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NON-COMMUTATIVE SPHERES II: RATIONAL ROTATIONS

O. BRATTELI, G. A. ELLIOTT, D. E. EVANS, A. KISHIMOTO

Dedicated to the memory of Raphael Høegh-Krohn.

1. INTRODUCTION

In this paper we continue the study, begun in [2], of the fixed point subalgebra of the rotation algebra under the flip. Recall that the rotation algebra $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries U, V satisfying

$$VU = \rho UV \tag{1.1}$$

where $\rho = e^{2\pi i\theta}$ and $0 \leqslant \theta \leqslant 1$. The flip σ is the automorphism of this algebra defined through the requirements

$$\sigma(U) = U^{-1},\ \sigma(V) = V^{-1}. \tag{1.2}$$

Let $\mathcal{B}_\theta = \mathcal{A}_\theta^\sigma$ denote the fixed point algebra under the flip. The main problem in our study is the question whether $\mathcal{B}_\theta$ is an AF algebra or not, when θ is irrational. As a possible prelude to the settlement of this question we here make a detailed study of $\mathcal{B}_\theta$, and of the related algebra $\mathcal{A}_\theta \times_\sigma \mathbf{Z}_2$, in the case that θ is rational. Here $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. One may hope that this will lead to a solution of the problem by techniques related to the folding techniques in [3], [5], [10]. Note that examples of (non-AF)$^\sigma$ = AF were given in this articles — beginning with [5].

We now describe the main results of this paper. Let $\mathcal{P}(U,V)$ be the polynomial $*$-algebra in the unitaries U, V and let $\mathcal{P}^\sigma = \mathcal{P}^\sigma(U,V)$ be the algebra of flip invariant elements of $\mathcal{P}_\theta(U,V)$. The algebra $\mathcal{P}^\sigma$ is spanned linearly by the elements $[n,m]$ defined by

$$[n,m] = \rho^{\frac{nm}{2}}\left(U^nV^m + U^{-n}V^{-m}\right), \tag{1.3}$$

where $\rho^{\frac{1}{2}}$ is an arbitrary (but fixed) square root of ρ. $\mathcal{P}^\sigma$ can in fact be characterized as the $*$-algebra generated by the elements $[n,m]$, $n,m \in \mathbb{Z}$, with the relations

$$\begin{aligned} &[n,m][k,l] = \rho^{\frac{mk-nl}{2}}[n+k,m+l] + \rho^{\frac{nl-mk}{2}}[n-k,m-l] \\ &[n,m]^* = [n,m], \\ &[-n,-m] = [n,m]. \end{aligned} \tag{1.4}$$

In [2], Theorem 3.6, it was proved that the enveloping C^*-algebra of the $*$-algebra $\mathcal{P}^\sigma$ is canonically isomorphic to $\mathcal{B}_\theta$ when θ is irrational. Canonicity means that the isomorphism extends the map

$$[n,m] \mapsto \rho^{\frac{nm}{2}}\left(U^nV^m + U^{-n}V^{-m}\right).$$

Our first main result is

THEOREM 1.1. *If θ is rational, but $\theta \notin \{0, \frac{1}{2}\}$, then the enveloping C^*-algebra of $\mathcal{P}^\sigma$ exists and is canonically isomorphic to $\mathcal{B}_\theta$.*

During the proof we will see that the result fails when $\theta \in \{0, \frac{1}{2}\}$, in the strong sense that the enveloping C^*-algebra of $\mathcal{P}^\sigma$ does not exist. In these two cases, indeed, there exists representations π of (1.4) (of dimension 1 if $\theta = 0$ and of dimension 2 if $\theta = \frac{1}{2}$), such that $\|\pi([n,m])\|$ grows exponentially with the pair (n,m). We can also choose π such that $\|\pi([n,m])\|$ is arbitrarily large for fixed $(n,m) \neq (0,0)$. However, these spurious representations do not extend to continuous representations of the $*$-algebra of infinite sums $\sum_{n,m} \lambda_{n,m}[n,m]$ with rapidly decreasing coefficient sequences, equipped with its natural topology. Thus the enveloping C^*-algebra of this topological algebra is canonically isomorphic to $\mathcal{B}_\theta$ even when $\theta \in \{0, \frac{1}{2}\}$. All these facts, as well as Theorem 1.1, will be proved in Section 2.

Our main result is a simple description of the C^*-algebra $\mathcal{B}_\theta$.

THEOREM 1.2. *If $\theta = \frac{p}{q}$ where p, q are mutually prime positive integers, then $\mathcal{B}_\theta$ is a subalgebra of the C^*-algebra $C(\mathbb{S}^2, \mathrm{M}_q)$ of continuous functions from the 2-sphere $\mathbb{S}^2$ into the algebra of complex $q \times q$ matrices M_q. The subalgebra is determined up to isomorphism as follows: There are four distinct points $\omega_0, \omega_1, \omega_2, \omega_3$ in $\mathbb{S}^2$ and to each point ω_i is associated a self-adjoint projection P_i in M_q. The dimension of P_i is as follows:*

When q is odd, then

$$\dim(P_i) = \frac{q-1}{2} \tag{1.5}$$

for $i = 0, 1, 2, 3$.

When q is even, then

$$\dim\,(P_0) = \frac{q-2}{2}, \tag{1.6}$$

and

$$\dim\,(P_i) = \frac{q}{2} \tag{1.7}$$

for $i = 1, 2, 3$.

The algebra $\mathcal{B}_\theta$ consists of those functions $f \in C\left(\mathbb{S}^2, \mathrm{M}_q\right)$ such that $f(\omega_i)$ commutes with P_i for $i = 0, 1, 2, 3$.

This theorem will be proved in Section 3.

Thus, if $q = 1$, $\mathcal{B}_\theta$ is the algebra of continuous functions on $\mathbb{S}^2$; if $q = 2$, $\mathcal{B}_\theta$ is the algebra of continuous functions from $\mathbb{S}^2$ into M_2 such that the functions take values in a subalgebra of the form $\mathrm{M}_1 \oplus \mathrm{M}_1$ at ω_1, ω_2 and ω_3, and with no restriction at ω_0. When $q \geqslant 3$, the algebra has a proper splitting into the sum of two full matrix algebras at each of the points $\omega_0, \omega_1, \omega_2$ and ω_3.

Note that Theorem 1.2 is consistent with the classification of the one-dimensional representations of $\mathcal{B}_\theta$ given in [2], Lemma 2.6: For example; when $q = 3$, there are four one-dimensional representations, corresponding to the one-dimensional projection P_i at $\omega_i, i = 0, 1, 2, 3$, and when $q = 4$ there is only one one-dimensional representation, corresponding to the one-dimensional projection P_0 at ω_0; when $q \geqslant 5$ there are no one-dimensional representations. The spectrum $\widehat{\mathcal{B}_\theta}$ is easily described from Theorem 1.2: If $q \geqslant 3$ then $\widehat{\mathcal{B}_\theta}$ consists of $\mathbb{S}^2$ with each of the four points $\omega_0, \ldots, \omega_3$ replaced by two points which are both limit points of the neighbouring points.

Note also that when $\theta = \frac{p}{q}$, the isomorphism class of $\mathcal{B}_\theta$ depends only on q – not on p. This is different from the case of $\mathcal{A}_\theta$, where $\mathcal{A}_{\frac{p}{q}}$ and $\mathcal{A}_{\frac{(q-p)}{q}}$ are isomorphic, but $\mathcal{A}_{\frac{p'}{q}}$ is not isomorphic to $\mathcal{A}_{p/q}$ when $p' \neq p$, $q - p$. In that case, $\mathcal{A}_{\frac{p}{q}}$ is a homogeneous C^*-algebra over the two-torus $\mathbb{T}^2$ with fibre M_q, and these are distinguished by the K_0-class of the unit; cf. [9].

Actually, there are also q isomorphism classes of homogeneous C^*-algebras over $\mathbb{S}^2$ with values in M_q, indexed by

$$\pi_1\left(U(q)/\mathbb{T}\right) = \mathbb{Z}/q\mathbb{Z} = \mathbb{Z}_q \tag{1.8}$$

where π_1 is the first homotopy group and $U(q)/\mathbb{T}$ is the group of unitary $q \times q$ matrices modulo its centre. Note that

$$U(q)/\mathbb{T} = \mathrm{Aut}(\mathrm{M}_q). \tag{1.9}$$

This is because any element in a homogeneous algebra over $\mathbb{S}^2$ with values in M_q can be represented by two functions, one each from the northern and southern hemispheres into M_q; on the equator the two functions have to be matched up by a function form $\mathbb{T}$ into $\mathrm{Aut}(M_q)$. If U_1 and U_2 are two such functions from $\mathbf{T}$ into $\mathrm{Aut}(M_q)$, then the two corresponding algebras are $*$-isomorphic if, and only if, the map $e^{it} \in \mathbf{T} \mapsto U_2(e^{it})^* U_1(e^{it})$ can be extended to a map from the unit disc into $\mathrm{Aut}(M_q)$, and this is the case if and only if $\pi_1(U_1) = \pi_1(U_2)$.

Nevertheless, the algebra $\mathcal{B}_{\frac{p}{q}}$ depends only on q and is a trivial bundle. In the course of the proof of Theorem 1.2 we shall in fact show that any fibre bundle over $\mathbb{S}^2$ with fibres contained in M_q, and with the given special behaviour at the four points $\omega_0, \ldots, \omega_3$, is trivial in the sense that there exists a global family of matrix units. For this it is important that the dimensions of the projections P_i are as specified in Theorem 1.2. If, for example, q is even and $\dim(P_i) = \frac{q}{2}$ for all i, there would be $\frac{q}{2}$ isomorphism classes in lieu of merely one.

In addition to $\mathcal{B}_\theta$ we will consider the closely related crossed product algebra,

$$\mathcal{C} = \mathcal{A}_\theta \times_\sigma \mathbf{Z}_2, \tag{1.10}$$

and we shall prove

THEOREM 1.3. *If $\theta = \frac{p}{q}$, the algebra $\mathcal{C}_\theta$ is isomorphic to a subalgebra of the C^*-algebra $C(\mathbb{S}^2, M_{2q})$, determined as follows: There are four distinct points $\omega_0, \omega_1, \omega_2$ and ω_3 in $\mathbb{S}^2$ and an orthogonal projection P in M_{2q} of dimension q, such that the subalgebra consists of those functions $f \in C(\mathbb{S}^2, M_q)$ such that $f(\omega_i)$ commutes with P for $i = 0, 1, 2, 3$.*

This will be proved in Section 4.

In particular, $\mathcal{C}_\theta$ and $\mathcal{B}_\theta$ are Morita equivalent if $q \geqslant 3$, but not for $q = 1$ and $q = 2$ (see [6]).

In Section 5 we will study the trace functionals on the algebraic crossed product $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$, which is the $*$-algebra generated by three unitaries U, V, W with the relations

$$\begin{aligned} VU &= \rho UV, \\ WU &= U^{-1}W, \\ WV &= V^{-1}W, \\ W^2 &= 1. \end{aligned} \tag{1.11}$$

When θ is irrational, the space of trace functionals is five dimensional and is spanned by a trace state τ together with four tracial functionals $\tau_{p_1 p_2}$ where $p_1, p_2 \in$

$\in$ {even, odd}. These are determined by

$$\tau\left(\rho^{\frac{nm}{2}}U^nV^m\right) = \begin{cases} 1 & \text{if } n=m=0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.12}$$
$$\tau\left(\rho^{\frac{nm}{2}}U^nV^mW\right) = 0,$$

and

$$\tau_{p_1p_2}\left(\rho^{\frac{nm}{2}}U^nV^m\right) = 0,$$
$$\tau_{p_1p_2}\left(\rho^{\frac{nm}{2}}U^nV^mW\right) = \begin{cases} 1 & \text{if parity}(n) = p_1 \text{ and parity}(m) = p_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

These formulae also define a trace state τ and tracial functionals $\tau_{p_1p_2}$ when θ is rational.

