,t}) = \theta(st^{-1})\varphi(\chi_{s,t})$, $\forall s,t \in P$ (Proposition 3 of 4.5); this relation can also be written $\psi(\chi_{s,t}) = \varphi(L_\theta(\chi_{s,t}))$, $\forall s,t \in P$, and it obviously entails $\psi(f) = \varphi(L_\theta(f))$. Hence $|\varphi(L_\theta(f))| \leqslant \|\psi\|\,\|f\|$; but $\|\psi\| = \psi(\chi_{e,e}) = \theta(e)\varphi(\chi_{e,e}) = \theta(e)$, so what we have is $|\varphi(L_\theta(f))| \leqslant \theta(e)\|f\|$ (for any state $\varphi$ of $C^*(G,P)$). This clearly ends the proof. ■

COROLLARY. *In the same conditions as in the previous proposition, an ideal $\mathcal{J}$ of $\mathcal{W}(G,P)$ is induced from $\mathcal{D}$ if and only if it is closed under the conditional expectation.*

*Proof.* "⇒" If $\mathcal{J} = \operatorname{Ind}\mathcal{I}$, then $T \in \mathcal{J} \Rightarrow E(T) \in \mathcal{I} \subseteq \mathcal{J}$ due to the characterization 1° of the previous proposition ($E(T) = W(\sigma(e))^* E_e(T) W(\tau(e))$).

"⇐" Let us denote $\mathcal{J} \cap \mathcal{D}$ by $\mathcal{I}$. $\mathcal{I}$ clearly is an ideal of $\mathcal{D}$, and it is invariant because the $*$-endomorphisms $\{\alpha_{s,t} | s,t \in P\}$ are defined by multiplication operators. For any $T$ in $\mathcal{J}$ we have; $T^*T \in \mathcal{J} \Rightarrow E(T^*T) \in \mathcal{J} \cap \mathcal{D} = \mathcal{I} \Rightarrow T \in \operatorname{Ind}\mathcal{I}$, so that $\mathcal{J} \subseteq \operatorname{Ind}\mathcal{I}$. On the other hand $\mathcal{J}$ is an ideal of $\mathcal{W}(G,P)$ which contains $\mathcal{I}$, so it also contains $\operatorname{Ind}\mathcal{I}$ by assertion 2° of the previous proposition. ∎

6.2. THE SPECTRUM OF $\mathcal{D}$. Since the ideals of $\mathcal{D}$ are given by the closed subsets of the spectrum of $\mathcal{D}$, it is useful (if we want to know: what are we inducing?) to have an explicit description of the spectrum. This is the goal of the present subsection.

DEFINITIONS. A subset $A$ of $P$ will be called *hereditary* if "$s,t \in P$, $s \leqslant t$, $t \in A \Rightarrow s \in A$", and will be called *directed* if any two elements of $A$ have c.u.b. in $A$. We shall denote by $\Omega$ the set of all non-void hereditary and directed subsets of $P$ (remark that for $A \in \Omega$ we have $A \ni e$ and "$s,t \in A \Rightarrow \sigma(s,t)$ exists and is in $A$"). Identifying every subset of $P$ with its characteristic function and considering the product topology on $\{0,1\}^P$ we get a canonical compact Hausdorff topology on the space of subsets of $P$. It is clear that $\Omega$ is closed into this topology; hence $\Omega$ is a compact Hausdorff space.

For any $t$ in $P$ the "interval" $\{a \in P | a \leqslant t\}$ will be denoted by $[e,t]$. Clearly $\{[e,t] | t \in P\} \subseteq \Omega$; this is a dense subset, because for any $A$ in $\Omega$ the net $([e,t])_{t\in A}$ directed by $(A, \leqslant)$ converges to $A$ (immediate verification). Moreover, $[e,s] = [e,t] \Rightarrow s \leqslant t$ and $t \leqslant s \Rightarrow s = t$, so that $\Omega$ is a compactification of $P$.

