

ALGEBRAIC ORDERS ON K_0 AND APPROXIMATELY FINITE OPERATOR ALGEBRAS

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Approximately finite (AF) C^* -algebras are classified by approximately finite (r -discrete principal) groupoids. Certain natural triangular subalgebras of AF C^* -algebras are similarly classified by triangular subsemigroupoids of AF groupoids [15]. Putting this in a more intuitive way, such subalgebras A are classified by the topologised fundamental binary relation $R(A)$ induced on the Gelfand space of the masa $A \cap A^*$ by the normaliser of $A \cap A^*$ in A . (This relation $R(A)$ is also determined by any matrix unit system for A affiliated with $A \cap A^*$.) The fundamental relation $R(A)$ has been useful both in understanding the isomorphism classes of specific algebras and in the general structure theory of triangular and chordal subalgebras of AF C^* -algebras ([7], [15], [14], [21], [22]). Nevertheless it is desirable to have more convenient and computable invariants associated with the K_0 group, and we begin such an inquiry in this paper.

We introduce the algebraic order and the strong algebraic order on the scale of the K_0 group of a non-self-adjoint subalgebra of a C^* -algebra. Analogues (and generalisations) of Elliott's classification of AF C^* -algebras are obtained for limit algebras of direct systems

$$A_1 \longrightarrow A_2 \longrightarrow \dots$$

of finite-dimensional CSL algebras (poset algebras) with respect to certain embeddings with C^* -extensions which, in a certain sense, preserve the algebraic order. We also require that the systems have a certain conjugacy property. Despite the restrictions there are many interesting applications. For example conjugacy properties prevail for certain embeddings of finite-dimensional nest algebras (block upper triangular matrix algebras) and for systems associated with ordered Bratteli diagrams.

Hitherto the study of non-self-adjoint subalgebras of AF C^* -algebras has focused on triangular subalgebras, where $A \cap A^*$ is a certain approximately finite regular maximal abelian self-adjoint algebra ([1], [7], [9], [11], [14], [15]). See also [8]. However from the point of view of identifying the algebraically ordered scaled ordered dimension group, the viewpoint of this paper, it is the *nontriangular* subalgebras which are particularly interesting since in this case $K_0(A)$ (which agrees with $K_0(A \cap A^*)$) can be a "small group", such as \mathbf{Z}^5 or \mathbf{Q}^2 . In such settings, the algebraic orders can be revealed more explicitly. For example, in Section 4 we see situations in which $A \cap A^*$ is a simple C^* -algebra in the simple C^* -algebra $C^*(A)$, and A is one of only finitely many algebras between $A \cap A^*$ and $C^*(A)$. The algebraic order of such an algebra corresponds to partial orders on the fibres of the surjection $i_*: K_0(A) \longrightarrow K_0(C^*(A))$.

In Section 1 we define the reflexive transitive antisymmetric order $S(A)$ on the scale of the K_0 group of a subalgebra of a C^* -algebra, and we recall some basic facts concerning (regular) canonical subalgebras of AF C^* -algebras. In Section 2 we discuss various kinds of embeddings of finite-dimensional algebras of matrices, and we define the strong algebraic order $S_1(A)$ associated with a canonical masa. In Section 3 we obtain the main results, Theorems 3.1 and 3.2, together with various examples and associated remarks. In particular we consider a class of triangular algebras associated with *ordered* Bratteli diagrams. In Section 4 we consider examples of non-self-adjoint subalgebras of AF C^* -algebras with small K_0 group. In this connection we look at *stationary pairs* of AF C^* -algebras $D \subseteq B$ and their intermediate non-self-adjoint algebras.

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1. ALGEBRA ORDERS ON K_0

We start by recalling some terminology and properties of subalgebras of AF C^* -algebras.

A finite-dimensional commutative subspace lattice algebra A , or FDCSL algebra, is an operator algebra on a finite-dimensional Hilbert space which contains a maximal abelian self-adjoint algebra (masa). We say that a masa C in an AF C^* -algebra B is a *canonical* masa (or, more precisely, a regular canonical masa) if there is a chain of finite-dimensional C^* -subalgebras $B_1 \subseteq B_2 \subseteq \dots$, with dense union, such that the algebras $C_k = B_k \cap C$ are masas in the algebras B_k with dense union in C and, such that, for each k , the normaliser of C_k in B_k is contained in the normaliser of C_{k+1} in B_{k+1} . A closed subalgebra A of B is said to be a (regular) *canonical* subalgebra if

$C \subseteq A \subseteq B$ for some canonical masa C . In this case A is necessarily the closed union of the FDCSL algebras $A_n = B_n \cap A$ (See [16].) In particular the algebra A is an approximately finite operator algebra and is identifiable with the direct limit Banach algebra $\varinjlim A_n$ where the embeddings possess star extensions. Of course the converse is true; if $A_1 \rightarrow A_2 \rightarrow \dots$ is a direct system of FDCSL algebras with respect to embeddings, not necessarily unital, which have star extensions $C^*(A_k) \rightarrow C^*(A_{k+1})$, then the Banach algebra $A = \varinjlim A_k$ is completely isometrically isomorphic to a subalgebra of the AF C^* -algebra $\overline{B} = \varinjlim C^*(A_k)$. (However such a subalgebra need not to be a regular canonical subalgebra in the sense above.)

The masas above coincide with approximately finite Cartan subalgebras of AF C^* -algebras [18]. A useful discussion of them is given in the notes of Strătilă and Voiculescu [20].

We now give a definition of $K_0(A)$ for a not necessarily self-adjoint subalgebra A of a C^* -algebra. In the unital, or stably unital case, $K_0(A)$ coincides with the usual definition in terms of the stable algebraic equivalence of idempotents. (See Proposition 5.5.5 of [2].) Write $p \rightarrow q$, or $p \rightarrow_A q$, or $p \rightarrow_v q$, for p, q in $\text{Proj}(A)$, the set of self-adjoint projections of A , if there exists a partial isometry v in A with $v^*v = p, vv^* = q$. Write $p \sim q$ if v can be chosen in $A \cap A^*$. Define $K_0^+(A)$ as the set of (Murray von Neumann) equivalence classes $[p]$ of projections in $\text{Proj}(A \otimes M_n), n = 1, 2, \dots$ with the usual identifications $A \otimes M_n \subseteq A \otimes M_{n+1}, n = 1, 2, \dots$. A semigroup operation is given by $[p] + [q] = [p + q]$, where p and q are representative with $pq = 0$, and $K_0(A)$ is, by definition, the Grothendieck group of $K_0^+(A)$. For a canonical AF subalgebra $K_0^+(A)$ has cancellation and embeds injectively in $K_0(A)$. The scale of A in $K_0(A)$ is the partially ordered set $\Sigma(A) = \{[p]: p \in \text{Proj}(A)\}$. A celebrated theorem of G. Elliott [4] asserts that AF C^* -algebras B_1 and B_2 are isomorphic if there is a group isomorphism $\theta: K_0(B_1) \rightarrow K_0(B_2)$ with $\theta(\Sigma(B_1)) = \Sigma(B_2)$.