When θ is irrational, the algebra $\mathcal{C}_\theta$ has a unique trace state, namely, the canonical extension of τ to $\mathcal{C}_\theta$, see [2], Remark 4.6. It follows from the uniqueness of the Jordan decomposition of continuous hermitian functionals on $\mathcal{C}_\theta$ that the only continuous trace functionals on $\mathcal{C}_\theta$ are the scalar multiples of τ. It is all the more remarkable that, when θ is rational, the four other trace functionals on $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$ are continuous and extend to $\mathcal{C}_\theta$ as follows: There is a certain natural indexing of the four special points ω_i by $\mathbb{Z}_2 \times \mathbb{Z}_2$, say $\omega_{p_1p_2}, p_i \in \mathbb{Z}_2$. Now define four continuous trace functionals on $\mathcal{C}_\theta$ by

$$\tilde{\tau}_{p_1p_2}(f) = \frac{1}{q}\mathrm{Tr}_{2q}\left((2P-1)\, f\left(\omega_{p_1p_2}\right)\right) \tag{1.14}$$

where $\mathrm{Tr}2q$ is the unnormalized trace on M_{2q} and P is the projection in Theorem 1.3 $\tilde{\tau}_{p_1p_2}$, so defined, is unique up to a sign corresponding to the choice P or $1-P$ in Theorem 1.3. In the proof of Section 4 the sign will be fixed by a convention, and we shall use this convention now. Then $\tilde{\tau}$ is related to τ as follows: If q is even then

$$\tilde{\tau}_{p_1p_2} = 2(-1)^{p_1p_2}\tau_{p_2p_1}, \tag{1.15}$$

and if q is odd then

$$\tilde{\tau}_{p_1p_2} = \sum_{n,m\in\mathbb{Z}_2} (-1)^{pnm+p_1n+p_2m}\tau_{nm}, \tag{1.16}$$

where we identify {even, odd} with $\mathbb{Z}_2 = \{0,1\}$ in the standard manner. Inverting these relations (the latter by Fourier analysis on $\mathbb{Z}_2 \times \mathbb{Z}_2$) we obtain

$$\tau_{p_1p_2} = \frac{1}{2}(-1)^{p_1p_2}\tilde{\tau}_{p_2p_1} \tag{1.17}$$

when q is even, and

$$\tau_{nm} = \frac{1}{4}(-1)^{pnm}\left\{\sum_{p_1p_2\in\mathbf{Z}_2}(-1)^{np_1+mp_2}\tilde{\tau}_{p_1p_2}\right\} \tag{1.18}$$

when q is odd. This shows that the trace functionals $\tau_{p_1p_2}$ are continuous, and we even have

$$\|\tau_{p_1p_2}\| = 1 \tag{1.19}$$

if q is even, and

$$\|\tau_{p_1p_2}\| = 2 \tag{1.20}$$

when q is odd. (We have equality in the latter case because the four functionals $\tilde{\tau}_{p_1p_2}$ have disjoint supports.)

Finally, in Section 6, we compute the K-theory of $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ when θ is rational. We find that

$$\mathrm{K}_0(\mathcal{C}_\theta) = \mathbf{Z}^6,$$

and

$$\mathrm{K}_1(\mathcal{C}_\theta) = 0 = \mathrm{K}_1(\mathcal{B}_\theta)$$

for all rational θ, and $\mathrm{K}_0(\mathcal{C}_\theta) = \mathbf{Z}^6$ if $q \neq 1, 2$, but

$$\mathrm{K}_0\left(\mathcal{B}_{\frac{1}{2}}\right) = \mathbf{Z}^5$$

and

$$\mathrm{K}_0(\mathcal{B}_0) = \mathbf{Z}^2.$$

By using the trace functionals described above we can also show that $\mathbf{Z}^6 \subseteq \mathrm{K}_0(\mathcal{B}_\theta)$ when θ is irrational (see Section 6).

2. THE ENVELOPING C^*-ALGEBRA

In this section we will prove Theorem 1.1, i.e. that $\mathcal{B}_\theta$ is the enveloping C^*-algebra of $\mathcal{P}^\sigma$ when θ is rational and $\theta \neq 0$ or $\frac{1}{2}$.

In the case $\theta = 0$ or $\frac{1}{2}$, the enveloping C^*-algebra of $\mathcal{P}^\sigma$ does not exist, but in any case $\mathcal{B}_\theta$ is still the enveloping C^*-algebra of the topological $*$-algebra of sums $\sum_{nm} \lambda_{n,m}[n,m]$ with rapidly decreasing coefficients (used in the proof of Theorem 3.6

in [2]; the coefficients $\lambda_{n,m}$ are said to be rapidly decreasing if for all $k = 1, 2, \ldots$ there exist a constant $C_k > 0$ such that

$$|\lambda_{n,m}| \leqslant C_k(1 + |n| + |m|)^{-k}$$

for all $n, m \in \mathbb{Z}$).

These facts follow from a description of the characters (i.e. the one-dimensional representations) of $\mathcal{P}^\sigma$. A character of $\mathcal{P}^\sigma$ in the case $\theta = 0$ is determined by a nonzero family of scalars $(\mu_{n,m})_{n,m \in \mathbb{Z}}$ verifying the relations (1.4) with $\mu_{n,m}$ in place of $[n, m]$:

$$\mu_{n,m}\mu_{k,l} = \mu_{n+k,m+l} + \mu_{n-k,m-l}, \tag{2.1}$$

$$\overline{\mu_{n,m}} = \mu_{n,m}, \tag{2.2}$$

$$\mu_{-n,-m} = \mu_{n,m}. \tag{2.3}$$

By (2.1) and (2.3), $\mu_{00}\mu_{nm} = 2\mu_{nm}$, and so since not all μ_{nm} are zero,

$$\mu_{00} = 2 \tag{2.4}$$

(This corresponds to the fact $[0, 0] = 2$.) It is easily seen from (2.1) and (2.3) that all other μ_{nm} are determined by μ_{10}, μ_{01} and μ_{11}. It follows from (2.1) and (2.3) that

$$\begin{aligned}
\mu_{10}\mu_{01}\mu_{11} &= (\mu_{11} + \mu_{1,-1})\,\mu_{11} \\
&= \mu_{22} + \mu_{00} + \mu_{20} + \mu_{02} \\
&= \mu_{11}^2 - \mu_{00} + \mu_{00} + \mu_{10}^2 + \mu_{00} + \mu_{01}^2 - \mu_{00} \\
&= \mu_{11}^2 + \mu_{10}^2 + \mu_{01}^2 - 2\mu_{00},
\end{aligned}$$

i.e.,

$$2\mu_{00} - \mu_{01}^2 - \mu_{10}^2 - \mu_{11}^2 + \mu_{10}\mu_{01}\mu_{11} = 0. \tag{2.5}$$

It is not difficult to check that any quadruple $(\mu_{00}, \mu_{10}, \mu_{01}, \mu_{11})$ of complex numbers satisfying (2.4) and (2.5) has the form

$$(2, 2\cos\alpha, 2\cos\beta, 2\cos(\alpha + \beta)) \tag{2.6}$$

for some pair of complex numbers (α, β), unique modulo $2\pi\mathbb{Z}$ except for the transformation $(\alpha, \beta) \mapsto (-\alpha, -\beta)$. On the other hand, for any pair $(\alpha, \beta) \in \mathbb{C}^2$, the family

$$(\mu_{nm}) = (2\cos(n\alpha + m\beta)) \tag{2.7}$$

verifies (2.1) and (2.3). The relation (2.2) for the family $(2\cos(n\alpha+m\beta))$ just means that $\cos\alpha, \cos\beta$ and $\cos(\alpha+\beta)$ are all real. Equivalently, this means that either α and β are both real, or, modulo $\pi\mathbf{Z}$, they are both purely imaginary. In the latter case the coefficients μ_{nm} grow exponentially with (n, m), and hence only the characters corresponding to real α and β extend to the C^*-algebra $\mathcal{B}_0$ – or, for that matter, to the subalgebra of sums $\sum_{n,m} \lambda_{nm}[n, m]$ with rapidly decreasing coefficients.

Incidentally, the argument above showed that the relations (2.4) and (2.5) are the only relations among the coefficients $\mu_{00}, \mu_{10}, \mu_{01}$ and μ_{11}, apart from reality.

After having determined the characters of $\mathcal{P}_0^\sigma$, let us now consider the case that $\theta = \frac{p}{q}$ is rational but nonzero, and let us show that, provided also that $\theta \neq \frac{1}{2}$, $\mathcal{B}_\theta$ is the enveloping C^*-algebra of $\mathcal{P}_\theta^\sigma$. It is enough to show that any irreducible $*$-representation π of $\mathcal{P}^\sigma$ by bounded operators on a Hilbert space extends to $\mathcal{B}_\theta$. First note that (1.4) implies that the centre of $\mathcal{P}^\sigma$ is equal to the linear span of the elements $[nq, mq]$, $m, n \in \mathbf{Z}$. Furthermore, this subalgebra is isomorphic to the algebra $\mathcal{P}_0^\sigma$, – since the elements $[n, m]_q = [nq, mq]$ are linearly independent apart from the relation $[-n, -m]_q = [n, m]_q$ and satisfy the relations (1.4) with $\rho = 1$. Since π is irreducible, the restriction of π to the centre of $\mathcal{P}^\sigma$ is a character. It follows from the previous classification of characters on $\mathcal{P}_0^\sigma$ that there exist complex numbers α and β, either both real, or both imaginary modulo $\pi\mathbf{Z}$, such that

$$\pi([n, m]_q) = 2\cos(n\alpha + m\beta) \tag{2.8}$$

for $n, m \in \mathbf{Z}$.

