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CARTAN NEST SUBALGEBRAS OF HYPERFINITE FACTORS

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We say that a nest $\mathcal{N}$ of projections in a factor M is a Cartan nest if it is strongly closed and if the second commutant $\mathcal{N}''$ is a Cartan subalgebra in the sense of Feldman and Moore. In the II_1 case this means that $\mathcal{N}''$ is a maximal abelian self-adjoint subalgebra (masa) which is regular in the sense that the normalizer of $\mathcal{N}''$ in M generates M as a von Neumann algebra. In general one must add the condition that there exists a faithful normal expectation $M \to \mathcal{N}''$. Associated with $\mathcal{N}$ is the Cartan nest subalgebra $\mathcal{T}(\mathcal{N})$ consisting of the operators x in M such that $(1-p)xp = 0$ for p in $\mathcal{N}$. It will be shown that if there is an isometric weak star contionuous isomorphism

$$\mathcal{T}(\mathcal{N}_1) \otimes \ldots \otimes \mathcal{T}(\mathcal{N}_r) \to \mathcal{T}(\mathcal{M}_1) \otimes \ldots \otimes \mathcal{T}(\mathcal{M}_s)$$

where $\mathcal{N}_1, \ldots, \mathcal{N}_r$ and $\mathcal{M}_1, \ldots, \mathcal{M}_s$ are Cartan nests in hyperfinite factors (of any type) then $r = s$ and there is a permutation π such that $\mathcal{N}_i$ and $\mathcal{M}_{\pi(i)}$ are conjugate for all i. In a similar way we classify infinite tensor products, relative to the normalised product trace, of Cartan nest subalgebras of the hyperfinite II_1 factor. Furthermore, a general representation theorem for Cartan nests is obtained in terms of measured Borel equivalence relations on the unit interval, and the characterisation of Cartan nests up to conjugacy is obtained in terms of this spectral representation.

Our analisys is based, in part, on a result of independent interest which shows that if $\Phi : A_1 \to A_2$ is an isometric weak star continuous isomorphism between weakly closed subalgebras of hyperfinite factors, and if $\Phi(C_1) = C_2$ where $C_i \subseteq \subseteq A_i$, for $i = 1, 2$, are Cartan masas, then there is a unique extension isomorphism $\tilde{\Phi} : W^*(A_1) \to W^*(A_2)$. See [9] for a generalisation of this to nonhyperfinite factors.

Recall that Feldman and Moore [5], [6] have characterised Cartan subalgebras of factors (and general von Neumann algebras) in terms of measured equivalence relations R. In fact it is an elementary construction, which we describe in Section

], which attaches to each hyperfinite equivalence relation R, with quasi-invariant measure μ, a hyperfinite von Neumann algebra $M(R)$ on $L^2(R,\nu)$ (where ν is a measure constructed from μ), such that $M(R)$ naturally contains $L^\infty(X,\mu)$ as a Cartan masa. Here X is the measure space underlying R. Feldman and Moore showed that in the hyperfinite case all Cartan subalgebras arise in this way. In the general case, which we will not consider here, it is necessary to involve a 2-cocycle on R. We have endeavoured to make this paper reasonably self-contained and so we shall not rely too much on results established in [5] and [6]. An exception is that the Feldman Moore characterisation must be invoked to see that the specific Cartan nests that we construct constitute all possible Cartan nests in hyperfinite factors (Theorem 1.3).

Following on from the work of Feldman and Moore, in the direction of non-self--adjoint operator algebras, Muhly, Saito and Solel [10] have given an extensive analysis of maximal triangular subalgebras A, in factors, for which $A \cap A^*$ is a Cartan subalgebra. In particular they obtain a definitive characterisation of the isometric weak star continuous isomorphism classes of maximal triangular subalgebras ([10], Theorem 5.2]). There is some overlap between this result and our considerations of isometric isomorphisms, but our considerations relate to quite general subalgebras containing a Cartan subalgebra. Furthermore, since we restrict to the hyperfinite context, the cocycle considerations are absent, and elementary matricial arguments are at hand. In fact our analysis of isometric isomorphism is the natural hyperfinite Cartan bimodule generalization of arguments given in Davidson and Power [4].

With regard to the uniqueness of tensor product factorisations we point out that Arveson has studied this in the context of infinite tensor products of upper triangular matrix algebras [1, section 3]. More recently the second author has obtained unique factorisation for tensor products of certain irreducible triangular CSL algebras (reflexive algebras with commutative invariant subspace lattice) and for certain triangular subalgebras of approximately finite C^*-algebras [14]. See also [11]. However, the analysis below is independent of the deeper refinement theory used in [14] and rests on the structure of direct products of totally ordered measure spaces. Unique tensor product factorisation should be viewed as a topic which goes beyond the general identification of complete invariants, up to outer conjugacy, and addresses a more detailed analysis of indecomposable tensor product factorisation.

1. CARTAN NESTS

First we show how to construct Cartan nests.