For a canonical subalgebra A of an AF C^* -algebra note that

$$K_0(A) = \varinjlim K_0(A_n) = \varinjlim K_0(A_n \cap A_n^*) = K_0(A \cap A^*).$$

Define the algebraic order $S = S(A)$ on $\Sigma(A)$ to be the reflexive transitive anti-symmetric relation such that $[p]S[q]$ if and only if $q \rightarrow_v p$ for some partial isometry v in some $A \otimes M_n$ for some n . For canonical subalgebras we can take p, q, v in A , because $K_0^+(A)$ has cancellation. The pair $(\Sigma(A), S(A))$ does not form a complete invariant for such subalgebras of AF C^* -algebras, but we shall see that it is complete for certain subclasses.

2. EMBEDDINGS AND NORMALISERS

Let $C \subseteq A \subseteq B$ be as in the second paragraph of Section 1. Every self-adjoint projection in A is equivalent in $A \cap A^*$ to a projection in $A_n \cap A_n^*$ for some n , and so is equivalent to a projection in C_n for some n . Furthermore the algebraic orders can be understood in terms of the partial isometries which normalise C , as we now indicate.

DEFINITION 2.1. The *normaliser* of C in A is the semigroup $N_C(A)$ of partial isometries v in A such that $vCv^* \subseteq C$ and $v^*Cv \subseteq C$. The *strong normaliser* of C in A is the subsemigroup $N_C^s(A)$ of elements v which preserve the relation \longrightarrow_A in the sense that if $p_1 \longrightarrow_A p_2$, with $p_1 \leq v^*v$, and $p_2 \leq v^*v$, then $vp_1v^* \longrightarrow_A vp_2v^*$, and if $p_1 \longrightarrow_A p_2$, with $p_1 \leq vv^*$, and $p_2 \leq vv^*$, then $v^*p_1v \longrightarrow_A v^*p_2v$.

The normaliser $N_C(A)$ has the following important property. Each v in $N_C(A)$ has the form cw with c a partial isometry in C and w an element of $N_{C_k}(A_k)$ for some k . (Also, every operator of this form is in $N_C(A)$.) We use this below without further explanation. For details see [14] or [16].

LEMMA 2.2. Let p and q be projections in A . Then $[p]S(A)[q]$ if and only if there exist projections p' and q' in C and a partial isometry v in $N_C(A)$ such that $p \sim p'$, $q \sim q'$ and $v^*v = q'$, $vv^* = p'$.

Proof. Suppose that $[p]S(A)[q]$. For some large k there are projections in A_k which are close to p and q , and so it follows that there exist projections p' and q' in C_k with $p \sim p'$, $q \sim q'$. By the hypothesis it follows that there is a partial isometry w in A with $w^*w = q'$ and $ww^* = p'$. Increasing k if necessary, choose an operator x in A_k , close to w , with $x = p'xq'$, such that x is invertible when viewed as an operator from $q'H$ to $p'H$ where H is the finite-dimensional Hilbert space underlying A_k . Let $p' = p'_1 + \dots + p'_r$, $q' = q'_1 + \dots + q'_r$ be the decompositions into minimal projections of C_k . By the invertibility of x it follows that there is a permutation π of $1, \dots, r$ such that $p'_i x q'_{\pi(i)}$ is nonzero for each i . By the minimality of the p'_i and q'_j it follows that there is a partial isometry v_i with initial projection $q_{\pi(i)}$ and final projection p_i . The partial isometry $v = v_1 + \dots + v_r$ satisfies $v^*v = q'$, $vv^* = p'$, and since v belongs to $N_{C_k}(A_k)$ it follows that v belongs to $N_C(A)$. ■

In the next section we consider embeddings affiliated with maximal abelian subalgebras and the following terminology will be useful.

DEFINITION 2.3. Let A, A' be FDCSL algebras containing masas C, C' respectively and let $\alpha: A \longrightarrow A'$ be an injective algebra homomorphism with $C \longrightarrow C'$. The embedding is said to be

- (i) *star-extendible*, if there is an extension $C^*(A) \longrightarrow C^*(A')$,
- (ii) *regular* (with respect to C, C') if $N_C(A) \longrightarrow N_{C'}(A')$,
- (iii) *strongly regular* (with respect to C, C') if $N_C^s(A) \longrightarrow N_{C'}^s(A')$.

If $A \subseteq M_n, A' \subseteq M_n \otimes M_m$, identified with $M_n(M_m)$, and if $A \otimes CI \subseteq A'$, then we refer to the natural embedding $\rho: A \longrightarrow A'$, given by $\rho(a) = a \otimes 1$, as a refinement embedding. In particular, viewing T_{nm} as the upper triangular matrix subalgebra of $M_n(M_m)$ we have the *refinement* embedding $\rho: T_n \longrightarrow T_{nm}$. In contrast, if $CI \otimes A \subseteq A'$, then we refer to the embedding $\sigma: A \longrightarrow A'$, given by $\sigma(a) = 1 \otimes a = a \otimes \dots \otimes a$ (n times) as a standard embedding. In particular, with the same identification $T_{nm} \subseteq M_n(M_m)$, we have the *standard* embeddings $\sigma: T_m \longrightarrow T_{nm}$. Standard embeddings, refinement embeddings, and many other hybrid embeddings are strongly regular. In contrast the embedding $T_2 \longrightarrow T_4$ given by

$$\begin{bmatrix} a & b \\ & c \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & 0 & b \\ & a & b & 0 \\ & & c & 0 \\ & & & c \end{bmatrix}$$

is a regular star-extendible embedding which is not strongly regular.

It is elementary to check that a strongly regular star-extendible embedding $T_n \longrightarrow T_m$ is determined, up to conjugation by a unitary in the diagonal algebra $D_m = T_m \cap (T_m)^*$, by its restriction to the diagonal algebra D_n . Recall that a finite-dimensional nest algebra A is an FDCSL algebra whose lattice of invariant projections is totally ordered. Similarly it can be shown that a strongly regular embedding $\alpha: A \longrightarrow A'$ between direct sums of such algebras is determined, up to conjugacy by a unitary in $A' \cap (A')^*$, by its restriction to $A \cap A^*$.

DEFINITION 2.4. Let $C \subseteq A \subseteq B$ be as in the second paragraph of Section 1. The *strong algebraic order* $S_1(A)$ is the subrelation of the algebraic order $S(A)$ such that $[p]S_1(A)[q]$ if and only if there are representatives p, q in $\text{Proj}(C)$ and a partial isometry v in $N_C^s(A)$ with $q \longrightarrow_v p$.