Let us use the assumption $\theta \notin \{0, \frac{1}{2}\}$ to show that α and β are real. Forgetting for the moment about the involution and the restrictions on α and β, any character of the centre of $\mathcal{P}^\sigma$, corresponding to the pair (α, β), extends to a representation of $\mathcal{P}^\sigma$ on $\mathbf{C}^q$, namely,

$$\pi' \colon [n, m] \mapsto \rho^{\frac{nm}{2}} \left[\left(e^{i\frac{\alpha}{q}} U_0\right)^n \left(e^{i\frac{\beta}{q}} V_0\right)^m + \left(e^{i\frac{\alpha}{q}} U_0\right)^{-n} \left(e^{i\frac{\beta}{q}} V_0\right)^{-m} \right], \tag{2.9}$$

where U_0, V_0 are $q \times q$ matrices with q'th power equal to 1 such that $V_0U_0 = \rho U_0V_0$. (For an example of such matrices, see Section 3.) The eigenvalues of U_0 and V_0 are then the q'th roots of 1, and hence the eigenvalues of $\pi'([1, 0])$ are $e^{i\frac{\alpha}{q}+2\pi i\frac{k}{q}} + \left(e^{i\frac{\alpha}{q}+2\pi i\frac{k}{q}}\right)^{-1}$, $k = 0, 1, \ldots$. If α is purely imaginary and nonzero modulo $\pi\mathbf{Z}$, it follows that $\pi'([1, 0])$ has q distinct eigenvalues. Correspondingly, if β is purely imaginary and nonzero modulo $\pi\mathbf{Z}$, $\pi'([1, 0])$ has q distinct eigenvalues. Thus, if both α and β are purely imaginary and nonzero modulo $\pi\mathbf{Z}$, then $\pi'(\mathcal{P}^\sigma)$ contains all the spectral projections of V_0 and U_0 and hence $\pi'(\mathcal{P}^\sigma)$ consists of all operators on

$\mathbb{C}^q$. If only one of α, β, say β, is purely imaginary and nonzero modulo $\pi\mathbb{Z}$ then the eigenvalues of $\pi'([1,0])$ become pairwise equal except for the eigenvalues ± 1 (where -1 occurs only if q is even). Thus $\pi'(\mathcal{P}^\sigma)$ contains V_0 and the one-dimensional projection P onto the eigenspace of U_0 corresponding to the eigenvalue 1. But $V_0^k P V_0^{-k}$, where $k = 0, 1, \ldots, q-1$, ranges over all the eigenprojections of U_0, and thus $\pi'(\mathcal{P}^\sigma)$ again consists of all operators on $\mathbb{C}^q$. We conclude that in all cases where α and β are imaginary modulo $\pi\mathbb{Z}$, and at least one of them is not real, then $\pi'(\mathcal{P}^\sigma)$ consists of all operators on $\mathbb{C}^q$.

Since (as is seen by careful inspection of (1.4)) the q^2 elements $[n,m]$, $n, m = 1, \ldots, q-1$ span $\mathcal{P}^\sigma$ linearly over the centre of $\mathcal{P}^\sigma$, it follows that $\pi'(\mathcal{P}^\sigma)$ is the algebra generated freely by the generators $\pi'([n,m])$ subject to the first and third of the relations (1.4) together with the relation

$$\pi'([nq, mq]) = 2\cos(n\alpha + m\beta)1. \tag{2.10}$$

(It has the maximal dimension permitted by these relations.) Therefore, there is a homomorphism of $\pi'(\mathcal{P}^\sigma)$ to $\pi(\mathcal{P}^\sigma)$ taking $\pi'([n,m])$ to $\pi([n,m])$, and since $\pi'(\mathcal{P}^\sigma)$ is a simple ring this is an isomorphism. Since $\pi(\mathcal{P}^\sigma)$ is also all operators, this shows that π' is similar to a $*$-representation. But since spectrum is a similarity invariant, and since $[1,0]^* = [1,0]$ and $[0,1]^* = [0,1]$, it follows that the spectrum of $\pi'([1,0]) = e^{i\frac{\alpha}{q}} U_0 + \left(e^{i\frac{\alpha}{q}} U_0\right)^{-1}$ and of $\pi'([0,1]) = e^{i\frac{\beta}{q}} V_0 + \left(e^{i\frac{\beta}{q}} V_0\right)^{-1}$ must be real. But the spectrum of $\pi'([1,0])$ is

$$\left\{ e^{i\frac{\alpha}{q} + 2\pi i\frac{k}{q}} + \left(e^{i\frac{\alpha}{q} + 2\pi i\frac{k}{q}}\right)^{-1} | k = 0, 1, \ldots, q \right\} \tag{2.11}$$

and this cannot be real if α has a nonzero imaginary part, unless $q = 1$ or $q = 2$. A similar argument for β establishes that α and β must be real when $q \geqslant 3$.

In this case, π extends to a (continuous) $*$-representation of the norm closure of the centre of $\mathcal{P}^\sigma$ inside $\mathcal{B}_\theta$. Since $\mathcal{B}_\theta$ has a finite basis over this subalgebra consisting of elements of $\mathcal{P}^\sigma$, namely, the elements $[n,m]$, $0 \leqslant n, m \leqslant q-1$ (see Section 3), it follows immediately that π extends to a $*$-representation of $\mathcal{B}_\theta$, as desired.

Note in particular that when $\theta = \frac{1}{2}$, two-dimensional $*$-representations of $\mathcal{P}^\sigma$ with exponential increase of $\|\pi([n,m])\|$ are given by

$$\pi([n,m]) = 2i^{nm}\cosh(na + mb) U_0^{[n]} V_0^{[m]} \tag{2.12}$$

where

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.13}$$

and

$$[n] = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{of } n \text{ is odd,} \end{cases} \tag{2.14}$$

and a, b are real constants. These representations appear by choosing $\mathrm{i}\frac{\alpha}{q} = a$ and $\mathrm{i}\frac{\beta}{q} = b$ in (2.9).

3. STRUCTURE OF THE FIXED POINT ALGEBRA

In this section we shall prove Theorem 1.2. The starting point for the description of $\mathcal{B}_\theta = \mathcal{A}_\theta^\sigma$ and $\mathcal{C}_\theta = \mathcal{A}_\theta \times_\sigma \mathbf{Z}_2$ is the characterization of $\mathcal{A}_\theta$ given in [9], which we will develop in a form suitable for our purposes.

Assume that $\theta = \frac{p}{q}$, where p, q are mutually prime positive integers and $1 \leqslant p \leqslant q-1$. Put

$$\rho = \mathrm{e}^{\pi \mathrm{i}\theta}, \quad \omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}}. \tag{3.1}$$

Define $q \times q$ matrices U_0, V_0 and Γ_0 by

$$U_0 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \rho & 0 & \dots & 0 \\ 0 & 0 & \rho^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \rho^{q-1} \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

The following relations are valid:

$$\begin{aligned} &V_0 U_0 = \rho U_0 V_0, \quad U_0 \Gamma_0 = \Gamma_0 U_0^{-1}, \quad V_0 \Gamma_0 = \Gamma_0 V_0^{-1}, \\ &\Gamma_0^2 = 1, \quad \Gamma_0 = \Gamma_0^*, \quad U_0, V_0 \text{ are unitary.} \end{aligned} \tag{3.3}$$

Using that $\frac{1}{q} \sum_{n=0}^{q-1} \rho^{-nk} U_0^n$ is equal to the matrix element e_{kk} in M_k, $0 \leqslant k \leqslant q$, it is not hard to verify that

$$\Gamma_0 = \frac{1}{q} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} \rho^{nm} U_0^m V_0^{2n}. \tag{3.4}$$

We now define three automorphisms α_1, α_2 and γ_0 of M_q by the requirements

$$\begin{aligned} \alpha_1(U_0) &= U_0, & \alpha_1(V_0) &= \omega V_0, \\ \alpha_2(U_0) &= \omega U_0, & \alpha_2(V_0) &= V_0, \\ \gamma_0(U_0) &= U_0^{-1}, & \gamma_0(V_0) &= V_0^{-1}. \end{aligned} \tag{3.5}$$

Then

$$\alpha_1 = \operatorname{Ad} W_1 \tag{3.6}$$

where

$$W_1 = U_0^{p'} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \omega^{-1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \omega^{-(q-1)} \end{bmatrix} \tag{3.7}$$

and

$$pp' = -1 \bmod q \text{ (and } 0 < p' < q). \tag{3.8}$$

Furhermore,

$$\alpha_2 = \operatorname{Ad} W_2 \tag{3.9}$$

where

$$W_2 = V_0^{p''} = \begin{array}{l} \left[\begin{array}{cccc|cccc} & & & & 1 & 0 & \dots & 0 \\ & & & & 0 & 1 & \dots & 0 \\ & 0 & & & \vdots & & \ddots & \vdots \\ & & & & 0 & 0 & \dots & 1 \\ \hline 1 & 0 & \dots & 0 & & & & \\ 0 & 1 & \dots & 0 & & & & \\ \vdots & & \ddots & \vdots & & 0 & & \\ 0 & 0 & \dots & 1 & & & & \end{array}\right] \\ \underbrace{\hspace{3.5cm}}_{p''} \end{array} \tag{3.10}$$

and

$$pp'' = 1 \bmod q \text{ (and } 0 < p'' < q). \tag{3.11}$$

Finally,

$$\gamma_0 = \operatorname{Ad} \Gamma_0. \tag{3.12}$$

The algebra $\mathcal{A}_\theta$ can be described as the algebra of functions f from $[0,1] \times [0,1]$ into M_q such that

$$\begin{aligned} f(x,1) &= \alpha_1(f(x,0)) \quad 0 \leqslant x \leqslant 1, \\ f(1,y) &= \alpha_2(f(0,y)) \quad 0 \leqslant y \leqslant 1, \end{aligned} \tag{3.13}$$

with pointwise matrix multiplication and involution.

In particular, the generators U and V correspond to the functions given by

$$\begin{aligned} U(x,y) &= \omega^x U_0, \\ V(x,y) &= \omega^y V_0 \end{aligned} \tag{3.14}$$

for $(x,y) \in [0,1] \times [0,1]$, where $\omega^x = e^{2\pi i \frac{x}{q}}$. The flip

$$\sigma(U) = U^{-1}, \quad \sigma(V) = V^{-1} \tag{3.15}$$

can be described as

$$(\sigma f)(x,y) = \sigma_0(f(1-x, 1-y)), \tag{3.16}$$

where σ_0 is the automorphism of M_q determined by

$$\sigma_0(U_0) = \omega^{-1} U_0^{-1}, \quad \sigma_0(V_0) = \omega^{-1} V_0^{-1}. \tag{3.17}$$

One checks that

$$\begin{aligned} U(x,y)^{-1} &= \omega^{-x} U_0^{-1} = \omega^{1-x} \omega^{-1} U_0^{-1} = \\ &= \omega^{1-x} \sigma_0(U_0) = \sigma_0(\omega^{1-x} U_0) = \sigma_0(U(1-x, 1-y)) \end{aligned}$$

and similarly

$$V(x,y)^{-1} = \omega^{-y} V_0^{-1} = \sigma_0(V(1-x, 1-y)),$$

so σ does really represent the flip.