A countable (standard) Borel equivalence relation on [0, 1] is an equivalence relation R, with countable equivalence classes, which is a Borel set in the product Borel σ-algebra on $[0, 1] \times [0, 1]$. Let $\pi_r(x,y) = y$ and $\pi_l(x,y) = x$ be the right and left co-ordinate projections and define the right counting measure ν_r on R, with the relative Borel structure, by

$$\nu_r(E) = \int \left|\pi_r^{-1}(x) \cap E\right| \mathrm{d}\lambda(x),$$

where λ is some σ-finite Borel measure on $[0, 1]$. Left counting measure ν_l is defined analogously. The integrands here are Borel functions of x. Also, if $A \subseteq [0, 1]$ is a Borel set then so too is its saturation

$$R(A) = \{\, y \, : \, (x,y) \in R \ \text{ for some } x \in A \,\}.$$

These two facts follow since π_r is countable-to-one (see [5], and [6]). We shall assume that the measures ν_r and ν_l are mutually absolutely continuous. This is equivalent to λ being quasi-invariant for R. This means that whenever $A \subseteq [0, 1]$ is a null Borel set then so is its saturation (see [5]).

We now define a von Neumann algebra on $L^2(R, \nu_r)$. Let $a(x,y)$ be a Borel function on R, with $|a(x,y)| \leqslant \alpha$ everywhere, such that for each x there is at most a single point $z = z(x)$ with $|a(x,z)| \neq 0$, and such that such points $z(x)$ are distinct. For ζ in $L^2(R, \nu_r)$ define $(L_a\zeta)(x,y) = \sum_z a(x,z)\zeta(z,y)$. Then

$$\begin{aligned}
\|L_a\zeta\|^2 = \int_R |(L_a\zeta)(x,y)|^2 \mathrm{d}\nu_r(x,y) = \\
= \int \sum_x |(L_a\zeta)(x,y)|^2 \mathrm{d}\lambda(y) = \\
= \sum_x \int |a(x,z(x))\zeta(z(x),y)|^2 \mathrm{d}\lambda(y) \leqslant \\
\leqslant \alpha^2 \sum_x \int |\zeta(z(x),y)|^2 \mathrm{d}\lambda(y) \leqslant \\
\leqslant \alpha^2 \|\zeta\|^2.
\end{aligned}$$

More generally, if $a(x,y)$ is a finite linear combination of functions of the above type then the associated operator L_a, similarly defined, is a bounded operator on $L^2(R, \nu_r)$, and the collection of these form a self-adjoint operator algebra. Let $M(R)$ be the von Neumann algebra generated by the operators L_a. (It can be shown ([6, Proposition 2.3]) that the functions $a(x,y)$ coincide with the class of *left finite functions*; the bounded functions $a(x,y)$ for wich there exists an integer n such that the cardinality of the support of the coordinate functions $a(\cdot, y)$ and $a(x, \cdot)$ is no greater than n for every x and y.)

A useful class of examples to bear in mind are the relations R_G arising from a countable subgroup G of the real line: $(x,y) \in R_G$ if and only if $x - y \in G$.

The following two conditions ensure that $M(R)$ is a hyperfinite factor:

(i) R is hyperfinite in the sense of being the union of a sequence $R_1 \subseteq R_2 \subseteq \ldots$ of finite Borel equivalence relations on [0,1]. (Recall that finiteness here means that there is an upper bound to the cardinality of each equivalence class.) This is easily seen to imply that $M(R)$ is the weakly closed union of the subalgebras $M(R_n)$. Since $M(R_n)$ is isomorphic to $M_k(\mathbb{C} \otimes L^\infty([0,1],\mu)$, for some k and some *sigma*-finite measure μ, it follows that each $M(R_n)$ is hyperfinite, and also $M(R)$ is also hyperfinite.

(ii) For every Borel set $D \subseteq [0,1]$ the saturation $R(D)$ is either null or conull. This is the condition that λ is ergodic with respect to R, and it ensures that $M(R)$ is a factor.

If, in addition, the Radon-Nikodym derivate $d\nu_r/d\nu_l$ is 1 almost everywhere, then the functional $\tau(L_a) = \int a(x,x)d\lambda(x)$ extends to a trace on $M(R)$, and so in this case $M(R)$ is the hyperfinite II_1 factor.

Let $C(R)$, or simply C, if there is no possibility of confusion, be the abelian subalgebra of $M(R)$ generated by the left finite operators L_a for which $a(x,y) = = \delta_{xy}\varphi(x)$ for some function φ in $L^\infty([0,1],\lambda)$. Then the linear map $\varphi \to L_a$ is a von Neumann algebras isomorphism from $L^\infty([0,1],\lambda)$ to C. It is elementary to see that the normalizer of C (the group $N_C(M(R))$ of unitaries u in $M(R)$ with $uCu^* = C$ generates $M(R)$, and that C is a masa in $M(R)$. Furthermore there is a faithful normal expectation from $M(R)$ onto C. (This is elementary in the II_1 case.) For these last two facts the reader should consult Feldman and Moore [6]. That C is a masa comes about throught the description of the commutant of $M(R)$ in terms of a von Neumann algebra generated by right finite operators.

We now know that C is a regular masa and the image of a faithful normal expectation, and so, by definition, C is a *Cartan subalgebra* of $M(R)$. In the hyperfinite case the characterization in [6] can be started as follows.

THEOREM 1.1. *Let A be a Cartan subalgebra of a hyperfinite factor M (of type* I, II *or* III*). Then there is a hyperfinite countable Borel equivalence relation R on* $[0,1]$ *with an ergodic quasi-invariant σ-finite Borel measure such that M is isomorphic to $M(R)$ by a von Neumann algebra isomorphism which carries A onto the canonical Cartan subalgebra C of $M(R)$.*

Apart from some technical aspects of countable standard relations, the verification of conditions (i), (ii), and the verification that $C \subseteq M(R)$ is a masa, we have given the details necessary for the construction of the pair $(M(R), C)$. On the other hand we have given no details of the proof of Theorem 1.1, and for these we refer the reader to [6].