In general $S_1(A)$ is a proper subrelation of $S(A)$. This can be seen for elementary FDCSL algebras. On the other hand these relation agree in the case of triangular nest algebras. Note that $S_1(A)$ depends on an implicit choice of canonical masa, so it is not clear, a priori, whether $S_1(A)$ is even an invariant for isometric isomorphism. However in the context of limits of nest algebras we have:

LEMMA 2.5. Let A be the limit of the system $A_1 \longrightarrow A_2 \longrightarrow \dots$ consisting of direct sums of triangular finite-dimensional nest algebras and strongly regular embeddings. Then $S_1(A) = S(A)$, where $S_1(A)$ is the strong algebraic order of A .

Proof. If A_k is a triangular finite-dimensional nest algebra then $S_1(A_k) = S(A_k)$. Indeed, if p, q are projections in a masa C_k of A_k , and $q \rightarrow p$, then we can order the minimal subprojections (in C_k) of q and p and obtain $q \rightarrow_v p$ where $v \in N_{C_k}(A_k)$ is a partial isometry which matches these subprojections in order. Since v preserves the partial ordering on minimal projections (induced by A_k) v belongs to $N_{C_k}^s(A_k)$.

If $[e]S(A)[f]$, and C is the limit of the subsystem $C_1 \rightarrow C_2 \rightarrow \dots$, choose p, q in $\text{Proj}(C_k)$ for some large k with $[e] = [p]$, $[f] = [q]$ and with $q \rightarrow_{A_k} p$. Choose v as above in $N_{C_k}^s(A_k)$. The embeddings are strongly regular and so it follows that v is in $N_C^s(A)$. ■

3. CLASSIFYING LIMIT ALGEBRAS

Let $A_1 \rightarrow A_2 \rightarrow \dots$ be a direct system of FDCSL algebras with star-extendible injective embeddings. We say that the system has the *conjugacy property* if whenever $\alpha: A_n \rightarrow A_{n+k}$ is a star-extendible embedding whose restriction $\alpha|_{C_n}$ to a masa in A_n is equal to the given injection $i: C_n \rightarrow A_{n+k}$, then there is a unitary operator u in $A_{n+k} \cap A_{n+k}^*$ such that $\alpha = (\text{Ad } u) \circ i$, where i is the given injection of A_n . Similarly we say that the system has the *conjugacy property for strongly regular maps* if the same conclusion holds just for maps $\alpha: A_n \rightarrow A_{n+k}$ which are strongly regular relative to two masas. This is the appropriate concept for direct systems with strongly regular embeddings.

We noted above that a strongly regular direct system $A_1 \rightarrow A_2 \rightarrow \dots$, with each A_k a direct sum of triangular finite-dimensional nest algebras, has the conjugacy property for strongly regular maps. On the other hand it can be shown that the natural direct system

$$A_1 \rightarrow A_1 \otimes A_2 \rightarrow A_1 \otimes A_2 \otimes A_3 \rightarrow \dots,$$

where each A_k is an FDCSL algebra, has the ordinary conjugacy property.

In the theorem below we obtain a generalisation of Elliot's classification of AF C^* -algebras, and the proof is modelled on the self-adjoint case. However, in our generalisation it is necessary to do extra work to lift relations on the scale of $K_0(A)$ to normalising partial isometries in such a way that we can construct star-extendible embeddings. (We remark that even strongly regular embeddings of FDCSL algebras need not to be star-extendible.)

Let A be a canonical subalgebra with canonical masa C , and let $R \subseteq S(A)$ be a connected transitive reflexive finite subrelation. We view this as a binary relation

on the set $\{1, \dots, n\}$. It follows from Lemma 2.2 that we can find orthogonal projections p_1, \dots, p_n in $M_\infty(C)$ and partial isometries v_{ij} in $M_\infty(N_C(A))$, with $p_j \rightarrow_{v_{ij}} p_i$, whenever $(i, j) \in R$. We say that $S(A)$ has the *star realisation property* if for every such subrelation there is a choice with $\{v_{ij}: (i, j) \in R\}$ a subset of a complete matrix unit system. By this we mean that the natural map $A(R) \rightarrow M_\infty(A)$ given by $(a_{ij}) \rightarrow (a_{ij} v_{ij})$ is a star-extendible injection from the FDCSL algebra $A(R)$ associated with R . The star realisation property is rather restricted, as we observe below in Remark 3.3. Nevertheless it holds in several contexts of interest and allows for the statement of a general theorem.

In an exactly analogous way we can define when $S_1(A)$ has the star realisation property, and there is a corresponding variant of the following theorem for strongly regular systems.

THEOREM 3.1. *Let A and A' be the limits of the systems $A_1 \rightarrow A_2 \rightarrow \dots$ and $A'_1 \rightarrow A'_2 \rightarrow \dots$ consisting of FDCSL algebras and injective star-extendible regular embeddings. Suppose further that the systems have the conjugacy property and that the algebraic orders $S(A)$ and $S(A')$ have the star realisation property. Then A and A' are isometrically isomorphic if and only if there is a scaled order group isomorphism $\theta: K_0(A) \rightarrow K_0(A')$ which gives an isomorphism of the algebraic orders.*

Proof. Assume that θ exists. Thus it is assumed that θ gives a bijection between the algebraic orders $S(A), S(A')$ associated with canonical masas C, C' , respectively, affiliated with the given direct systems. It will be enough to construct a system of embeddings

$$A_1 \xrightarrow{\varphi_1} A'_{n_1} \xrightarrow{\psi_1} A_{m_1} \xrightarrow{\varphi_2} \dots$$

which commute with the given embeddings $A_1 \rightarrow A_{m_1}, A'_{n_1} \rightarrow A'_{n_2}, \dots$. In particular the constructed isomorphism maps $\cup A_n$ onto $\cup A'_n$. This isomorphism, namely $\varphi = \varinjlim \varphi_k$, will also implement the given isomorphism θ .

The algebra A is a regular canonical subalgebra of the AF C^* -algebra $C^*(A) = \varinjlim C^*(A_n)$, and from this it follows that we can choose systems $\{e_{ij}^n\}$ of matrix units for $C^*(A_n)$, for $n = 1, 2, \dots$, such that each e_{ij}^n is a sum of matrix units in $\{e_{ij}^{n+1}\}$, and such that $\{e_{ii}^n\}$ spans the masa C_n in A_n . (See [16] for example.) Let $\{e_{ij}^n: (i, j) \in \Omega_n\}$ be the set of matrix units in A_n . Similarly choose the matrix unit system $\{f_{ij}^n\}$ for $C^*(A'_n)$, with $\{f_{ii}^n\}$ spanning C'_n , and let $\{f_{ij}^n: (i, j) \in \Omega'_n\}$ be the set $A'_n \cap \{f_{ij}^n\}$.