PROPOSITION 1°. *Let $\gamma$ be a character of $\mathcal{D}$. Then $\gamma(M(t)) \in \{0,1\}$, $\forall t \in P$, and $A_\gamma = \{t \in P | \gamma(M(t)) = 1\}$ belongs to $\Omega$.*

2° *For any $t$ in $P$, the vector state $\langle \cdot \delta_t | \delta_t \rangle$ on $\mathcal{D}$ is a character and $A_{\langle \cdot \delta_t | \delta_t \rangle} = [e,t]$.*

3° *$\gamma \to A_\gamma$ is a homeomorphism from the spectrum of $\mathcal{D}$ onto $\Omega$.*

*Proof.* 1° $\gamma(M(t)) \in \{0,1\}$ because $M(t)^2 = M(t)$. $\gamma(M(e)) = \gamma(I) = 1$, hence $e \in A$ (and hence $A_\gamma \neq \emptyset$). Using (4) of 3.1 and the positivity of $\gamma$ we immediately infer that $s \leqslant t \Rightarrow \gamma(M(s)) \geqslant \gamma(M(t))$, and this implies that $A_\gamma$ is hereditary. Let us also prove that $A_\gamma$ is directed. If $s,t$ are any two elements of $A_\gamma$, from $1 = \gamma(M(s))\gamma(M(t)) = \gamma(M(s)M(t))$ we see that $s$ and $t$ have c.u.b. (otherwise we would obtain $\gamma(M(s)M(t)) = \gamma(0) = 0$); we can further write, by (4) of 3.1: $1 = \gamma(M(\sigma(s,t)))$, so that $\sigma(s,t) \in A_\gamma$.

2° The fact that $\langle \cdot \delta_t | \delta_t \rangle$ is a character of $\mathcal{D}$ follows from 3.3; the equality $A_{\langle \cdot \delta_t | \delta_t \rangle} = [e,t]$ is a consequence of the relation (6) of the same subsection.

3° If $\gamma_i \xrightarrow[i]{w^*} \gamma$ in the spectrum of $\mathcal{D}$, then $\gamma_i(M(t)) \xrightarrow[i]{} \gamma(M(t))$, $\forall t \in P$, which means exactly that the characteristic functions of $(A_{\gamma_i})_i$ converge pointwisely to the one of $A_\gamma$. This makes clear that the map $\gamma \to A_\gamma$ is continuous. It is also clear from 1° that this map is one-to-one, and from 2° that it has dense range; but its range is compact, hence closed in $\Omega$, and so we obtain surjectivity. Since we are dealing with compact Hausdorff spaces, the continuous bijection $\gamma \to A_\gamma$ is a homeomorphism. ∎

REMARK ON INVARIANCE. Due to the previous proposition, $\mathcal{D}$ can be canonically identified with $C(\Omega)$ and consequently the ideals of $\mathcal{D}$ can be canonically identified with the closed subsets of $\Omega$. There exists an appropriate notion of invariance for closed subsets of $\Omega$, such that the invariant ideals of $\mathcal{D}$ correspond to closed invariant subsets. Its precise definition is given as follows:

1° it can be shown that for any $A$ in $\Omega$, there are still in $\Omega$ the sets: $A_t = \{a \in \in P \mid a$ has upper bounds in $tA\}$, for every $t$ in $P$, and $A_{t^{-1}} = t^{-1}A \cap P$, for every $t$ in $A$;

2° the closed subset $\Omega_0$ of $\Omega$ is invariant if for any $A$ in $\Omega_0$ we have $A_t \in \Omega_0$, $\forall t \in \in P$, and $A_{t^{-1}} \in \Omega_0$, $\forall t \in A$. (These facts will not be used in the sequel, and that is why we do not enter into details.)

6.3. FIRST APPLICATION: WHEN DOES $\mathcal{W}(G,P)$ CONTAIN THE COMPACT OPERATORS?

PROPOSITION. *The following are equivalent:*

*1°* $\mathcal{W}(G,P) \supseteq \mathcal{K}(\ell^2(P))$.

*2° For any $t$ in $P$, $[e,t]$ is an open point of $\Omega$.*

*3° $[e,e]$ is an open point of $\Omega$.*

*4° There exists a finite subset $F$ of $P \setminus \{e\}$ such that every $t$ in $P \setminus \{e\}$ has lower bounds in $F$.*

REMARKS 1°. The condition 2° can be rephrased: "$\Omega$ is a regular compactification of $P$" (i.e. $P \ni t \to [e,t] \in \Omega$ has open dense range and is a homeomorphism onto the range). The implication 2° ⇒ 1° was proved by P. Muhly and J. Renault in a more general context ([9], Corollary 3.7.2); they conjecture that 1° ⇒ 2° also holds in general (see [9], 3.7.3).

2° Condition 4° depends only on the order relation on $P$. We note that it is always fulfilled when $P$ is finitely generated, because in this case any finite set of generators of $P \setminus \{e\}$ can be taken as $F$.