In the next section we will make use of the following more elementary facts. Let R, S be equivalence relations on $[0,1]$, as in Theorem1.1, with quasi-invariant measures μ_R, μ_S, and right counting measures ν_R, ν_S respectively. If R and S are isomorphic, that is, if there is a Borel isomorphism τ of $[0,1]$ with $\mu_S \circ \tau$ and μ_R mutually absolutely continuous and $\tau \times \tau(R) = S$, ν_S-almost everywhere, then $M(R)$ and $M(S)$ are naturally isomorphic by an isomorphism Φ_τ which carries $C(R)$ onto $C(S)$. Conversely, if Φ is an isomorphism with this property, then there is a unitary element x in $C(S)$, and an isomorphism τ of R and S, such that $\Phi = Adx \circ \Phi_\tau$. For further details see [10].

DEFINITION 1.2. A *Cartan nest* in a factor is a strongly closed totally ordered family of projections which generates a Cartan subalgebra.

Let R, λ be as in Theorem 1.1 and define

$$\mathcal{N}(R) = \{L_a : a(x,y) = \delta_{x,y}\chi_{[0,t]}(x), 0 \leqslant t \leqslant 1\}$$

to be the nest in the Cartan subalgebra $C(R)$ of $M(R)$ associated with the intervals $[0,t]$, $0 \leqslant t \leqslant 1$. Clearly $\mathcal{N}(R)$ is a Cartan nest in $M(R)$. The next theorem shows that all Cartan nests in hyperfinite factors arise this way.

THEOREM 1.3. *Let $\mathcal{N}$ be a Cartan nest in a hyperfinite factor M (of any type). Then there is a pair R, λ, as in Theorem 1.1, together with a von Neumann algebra isomorphism from M to $M(R)$ which maps $\mathcal{N}$ onto $\mathcal{N}(R)$. Furthermore if M is of type II or III then λ can be taken to be Lebesgue measure.*

Proof: In the proof of the characterisation of Cartan subalgebras given in [6, Theorem 1] the first step is to identify the Cartan masa $C = \mathcal{N}''$ as $L^\infty(X, \lambda)$ where λ is a probability measure coming from a faithful normal state on C. The subsequent arguments obtain a standard Borel equivalence relation R so that the pair $(M(R), L^\infty(X,\lambda))$ is isomorphic to the pair (M, C). It suffices then to show that we can choose $X = [0,1]$ so that $\mathcal{N}$ appears as the standard projection nest in $L^\infty([0,1],\lambda)$. If $\mathcal{N}$ is not continuous then there exists an atomic interval $p - q$ with p, q in $\mathcal{N}$ which is necessarily a minimal projection of M since $\mathcal{N}''$ is a masa. It follows that M is the I_∞ factor in this cases. Cartan nests in I_∞ factors are necessarily purely atomic (see [3, Theorem8.6] for example) and the desired realisation of C and $\mathcal{N}$ is elementary in this case. On the other hand if $\mathcal{N}$ is continuous and σ is a faithful normal state on C then we obtain the desired realisation by considering $\Phi : C \to L^\infty([0,1],\lambda)$, where λ is Lebesgue measure, defined by $\Phi(p) = \chi_{[0,\sigma(p)]}$, for $p \in \mathcal{N}$, and by weak star linear extension.

We finish this section with some examples.

MATRICIAL NESTS. Let λ be Lebesgue measure on $[0,1]$ and let R_G be the countable Borel equivalence relation on $[0,1]$ associated with a countable subgroup G of the unit interval. Thus $(x,y) \in R_G$ if and only if $0 \leqslant x, y \leqslant 1$ and $y - x \in G$. Clearly λ is invariant for R_G and, by classical results of Dye, for example, $M(R_G)$ is the hyperfinite II_1 factor. If $G \subseteq \mathbb{Q}$ then the Cartan nest $\mathcal{N}(R_G)$ in $M(R_G)$ is is naturally the weakly closed union of finite subnests, and it is natural to make the following general definition of a matricial Cartan nest in a general factor M.

DEFINITION 1.4. A projection nest $\mathcal{N}$ in a hyperfinite factor M is called a *matricial nest* if there exist finite-dimensional subfactors $M_1 \subseteq M_2 \subseteq \ldots$ containing maximal projections nests $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots$ such that $\mathcal{N}$ is the weakly closed union of this chain of nests.

Although all Cartan subalgebras of the hyperfinite II_1 factor are conjugated by an automorphism, for Cartan nests we see in Corollary 2.4 that there is an extreme form of nonconjugacy. For example using this corollary it can be shown that the Cartan nests $\mathcal{N}(R_{G_1})$ and $\mathcal{N}(R_{G_2})$ are conjugate by an automorphism of the hyperfinite II_1 factor if and only if $G_1 = G_2$. This is no surprise since in many contexts in no-self-adjoint operator theory it happens that triangular algebras can serve as complete invariants for underlying relations or homeomorphism used in their construction. (We have in mind results in [2], [10], and [13], for example.)