Choose n_1 large enough so that there are orthogonal projections g_{ii} in C'_{n_1} such that $\theta([e_{ii}^1]) = [g_{ii}]$ for all i . (We need not be precise about the range of i .) Let $(i, j) \in \Omega_1$. Since θ preserves the algebraic order, and since $S(A')$ has the star realisation property, there is a choice of partial isometries v_{ij} in $M_\infty(N_{C'}(A'))$, for

$(i, j) \in \Omega_1$, such that $g_{jj} \xrightarrow{v_{ij}} g_{ii}$, and such that the induced map $\varphi_1: A_1 \longrightarrow M_\infty(A')$ is star-extendible. Since the orthogonal projections g_{ii} lie in C'_{n_1} , the range of φ_1 is actually in A' . In view of the remarks preceding Lemma 2.2, φ_1 has the form

$$\varphi_1 \left(\sum_{\Omega_1} a_{ij} e_{ij}^1 \right) = \sum_{\Omega_1} a_{ij} c_{ij} w_{ij}$$

where, for some large enough k , w_{ij} is a partial isometry which is a sum of some of the matrix units in $\{f_{ij}^k: (i, j) \in \Omega'_k\}$, and where $c_{ij} \in C'$ for all $(i, j) \in \Omega_1$. However, by the star-extendibility of φ_1 the set $\{c_{ij} w_{ij}: (i, j) \in \Omega_1\}$ is a subset of a complete matrix unit system. From this it follows that $\{\omega_{ij}: (i, j) \in \Omega_1\}$ is necessarily a subset of a complete matrix unit system. It can now be shown that $c_{ij} w_{ij} = c w_{ij} c^*$ for some partial isometry c in C' . So, replacing φ_1 by $(\text{Ad } c^*) \circ \varphi_1$, and replacing n_1 by k , we obtain the desired map φ_1 . It is regular because the images of the matrix units of A_1 lie in the normaliser of a masa.

We have obtained a regular star-extendible embedding $\varphi_1: A_1 \longrightarrow A'_{n_1}$ such that $[\varphi_1(e_{ii}^1)]S(A')[\varphi_1(e_{jj}^1)]$ for each matrix unit e_{ij}^1 in A_1 . We now construct the desired map $\psi_1: A'_{n_1} \longrightarrow A_{m_1}$. Choose orthogonal projections h_{ii} in C_{m_1} , for suitably large m_1 , so that $[h_{ii}] = \theta^{-1}([f_{ii}^{n_1}])$ for all i . We can do this in such a way so that, for each i , if $g_{ii} = \sum_{J_i} f_{ii}^{n_1}$ then $\sum_{J_i} h_{ii}$ coincides with $i(e_{ii}^1)$. In other words, the choice of the projections h_{ii} determines an injection $\omega: C'_{n_1} \longrightarrow C_{m_1}$ and we can arrange this so that $\omega \circ \varphi_1$ agrees with the given injection $i: C_1 \longrightarrow C_{m_1}$. By our earlier arguments, increasing m_1 if necessary, there is a strongly regular star-extendible embedding $\hat{\omega}: A'_{n_1} \longrightarrow A_{m_1}$ which extends ω . Because of the hypothesised conjugacy property there is a unitary element u in $A_{m_1} \cap A_{m_1}^*$ so that the map $q_1 = (\text{Ad } u) \circ \hat{\omega}$ is the desired injection from A'_{n_1} to A_{m_1} , with $q_1 \circ \varphi_1 = i$. We can now continue to obtain the desired system. ■

It will be noticed that the star realisation property is much stronger than is necessary for the proof of Theorem 3.1. The essential point is that any finite transitive reflexive subrelation of $S(A')$ (respectively $S(A)$) which is isomorphic to the relation for A_k (respectively A'_k) for some k , is star realisable.

THEOREM 3.2. *Let A and A' be limits of direct sums of triangular finite-dimensional nest algebras with respect to injective strongly regular star-extendible embeddings associated with ordered Bratteli diagrams (as in 3.8). Then A and A' are isometrically isomorphic if and only if there is a scaled group isomorphism $\theta: K_0(A) \longrightarrow K_0(A')$ which preserves the algebraic order.*

Proof. The proof above applies with simplifications. Firstly, note that the maps

φ_1, ψ_1, \dots are easily defined by specifying images for the superdiagonal matrix units (the matrix units of the first superdiagonal). Secondly, observe that ordered Bratteli diagram systems have the conjugacy property. Indeed, if $\phi: E \rightarrow F$ is an ordered Bratteli diagram embedding between triangular finite-dimensional nest algebras, and if e is a matrix unit in E , then in each summand of F the minimal subprojections of e^*e interlace those of $e\phi^*$. It follows that the partial isometries v in F with $e^*e \rightarrow v\phi\phi^*e$ agree modulo a multiplier of C . ■

REMARK 3.3. In general the algebraic order of a FD CSL algebra may not have the star realisation property. In fact more is true. There are FD CSL algebras A_1, A_2 and a scaled ordered group injection $\theta: K_0(A_1) \rightarrow K_0(A_2)$ with $\theta^{(2)}(\mathcal{R}(A_1)) \subseteq \mathcal{R}(A_2)$ which is not induced by any regular injective embedding. To see this consider the subalgebra A_2 of matrices (a_{ij}) in $M_4 \otimes M_2$ of the form

$$(a_{ij}) = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} & 0 & a_{17} & a_{18} \\ & a_{22} & 0 & a_{24} & 0 & a_{26} & a_{27} & a_{28} \\ & & a_{33} & 0 & 0 & 0 & a_{37} & 0 \\ & & & a_{44} & 0 & 0 & 0 & a_{48} \\ & & & & a_{55} & 0 & 0 & a_{58} \\ & & & & & a_{66} & a_{67} & 0 \\ & & & & & & a_{77} & 0 \\ & & & & & & & a_{88} \end{bmatrix}$$

Let $A_1 = T_2 \otimes T_2$ and consider the injection $\varphi: A_1 \cap A_1^* \rightarrow A_2 \cap A_2^*$ which is given by $c \rightarrow c \otimes I_2$. This in turn induces a map $\theta: \Sigma(A_1) \rightarrow \Sigma(A_2)$ with $\theta^{(2)}(\mathcal{R}(A_1)) \subseteq \mathcal{R}(A_2)$. Examination shows that there is no unital regular injection $A_1 \rightarrow A_2$, star-extendible or otherwise, which induces θ . We remark that in the case of infinite tensor products of proper finite-dimensional nest algebras the algebraic order contains arbitrary subrelations, and in particular subrelations isomorphic to $\theta^{(2)}(\mathcal{R}(A_1))$. Thus it does not seem that the methods of Theorem 3.1 are immediately applicable in the classification of infinite tensor products.

REMARK 3.4. If A and A' are isomorphic as Banach algebras then it can be shown that there is a scaled ordered group isomorphism from $K_0(A)$ to $K_0(A')$ which preserves the algebraic order (See [16]). As a consequence the algebraically ordered scaled ordered group $K_0(A)$ is a complete invariant for bicontinuous isomorphism within the classes subsumed by Theorems 3.1 and 3.2.