3° It can be shown that $\mathcal{W}(G,P)$ is irreducible; consequently, the equivalent conditions which appear in the proposition are necessary for $\mathcal{W}(G,P)$ to be type I. These conditions are not generally sufficient; for instance we saw in 5.1 that

$\mathcal{W}(F_n, SF_n) \supseteq \mathcal{K}(\ell^2(SF_n))$, but it has $O_n$ as a quotient (hence it cannot be type I).

4° A simple argument based on the minimality of $\mathcal{K}(\ell^2(P))$ can be invoked to prove that if $\mathcal{W}(G,P) \supseteq \mathcal{K}(\ell^2(P))$, then this ideal is induced from $\mathcal{D}$. (The proof given here does not explicitly use this fact.)

*Proof.* 1° $\Rightarrow$ 2° We consider the space $C_0(P) = \{\varphi : P \to \mathbb{C} | \forall \varepsilon > 0, \exists F \subseteq \subseteq P$ finite such that $|\varphi(t)| < \varepsilon$ for $t \in P \setminus F\}$. We fix for the moment a $\varphi$ in $C_0(P)$ and define $X_\varphi \in \mathcal{L}(\ell^2(P))$ to be the diagonal operator (relatively to the canonical basis) which has the $(t,t)$-entry of its matrix equal to $\varphi(t)$, for any $t$ in $P$. Clearly $X_\varphi$ is compact, hence it is in $\mathcal{W}(G,P)$ (by the hypothesis); but then, being diagonal, $X_\varphi$ must belong to $\mathcal{D}$ by 3.3. We can therefore define a continuous map $\tilde{\varphi} : \Omega \to \mathbb{C}$ by $\tilde{\varphi}(A) = \gamma(X_\varphi)$, $\forall A \in \Omega$, with $\gamma$ the character of $\mathcal{D}$ corresponding to $A$ by the canonical homeomorphism of 6.2. We have, in particular, $\tilde{\varphi}([e,t]) = \langle \cdot \delta_t | \delta_t \rangle (X_\varphi) = = \varphi(t)$, $\forall t \in P$. Taking into account that $\{[e,t] \,|\, t \in P\}$ is dense in $\Omega$, and that $\varphi \in C_0(P)$, we immediately obtain $\tilde{\varphi}(A) = 0$, $\forall A \in \Omega \setminus \{[e,t] \,|\, t \in P\}$.

The conclusion of the preceding paragraph is that for any $\varphi$ in $C_0(P)$ the function $\tilde{\varphi} : \Omega \Rightarrow \mathbb{C}$ defined by:

$$\tilde{\varphi}(A) = \begin{cases} \varphi(t), & \text{if } A = [e,t] \text{ with } t \in P, \\ 0, & \text{if } A \notin \{[e,t] \| t \in P\}, \end{cases}$$

is continuous. Particularizing $\varphi$ to be the characteristic function of $\{t\}$ we obtain that $[e,t]$ is an open point of $\Omega$.

2° $\Rightarrow$ 3° is clear.

3° $\Rightarrow$ 4°. Let us suppose that for any finite subset $F$ of $P \setminus \{e\}$ there exists $t_F$ in $P \setminus \{e\}$ which does not have lower bounds in $F$. We claim that the net $([e,t_F])_F$ converges to $[e,e]$ in $\Omega$ (the net is indexed by the family of finite subsets of $P \setminus \{e\}$, directed with inclusion). Since $\Omega$ is compact, it suffices to prove that an $A \neq [e,e]$ in $\Omega$ is not a cluster point of the considered net. And indeed, for any such $A$, we take a $t \neq e$ belonging to $A$ and we see that $\{B \in \Omega | B \ni t\}$ is an open neighborhood of $A$ in $\Omega$ which doesn't contain $[e,t_F]$ if $F \supseteq \{t\}$.

But $[e,t_F] \underset{F}{\to} [e,e]$ although $t_F \neq e$, $\forall F$, contradicts the hypothesis that $[e,e]$ is an open point of $\Omega$.

4° $\Rightarrow$ 1°. Let $F = \{a_1, \ldots, a_n\}$ be as in the hypothesis. We shall say about the non-void subset $I$ of $\{1, \ldots, n\}$ that it is "marked" if $\{a_j | j \in I\}$ have c.u.b.; for such an $I$ we make the notation $a_I = \sigma(\{a_j | j \in I\})$.