Now, the automorphisms α_1, α_2 and γ_0 verify the commutation relations

$$\begin{aligned} \alpha_1 \alpha_2 &= \alpha_2 \alpha_1, \\ \gamma_0 \alpha_1 &= \alpha_1^{-1} \gamma_0, \\ \gamma_0 \alpha_2 &= \alpha_2^{-1} \gamma_0, \end{aligned} \tag{3.18}$$

and the automorphism $\gamma_0\alpha_1^n\alpha_2^m$ maps U_0 into $\omega^m U_0^{-1}$ and V_0 into $\omega^n V_0^{-1}$. Hence

$$\sigma_0 = \gamma_0\alpha_1^{-1}\alpha_2^{-1} = \alpha_1\alpha_2\gamma_0. \tag{3.19}$$

We are now ready to describe the fixed point algebra $\mathcal{A}_\theta^\sigma = \mathcal{B}_\theta$. Using the description (3.13) of $\mathcal{A}_\theta$ land the description (3.16) of σ one easily identifies $\mathcal{B}_\theta$ as the algebra of continuous functions f from the triangle

$$\{(x,y)|0 \leqslant x, y \leqslant 1,\ x-y \geqslant 0\}: \tag{3.20}$$

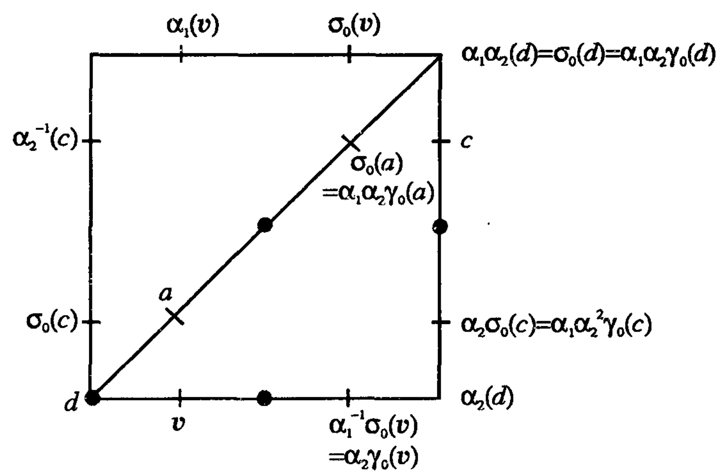
into M_q with the following relations at the boundary:

$$\begin{aligned} f(x,0) &= \alpha_2\gamma_0(f(1-x,0)), && 0 \leqslant x \leqslant 1, \\ f(1,y) &= \alpha_1\alpha_2^2\gamma_0(f(1,1-y)), && 0 \leqslant y \leqslant 1, \\ f(x,x) &= \alpha_1\alpha_2\gamma_0(f(1-x,1-x)), && 0 \leqslant x \leqslant 1. \end{aligned} \tag{3.21}$$

The identification is by restricting $f \in \mathcal{A}_\theta$ from the square to the triangle. This can be seen from the following picture of $f \in \mathcal{A}_\theta^\sigma$, where

$$a = f(x,x),\ b = f(x,0),\ c = f(1,y) \text{ and } d = f(0,0):$$

(3.22)



In particular, the points $(0,0)$, $\left(\frac{1}{2},0\right)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$, which correspond to the four fixed points of the homeomorphism defined by σ on the spectrum T^2 of $\mathcal{A}_\theta$, are exceptional in that the values of $f \in \mathcal{A}_\theta^\sigma$ are restricted to lie in a subalgebra of M_q

which is direct sum of two full matrix algebras. The subalgebra depends on the point, and can be described as follows:

$$\begin{aligned}
&\text{At } (0,0): \{d \in \mathrm{M}_q \mid \gamma_0(d) = d\}.\\
&\text{At } \left(\frac{1}{2},0\right): \{b \in \mathrm{M}_q \mid \alpha_2\gamma_0(b) = b\}.\\
&\text{At } \left(\frac{1}{2},\frac{1}{2}\right): \{a \in \mathrm{M}_q \mid \alpha_1\alpha_2\gamma_0(a) = a\}.\\
&\text{At } \left(1,\frac{1}{2}\right): \{c \in \mathrm{M}_q \mid \alpha_1\alpha_2^2\gamma_0(c) = c\}.
\end{aligned} \tag{3.23}$$

(Note that all automorphisms of the form $\alpha_1^n\alpha_2^m\gamma_0$ are involutions, as a consequence of (3.3), (3.12) and (3.18).) Thus, at the point (n,m) where n,m are half-integers (or integers), the involutive automorphism defining the subalgebra is

$$\alpha_1^{2m}\alpha_2^{2n}\gamma_0, \tag{3.24}$$

and this automorphism is implemented by the unitary

$$W_{n,m} = \rho^{\frac{1}{2}(p'p''2n2m)}U_0^{2mp'}V_0^{2np''}\Gamma_0 \tag{3.25}$$

where $pp' = -1 \mod q$, $pp'' = 1 \mod q$, and the phase factor $\rho^{\frac{1}{2}(p'p''2n2m)}$ is inserted in order to make the unitary $W_{n,m}$ self-adjoint. Here we have used (3.6) to (3.12).

The dimensions of the two subalgebras of M_q at the four exceptional points are equal to the dimensions of the two eigensubspaces of the self-adjoint unitary operator W corresponding to the point. These dimensions are determined by $\mathrm{Tr}_q(W)$ where Tr_q is the unnormalized trace on M_q, given by

$$\mathrm{Tr}_q\left(U_0^nV_0^m\right) = \begin{cases} q & \text{if } n,m = 0 \mod q,\\ 0 & \text{otherwise.}\end{cases} \tag{3.26}$$

Using this together with the expansion (3.4) for Γ_0, one computes

$$\begin{aligned}
&\mathrm{Tr}_q\left(\rho^{\frac{nm}{2}}U_0^nV_0^m\Gamma_0\right) =\\
&= \frac{1}{q}\sum_{k,l\in\mathbb{Z}_q}\rho^{\frac{nm}{2}}\rho^{kl}\mathrm{Tr}_q\left(U_0^nV_0^mU_0^kV_0^{2l}\right) =\\
&= \frac{1}{q}\sum_{k,l}\rho^{\frac{nm}{2}+kl+km}\mathrm{Tr}_q\left(U_0^{n+k}V_0^{m+2l}\right) =\\
&= \frac{1}{q}\sum_{l}\rho^{\frac{nm}{2}-n(m+l)}\mathrm{Tr}_q\left(V_0^{m+2l}\right) =
\end{aligned} \tag{3.27}$$

$$= \frac{1}{q}\sum_{l} \rho^{-\frac{n}{2}(m+2l)} \mathrm{Tr}_q \left(V_0^{m+2l}\right).$$

In the latter sum, only terms where $m + 2l = 0 \mod q$ survives, so there will be two, one or none surviving terms depending on the parity of q and m. We thus obtain the following table

(3.28)

q	m	pn	$\rho^{\frac{nm}{2}} \mathrm{Tr}_q \left(U_0^n V_0^m \Gamma_0\right)$
odd	odd	odd	-1
		even	1
	even		1
even	odd		0
	even	odd	0
		even	2

(An alternative way of deriving this table, suggested by the referee, is to note that it follows from (3.2) that Γ_0 represents the permutation $k \longrightarrow -k \,(\mathrm{mod})\, q$ and V_0^m represents the permutation $k \longrightarrow -k - m \,(\mathrm{mod})\, q$ of $\mathbf{Z}/q\mathbf{Z}$. Thus $V_0^m \Gamma_0$ represents the permutation $k \longrightarrow -k - m\,(\mathrm{mod})\, q$, and therefore the non-zero diagonal elements of the permutation matrix $V_0^m \Gamma_0$ are ones on the sites corresponding to the fixed points of this permutation, which are the integers among the two numbers $\frac{q-m}{2}$ and $-\frac{m}{2}$, $\mathrm{mod}\, q$. Since U_0^m multiplies the k'th row of $V_0^m \Gamma_0$ by ρ^{nk} we obtain

$$\mathrm{Tr}\left(U_0^n V_0^m \Gamma_0\right) = \sum \left\{ \rho^{nk} \middle| k \in \left\{ -\frac{m}{2}, \frac{q-m}{2} \right\} \cap \mathbf{Z} \right\}$$

and (3.28) follows immediately from this.)

We now use this table to compute $\mathrm{Tr}_q\left(W_{n,m}\right)$, where $W_{n,m}$ is given by (3.25). There are two cases:

Case 1: q is odd. Then pp' and pp'' are even, so $p{\cdot}2mp'$ is even for any half-integer m; thus

$$\mathrm{Tr}\left(W_{n,m}\right) = 1 \tag{3.29}$$

for all half-integers n, m in this case.

Case 2: q is even. Then pp' and pp'' are odd, and hence the two numbers

$$p \cdot 2mp', \quad 2np''$$

are even or odd according to whether the two numbers

$$2m, \quad 2n$$

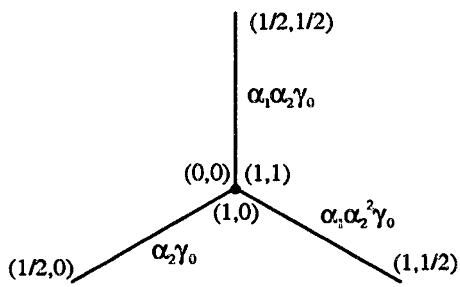
are even or odd. Thus

$$\mathrm{Tr}\,(W_{n,m}) = \begin{cases} 2 & \text{if } 2m \text{ and } 2n \text{ are both even,} \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

This ends the proof of Theorem 1.2 as far as the dimensions of the projections P are concerned. It remains to establish that $\mathcal{B}_\theta$ is a trivial bundle over the sphere.

To this end, we see from figure (3.22) that we may obtain the spectrum of $\mathcal{B}_\theta$ by folding the triangle along the three axes $\left(\frac{1}{2},0\right)-\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{1}{2}\right)-\left(1,\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)-\left(\frac{1}{2},0\right)$, joining the three corners at the top and joining the edge $(0,0)-\left(\frac{1}{2},0\right)$ to $(1,0)-\left(\frac{1}{2},0\right)$, etc.. Thus, $\mathcal{B}_\theta$ may be viewed as an algebra of functions f from the sphere $\mathbb{S}^2$ into M_q. These are continuous except on a tree on $\mathbb{S}^2$ with three edges emanating from a central vertex, corresponding to $(0,0)$, and ending at three vertices, corresponding to the points $\left(\frac{1}{2},0\right)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$. This tree corresponds to the perimeter of the triangle in figure (3.22). At the edges of the tree the function f has jump discontinuities, such that the limit values on one side of the edge are equal to the limit values on the other side modified by an involutory automorphism which is constant along the edge, but dependent upon which of the three edges we are at:

(3.31)

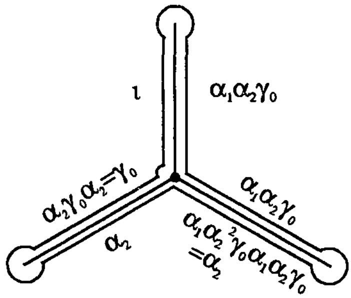


To finish the proof of Theorem 1.2, it is sufficient to show that this algebra is isomorphic to the algebra $\mathcal{B}_\theta^{\mathbb{T}}$ of continuous functions from $\mathbb{S}^2$ into M_q with the restrictions

$$\begin{aligned} &\alpha_2\gamma_0\left(f\left(\frac{1}{2},0\right)\right)=f\left(\frac{1}{2},0\right),\\ &\alpha_1\alpha_2\gamma_0\left(f\left(\frac{1}{2},\frac{1}{2}\right)\right)=f\left(\frac{1}{2},\frac{1}{2}\right),\\ &\alpha_1\alpha_2^2\gamma_0\left(f\left(1,\frac{1}{2}\right)\right)=f\left(1,\frac{1}{2}\right),\\ &\alpha_2\gamma_0\alpha_1\alpha_2^2\gamma_0\alpha_1\alpha_2\gamma_0\left(f\left(0,0\right)\right)=\gamma_0\left(f\left(0,0\right)\right)=f(0,0) \end{aligned} \tag{3.32}$$

at the four special points. This amounts to showing that there exists a map β from $\mathbb{S}^2$ into $\mathrm{Aut}\,(\mathrm{M}_q)=U(q)/\mathbb{T}$ which is continuous except at the tree, and is continuous on each side of the edges of the tree with limits as follows:

(3.33)



Furthermore, on going through the four circle sectors on the figure, the function has to take values in automorphisms with the same fixed point algebra as, or a larger one than, the quotient of the two automorphisms on the two sides on the circle sector. This ensures that, although the map from $\mathbb{S}^2$ into $\mathrm{Aut}(\mathrm{M}_q)$ is necessarily discontinuous at the four special points, when this map is applied to a function in $\mathcal{B}_\theta^{\mathbf{T}}$, a function which is continuous at the four special points is obtained. The isomorphism from $\mathcal{B}_\theta^{\mathbf{T}}$ to $\mathcal{B}_\theta$ is then given by

$$(\beta f)(\omega) = \beta(\omega)(f(\omega)) \tag{3.34}$$

for $f \in \mathcal{B}_\theta^{\mathbf{T}}$, $\omega \in \mathbb{S}^2 \backslash \text{tree}$.