It seems plausible that any two canonical subalgebras of an AF C^* -algebra are isometrically isomorphic if they are algebraically isomorphic. Settling this problem will be a good test of the effectiveness of any future methods in the study of canonical subalgebras of AF C^* -algebras.

REMARK 3.5. There exist triangular canonical subalgebras A and A' , with $S(A) = S_1(A)$, with $S(A') = S_1(A')$, and with an algebraic order preserving isomorphism $\theta: K_0(A) \rightarrow K_0(A')$, preserving the algebraic orders, which are nevertheless not isometrically isomorphic. To see this let $B = \varinjlim(M_{2^k}, \rho_k)$, where $\rho_k: M_{2^k} \rightarrow M_{2^{k+1}}$, $k = 1, 2, \dots$ are refinement embeddings, and let B' be the subspace $\varinjlim(M'_{2^k}, \rho_k)$ where M'_{2^k} is the subspace of matrices with zero diagonal. Furthermore, let $D = \varinjlim(D_{2^k}, \rho_k)$ be the canonical diagonal subalgebra of B . Adopting a little notational distortion, define

$$A = \begin{bmatrix} D & B \\ 0 & D \end{bmatrix}, \quad A' = \begin{bmatrix} D & B' \\ 0 & D \end{bmatrix}.$$

These canonical subalgebras of $M_2 \otimes B$ are not isometrically isomorphic. This can be deduced from the fact that A and A' do not have topologically isomorphic fundamental relations. See [15]. We leave the verification of the other assertion as a simple exercise.

REMARK 3.6. Theorem 3.2 is not true if the embedding condition is relaxed. Let $A = \varinjlim(T_{2^n}, \rho)$ be the limit of upper triangular matrix algebras with respect to refinement embeddings, and let $A' = \varinjlim(T_{2^n}, \theta_n)$ be the limit algebra where $\theta_n(e_{ij}^n) = \rho(e_{ij}^n)$ if $j < 2^n$ or $(i, j) = (2^n, 2^n)$, and $\theta_n(e_{i, 2^n}^n) = e_{2^i, 2^{n+1}-1}^{n+1} + e_{2^i-1, 2^{n+1}}^{n+1}$, otherwise. These embeddings, in which the final column of matrix units is embedded with twisted orientation, are not strongly regular. Despite the fact that the binary relations $(\Sigma(A), S(A)), (\Sigma(A'), S(A'))$ are naturally isomorphic, A and A' are not isomorphic. This observation is essentially due to Peters, Poon and Wagner [9]. See also [15].

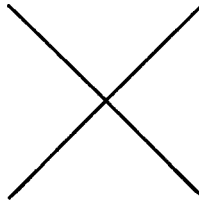
In [9] a partial order $<_A$ on $\text{Proj}(C)$ is defined by $p <_A q$ if and only if $q \rightarrow_v p$ for some v in $N_C(A)$. If A is triangular, so that $A \cap A^* = C$, then $<_A$ can be identified with $S(A)$. It is shown in [9], using the invariant $<_A$, that there are uncountably many isomorphism classes of limit algebras of the form $\varinjlim(T_2^k, \varphi_k)$. In [11] related invariants are exploited in the study of nest subalgebras. In particular it is shown that there are uncountably many nonisomorphic triangular nest algebras A in any given UHF algebra, all having the same trace invariant $\{\text{trace}(p): p \in \text{Lat } A\}$. Here $\text{Lat } A$ is the projection nest in C determining A .

REMARK 3.7. Baker [1] has shown that the unital limit algebras $\varinjlim(T_{n_k}, \sigma)$, associated with standard embeddings and the sequences (n_k) , with n_k dividing n_{k+1} for all k , are classified by their enveloping UHF C^* -algebras. In [9], [14] and [15] other proofs are given and the limit algebras $\varinjlim(T_{n_k}, \rho)$ are similarly classified. These standard limit algebras are special cases of the triangular algebras of Theorem 3.2.

REMARK 3.8. The limit algebras of Theorem 3.2 are determined by ordered

Bratteli diagrams as in the following discussion.

Consider, as an illustrative example, the two stationary direct systems $A = \varinjlim(T_{n_k} \oplus T_{n_{k+1}}, \theta_k)$, $A' = \varinjlim(T_{n_k} \oplus T_{n_{k+1}}, \psi_k)$ with the strongly regular embeddings $\theta_k(x \oplus y) = y \oplus (x \oplus y)$, and $\psi(x \oplus y) = y \oplus (y \oplus x)$ where $(n_k) = (1, 1, 2, 3, \dots)$ is the Fibonacci sequence. Then $C^*(A)$ and $C^*(A')$ are isomorphic, with stationary Bratteli diagram generated by the diagram



However, the ordered Bratteli diagrams representing A and A' are generated by the graphs



and A and A' are not isometrically isomorphic. The easiest way to see this is to note that there is a special point, x say, in the Gelfand space $M(A \cap A^*)$ with the following maximality property: if $y \neq x$ then there do not exist orthogonal projections p_y, p_x in $A_k \cap A_k^*$ for any k , such that $y(p_y) = 1, x(p_x) = 1$ and $p_y \rightarrow_{A_k} p_x$. This point is the intersection of the supports of the "right-most" minimal projections in the right summands of the A_k . An isometric isomorphism would transfer this property to $M(A' \cap A'^*)$, and it is easy to check that there is no such point.

The examples above fall into a class of limit algebras associated with what we call *standard ordered Bratteli diagrams*. These are the Bratteli diagrams for which at each vertex there is a specification of the order of the incident edges. Such a diagram, together with a specification of the size of the summands of A_1 , gives rise to a unital direct system $A_1 \rightarrow A_2 \rightarrow \dots$ in which each A_k is a direct sum of upper triangular matrix algebras. The resulting embeddings are in fact strongly regular. The examples

above illustrates the fact that these algebras are highly dependent on the specified orderings.

In a similar way one can consider ordered Bratteli diagrams for direct sums of general finite-dimensional nest algebras.

REMARK 3.9. It seems quite likely that Theorem 3.2 remains true without the assumption of triangularity. Unfortunately the proof of this given in [16] is incomplete because locally strongly regular maps (ones that map matrix units into the strong normaliser) are not necessarily strong regular.

4. FURTHER EXAMPLES

It is particularly interesting to calculate the algebraic orders for nontriangular subalgebras of AF C^* -algebras. In the examples below we have a canonical subalgebra A of an AF C^* -algebra $B = C^*(A)$, and we write $D = A \cap A^*$ for the diagonal subalgebra. By hypothesis, D contains a canonical masa of B , from which it follows that the inclusion $i: D \rightarrow B$ induces a surjection $i_*: K_0(D) \rightarrow K_0(B)$. We shall identify this map and the algebraic order on the scale $\Sigma(A) = \Sigma(D)$ for various examples.