Let $\mathcal{N}$ be a matricial Cartan nest in the hyperfinite II_1 factor M. Then M is the weakly closed union of finite-dimensional factors $M_1 \subseteq M_2 \subseteq \ldots$, with unital inclusions, and $\mathcal{N}$ is the weakly closed union of the finite nests $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots$. Let $B \subseteq M$ be the UHF C^*-algebra generated by $M_1, M_2, \ldots$. Then M is the weak closure of B and $\mathcal{T}(\mathcal{N})$ is the weak closure of the nest subalgebra $A = B \cap \mathcal{T}(\mathcal{N}) = B \cap \operatorname{Alg}\mathcal{N}_0$, where $\mathcal{N}_0$ is the union of the finite nests. There are many examples of matricial Cartan nests other than the examples $\mathcal{N}(R_G)$ above with $G \subseteq \mathbb{Q}$. Perhaps the simplest example is the following. Let T_n be the algebra of upper triangular matrices, let $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and define

$$B = \varinjlim(M_{2^n}, \varphi_n) \quad , A = \varinjlim(T_{2^n}, \varphi_n) \quad ,$$

where

$$\varphi_n((a_{ij})) = (a_{ij}U^{j-i}).$$

Let M be the weak closure of B in the standard left representation on $L^2(B,\tau)$, with τ the normalised trace, and let $\mathcal{N}$ be the weak closure of the (countable) nest of invariant projections for A. Then $\mathcal{N}$ is a Cartan nest, and it is straightforward to identify the associated Borel equivalence relation on $[0,1]$. (It is the union of the graphs of the partial homeomorphisms associated with the natural matrix unit system for B.) Corollary 2.3. shows that for any G the nests $\mathcal{N}$ and $\mathcal{N}(R_G)$ are not conjugate.

We should remark, however, that if $\mathcal{N}$ and $\mathcal{N}'$ are matricial Cartan nests in the hyperfinite II_1 factor, with M, $\mathcal{N}_0$, B, A and M', $\mathcal{N}_0'$, B', A' as above, then it can happen that $\mathcal{N}$ and $\mathcal{N}'$ are conjugate in M, even though $\mathcal{N}_0$ and $\mathcal{N}_0'$ are not equivalent by a $*$-automorphism $B \to B'$. Thus the classification of matricial Cartan nests is courser than the classification of their associated countable subnests in their UHF C^*-algebras. This is because the subnests are classified by their topological (fundamental) relation (by [13]) whereas the Cartan nests themselves are classified by their Borel relation on $[0,1]$ (by Corollary 2.3 below). This is most simply illustrated by considering the matricial Cartan nest $\mathcal{N}$ associated with

$$B = \varinjlim(M_{2^n}, \theta_n), \quad A = \varinjlim(T_{2^n}, \theta_n),$$

where $\theta(e_{ij}) = e_{ij} \otimes U$ if $j = 2^n$ and $(i,j) \neq (2^n, 2^n)$, and $\partial(e_{ij}) = e_{ij} \otimes I$ otherwise.

There is a natural von Neumann algebra isomorphism Φ of the hyperfinite II_1 factor such that $\Phi(\mathcal{N}) = \mathcal{N}(R_G)$, where G is the dyadic subgroup, because computation shows that $R_G = R(\mathcal{N})$ almost everywhere. Nevertheless the subnests are not conjugate within the associated UHF algebra. (For this see [12], where these twisted embeddings were first discussed in the C^*-algebra context, or see [13].)

There are many natural questions arising for the Cartan nests and their algebras. We mention the following.

Question 1. If $\mathcal{N}$ is a matricial Cartan nest, then is it equal to the weakly closed union of a chain $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots$, as above, with the additional property that the normaliser of $C^*(\mathcal{N}_k)$ in M_k is contained in the normaliser of $C^*(\mathcal{N}_{k+1})$ in M_{k+1} for all k? (This condition is a sufficient condition for the matricial nest to be a Cartan nest.)

Question 2. Which of the nests $\mathcal{N}(R_G)$ fail to be matricial nests?

2. EXTENDING ISOMETRIC ISOMORPHISMS

Let λ be Lebesgue measure on $[0,1]$ and let R be a hyperfinite countable ergodic Borel equivalence relation for which λ is quasi-invariant, as in Theorem 1.1. We pass over the I_∞ case since the needed modifications will be clear. Let R be the union of the finite Borel equivalence relations $R_n, n = 1, 2, \ldots$. Each of the subalgebras $M(R_n)$ is a type I von Neumann algebra and admits a representation

$$M(R_n) = (M_{k(n,1)} \otimes C_{n,1}) \oplus \ldots \oplus (M_{k(n,r_n)} \otimes C_{n,r_n})$$

where

$$C = (D_{k(n,1)} \otimes C_{n,1}) \oplus \ldots \oplus (D_{k(n,r_n)} \otimes C_{n,r_n})$$

is the direct sum decomposition of the masa C corresponding to the Borel partition of $[0,1]$ into the r_n sets where the R_n-orbits have the same finite cardinality $k(n,1),\ldots,k(n,r_n)$. Similarly, if $A_n \subseteq M(R_n)$ is a (non-self-adjoint) subalgebra containing C, then $W^*(A_n)$ is a type I subalgebra of the form $M(S_n)$, for some finite equivalence relation, and it follows that A_n admits a decomposition

$$A_n = (A_{n,1} \otimes C_{n,1}) \oplus \ldots \oplus (A_{n,s_n} \otimes C_{n,s_n})$$

where each $A_{n,i}$ is an irreducible subalgebra of a full complex matrix algebra containing the diagonal subalgebra for $1 \leqslant i \leqslant s_n$. In other words each $A_{n,i}$ is an irreducible finite-dimensional CSL algebra.