To obtain the map $\omega \mapsto \beta(\omega)$ we will actually define a map

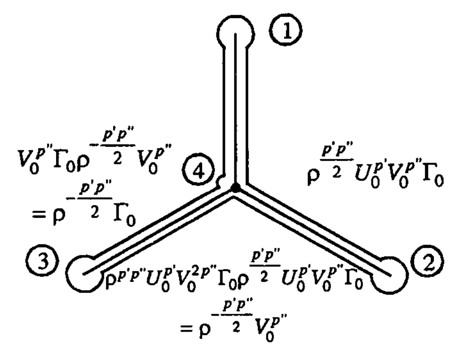
$$\omega \mapsto U(\omega) \in U(q) \tag{3.35}$$

and put

$$\beta(\omega) = \mathrm{Ad}\,(U(\omega))\,. \tag{3.36}$$

Along the edges of the tree, the values of U are given as follows:

(3.37)



Here $pp' = -1 \bmod q$ and $pp'' = -1 \bmod q$. In going around the circle sectors at 1,2,3, and 4, we have to choose the map U so that the value of U commutes with the quotient of the values of U on both sides of the circle sector. For example, going around 1, let P_1 be the spectral projection of $\rho^{\frac{p'p''}{2}} U_0^{p'} V_0^{p''} \Gamma_0$ corresponding to the eigenvalue -1 so that

$$\rho^{\frac{p'p''}{2}} U_0^{p'} V_0^{p''} \Gamma_0 = (1 - P_1) - P_1 \tag{3.38}$$

is the spectral decomposition of this self-adjoint unitary operator. If t is a real parameter varying from 0 to 1 in going around the circle sector near 1 clockwise, then the map U along the circle sector can be taken to be

$$t \mapsto (1 - P_1) + e^{2\pi i(\frac{t}{2}+n_1 t)} P_1 \tag{3.40}$$

where n_1 is an arbitrary integer. In a similar way we use the spectral decompositions

$$\begin{aligned} &\rho^{p'p''} U_0^{p'} V_0^{2p''} \Gamma_0 = (1 - P_2) - P_2, \\ &V_0^{p''} \Gamma_0 = (1 - P_3) - P_3, \\ &\Gamma_0 = (1 - P_4) - P_4 \end{aligned} \tag{3.41}$$

when going around the semicircles at 2, 3, and 4. At 4 we have in addition to unwind the phase factor $\rho^{-\frac{p'p''}{2}}$. Altogether, when going around the contour, we thus define a map from the circle into $U(q)$ with winding number (in the sense of [1]; in this case, the winding number of the determinant)

$$\begin{aligned} \# = &\left(\frac{1}{2} + n_1\right) \dim (P_1) + \left(\frac{1}{2} + n_2\right) \dim (P_2) + \\ &+ \left(\frac{1}{2} + n_3\right) \dim (P_3) - \left(\frac{1}{2} + n_4\right) \dim (P_4) + q\left(\frac{pp'p''}{2q} + n_5\right), \end{aligned} \tag{3.42}$$

where the last term comes from unwinding the phase factor, and $n_1, \ldots, n_5$ are arbitrary integers. This map now defines a map β from the circle into $U(q)/\mathbb{T} = \operatorname{Aut}(M_q)$, and a necessary and sufficient condition for this map to be extendible to a continuous map β on all of $\mathbb{S}^2\backslash$tree is that the winding number of the map into $U(q)$ be an integer multiple of q. (To prove sufficiency, which is all we shall need, it is easily seen to be enough to consider the case that the winding number is zero, and then, for instance by [1], the path is contractible in $U(q)$.) To see that it is possible to choose $n_1, \ldots, n_4$ (n_5 is irrelevant) such that this is the case we divide the discussion into two cases:

Case 1: q is odd. Then

$$\dim (P_1) = \frac{q-1}{2} \tag{3.43}$$

for $i = 1, 2, 3, 4$ by (3.29). Thus, modulo q,

$$\# = \frac{1}{2}pp'p'' + \frac{q-1}{2} + (n_1 + n_2 + n_3 - n_4)\frac{q-1}{2} = \frac{1}{2}pp'p'' + n\frac{q-1}{2}, \tag{3.44}$$

where n is an arbitrary integer. But since

$$pp' = -1 \bmod q,$$
$$pp'' = 1 \bmod q,$$

and $1 \leqslant p', p'' \leqslant q-1$, we have $p' = q - p''$ and hence p' and p'' have distinct parity. It follows that $\frac{1}{2}pp'p''$ is an integer. But as $\frac{q-1}{2}$ and q are mutually prime, it follows that $\#$ can take any integral value modulo q as n varies, and in particular we may choose n so that

$$\# = 0 \bmod q.$$

Case 2: q is even. Then

$$\begin{aligned} \dim(P_i) &= \frac{q}{2}, \ i = 1, 2, 3, \\ \dim(P_4) &= \frac{q-2}{2}, \end{aligned} \tag{3.45}$$

by (3.30). Thus, modulo q,

$$\# = \frac{1}{2}pp'p'' + (1 + n_1 + n_2 + n_3 - n_4)\frac{q}{2} - \left(\frac{1}{2} + n_4\right). \tag{3.46}$$

In this case p, p' and p'' are necessarily all odd; hence $\frac{1}{2}pp'p'' - \frac{1}{2} = m$ is an integer, and hence the possible range of values of $\#$ is

$$\# = m + n\frac{q}{2} + k$$

where n, k are arbitrary integers. Now choose $n = 0, k = -m$.

It follows that β does indeed extend to a continuous map from $\mathbb{S}^2\backslash$tree into $\mathrm{Aut}(\mathrm{M}_q)$. This ends the proof of Theorem 1.2.

REMARK: As a matter of fact, it is necessary to compute the winding number of the path U in $U(q)$ as defined in the proof of Theorem 1.2. Since blocks at two of the four double points have relatively prime order, the choice of U can be modified near these points so that U has winding number zero.

More generally, we see in this way that any bundle over $\mathbb{S}^2$ with fibre M_q except at some finite number of points, where it is the commutant of some finite family of orthogonal projections, such that some integral combination of the dimensions of all the projections is equal to 1(i.e., the dimensions are relatively prime), is trivial.

(One first expresses such a bundle in terms of a sewing along some tree joining the singular points, determined by some continuous path of unitaries around the tree, commuting with the projections at each vertex in the neighborhood of that vertex, and then modifies the path, near the vertices, so that it has winding number zero, and so extends to a continuous family on the whole complement of the tree.)

4. STRUCTURE OF THE CROSSED PRODUCT

In this section we shall prove Theorem 1.3. We first note that if σ is an involutive automorphism of a C^*-algebra $\mathcal{A}$, the crossed product $\mathcal{A} \times_\sigma \mathbb{Z}_2$ can be described in a simple fashion which in particular makes the connection between $\mathcal{A} \times_\sigma \mathbb{Z}_2$ and $\mathcal{A}^\sigma$ clear. If W is the canonical unitary in $\mathcal{A} \times_\sigma \mathbb{Z}_2$ implementing σ, the usual left regular representation of $\mathcal{A} \times_\sigma \mathbb{Z}_2$ is given by

$$A \in \mathcal{A} \mapsto \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix},$$
$$W \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, a general element in the crossed product is

$$A + BW = \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \sigma(B) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ \sigma(B) & \sigma(A) \end{bmatrix}.$$

Changing coordinates such that W becomes diagonal, i.e. conjugating the expression above with the self-adjoint unitary

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

we obtain the following representation of the crossed product:

$$A + BW = \frac{1}{2} \begin{bmatrix} (1+\sigma)(A+B) & (1-\sigma)(A-B) \\ (1-\sigma)(A+B) & (1+\sigma)(A-B) \end{bmatrix}. \tag{4.1}$$

Consider the spectral subspace

$$\mathcal{A}^\sigma(-1) = \{A \in \mathcal{A} | \sigma(A) = -A\} \tag{4.2}$$

of $\mathcal{A}$ corresponding to the eigenvalue -1. Then, as A, B vary independently over $\mathcal{A}$, the elements $(1+\sigma)(A+B)$ and $(1+\sigma)(A-B)$ vary independently over $\mathcal{A}^\sigma$, and the elements $(1-\sigma)(A+B)$ and $(1-\sigma)(A-B)$ vary independently over $\mathcal{A}^\sigma(-1)$. It follows that the crossed product can be described as

$$\mathcal{A} \times_\sigma \mathbb{Z}_2 = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \middle| A, D \in \mathcal{A}^\sigma,\ B, C \in \mathcal{A}^\sigma(-1) \right\} \tag{4.3}$$

with standard matrix multiplication and involution. In this representation

$$W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.4}$$

and the projection $P = \frac{1}{2}(1+W)$ such that (by [13]) $P(\mathcal{A} \times_\sigma \mathbb{Z}_2)P \cong \mathcal{A}^\sigma$ is

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

Let us now specialize this to A_θ, with $\theta = \frac{p}{q}$. Using (3.16) and the technique leading to (3.21), one may identify $\mathcal{A}^\sigma(-1)$ with the set of functions f from the triangle

$$\{(x,y) | 0 \leqslant x,\, y \leqslant 1,\, x - y \geqslant 0\} \tag{4.6}$$

into M_q with the following relations at the boundary:

$$\begin{aligned} f(x,0) &= -\alpha_2\gamma_0(f(1-x,0)), && 0 \leqslant x \leqslant 1, \\ f(1,y) &= -\alpha_1\alpha_2^2\gamma_0(f(1,1-y)), && 0 \leqslant y \leqslant 1, \\ f(x,x) &= -\alpha_1\alpha_2\gamma_0(f(1-x,1-x)), && 0 \leqslant x \leqslant 1. \end{aligned} \tag{4.7}$$

Hence, on using (4.3), the crossed product $\mathcal{A} \times_\sigma \mathbb{Z}_2$ may be identified with the set of functions f from the triangle (4.6) into $\mathrm{M}_{2q} = \mathrm{M}_q \otimes \mathrm{M}_2$ with the following relations at the boundary:

$$\begin{aligned} f(x,0) &= \left(\alpha_2\gamma_0 \otimes \mathrm{Ad}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)(f(1-x,0)), && 0 \leqslant x \leqslant 1, \\ f(1,y) &= \left(\alpha_1\alpha_2^2\gamma_0 \otimes \mathrm{Ad}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)(f(1,1-y)), && 0 \leqslant y \leqslant 1, \\ f(x,x) &= \left(\alpha_1\alpha_2\gamma_0 \otimes \mathrm{Ad}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)(f(1-x,1-x)), && 0 \leqslant x \leqslant 1. \end{aligned} \tag{4.8}$$

Thus the value of f ranges over all of M_{2q} except at the four exceptional points $(0,0)$ (and $(1,0),(1,1)$), $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$ and $(1,\frac{1}{2})$, where the value is restricted to the $2q \times 2q$ matrices commuting with a unitary matrix of the form $V \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ where V is a self-adjoint unitary matrix in M_q. But the dimensions of both eigenspaces of $V \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are q, and this establishes Theorem 1.3 as far as the dimension of P is concerned. To prove triviality of the bundle, one notes that if U is the map from the circle into $U(q)$ defined by Figure (3.37) and the subsequent remark, one may immediately define a similar map by replacing U by

$$V = U \otimes \Gamma$$

where Γ is a map from the circle into $U(2)$ which varies from 1 to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on the circle sector 1, from $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to 1 on sector 2, from 1 to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on sector 3, and back to 1 on sector 4. The winding number of this new map can be taken to be zero, and hence the winding number of V is two times the winding number of U, and therefore V may be chosen to have winding number zero modulo $2q$ by the reasoning around (3.43) and (3.46). The remaining details of the proof of Theorem 1.3 are like those of Theorem 1.2.