We claim that the operator:

$$X = I + \sum_{I \text{marked}} (-1)^{\operatorname{card} I} M(a_I)$$

is $\langle\cdot|\delta_e\rangle\delta_e$. It is clear firstly, from the formula (6) of 3.3, that $X\delta_e = \delta_e$. Let us further fix a $t \neq e$ in $P$ and denote the set $\{j|1 \leqslant j \leqslant n,\ a_j \leqslant t\}$, which is non-void by the hypothesis, by $I_0$. For any $I \subseteq \{1,\dots,n\}$ we obviously have: "$I$ marked and $a_I \leqslant t \Leftrightarrow \emptyset \neq I \subseteq I_0$". But then, again by (6) of 3.3:

$$X\delta_t = (1 + \sum_{I \text{marked},\ a_I \leqslant t} (-1)^{\operatorname{card} I})\delta_t = (1 + \sum_{\emptyset \neq I \subseteq I_0} (-1)^{\operatorname{card} I})\delta_t = 0.$$

Hence $\langle\cdot|\delta_e\rangle\delta_e \in \mathcal{W}(G,P)$. But then for any $s,t$ in $P$ : $\langle\cdot|\delta_s\rangle\delta_t = W(t)(\langle\cdot|\delta_e\rangle\delta_e)\cdot \cdot W(s)^* \in \mathcal{W}(G,P)$, and finally $\mathcal{K}(\ell^2(P)) = \text{clos sp}\{\langle\cdot|\delta_s\rangle\delta_t|s,t \in P\} \subseteq \mathcal{W}(G,P)$. ∎

6.4. Second application: a Converse to a Theorem of R. Douglas. In this subsection we particularize and assume that $(G,P)$ is a totally ordered Abelian group, written additively.

Proposition. *The following are equivalent:*

1° *$P$ is Archimedean.*

2° *Every non-unitary representation by isometries of $P$ extends to an isometric representation of $\mathcal{W}(G,P)$.*

3° *The commutator ideal $\mathcal{C}$ of $\mathcal{W}(G,P)$ is simple.*

Remarks. 1° The implication 1° $\Rightarrow$ 2° is a reformulation of the result of R. Douglas discussed in 5.2. 2° $\Rightarrow$ 3° was also proved by R. Douglas in the paper [5] (see the Corollary to the Theorem 1 of [5]). We shall prove here only 3° $\Rightarrow$ 1°.

2° The proposition shows that the generalization (M) discussed in 5.2 imposes an effective restriction on the representation, since in general for a given representation by isometries of the semigroup, $V$, some of the $V(t)$'s are unitary and some are not. A typical example is $\mathbb{Z}^2$ with lexicographic order (i.e. $G = \mathbb{Z}^2$ and $P = (\{0\} \times \mathbb{N}) \cup ((\mathbb{N} \setminus \{0\}) \times \mathbb{Z}))$, which clearly is totally ordered but not Archimedean. $\mathcal{W}(\mathbb{Z}^2, \text{lex})$ contains $\mathcal{K}$, the ideal of compact operators, by 6.3 and the simple remark that $(0,1)$ is the smallest element of $P \setminus \{(0,0)\}$. We have $I - M(t) \in \mathcal{K} \Leftrightarrow I - M(t)$ has finite rank $\Leftrightarrow t \in \{0\} \times \mathbb{N}$; hence considering an isometric representation $\pi$ of $\mathcal{W}(\mathbb{Z}^2,\text{lex})/\mathcal{K}$ and defining $V(t) = \pi(W(t) + \mathcal{K})$, $\forall t \in P$, we obtain a representation by isometries $V$ of $P$ such that $V(t)$ is unitary if and only if $t \in \{0\} \times \mathbb{N}$.

We mention that using the groupoid interpretation, it can be shown that $\mathcal{K} \subseteq \subseteq \mathcal{C} \subseteq \mathcal{W}(\mathbb{Z}^2,\text{lex})$ is a decomposition series of $\mathcal{W}(\mathbb{Z}^2,\text{lex})$, which is hence type I.

3° It can be shown that the ideal $\mathcal{C}$ is induced from the diagonal subalgebra; the (invariant) inducing ideal of $\mathcal{D}$ can be precisely described as clos sp $\{I - M(t)|t \in P\}$.

*Proof of* 3° $\Rightarrow$ 1°. We fix an arbitrary $u \neq 0$ in $P$; our task is to prove that $\{a \in P|a \leqslant nu \text{ for some } n \text{ in } \mathbb{N}\} = P$. For any $v$ in $P$ we define $A_v = \{a \in P|a \leqslant$

$\leqslant nu + v$ for some $n$ in $\mathbb{N}\}$, which clearly belongs to $\Omega$, and we denote by $\gamma_v$ the character of $\mathcal{D}$ canonically corresponding to $A_v$ (see Proposition 6.2).