If $A \subseteq M(R)$ is a weakly closed subalgebra containing the Cartan masa C then, by the spectral theorem for bimodules, ([10],[8]) A is the weakly closed union of the subalgebras $A_n = A \cap M(R_n), n = 1,2,\ldots$. These subalgebras have the form just mentioned, and so are amenable to analysis by finite-dimensional methods.

LEMMA 2.1. *Let A be a finite-dimensional* CSL *algebra in M_n, let D be a masa on the separable Hilbert space H, let C be a Cartan subalgebra of the factor M, and let $\Phi : A \bar{\otimes} D \to M$ be an isometric weak star continuous algebra injection with $\Phi(D_n \bar{\otimes} D) = C$, where D_n is the diagonal subalgebra of M_n. Then there is a unique star extension $\tilde{\Phi} : C^*(A) \otimes D \to M$.*

Proof: The proof of this lemma is a variation of an argument used in Davidson and Power [3] where isometric isomorphisms of non-self-adjoint algebras were considered in a various situations, including the most basic context of finite-dimensional CSL algebras. It was noted there that if the unitary operator $U = (u_{ij})$ in $M_k(\mathbf{C})$ has the form

$$U = \begin{bmatrix} 0 & u_1 & 0 & . & 0 \\ 0 & 0 & u_2 & & \\ . & & . & & \\ . & & & & \\ 0 & & & & u_{k-1} \\ u_k & 0 & & & 0 \end{bmatrix}$$

with unimodular entries $u_1,\ldots,u_k$, and if S is the subspace of M_k consisting of matrices whose supports are contained in the support of $(v_{ij}) = V = I + U$, then the Schur product map $\Psi_V : S \to S$ given by $(a_{ij}) \to (a_{ij}v_{ij})$ is isometric if and only if $u_1u_2\ldots u_k = 1$. Similarly, if $u_1,\ldots,u_k$ are unitary operators on a separable Hilbert space H, then the associated Schur product map from S to $S \otimes \mathcal{L}(H)$ is isometric if and only if $u_1u_2\dots u_k = I_H$. Indeed, reduce to the case with $u_1 = u_2 = \dots = u_{k-1} = 1$, by conjugation with a block diagonal unitary, and obtain this case by considering the spectral representation of u_k.

To apply the above, consider the isometric representation $\Psi : A \to M$ given by $\Psi = \Phi \circ \delta$ where $\delta : A \to A \otimes D$ is the canonical injection. This map is an inflated Schur product map of the form $\Psi : (a_{ij}) \to (a_{ij}u_{ij})$ where $u_{ij} = \Psi(e_{ij})$ are the images of the matrix units e_{ij} which span A. Let $\{ e_{ij} : (i,j) \in T \}$ be the set of these matrix units. It will be sufficient to show that there exist partial isometries v_i, belonging to the normalizer of C in M, such that $u_{ij} = v_iv_j^*$ for all (i,j) with e_{ij} in S. Clearly we may assume that A is irreducible.

First we observe that the operators u_{ij} are partial isometries in the normaliser of C. Fix $x = u_{ij}$. There is an identifying isomorphism from $\Phi(e_{ii} \otimes D)$ to $\Phi(e_{jj} \otimes D)$ given by $\Phi \circ \mathrm{Ad}(e_{ij} \otimes 1) \circ \Phi^{-1}$. Let $E = \Phi(e_{ii} \otimes D)$ and identify $\Phi(e_{jj} \otimes D)$ with E by this isomorphism. Since Φ is isometric we obtain $\|e_1xe_2\| = \|e_1e_2\|$ for all projections e_1, e_2 in E. Since E is maximal abelian self-adjoint algebra in $\Phi(e_{ii} \otimes D)M\Phi(e_{ii} \otimes D)$, it follows that x is identified with a unitary element of E. Thus u_{ij} normalises C.

By our earlier remarks, since Ψ is isometric we know that

$$u_{i_1j_1}u^*_{i_2j_1}u_{i_2j_2}\dots u_{i_kj_k}u^*_{i_1j_k} = I$$

whenever there is a cycle of indices $(i_1j_1), (i_2j_1), \dots (i_1j_k)$ in T. Fix $v_1 = u_{11}$ and define $v_j^* = u_{1j}$ whenever $(1,j) \in S$. Continue, defining $v_i = u_{ij}v_j$, if $(i,j) \in S$ and v_j is defined, and let $v_j^* = v_i^*u_{ij}$ if $(i,j) \in S$ and v_i is defined. The cyclic identity above is precisely the condition needed to ensure that $v_1, \dots, v_k$ are well-defined and have the desired property. By the irreducibility of A all these elements will be defined and so the lemma follows. ∎

THEOREM 2.2. *Let A^1 and A^2 be weakly closed subalgebras of hyperfinite factors, containing the Cartan subalgebras C^1 and C^2 respectively, and let $\Phi : A^1 \to A^2$ be an isometric weak star continuous algebra isomorphism with $\Phi(C^1) = = C^2$. Then there exists a unique weak star continuous star extension $\tilde{\Phi} : W^*(A^1) \to \to W^*(A^2)$.*