5. TRACES AND PROJECTIONS ASSOCIATED TO THE CROSSED PRODUCT

In [2], Proposition 4.1, all the trace functionals on $\mathcal{P}^\sigma(U,V)$ were determined in the case that θ is irrational. These are spanned by five functionals $\tau, \tau_{ee}, \tau_{eo}, \tau_{oe}, \tau_{oo}$ (e=even, o=odd), where τ is the trace coming from $\mathcal{P}(U,V)$, i.e.

$$\tau([n,m]) = \begin{cases} 2 & \text{if } n = m = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

and

$$\tau_{p_1p_2}([n,m]) = \begin{cases} 1 & \text{if parity}(n) = p_1,\ \text{parity}(m) = p_2, \\ 0 & \text{otherwise}. \end{cases} \tag{5.2}$$

In particular, this incorporates the convention

$$\tau_{ee}([0,0]) = 1, \tag{5.3}$$

which is essential for (5.4), below. These trace functionals are of course also defined on $\mathcal{P}^\sigma(U,V)$ when θ is rational.

The four latter traces have the additional property that

$$\tau_{p_1p_2}(CD) = -\tau_{p_1p_2}(DC) \tag{5.4}$$

whenever C, D are in the odd subspace $\mathcal{P}^\sigma(-1)$. This can be verified by explicit calculations on C, D of the form

$$[n,m]_- = \rho^{\frac{nm}{2}} \left(U^n V^m - U^{-n} V^{-m}\right). \tag{5.5}$$

The latter elements span $\mathcal{P}^\sigma(-1)$ linearly, and satisfy the relations

$$[n,m]_-[k,l]_- = \rho^{\frac{mk-nl}{2}}[n+k, m+l] - \rho^{\frac{nl-mk}{2}}[n-k, m-l], \tag{5.6}$$

$$[n,m]_-^* = -[n,m]_-, \tag{5.7}$$

$$[-n,-m]_- = -[n,m]_-. \tag{5.8}$$

Using that $n+k$ and $n-k$ have the same parity, and also $m+1$ and $m-1$, one immediately obtains (5.4) from (5.6).

Now let $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$ be the algebraic crossed product of $\mathcal{P}_\theta$ by $\mathbf{Z}_2$. Then $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$ is the $*$-algebra generated by three unitaries U, V, W with the relations

$$\begin{aligned} VU &= \rho UV, \\ WU &= U^{-1}W, \\ WV &= V^{-1}W, \\ W^2 &= 1. \end{aligned} \tag{5.9}$$

Since $\mathcal{A}_\theta$ is the enveloping C^*-algebra of a pair U, V of unitaries with the first relation, it follows from the universal properties of the crossed product that the enveloping C^*--algebra of $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$ exists and is canonically isomorphic to $\mathcal{A}_\theta \times_\sigma \mathbf{Z}_2$. (Incidentally, Alexander Kumjian (E-mail correspondence, May 24, 1989) has pointed out that $\mathcal{A}_\theta \times_\sigma \mathbf{Z}_2$ is the universal C^*-algebra generated by three self-adjoint unitaries x, y, z with the relation

$$xyz = \rho zyx.$$

The correspondence with (5.9) is

$$x = VW, \quad y = W, \quad z = UW,$$

or

$$U = zy, \quad V = xy, \quad W = y.)$$

Furthermore, it follows from (4.3) that $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$ can be realised as an algebra of 2×2 matrices:

$$\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2 = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \middle| A, D \in \mathcal{P}^\sigma,\ B, C \in \mathcal{P}^\sigma(-1) \right\}. \tag{5.10}$$

PROPOSITION 5.1. *If θ is irrational, any trace functional on $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$ has the form*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \lambda\tau(A+D) + \sum_{p_1p_2} \lambda_{p_1p_2}\tau_{p_1p_2}(A-D) \tag{5.11}$$

where $\lambda, \lambda_{ee}, \lambda_{oe}, \lambda_{eo}, \lambda_{oo}$ are complex numbers, $A, D \in \mathcal{P}^\sigma$, $B, C \in \mathcal{P}^\sigma(-1)$ and the last sum is over all four possible combinations of parities p_1, p_2.

In terms of the standard description of the crossed product, the same functional is given by

$$A + BW \mapsto \lambda\tau\left(\frac{1}{2}(1+\sigma)(A)\right) + \sum_{p_1p_2} \lambda_{p_1p_2}\tau_{p_1p_2}\left(\frac{1}{2}(1+\sigma)(B)\right) \tag{5.12}$$

for $A, B \in \mathcal{P}(U, V)$.

Before proving this proposition we note from Section 4 that when θ is rational the algebra $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$, and even $\mathcal{A}_\theta \times_\sigma \mathbb{Z}_2$, has many more traces. The space of extremal trace states of $\mathcal{A}_\theta \times_\sigma \mathbb{Z}_2$ is homeomorphic to the sphere $\mathbb{S}^2$ with four points deleted, joined disjointly with a discrete eight point set (see Theorem 1.3).

To prove Proposition 5.1, let φ be a trace functional on $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$, and decompose φ as

$$\varphi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \varphi_1(A) + \varphi_2(D) + \varphi_3(B) + \varphi_4(C) \tag{5.16}$$

where φ_1, φ_2 are linear functionals on $\mathcal{P}^\sigma$ and φ_3, φ_4 are linear functionals on $\mathcal{P}^\sigma(-1)$. Now the identity

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -B \\ 0 & 0 \end{bmatrix} \tag{5.14}$$

and the trace property of φ imply $\varphi_3 = 0$, and similarly $\varphi_4 = 0$. Furthermore, restricting to elements of the form $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ one sees that φ_1 is a trace functional on $\mathcal{P}^\sigma$, and similarly so is φ_2. Finally, using this, and checking the trace property on the product of two general elements of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, one deduces:

LEMMA 5.2 *The functional φ on $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$ defined by (5.10), is a trace functional if, and only if,*

$$\varphi_3 = \varphi_4 = 0, \tag{5.15}$$

$$\varphi_1, \varphi_2 \text{ are trace functionals on } \mathcal{P}^\sigma, \tag{5.16}$$

$$\varphi_1(BC) = \varphi_2(CB) \quad \text{for } C, B \in \mathcal{P}^\sigma(-1). \tag{5.17}$$

Now, by (5.16) and the first paragraph, above, which uses the irrationality of θ, each φ_i has the form

$$\varphi_i = \lambda_i\tau + \sum_{p_1p_2} \lambda_{ip_1p_2}\tau_{p_1p_2}, \quad i = 1, 2. \tag{5.18}$$

Using (5.4), the fact that $\tau(BC) = \tau(CB)$ for $B, C \in \mathcal{P}^\sigma(-1)$, and the fact that $\mathcal{P}^\sigma(-1)^2 = \mathcal{P}^\sigma$ (see [2], Lemma 2.10 and its proof), one deduces from (5.17) that

$$\begin{aligned} &\lambda_1 = \lambda_2 \equiv \lambda \\ &\lambda_{1,p_1,p_2} = -\lambda_{2,p_1,p_2} \equiv \lambda_{p_1,p_2} \end{aligned} \tag{5.19}$$

Conversely, one can deduce (5.16) and (5.17) from these relations, and this e-stablishes Proposition 5.1 execpt for the final remark, which is immediate from (4.1).

Now, define five trace functionals on $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$ by

$$\tau^0(A + BW) = \frac{1}{2}\tau((1+\sigma)A) \tag{5.20}$$

and

$$\tau^0_{p_1p_2}(A + BW) = \tau_{p_1p_2}((1+\sigma)B) \tag{5.21}$$

for p_1, p_2 =odd, even. By Proposition 5.1, if θ is irrational these functionals span all trace functionals.

Let us now consider the following projections in $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$ introduced by Nest in [11]:

$$P(n,m) = \frac{1}{2}\left(1 + \rho^{\frac{nm}{2}} U^n V^m W\right) = \frac{1}{4}\begin{bmatrix} 2 + [n,m] & -[n,m]_- \\ [n,m]_- & 2 - [n,m] \end{bmatrix}. \tag{5.22}$$

It is easily verified that these elements are indeed self-adjoint projections, and

$$\begin{aligned} &\tau^0(P(n,m)) = 1, \\ &\tau^0_{p_1p_2}(P(n,m)) = \begin{cases} \frac{1}{2} & \text{if parity}(n) = p_1 \text{ and parity}(m) = p_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.23}$$

Thus, the five projections $1, P(0,0), P(0,1), P(1,0)$ and $P(1,1)$ separate the five trace functionals, as

$$\begin{aligned} &\tau^0(1) = 1, \\ &\tau^0 p_1 p_2(1) = 0. \end{aligned} \tag{5.24}$$

In particular, whether θ is irrational or not, these traces are independent in the periodised cyclic cohomology of $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$. A complete computation of the cyclic cohomology is given in [11].

We have seen that whether θ is irrational or not, the poliomial algebra $\mathcal{P}_\theta \times_\sigma \mathbb{Z}_2$ has the five, "abstract", trace functionals $\tau, \tau_{p_1p_2}, p_1, p_2 \in \{\text{odd,even}\} = \{o, e\}$. (We omit the super-script 0 from now on). Explicitely, these are given by

$$\begin{aligned} &\tau\left(\rho^{\frac{nm}{2}} U^n V^m\right) = \begin{cases} 1 & \text{if } n = m = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &\tau\left(\rho^{\frac{nm}{2}} U^n V^m W\right) = 0, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} &\tau_{p_1p_2}\left(\rho^{\frac{nm}{2}}U^nV^m\right) = 0, \\ &\tau_{p_1p_2}\left(\rho^{\frac{nm}{2}}U^nV^mW\right) = \begin{cases} 1 & \text{if parity}(n) = p_1 \text{ and parity}(m) = p_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.26}$$

When $\theta = \frac{p}{q}$ is rational, it follows from Theorem 1.3 that the space of extremal trace states on $\mathcal{A}_\theta \times_\sigma \mathbf{Z}_2$ is the disjoint union of the two-sphere $\mathbb{S}^2$ with four points deleted and a discrete eight point set. Using 3.22, a point on the sphere outside the four points can be specified by a pair (x, y) with $0 \leqslant x$, $y \leqslant 1$, $x \geqslant y$ (with some identifications at the boundary on this region), where $(x, y) \neq (0,0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$. The corresponding trace state is defined by

$$\tau_{(x,y)}(f) = \frac{1}{2q}\operatorname{Tr}_{2q}(f(x,y)) \tag{5.27}$$

in the function representation given by (4.8). We will compute $\tau_{x,y}$ on the linear generators of $\mathcal{P}_\theta \times_\sigma \mathbf{Z}_2$. These are given by

$$\begin{aligned} \rho^{\frac{nm}{2}}U^nV^m &= \frac{1}{2}\begin{pmatrix} [n,m] & [n,m]_- \\ [n,m]_- & [n,m] \end{pmatrix}, \\ \rho^{\frac{nm}{2}}U^nV^mW &= \frac{1}{2}\begin{pmatrix} [n,m] & -[n,m]_- \\ [n,m]_- & -[n,m] \end{pmatrix}, \end{aligned} \tag{5.28}$$

by (4.1).