Let us prove that for any $t, v$ in $P$:

$$(23) \qquad \gamma_v \circ \alpha_t = \begin{cases} 0, & \text{if } t \notin A_v, \\ \gamma_{ku+v-t} \text{ with } k \in \mathbb{N} \text{ such that } t \leqslant ku + v, & \text{if } t \in A_v, \end{cases}$$

where $\alpha = \{\alpha_x | x \in PQ\}$ is the action of $PQ$ on $\mathcal{D}$ defined at 3.7. (Note that on the right side of this equality, $\gamma_{ku+v-t}$ does not depend on $k$, since it is generally true that $A_v = A_{v+lu}$, $\forall v \in P$, $l \in \mathbb{N}$.) Indeed, fixing for the moment $t$ and $v$ we see that for any $s$ in $P$:

$$(\gamma_v \circ \alpha_t)(M(s)) \overset{(11)\text{of } 3.7}{=\!=\!=} \gamma_v(M(t+s)) = \begin{cases} 1, & \text{if } t + s \in A_v, \\ 0, & \text{if } t + s \notin A_v. \end{cases}$$

If $t \notin A_v$, then $t + s \notin A_v$, $\forall s \in P$, hence $\gamma_v \circ \alpha_t$ vanishes on $\{M(s)|s \in P\}$ and is therefore identically zero. If $t \in A_v$, let us consider a $k$ in $\mathbb{N}$ such that $t \leqslant ku + v$. We have $t + s \in A_v \Leftrightarrow \exists n \geqslant k$ such that $t + s \leqslant nu + v \Leftrightarrow \exists n \geqslant k$ such that $s \leqslant (n-k)u + (ku + v - t) \Leftrightarrow s \in A_{ku+v-t}$, which implies that $\gamma_v \circ \alpha_t$ and $\gamma_{ku+v-t}$ coincide on $\{M(s)|s \in P\}$ and are hence equal.

In a similar manner it can be shown that:

$$(24) \qquad \gamma_v \circ \alpha_{t^{-1}} = \gamma_{v+t}, \qquad \forall v, t \in P.$$

The relations (23), (24) and the fact that $\alpha_{s,t} = \alpha_s \circ \alpha_{t^{-1}}$, $\forall s, t \in P$ (see 3.8) immediately imply that the ideal $\mathcal{I} = \bigcap_{v \in P} \operatorname{Ker} \gamma_v$ of $\mathcal{D}$ is invariant. Observe that for any $t$ in $P : I - M(t) \in \operatorname{Ind} \mathcal{I} \Leftrightarrow I - M(t) \in \mathcal{I}$ (because $(\operatorname{Ind}\mathcal{I}) \cap \mathcal{D} = \mathcal{I}$, by 6.1) $\Leftrightarrow \gamma_v(M(t)) = 1$, $\forall v \in P \Leftrightarrow t \in \bigcap_{v \in P} A_v = A_0$ (with 0 the unit of $G$).

Now $(\operatorname{Ind}\mathcal{I}) \cap \mathcal{C}$ is an ideal of $\mathcal{C}$, which is non-zero (it contains for instance $I - M(u) = W(u)^* W(u) - W(u) W(u)^*$), hence it must be $\mathcal{C}$, because $\mathcal{C}$ is simple. It results that $\operatorname{Ind}\mathcal{I} \supseteq \mathcal{C}$; but $I - M(t) \in \operatorname{Ind}\mathcal{I} \Leftrightarrow t \in A_0$, while $I - M(t) = W(t)^* W(t) - W(t) W(t)^* \in \mathcal{C}$, $\forall t \in P$, and it is thus clear that $A_0 = P$. ■

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Received December 28, 1989.

*Added in proof*: a) The implication $1^\circ \Rightarrow 2^\circ$ of Proposition 6.3 was proved in the general case by A. L.-J. Sheu ("On the type of Wiener-Hopf $C^*$-algebras", *Proc. Amer. Math. Soc.*, **109**(1990), 1053–1058).

b) Let $(G, P)$ be a quasi-lattice ordered group. Assume that every two elements of $P$ have common upper bounds; this is easily seen to be equivalent to the fact that $H = PP^{-1}$ is a subgroup of $G$. From the argument of 4.5 it is clear that "$H$ amenable $\Rightarrow (G, P)$ amenable". The converse can also be shown to hold. Moreover, $H$ is the maximal homomorphic group image of the semigroup $P \times P$ appearing in Section 4.1; hence by a theorem of J. Duncan and I. Namioka (*Proc. Royal Soc. of Edinburgh*, **80A**(1978), 309-321), the amenability of $(G, P)$ is in this case also equivalent to the one of $P \times P$.