Proof: By our earlier discussion A^1 is the weakly closed union of algebras A^1_n, $n = 1, 2, \dots$, each of which is the direct sum of algebras of the form $A \otimes D$, as in Lemma 2.1. It follows that Φ has a (unique) star extension $\tilde{\Phi}$ to the star subalgebra B^1_∞ which is the union of subalgebras $B^1_n = W^*(A^1_n), n = 1, 2, \dots$. It now follows from hyperfiniteness that $\tilde{\Phi}$ has a unique weak star continuous extension to the weak closure, which gives us the desired extension. Indeed let $E^1_n : W^*(A^1) \to B^1_n$ and $E^2_n : W^*(A^2) \to B^2_n$ be the conditional expectations onto B^1_n and $B^2_n = \tilde{\Phi}(B^1_n)$. Let

x_α be a net in the unit ball of B^1_∞ with the weak star limit 0. We have $E^2_n(\tilde{\Phi}((x_\alpha)) = \tilde{\Phi}(E^1_n(x_\alpha))$ and so $\lim_\alpha E^2_n(\tilde{\Phi}((x_\alpha))) = 0$ for each n. Thus if y is a weak star limit point of the net $\tilde{\Phi}(x_\alpha)$ it follows that $E^2_n(y) = 0$ for all n. Since $W^*(A^2)$ is the weakly closed union of the algebras B^2_n this means that $y = 0$. Thus if $\lim_\alpha x_\alpha = 0$ then $\lim_\alpha \tilde{\Phi}(x_\alpha) = 0$, and so $\tilde{\Phi}$ has an extension to the weak closure, as desired. ∎

COROLLARY 2.3. *Let $\mathcal{N}(R)$ and $\mathcal{N}(S)$ be Cartan nests for the hyperfinite Borel equivalence relations R, S on $[0,1]$ with ergodic quasi-invariant σ-finite measures λ_1, λ_2 respectively. Then $\mathcal{N}(R)$ and $\mathcal{N}(S)$ are conjugate if and only if there is an order preserving Borel isomorphism $\tau : [0,1] \to [0,1]$ such that $\lambda_2 \circ \tau$ and λ_1 are mutually absolutely continuous and $\tau \times \tau(R) = S$.*

COROLLARY 2.4. *Let $M(R)$ and $M(S)$ be realisations of the hyperfinite* II_1 *factor arising from hyperfinite Borel equivalence relations on $[0,1]$ which have Lebesgue measure as invariant measure. Then the Cartan nests $\mathcal{N}(R)$ and $\mathcal{N}(S)$ are conjugate if and only if $R = S$ almost everywhere with respect to left counting measure.*

COROLLARY 2.5. *Let $M(R)$ be a hyperfinite factor associated with the measured Borel equivalence relation $([0,1], R, \lambda)$ as in Theorem 1.1, and let S_1, S_2 be Borel subsets of R which are reflexive, antisymmetric, transitive and contain the diagonal Δ (perhaps after deletion of a ν_r-null set). Then the associated non-self-adjoint subalgebras $A(S_1)$ and $A(S_2)$ are isometrically weak star isomorphic if and only if there is a Borel isomorphism $\tau : [0,1] \to [0,1]$, with $\lambda \circ \tau$ and λ mutually absolutely continuous, such that $\tau \times \tau(S_1) = S_2$ almost everywhere.*

Notice that Corollary 2.4. also serves to classify Cartan nest subalgebras of the hyperfinite II_1 factor up to conjugacy, simply because the nests are conjugate if and only if the algebras are conjugate.

The proofs of the Corollaries follow quickly from Theorem 2.2 and the remarks preceding Definition 1.2. In Corollary 2.4 τ is necessarily the trivial Borel isomorphism of $[0,1]$ since it is both trace preserving and order preserving. The crucial point that τ is order preserving, in Corollary 2.3, is simply a consequence of the fact that there must be an induced bijection between the Cartan nests.

A different approach to Corollary 2.3 and Corollary 2.4, which are weaker than Corollary 2.5, follows from the analysis of maximal triangular algebras of Muhly, Saito and Solel. The assumptation of maximal triangularity there means that the arguments of Lemma 2.1 are not needed.

3. PRODUCTS OF CARTAN NESTS

We use the main result of the last section, together with an elementary unique direct product factorisation theory for totally ordered measure algebras, to obtain the following two unique factorisation results, whose proofs are presented later in this section.

THEOREM 3.1. *Let $\mathcal{N}_1, \ldots, \mathcal{N}_r$ and $\mathcal{M}_1, \ldots, \mathcal{M}_s$ be Cartan nests in hyperfinite factors of any type (except I_1) and suppose that the product nest algebras $\mathcal{T}(\mathcal{N}_1) \otimes \otimes \ldots \otimes \mathcal{T}(\mathcal{N}_r)$ and $\mathcal{T}(\mathcal{M}_1) \otimes \ldots \otimes \mathcal{T}(\mathcal{M}_s)$ are isomorphic by an isometric weak star continuous isomorphism. Then $r = s$, and there is a permutation π such that $\mathcal{N}_i$ and $\mathcal{M}_{\pi(i)}$ are conjugate for $1 \leqslant i \leqslant r$.*