In the function representation, by (3.14),

$$\begin{aligned} [n,m](x,y) &= \rho^{\frac{nm}{2}}\left(U(x,y)^nV(x,y)^m + U(x,y)^{-n}V(x,y)^{-m}\right) = \\ &= \rho^{\frac{nm}{2}}\left(\omega^{nx+my}U_0^nV_0^m + \omega^{-nx-ny}U_0^{-n}V_0^{-m}\right). \end{aligned} \tag{5.29}$$

Combining this with (3.26), we see that

$$\begin{aligned} &\tau_{(x,y)}\left(\rho^{\frac{nm}{2}}U^nV^m\right) = \\ &= \rho^{\frac{nm}{2}}\cos\left(\frac{2\pi}{q}(nx+my)\right)\cdot\begin{cases} 1 & \text{if } n, m = 0 \bmod q, \\ 0 & \text{otherwise,} \end{cases} \\ &\tau_{(x,y)}\left(\rho^{\frac{nm}{2}}U^nV^mW\right) = 0. \end{aligned} \tag{5.30}$$

Of course, $\tau_{(x,y)}$ is also a trace state when (x, y) is in the half-integer lattice, but then it is not extremal, — it decomposes into the average of the two extremal trace states given by

$$\tau^{\pm}_{(x,y)}(f) = \frac{1}{q}\operatorname{Tr}_{2q}\left(\frac{1}{2}\left(1 \pm W_{x,y}\right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) f(x,y)\Bigg) \tag{5.31}$$

where $W_{(x,y)}$ is the self-adjoint unitary implementing the flip at (x,y), i.e.,

$$W_{(x,y)} = \rho^{\frac{1}{2}(p'p''2x2y)} U_0^{2yp'} V_0^{2xp''} \Gamma_0 \tag{5.32}$$

by (3.23) and (3.25), where $pp' = -1 \bmod q$, $pp = 1 \bmod q$. Since

$$\frac{1}{2}\left(1 + W_{(x,y)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \pm W_{(x,y)} & 0 \\ 0 & 1 \mp W_{(x,y)} \end{bmatrix} \tag{5.33}$$

we obtain

$$\tau^{\pm}_{(x,y)}\left(\rho^{\frac{nm}{2}} U^n V^m\right) = \rho^{\frac{nm}{2}} \cos\left(\frac{2\pi}{q}(nx+my)\right) \cdot \begin{cases} 1 & \text{if } n,m = 0 \bmod q, \\ 0 & \text{otherwise,} \end{cases} \tag{5.34}$$

while

$$\begin{aligned} &\tau^{\pm}_{(x,y)}\left(\rho^{\frac{nm}{2}} U^n V^m W\right) = \\ &= \pm\frac{1}{2q} \operatorname{Tr}_q \left(\rho^{\frac{nm}{2}}\left(\omega^{nx+my} U_0^n V_0^m + \omega^{-nx-my} U_0^{-n} V_0^{-m}\right) W_{(x,y)}\right) = \\ &= \pm\frac{1}{2q} \operatorname{Tr}_q \Big(\omega^{nx+my} \rho^{myp'} \rho^{-nxp''} \rho^{\frac{1}{2}(2yp'+n)(2xp''+m)} \cdot \\ &\cdot U_0^{2yp'+n} V_0^{2xp''+m} \Gamma_0 + \text{similar term with } n \mapsto -n,\ m \mapsto -m\Big). \end{aligned} \tag{5.35}$$

But according to (3.28), $\operatorname{Tr}_q\left(\rho^{\frac{nm}{2}} U_0^n V_0^m \Gamma_0\right)$ depends only on n and m through the parity of n and m, and as the two integers $2yp' \pm n$ have the same parity, and also $2xp'' \pm m$, we obtain

$$\begin{aligned} &\tau^{\pm}_{(x,y)}\left(\rho^{\frac{nm}{2}} U^n V^m W\right) = \\ &= \pm\frac{1}{q} \cos\left(\frac{2\pi}{q}(nx(1-pp'') + my(1+pp'))\right) \cdot \\ &\cdot \operatorname{Tr}_q \left(\rho^{\frac{1}{2}(2yp'+n)(2xp''+m)} U_0^{2yp'+n} V_0^{2xp''+m} \Gamma_0\right). \end{aligned} \tag{5.36}$$

Since $1 - pp'' = 0 \bmod q$, $1 + pp' = 0 \bmod q$ and x, y are half-integers, we deduce that

$$\begin{aligned} &\tau^{\pm}_{(x,y)}\left(\rho^{\frac{nm}{2}} U^n V^m W\right) = \\ &= \pm\frac{1}{q}(-1)^{\frac{1}{q}(n2x(1-pp'')+m2y(1+pp'))} \cdot \phi(q, 2xp''+m, p(2yp'+n)) \end{aligned} \tag{5.37}$$

where $\phi(\cdot,\cdot,\cdot)$ is the integer-valued function tabulated in (3.28).

Now, for $(x,y) = (0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$, introduce the trace functionals

$$\tilde{\tau}_{(x,y)} = \frac{q}{2}\left(\tau^{+}_{(x,y)} - \tau^{-}_{(x,y)}\right). \tag{5.38}$$

Then

$$\tilde{\tau}_{(x,y)}\left(\rho^{\frac{nm}{2}}U^nV^m\right)=0, \tag{5.39}$$

$$\begin{aligned}&\tilde{\tau}_{(x,y)}\left(\rho^{\frac{nm}{2}}U^nV^mW\right)=\\&=(-1)^{\frac{1}{q}(n2x(1-pp'')+m2y(1+pp'))}\cdot\phi(q,2xp''+m,p(2yp'+n)).\end{aligned} \tag{5.40}$$

Hence,

$$\tilde{\tau}_{(0,0)}\left(\rho^{\frac{nm}{2}}U^nV^mW\right)=\phi(q,m,pn), \tag{5.41}$$

$$\tilde{\tau}_{(\frac{1}{2},0)}\left(\rho^{\frac{nm}{2}}U^nV^mW\right)=(-1)^{\frac{1}{q}n(1-pp'')}\phi(q,p''+m,pn), \tag{5.42}$$

$$\tilde{\tau}_{(\frac{1}{2},\frac{1}{2})}\left(\rho^{\frac{nm}{2}}U^nV^mW\right)=(-1)^{\frac{1}{q}(n(1-pp'')+m(1+pp'))}\phi(q,p''+m,p(p'+n)), \tag{5.43}$$

$$\tilde{\tau}_{(1,\frac{1}{2})}\left(\rho^{\frac{nm}{2}}U^nV^mW\right)=(-1)^{\frac{1}{q}m(1+pp')}\phi(q,m,p(p'+n)), \tag{5.44}$$

where we have used that ϕ depends only on the parity of its arguments.

To obtain a more tractable expression for $\tilde{\tau}$, we again divide the discussion into two subcases: q even or odd.

Case 1: q even: Then p,p' and p'' are all odd and as $p'+p''=q$ we derive

$$(1+pp')-(1-pp')=pq=\text{odd multiple of } q.$$

Then

$$\text{parity}\left(\frac{1-pp'}{q}\right)\neq\text{parity}\left(\frac{1+pp''}{q}\right).$$

If the parity of $\frac{1+pp''}{q}$ is odd, one uses the table (3.28) to derive (putting $\tilde{\tau}_{(x,y)}\left(\rho^{\frac{nm}{2}}U^n\cdot\right.$ $\left.\cdot V^mW\right)=\tilde{\tau}_{(x,y)}(n,m)$)

$$\begin{aligned}&\tilde{\tau}_{(0,0)}(n,m)=\phi(q,m,n)=\begin{cases}2 & \text{if } m \text{ and } n \text{ are even},\\ 0 & \text{otherwise},\end{cases}\\&\tilde{\tau}_{(\frac{1}{2},0)}(n,m)=\phi(q,m+1,n)=\begin{cases}2 & \text{if } m \text{ is odd and } n \text{ is even},\\ 0 & \text{otherwise},\end{cases}\\&\tilde{\tau}_{(\frac{1}{2},\frac{1}{2})}(n,m)=(-1)^m\phi(q,m+1,n+1)=\begin{cases}-2 & \text{if } m,n \text{ are odd},\\ 0 & \text{otherwise},\end{cases}\\&\tilde{\tau}_{(1,\frac{1}{2})}(n,m)=(-1)^m\phi(q,m,n+1)=\begin{cases}2 & \text{if } m \text{ is even and } n \text{ is odd},\\ 0 & \text{otherwise}.\end{cases}\end{aligned} \tag{5.45}$$

If $\frac{1+pp'}{q}$ is even, one derives exactly the same expression for $\tilde{\tau}_{(x,y)}$ as above. Comparing this table with (5.26) we see that

$$\tilde{\tau}_{(x,y)} = 2(-1)^{4xy}\tau_{p(2y)p(2x)} \tag{5.46}$$

for $(x,y) = (0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$ where

$$p(k) = \text{parity of}\, k. \tag{5.47}$$

This concludes the discussion of even q.

Case 2: q is odd: In this case the table (3.28) implies

$$\phi(q,m,pn) = (-1)^{pmn}.$$

Using (5.38) to (5.41) we obtain

$$\begin{aligned}
\tilde{\tau}_{00}(n,m) &= (-1)^{pnm},\\
\tilde{\tau}_{(\frac{1}{2},0)}(n,m) &= (-1)^{pnm}\varphi_1(n),\\
\tilde{\tau}_{(\frac{1}{2},\frac{1}{2})}(n,m) &= (-1)^{pnm}\varphi_1(n)\varphi_2(m),\\
\tilde{\tau}_{(1,\frac{1}{2})}(n,m) &= (-1)^{pnm}\varphi_2(m)
\end{aligned}$$

where

$$\begin{aligned}
\varphi_1(n) &= (-1)^{\left(\frac{1}{q}(1-pp'')+pp''\right)n},\\
\varphi_2(m) &= (-1)^{\left(\frac{1}{q}(1+pp')+pp'\right)m}.
\end{aligned}$$

But as $pp'' = 1 \bmod q$ one has

$$\frac{1}{q}(1-pp'') + pp'' = 1 \bmod (q-1),$$

and as $q-1$ is even, this number is odd; hence

$$\varphi_1(n) = (-1)^n.$$

Similarly,

$$\varphi_2(m) = (-1)^m,$$

and hence

$$\tilde{\tau}_{(x,y)}(n,m) = (-1)^{pnm}(-1)^{np(2x)}(-1)^{mp(2y)}(-1)^{pnm+2xn+2ym} \tag{5.48}$$

for $x,y = (0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,\frac{1}{2}\right)$. Comparing with (5.23) we deduce that

$$\tilde{\tau}_{(x,y)} = \sum_{n,m\in\mathbb{Z}_2} (-1)^{pnm+2xn+2ym}\tau_{p(n)p(m)} \tag{5.49}$$

for $(x,y) = (0,0), (\frac{1}{2},0), (\frac{1}{2}.\frac{1}{2})$ and $(1,\frac{1}{2})$. Conversely, one could obtain $\tau_{p_1p_2}$ from the $\tilde{\tau}_{(x,y)}$'s by applying the inverse Fourier transform over $\mathbf{Z}_2 \times \mathbf{Z}_2$.