EXAMPLE 4.1. The simplest example is finite-dimensional. Let $B = M_n$, $A = T(n_1, \dots, n_r)$, $D = A \cap A^* = M_{n_1} \oplus \dots \oplus M_{n_r}$ where $n = n_1 + \dots + n_r$, and A is the block upper triangular subalgebra of M_n associated with the ordered r -tuple n_1, \dots, n_r . Then $K_0(A) = \mathbf{Z}^r$ with scale $[0, n_1] \times \dots \times [0, n_r]$ (with the product order), and $(a_1, \dots, a_r)S(A)(b_1, \dots, b_r)$ if and only if $a_1 + \dots + a_r = b_1 + \dots + b_r$ and $b_k + \dots + b_r \geq a_k + \dots + a_r$ for $1 \leq k \leq r$. The map $i_*: \mathbf{Z}^r \rightarrow \mathbf{Z}$ is simply addition. Let \tilde{S} be the equivalence relation on $\Sigma(A)$ generated by $S = S(A)$. Then the sets $i_*^{-1}(x)$ for x in $\Sigma(B) = [0, n]$ are precisely the \tilde{S} equivalence classes.

EXAMPLE 4.2. In analogy with the last example let B be the UHF C^* -algebra associated with the generalised integer 2^∞ , let C be a canonical masa in B , and consider a finite nest $0 < p_1 < \dots < p_r = 1$ of projections in C , and its associated nest subalgebra $A = \{b \in B: (1 - p_j)bp_j = 0, 1 \leq j \leq r\}$. Let τ be the normalised trace on B and set $d_i = \tau(p_i - p_{i-1})$, $1 < i < r$. Then $K_0(B) = \mathbb{Q}_d$, the binary rationals, with the ordinary ordering, $K_0(A) = \mathbb{Q}_d^r$, and $\Sigma(A) = \mathbb{Q}_d^r \cap ([0, d_1] \times \dots \times [0, d_r])$. The algebraic order is exactly as in the finite-dimensional case, i_* is the addition map, and the fibres $i_*^{-1}(x)$ for x in $\Sigma(B)$ are the \tilde{S} equivalence classes. (This latter point is a general phenomenon.)

EXAMPLE 4.3. For a related example, let $B = \varinjlim (M_{2^k}, \rho)$, let A_k be the unital

subalgebra $T(n_k, 1, m_k) \subseteq M_{2^k}$ with $n_k 2^{-k} < \alpha < (n_k + 1) 2^{-k}$, for all k , where α is a fixed nondyadic point in $[0, 1]$. The refinement embeddings ρ restrict to strongly regular embeddings $\theta_k: A_k \rightarrow A_{k+1}$ and the limit algebra $A = \varinjlim(A_k, \theta_k)$ can be visualised as a subalgebra of B . We have $K_0(A) = \varinjlim K_0(A_k) = \varinjlim(\mathbb{Z}^3, (\theta_k)_*)$ where $(\theta_k)_*$ has the form

$$(\theta_k)_* = \begin{bmatrix} 2 & \delta_k & 0 \\ 0 & 1 & 0 \\ 0 & 1 - \delta_k & 2 \end{bmatrix}$$

where (δ_k) is a sequence of zeros and ones.

A direct argument can be given to show that $K_0(A)$ is the subgroup of $(\mathbb{Q}_d + \alpha\mathbb{Z}) \oplus (\mathbb{Q}_d + \alpha\mathbb{Z})$ consisting of pairs $a \oplus b$ with $a + b \in \mathbb{Q}_d$ and that $\Sigma(A)$ is the subset with $a \in [0, \alpha], b \in [0, 1 - \alpha]$. The algebraic order is given by $(a \oplus b)S(A)(c \oplus d)$ if and only if $a + b = c + d$ and $b \leq d$.

One can similarly compute $(\Sigma(A), S(A))$ for analogous algebras of the form

$$A = \varinjlim(T(n_{k,1}, \dots, n_{k,r_k}), \rho).$$

Notice that in all the examples above the diagonal algebra D has a certain block diagonal form. In contrast the examples below use more interesting embeddings which result in algebras for which D is simple. G. A. Elliott [5] and E. G. Effros and C. Y. Shen [3] have analysed the dimension groups of various stationary direct systems. Recall that the dimension group of a strictly positive stationary unimodular system is determined in terms of a distinguished Perron-Frobenius eigenvector for the matrix determining the system. We will not need detailed theory beyond this in the discussion below.

EXAMPLE 4.4. Let $A = \varinjlim(A_k, \lambda_k)$ where $A_k = T_2 \otimes M_{4^k}$ and $\lambda_k: A_k \rightarrow A_{k+1}$ is the regular embedding

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} x & & z & 0 \\ & x & & z \\ & & x & z \\ \hline & & y & 0 \\ & & & x \\ & & & & y \\ & & & & & y \\ & & & & & & y \end{array} \right]$$

where $x, y, z \in M_{4^k}$, and where unspecified entries are zero. Then $K_0(A) = K_0(D) =$

$= \mathbb{Q}_d^2$ since this is the limit of the stationary system

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}} \dots$$

Furthermore \mathbb{Q}_d^2 has the strict ordering from the first coordinate ($(a, b) \leq (c, d)$ if and only if $a < c$ or $(a, b) = (c, d)$) and the scale $\Sigma(A)$ is the order interval $[(0, 0), (1, 0)]$ (see [2, page 61]). The map $i_*: K_0(A \cap A^*) \rightarrow K_0(C^*(A))$ is $(a, b) \rightarrow a$ ($C^*(A)$ is the 2^∞ UHF algebra), and the algebraic order is such that $(a, b)S(A)(c, d)$ if and only if $a = c$ and $b \leq d$.

It is straightforward to check that the conjugacy property holds for the system above and, more generally, for systems over the algebras $T_2 \otimes B$ with B a finite-dimensional C^* -algebra. Thus it follows from Theorem 3.1, and the remark after the proof, that the associated limit algebras are classified by the algebraically ordered scaled K_0 group.

EXAMPLE 4.5. Consider the stationary system

$$T(1, 1) \oplus \mathbb{C} \xrightarrow{\mu} T(1, 2) \oplus M_2 \xrightarrow{\mu} T(1, 4) \oplus M_3 \xrightarrow{\mu} \dots$$

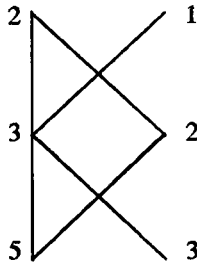
where

$$\mu: \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \oplus [w] \xrightarrow{\mu} \begin{bmatrix} x & 0 & z \\ 0 & w & 0 \\ 0 & 0 & y \end{bmatrix} \oplus \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$$

with direct limit $A = \varinjlim(A_k, \mu)$. Then $K_0(A) = K_0(A \cap A^*)$ is the limit of the stationary system

$$\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\dots}$$

The embedding matrix is in $GL(3, \mathbb{Z})$, and so $K_0(A) = \mathbb{Z}^3$. The enveloping C^* -algebra $B = C^*(A)$ has Bratteli diagram



and so $K_0(B)$ is \mathbf{Z}^2 and the surjection $K_0(A \cap A^*) \rightarrow K_0(B)$ can be identified with the map $(\ell, m, n) \rightarrow (\ell + m, n)$. The algebraic order is given by $(\ell, m, n)S(q, r, s)$ if and only if $n = s, \ell + m = q + r$ and $m \leq r$.