THEOREM 3.2. *Let $\mathcal{N}_1, \mathcal{N}_2, \ldots$ be Cartan nests in the hyperfinite II_1 factor R, and let $A = \mathcal{T}(\mathcal{N}_1) \otimes \mathcal{T}(\mathcal{N}_2) \otimes \ldots$ be the associated weakly closed subalgebra of $R = R \otimes R \otimes \ldots$ (with product trace as normalized trace). If α is an isometric weak star continuous automorphism of A then there is a permutation π which permutes conjugate Cartan nests, an associated automorphism β_π of R, and an isometric weak star continuous automorphism α_i of $\mathcal{T}(\mathcal{N}_i)$ such that $\alpha = (\alpha_1 \otimes \alpha_2 \otimes \ldots) \circ \beta_\pi$.*

Let $(X, P) = \prod_{i=1}^{\infty} (X_i, P_i)$ be the direct product lattice obtained from the total orderings P_i on the sets X_i, for $i = 1, 2, \ldots$. If $[x, y]$ is an order interval in (X, P) then we say that an *axis* for $[x, y]$ is a totally ordered subinterval which is maximal in the sense of set inclusion. (Note that in order for a subinterval to be totally ordered, its members can differ in only one coordinate.) Furthermore we define the dimension of $[x, y]$ to be the number of distinct axes it contains. Plainly $[x, y]$ has dimension one if x and y differ in exactly one coordinate, and it is easily checked that these are the only intervals of dimension one. The point of this discussion is that the dimension of an interval is preserved under order isomorphism. Using this we can obtain.

LEMMA 3.3. *Let $(X, P) = \prod_{i=1}^{\infty} (X_i, P_i)$ and $(Y, Q) = \prod_{j=1}^{\infty} (Y_j, Q_j)$ be products of total orders, and let $\alpha : (X, P) \to (Y, Q)$ be an order isomorphism. Then there is a permutation π and order isomorphisms*

$$\alpha_i : (X_i, P_i) \to (Y_{\pi(i)}, Q_{\pi(i)}) \quad , \quad i = 1, 2, \ldots,$$

such that $\alpha = \alpha_1 \times \alpha_2 \times \ldots$.

Proof: Consider a point $x = (x_i)$ in X with image $y = (y_i)$ in Y. If we fix k and set $x' = (x'_i)$ where $x'_i = x_i$ for $i \neq k$ and x'_k is strictly greater than x_k, then $[x, x']$ is a one-dimensional interval. Hence $[y, \alpha(y')]$ is a one-dimensional interval, from which it follows that y' and $\alpha(y')$ differs in only one coordinate. It is elementary now to use this to show that α has the desired form. ■

LEMMA 3.4. *Let $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_s$ be nonzero finite Borel measures on $[0,1]$, none of which is a point mass, and each of which is either purely atomic or nonatomic. Let τ be a measure space isomorphism from $([0,1]^r, \lambda_1 \times \ldots \times \lambda_r)$ to $([0,1]^r, \mu_1 \times \ldots \times \mu_s)$ such that τ is order preserving after restriction to some conull set. Then $r = s$ and there is a permutation π, and order preserving measure space isomorphism $\tau_i : ([0,1], \lambda_i) \to ([0,1], \mu_{\pi(i)})$, for $1 \leqslant i \leqslant r$, such τ is the product of the maps τ_i.*

Proof: Let B, B' be the Borel measure algebras associated with $\lambda_1 \times \ldots \times \lambda_r$ and $\mu_1 \times \ldots \times \mu_s$ respectively. Consider the subset $X \subseteq B$ consisting of classes $[E]$ determined by sets E of the form $E(t_1, \ldots, t_r) = [0, t_1] \times \ldots \times [0, t_r]$, for $0 \leqslant t_i \leqslant 1, 1 \leqslant i \leqslant r$. Observe that X carries a natural partial ordering, P say. Furthermore if $X_i \subseteq X$ is the subset of classes associated with the sets $E(t_1, \ldots, t_r)$ for which $t_j = 1$ for all $j \neq i$, then X_i is totally ordered by P. With these total orderings, P_i say, we can identify (X, P) with the direct product $(X_1, P_1) \times \ldots \times (X_r, P_r)$. Similarly we can identify the analogous subset $Y \subseteq B'$ for $\mu_1 \times \ldots \times \mu_s$ associated with the classes of sets $F(t_1, \ldots, t_r) = [0, t_1] \times \ldots [0; t_r]$, with its partial ordering Q, as a direct product $(Y_1, Q_1) \times \ldots \times (Y_r, Q_r)$.