This proves relations (1.15) to (1.18).

6. K-THEORY.

THEOREM 6.1. *Let $\omega_1, \omega_2, \ldots, \omega_r$ be a set of r distinct points on the 2-spere $\mathbf{S}^2$. If $q > 1$, let $P_1, P_2, \ldots, P_r$ be projections in M_q, with $0 < P_i < 1$. Let $\mathcal{C}_r$ denote the C^*-algebra of functions $f \in C(\mathbf{S}^2, \mathrm{M}_q)$ such that $f(\omega_i)$ commutes with P_i, for $r = 0, 1, 2, \ldots r$,*

$$\mathrm{K}_0(\mathcal{C}_r) \simeq \mathbf{Z}^{r+2},$$
$$\mathrm{K}_1(\mathcal{C}_r) \simeq \{0\}.$$

Proof. Let $\mathcal{J}_r$ denote the ideal of $\mathcal{C}_r$ consisting of those functions which vanish at $\omega_1, \omega_2, \ldots, \omega_r$. Then from the exact sequences, for $r = 0, 1, 2, \ldots$,

$$0 \mapsto \mathcal{J}_{r+1} \mapsto \mathcal{J}_r \mapsto \mathrm{M}_q \mapsto 0,$$

where $\mathcal{J}_0 \simeq C(\mathbf{S}^2) \otimes \mathrm{M}_q \simeq \mathcal{C}_0$, we deduce that

$$\mathrm{K}_0(\mathcal{J}_r) \simeq \mathbf{Z},$$
$$\mathrm{K}_1(\mathcal{J}_r) \simeq \mathbf{Z}^{r-1}$$

for $r > 1$. The identification of $\mathrm{K}_1(\mathcal{J}_r)$ with $\mathbf{Z}^{r-1}$ is as follows. Let γ be a unitary in $\mathcal{J}_r$, the unital extension of $\mathcal{J}_r$. Then the class of γ in $\mathrm{K}_1(\mathcal{J}_r)$ is identified with $\varphi[\gamma] = (n_i : i \in \mathbf{Z}/r\mathbf{Z})$ in $\mathbf{Z}^r/\mathbf{Z}(1,1,\ldots 1)$, where n_i is the winding number of $\det(\gamma)$ along a line from ω_i to ω_{i+1}.

Then from the exact sequence

$$0 \mapsto \mathcal{J}_r \mapsto \mathcal{C}_r \mapsto \bigoplus_{i=1}^{r} (P_i \mathrm{M}_q P_i \oplus (1 - P_i) \mathrm{M}_q (1 - P_i)) \mapsto 0$$

we have

$$\begin{array}{ccccc} \mathbf{Z} & \longrightarrow & \mathrm{K}_0(\mathcal{C}_r) & \longrightarrow & \mathbf{Z}^{2r} \\ \uparrow & & & & \downarrow{\scriptstyle \alpha} \\ 0 & \longleftarrow & \mathrm{K}_1(\mathcal{C}_r) & \longleftarrow & \mathbf{Z}^{r-1} \end{array}$$

We show that α is surjective, from which the assertion for $\mathrm{K}_*(\mathcal{C}_r)$ follows. Fix $j \in \in \{1, \ldots, r\}$. Take $\mathrm{e} = (\mathrm{e}_i) \in \bigoplus_{i=1}^{r} (P_i \mathrm{M}_q P_i \oplus (1 - P_i) \mathrm{M}_q (1 - P_i))$, where e_j is a minimal projection, and $\mathrm{e}_i = 0$ for $i \neq j$. Then we can extend e to a continuous function on $\mathbf{S}^2$, so that $\varphi[\exp 2\pi i \mathrm{e}] = [(n_i)]$ where

$$n_i = \begin{cases} -1 & \text{if } i = j - 1, \\ +1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus α is surjective.

By Theorem 1.2 and 1.3, this implies that the K-groups of $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ are as described in the introduction when θ is rational.

If θ is irrational, then, as shown in Section 5, there is a rank five subgroup of $\mathrm{K}_0(\mathcal{C}_\theta)$ which is separated by the five-dimensional space of traces on the dense subalgebra of rapidly decreasing sums. A sixth independent element is obtained by adjoining the class of the Rieffel projection, [12]. This is seen by noting that, while Nest's projections have rational values on a basis for the space of traces, Rieffel's projection has trace θ with respect to the canonical trace, τ. Thus, $\mathrm{K}_0(\mathcal{C}_\theta) \supseteq \mathbf{Z}^6$.

Since $\mathcal{B}_\theta$ is Morita equivalent to $\mathcal{C}_\theta$, also $\mathrm{K}_0(\mathcal{B}_\theta) \supseteq \mathbf{Z}^6$. It would be interesting to determine if Nest's projections in fact already belong to $\mathrm{M}_2(\mathcal{B}_\theta)$.

7. A SECOND APPLICATION OF THE DENSELY DEFINED TRACES ON $\mathcal{B}_\theta$ AND $\mathcal{C}_\theta$.

Throughout this section we will assume that θ is irrational. We shall show that in the dense subalgebra of $\mathcal{B}_\theta$ consisting of sums $\sum \lambda_{n,m}[n,m]$ with $(\lambda_{n,m})$ rapidly decreasing (see (1.3)), the linear combinations of the projections cannot be dense in the natural topology. We refer to the topology obtained from the seminorms $\|\delta_1^n\delta_1^m(\cdot)\|$ on $\mathcal{A}_\theta^\infty$, where δ_1 and δ_2 are the canonical derivations of $\mathcal{A}_\theta^\infty$ (see [4]). We shall denote this dense subalgebra of $\mathcal{B}_\theta$, which is a Fréchet algebra in this topology, by $\mathcal{B}_\theta^\infty$. Likewise we will consider the subalgebra of $\mathcal{C}_\theta$ consisting of linear combinations of the monomials U^nV^m and U^nV^mW with rapidly decreasing coefficients. This $*$--algebra is a Fréchet algebra in its obvious natural topology and is denoted by $\mathcal{C}_\theta^\infty$. It is the algebraic crossed product of $\mathcal{B}_\theta^\infty$ by the action σ of $\mathbf{Z}_2$. We will show that the linear combinations of the projections in $\mathcal{C}_\theta^\infty$ are not dense in $\mathcal{C}_\theta^\infty$ in the Fréchet topology.

These results should be contrasted with the fact, proved in [7], that the linear combinations of projections in the C^*-algebras $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ are dense in the algebras in the usual topology, at least for a dense set of irrational values of θ.

Another comment which seems appropriate at this point is that, while the result of [7] also holds for the rotation C^*-algebra, — that is to say, the linear combinations of the projections are dense in the C^*-algebra $\mathcal{A}_\theta$ for many values of θ —, our methods, which involve the four-dimensional spaces of spurious trace functionals on $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$, respectively, do not allow us to conclude that the linear combinations of the projections are not dense in $\mathcal{A}_\theta^\infty$ (in the natural topology of this algebra).

We shall prove the opening statement for the algebra $\mathcal{C}_\theta^\infty$; the statement for $\mathcal{B}_\theta^\infty$ can be deduced by minor modifications of the argument by using the embedding of $\mathcal{B}_\theta^\infty$ into $\mathcal{C}_\theta^\infty$ described in Section 4. Thus we shall establish that the linear combinations of the projections cannot be dense in C_θ^∞.

We note first that if τ is the canonical trace and τ_{ee} is the tracial functional on B_θ^∞ defined by (5.18), then the tracial functional $\tau + \frac{1}{2}\tau_{ee}$ is positive on any projection in B_θ^∞ for each rational value $\frac{p}{q}$ of θ. This follows from (1.14), (1.17) and (1.18), together with the fact that, on a projection, the value of τ can be obtained by evaluating at any point of $\mathbb{S}^2$ and then taking the normalized trace on M_{2q} — rather than doing this at all points and averaging. In particular, one can evaluate at one of the split points of $\mathbb{S}^2$; at such a point the relevant trace is the sum of the traces in the two blocks, normalized by dividing by $2q$.

It follows by continuity that $\tau + \frac{1}{2}\tau_{ee}$ is positive on each projection in C_θ^∞ when θ is irrational. More explicitly, each projection e_θ in C_θ^∞ is part of a family of projections

$$e_{\theta'} = \sum_{n,m} \left(\lambda_{n,m}^{\theta'} U^n V^m + \mu_{n,m}^{\theta'} U^n V^m W \right) \in C_{\theta'}^\infty,$$

θ' belonging to an interval containing θ, such that the coefficients $\lambda_{n,m}^{\theta'}$ and $\mu_{n,m}^{\theta'}$ are rapidly decreasing uniformly in θ', and are continuous in θ' for each n and m. This follows from continuous field techniques (see [8]) together with the fact that C_θ^∞ is closed under C^∞-calculus of the self-adjoint elements (see [4]), or it suffices to use that C_θ^∞ is closed under holomorphic calculus. Continuity of $\theta' \mapsto \left(\tau + \frac{1}{2}\tau_{ee}\right)(e_\theta)$ follows, and positivity at the irrational point $\theta' = \theta$ follows from positivity at rational points shown above.

Therefore, if the linear combinations of the projections are dense in C_θ^∞ in the natural topology, then, since τ_{ee} is continuous in this topology, and of course also τ, the trace functional $\tau + \frac{1}{2}\tau_{ee}$ is positive on C_θ^∞.

This is contrary to the uniqueness of τ as a tracial state of C_θ^∞ proved in Theorem 4.7 of [2].

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Note added in proof. Since this paper was submitted, several of the questions posed in the paper have been solved. A. Kumjian proved that $K_0(C_\theta) \cong \mathbb{Z}^6$ and $K_1(C_\theta) \cong 0$ for all θ, whether irrational or not, by using different techniques from ours, in KUMJIAN, A., Non-commutative spherical orbifolds, *C.R. Math. Rep. Sci. Canada*, 12(1990), 87–89. Also, using techniques completely different from those proposed in the introduction of this paper, it has been established that C_θ is an AF-algebra when θ is irrational, in BRATTELI, O.; EVANS, D. E.; KISHIMOTO, A., Crossed products of totally disconnected spaces by $\mathbb{Z}_2 * \mathbb{Z}_2$, *Ergodic theory and dynamical systems*, to appear, and BRATTELI, O.; KISHIMOTO, A., Non-commutative spheres III, Irrational rotations, *Comm. Math. Phys*, 147(1992), 605–624.

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OLA BRATTELI
Department of Mathematics,
University of Oslo,
PB.1053 Blirdern,
N-0316 Oslo 3,
Norway.

GEORGE A. ELLIOTT
Mathematics Institute,
University of Copenhagen,
DK-2100 Copenhagen Ø,
Denmark.

and

Department of Mathematics,
University of Toronto,
Toronto, Ontario,
Canada M5S 1A1.

DAVID E. EVANS
Mathematics Institute,
University College of Swansea,
Swansea SA2 8PP,
Wales, U. K.

AKITAKA KISHIMOTO
Department of Mathematics,
Hokkaido University,
Sapparo,
060 Japan.

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