The positive cone, P_α say, of $K_0(A \cap A^*) = \mathbf{Z}^2$, is $\{(m, n) = m\alpha + n \geq 0\}$ where $\alpha = (1 + \sqrt{5})/2$, and this, in turn, can be viewed geometrically as the positive cone of the subgroup $\mathbf{Z}\alpha + \mathbf{Z}$ of \mathbf{R} . We now indicate how to identify the positive cone \mathcal{C} of $\mathbf{Z}^3 = \mathbf{Z} \oplus (\mathbf{Z}\alpha + \mathbf{Z})$ with the set $\{(\ell, m, n) : \ell \in \mathbf{Z}_+, m\alpha + n \in \ell(1 - \alpha) + P_\alpha\}$. Let p_k be the Fibonacci sequence so that p_{2k+1}/p_k decreases to α and p_k/p_{2k+1} increases to α . With $Y = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$Y^k = \begin{pmatrix} p_{k+1} & p_k \\ p_k & p_{k-1} \end{pmatrix}$$

$$X^{-k} = \begin{pmatrix} 1 & 0 & 0 \\ -p_k & & Y^{-k} \\ p_{k+1} & & \end{pmatrix}.$$

The point (ℓ, m, n) lies in the positive cone \mathcal{C} if and only if for some u, v, w in \mathbf{Z}_+ and for some odd integer $k, (\ell, m, n)^t = X^{-k}(u, v, w)^t$. In particular, $(1, m, n)$ lies in the cone if and only if

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} -p_k \\ p_{k+1} \end{pmatrix} + Y^{-k} \begin{pmatrix} v \\ w \end{pmatrix}$$

for some odd k and u, v in \mathbf{Z}_+ . Since the smallest value of $-p_k\alpha + p_{k+1}$ is $1 - \alpha$, and since $(0, m', n')$ lies in \mathcal{C} for all m', n' with $m'\alpha + n' \geq 0$, it follows that $(1, m, n)$ is a point of \mathcal{C} if and only if $m\alpha + n \geq (1 - \alpha)$. The desired description of \mathcal{C} now follows.

STATIONARY PAIRS OF AF C^* -ALGEBRAS.

The last example is a special case of the following very general scheme.

Let $X = (a_{ijkl})$ be an $n \times n$ matrix of nonnegative integers, where $1 \leq k \leq k_i, 1 \leq \ell \leq k_j, 1 \leq i \leq r, 1 \leq j \leq r$, and $k_1 + \dots + k_r = n$. Assume that for each pair i, j the

partial column sum

$$b_{ij} = a_{ij1\ell} + \dots + a_{ijk_i\ell}$$

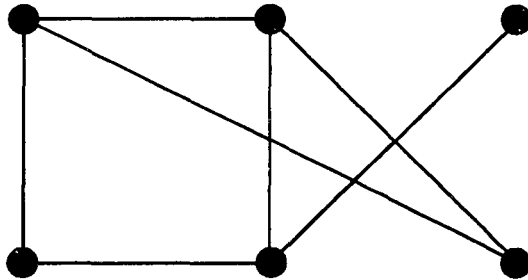
is independent of ℓ , and form the associated $r \times r$ matrix $Y = (b_{ij})$. We have the commuting square of group homomorphisms

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{X} & \mathbb{Z}^n \\ \downarrow S & & \downarrow S \\ \mathbb{Z}^r & \xrightarrow{Y} & \mathbb{Z}^r \end{array}$$

where S is the homomorphism associated with the partition of X , given by

$$(S\underline{x})_i = x_{\ell_i+1} + \dots + x_{\ell_{i+1}} \text{ where } \ell_1 = 0 \text{ and } \ell_{i+1} = \ell_i + k_i \text{ for } i = 1, \dots, r - 1.$$

For the stationary dimension groups $G_1 = \varinjlim(\mathbb{Z}^n, X), G_2 = \varinjlim(\mathbb{Z}^r, Y)$ we have the induced group homomorphism $S_\infty: G_1 \rightarrow G_2$. Choose order units \underline{u} in G_1 and $\underline{v} = S_\infty \underline{u}$ in G_2 and consider AF C^* -algebras D and B with $K_0(D) = G_1, K_0(B) = G_2$, with the chosen order units. Furthermore view D as a unital subalgebra of B so that the inclusion map $i: D \rightarrow B$ induces S_∞ ($i_* = S_\infty$). One way to visualise this inclusion is to form the stationary Bratteli diagram for X and to group together the summands associated with the partition of the n summands into r sets. In Example 4.5 this can be indicated by the following diagram



where the horizontal lines indicate the grouping. The partial summation condition above is precisely the condition needed so that we can enlarge the grouped summands to full matrix algebras, and extend the given embeddings to these matrix algebras, and so thereby obtain a stationary Bratteli diagram associated with Y .

We call the resulting pair of unital AF C^* -algebras $D \subseteq B$ a *stationary pair*. Actually it would be more precise to refer to the pair $D \subseteq B$ as a symmetrically partitioned pair, since the same partitioning is used for rows and columns. However

we restrict attention to this symmetric case and use the more relaxed terminology. Clearly D is a canonical subalgebra of B , and so too are all the closed intermediate algebras $D \subseteq A \subseteq B$. It should be noted that the pair $D \subseteq B$ is determined by the construction above, even though we make a choice of matrix units when we form a system $B_1 \rightarrow B_2 \rightarrow \dots$ which extends the system $D_1 \rightarrow D_2 \rightarrow \dots$.

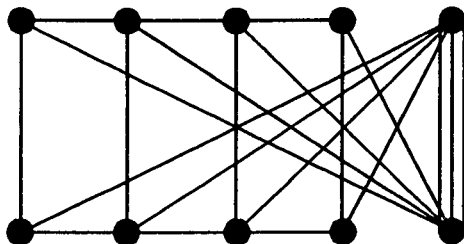
In the example above it is easy to see from the Bratteli diagrams that there are only two distinct proper intermediate algebras (namely Example 4.5 and its adjoint). The following simple pigeonhole argument shows that in general there are only a finite number of intermediate algebras. Let $B = \varinjlim B_n$ be the stationary unital direct system associated with the matrix Y (and a choice of order unit) so that $D = \varinjlim D_n$ is a direct system associated with X where $D_n \subseteq B_n$ for all n . Note that for each n there are exactly the same number of distinct proper intermediate algebras, E_n^1, \dots, E_n^s say, lying between D_n and B_n . Suppose that $D \subseteq E \subseteq B$. Then $E = \varinjlim E_n$, with $E_n = B_n \cap E$. If A^1, \dots, A^{s+1} are $s + 1$ such algebras, then there must exist distinct i and j so that $A^i \cap B_n = A^j \cap B_n$ for an infinity of values of n . Thus $A^i = A^j$. Simple examples reveal that the number of intermediate limit algebras can be strictly less than s .