The hypotheses on τ ensure that the map $\alpha : [E(t_1, \ldots, t_r)] \to [\tau(E(t_1, \ldots, t_r))]$ is an order isomorphism from X to Y. Moreover, except possibly a null set, we have $\alpha([E(t_1, \ldots, t_r)]) = [F(\tau(t_1, \ldots, t_r))]$. By Lemma 3.3. $r = s$ and there are order isomorphisms $\alpha_i : X_i \to Y_{\pi(i)}$, for all $1 \leqslant i \leqslant r$ and a fixed permutation π such that $\alpha = \alpha_1 \times \ldots \times \alpha_r$. For notational convenience we assume that π is trivial. If λ_i is nonatomic then μ_i is also nonatomic and we may assume that λ_i and μ_i have support $[0,1]$. In this case we can define a point realisation of α_i by α_i itself, so that $[F(1, \ldots, \alpha_i(t_i), 1, \ldots 1)] = \alpha_i([E(1, \ldots, t_i, 1, \ldots, 1)])$. On the other hand if λ_i is purely atomic then its support is a countable set, and we can identify X_i with this countable set. In this case Y_i is necessarily countable and α_i can be viewed as an order isomorphism from $X_i \subseteq [0,1]$ to $Y_i \subseteq [0,1]$. We have shown then, that almost every $(t_1, \ldots, t_r)$ in $[0,1]^r$, with the understanding that $t_i \in X_i$ in the discrete case, we have

$$\begin{aligned}
&[F(\tau(t_1, \ldots, t_r))] = \alpha([E(t_1, \ldots, t_r)]) = \\
&= \alpha_1([E(t_1, 1, \ldots, 1)]) \wedge \ldots \wedge \alpha_r([E(1, \ldots, 1, t_r)]) = \\
&= [F(\alpha_1(t_1), 1, \ldots, 1)] \wedge \ldots \wedge [F(1, \ldots, 1, \alpha_r(t_r))] =
\end{aligned}$$

$$= [F(\alpha_1(t_1), \ldots, \alpha_r(t_r))].$$

It follows that $\tau = \alpha_1 \times \ldots \times \alpha_r$ as a measure algebra isomorphism, as desired. ∎

In an exactly similar way we can use Lemma 3.3 to obtain the following lemma, which is needed for Theorem 3.2.

LEMMA 3.5. *Let τ be a measure preserving measure space automorphism of $([0,1]^\infty, \lambda \times \lambda \times \ldots)$, where λ is Lebesgue measure, such that after restriction to a conull set, τ preserves the product order. Then τ is a permutation automorphism.*

Proofs of Theorem 3.1. and 3.2. Let Φ be an isometric weak star continuous isomorphism from $A^{(1)} = \mathcal{T}(\mathcal{N}_1) \otimes \ldots \otimes \mathcal{T}(\mathcal{N}_r)$ to $A^{(2)} = \mathcal{T}(\mathcal{M}_1) \otimes \ldots \otimes \mathcal{T}(\mathcal{M}_r)$. By Theorem 2.2 there exists a star extension $\tilde{\Phi}$ from $W^*(A^1)$ to $W^*(A^2)$ which maps the Cartan masa $C(R_1) \otimes \ldots \otimes C(R_r)$, say, for A^1, onto the Cartan masa $C(S_1) \otimes \ldots \otimes C(S_s)$, say, of A^2. By the remarks preceeding Definition 1.2, there is a Borel isomorphism $\tau : ([0,1]^r, \lambda_1 \times \ldots \times \lambda_r) \to ([0,1]^s, \mu_1 \times \ldots \times \mu_s)$ with $\tau^{(2)}(R_1 \times \ldots \times R_r) = S_1 \times \ldots \times S_s$ almost everywhere with respect to left counting measure, such that $\tilde{\Phi} = \mathrm{Ad}x \circ \Phi_\tau$, where Φ_τ is the isomorphism implemented by τ, and where x is a unitary in the Cartan masa of A^2. Let $R_i^+ = R_i \cap \{(x,y) : : x \leqslant y, 1 \leqslant i \leqslant r\}$, and similarly define the sets $S_i^+, 1 \leqslant i \leqslant s$. Then A^1 can be viewed as the subalgebra of $W^*(A^1)$ with support $R_1^+ \times \ldots \times R_r^+$, and in view of the fact that $\tilde{\Phi}(A^1) = A^2$ it follows that $\Phi_\tau(A^1) = A^2$, and, by a simple application of the spectral theorem for bimodules ([9], [7]), that $\tau^{(2)}(R_1^+ \times \ldots \times R_r^+) = S_1^+ \times \ldots \times S_s^+$, almost everywhere. Let $\lambda = \lambda_1 \times \ldots \times \lambda_r, \mu = \mu_1 \times \ldots \times \mu_s$. In view of the definition of counting measure it follows that there is a λ-conull set N such that $\tau^{(2)}((N \times N) \cap \cap (R_1^+ \times \ldots \times R_r^+)) = (M \times M) \cap (S_1^+ \times \ldots \times S_s^+)$ where $\tau^{(2)}(N) = M$. We are not yet in a position to apply Lemma 3.4 to factorise τ, since we have not shown that τ preserves the product order on a conull set. Indeed, as the lemma indicates, this property is equivalent to the factorability of τ. However we know that Φ induces a projection lattice isomorphism from $\mathrm{Lat}A^{(1)}$ to $\mathrm{Lat}A^{(2)}$, the invariant projection lattices of $A^{(1)}$ and $A^{(2)}$, namely $\mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_r$ and $\mathcal{M}_1 \otimes \ldots \mathcal{M}_s$ respectively. The classes $[E(t_1, \ldots, t_r)]$ correspond precisely to join-irreducible projections of the lattice (see [14] for example), and so Φ, which is implemented by τ, effects a map from (X, P) to (Y, Q), namely

$$[E(t_1, \ldots, t_r)] \to [F(\tau(t_1, \ldots, t_r))].$$

The proof of Lemma 3.4 gives the desired factorisation.

The proof of Theorem 3.2 is similar, using Lemma 3.5 in place of Lemma 3.4. ∎

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