We say that a stationary pair $D \subseteq B$ is *unimodular* if $X \in GL(n, \mathbf{Z})$. In this case Y is necessarily in $GL(r, \mathbf{Z})$ and so both the stationary systems for X and Y are unimodular in the usual sense. To see this form the matrix X_1 which is X but with the rows for $i = k_1, k_1 + k_2, \dots, n$ replaced by the associated partial columns sums. The entries in the new rows are the numbers b_{ij} and the determinant of X' is equal to that of X . Each term in the expansion of the determinants of X' along a fixed unchanged row is divisible by $\det Y$ (We can assume there is at least one such row otherwise $X = Y$ is either zero or has the determinant of Y as a divisor.) This can be seen more clearly if the new rows are moved by row operations to occupy the first r rows of a new matrix X^n , still with unimodular determinant. Since $\det Y$ divides $\det X$ the argument is complete.

EXAMPLE 4.6. Let X, Y be the matrices in $GL(5, \mathbf{Z})$ and $GL(2, \mathbf{Z})$ given by

$$X = \begin{bmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

so that Y and X are related as above (with $k_1 = 4, k_2 = 1$). Associated with the pair X, Y , and a choice of order units, is the stationary pair $D \subseteq B$ which has joint Bratteli diagram generated by the graph



A choice of order units corresponds to the specification of the size of the 5 matrix algebra summands of D_1 corresponding to the first row of the graph. $K_0(D) = \mathbf{Z}^5$ and $K_0(B) = \mathbf{Z}^2$ with positive cones $P(\underline{\alpha})$ and $P(\underline{\beta})$, respectively, determined by the eigenvectors $\underline{\alpha} = (1, 1, 1, 1, \alpha), \underline{\beta} = (1, \beta)$ for the maximal positive eigenvalues of X and Y . Thus $P(\underline{\alpha}) = \{a \in \mathbf{Z}^5 : (a, \underline{\alpha}) \geq 0\}, P(\underline{\beta}) = \{a \in \mathbf{Z}^2 : (a, \underline{\beta}) \geq 0\}$. These facts follow since Y is strictly positive and X has a strictly positive power. (See [3] for more detail.)

We now wish to describe all the intermediate algebras $D \subseteq A \subseteq B$. The lattice of such subalgebras is in fact a copy of the lattice of algebras lying between $M_4(\mathbf{C})$ and its diagonal subalgebra \mathbf{C}^4 . Indeed an algebra E between \mathbf{C}^4 and $M_4(\mathbf{C})$ is determined by a directed graph $G(E)$ on four vertices. Fixing an assignment of these vertices to the four summands of D_1 , which are grouped in B_1 , we can generate an intermediate algebra $D_1 \subseteq E_1 \subseteq B_1$ by including matrix units from B_1 to belong to E_1 if there is an associated directed edge in $G(E)$. The image of E_1 in B_2 generates, with D_2 , the analogous algebra E_2 and we obtain the intermediate algebra $\tilde{E} = \varinjlim E_k$. On the other hand if $D \subseteq A \subseteq B$ is a closed algebra then $A = \varprojlim (A \cap B_k)$, from which it follows that $A = \tilde{E}$ for some E . The map $E \rightarrow \tilde{E}$ provides a bijection of intermediate algebras.

Considering the special case $E = T_4$, and some ordering of the grouped vertices (for definiteness, take the order corresponding to rows of X), we obtain an intermediate algebra $A = \tilde{E}$ which is an inductive limit of finite-dimensional nest algebras. Whilst the embeddings $A \cap B_n \rightarrow A \cap B_{n+1}$ are not strongly regular, they are nevertheless regular and the direct system has the conjugacy property. Thus A is an example of the algebras appearing in Theorem 3.1. The algebraic order is given by $aS(A)b$ if and only if $a_5 = b_5, a_1 + \dots + a_4 = b_1 + \dots + b_4$, and $a_i + \dots + a_4 \leq b_i + \dots + b_4$ for $i = 1, 2, 3, 4$.

It can be shown that all of the intermediate algebras $\tilde{E} = \varinjlim E_k$ are defined by systems with the conjugacy property and have algebraic orders with the star

realisation property. Thus they also fall within the influence of Theorem 3.1. As with T_4 , the algebraic order of such an algebra \tilde{E} is rather simply related to the algebraic order of E .

Finally it is interesting to pause to consider the fundamental relation $R(A)$ of one of these intermediate algebras. Recall that if C is a canonical masa in A then $xR(A)y$, for x, y in the Gelfand space $M(C)$, if and only if there exist v in $N_C(A)$ such that $x(c) = y(v^*cv)$ for all c in C . It can be shown that in general $A = A^*$ if and only if $R(A)$ is symmetric (see [21] for example). In the case of the intermediate algebra $A = \tilde{T}_4$ the equivalence relation generated by $R(A)$ is $R(B)$. The fact that there is very little room between A and B is reflected in the observation that $R(B) \setminus R(A)$ is finite. (With the exception of at most four easily identified points x in $M(C)$, the $R(A)$ orbit of x and the $R(B)$ orbit of x agree.)

FINAL REMARK

Recently, the interesting preprints of Skau [19] and Herman, Putnam, and Skau [6] have appeared. In these it is shown, roughly speaking, how ordered Bratteli diagrams provide models for minimal homeomorphisms φ of Cantor spaces and their associated crossed products $C(X) \times_{\varphi} \mathbf{Z}$. This work rests, in part, on earlier work of Putnam [17] who showed that the C^* -algebra B_x generated by $C(X)$ and the elements fu , with u the canonical unitary, and f in $C(X)$ vanishing at x , is an AF C^* -algebra. In fact the tower construction in [17] generates an ordered Bratteli diagram with associated system $B_1 \rightarrow B_2 \rightarrow \dots$ of finite-dimensional C^* -algebras. It is not hard to check that if A is the semicrossed product $C(X) \times_{\varphi} \mathbf{Z}_+$, if $A_k = B_k \cap A$, and if $A(\varphi, x) = A \cap B_x$, then $A(\varphi, x) = \varinjlim A_k$ and $A(\varphi, x)$ is a triangular canonical subalgebra of B_x determined by a ordered Bratteli diagram. The converse assertion is true, and easier to establish: if A is a canonical triangular algebra determined by a ordered Bratteli diagram, then $A = A(\varphi, x)$ for some Cantor space homeomorphism and some point x of the Cantor space.

For more on this circle of ideas see [10], [16] and the recent preprint [12].

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