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ON RESONANCES OF GENERALIZED N-BODY STARK HAMILTONIANS

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1. INTRODUCTION

The resonances of Schrödinger operators in a weak homogeneous electric field (i.e. Stark effect) have been studied by many authors. See for example [8], [9], [11], [12] and the references quoted there. In particular, precise exponential estimates on resonances and resonant states are obtained in [11] and [12] for *atomic type* N--body problems with Stark effect. However it seems difficult to directly apply the methods utilized by the both authors to the regular N-body problem. Remark that in the later case, the existence and the stability of resonances in Stark-effect have been established by Herbst-Simon in [8] by means of analytic dilation. On the other hand, we have given in [12] precise location of the essential spectrum of distorted Stark Hamiltonians. Thus a natural question is to estimate the number of the discrete spectrum of distorted Stark Hamiltonians, which are by definition, the resonances of the original Stark Hamiltonian. The purpose of this work is to give a unified treatment of Stark resonances, to establish results paralell to those in [11] and [12] in a more general setting and to estimate the number of resonances in a large complex domain. The results obtained in this paper can be applied, for instance, to the N--body Stark Hamiltonians obtained by removing the mass centre from the operators with Coulomb potentials:

$$P = \sum_{j-1}^{N} \left(-\frac{\Delta_{x_j}}{2m_j} + \varepsilon q_j e \cdot x_j \right) + \sum_{i<j} \frac{c_{ij}}{|x_i - x_j|},$$

$x_j \in \mathbb{R}^3$, $q_j, c_{ij} \in \mathbb{R}$ and $e \in \mathbb{S}^2$ being the field direction and $\varepsilon > 0$ the field strength, and hence can give precise exponential estimates over the width of resonances and resonant states. Remark that the existence of resonances for this class of operators has been obtained in [8] (see also [9]), but the exponential estimates on the widths are not contained in [11] and [12]. We think the proofs presented here are more transparent, althought the classes of potentials in this paper are slightly different from those in [12].

The methods used in this paper are similar to those in [12]: we use analytic distortion machinery to define the resonances and deduce the main results throught a precise parametrix for a suitable Grushin problem of the distorted Stark Hamiltonians (see [2], [6]). However there are two differences: the analytical distortion is modified to adapt to the generalized N-body operators and we substitute some quasi-inversibility estimates to the detailed analysis of the essential spectrum of the distorted Stark Hamiltonian in [12]. Although the result on the essential spectrum of the distorted Stark Hamiltonian obtained in this paper is not so precise as that obtained in [12], it is still global in nature and enables us to estimate the number of resonances in a large domain.

The organization of this work is as follows: In Section 2, we give the definition of generalized N-body Stark Hamiltonians and formulate the generalized Weinberg-van Winter equation, which is a fundamental tool in the analysis of generalized N-body Schrödinger operators. We give this result, just because we have not been able to find an exact reference needed. In Section 3, we give the assumptions on potentials and introduce the analytical distortion. As an important step in the construction of a parametrix for the Grushin problem of the distorted Stark Hamiltonian, we establish a quasi-inversibility estimate in Section 4 (Theorem 4.2). In Section 5, we give the main results of this paper: the existence of resonances generated by the discrete eigenvalues of the N-body operator without Stark effect, the exponentials bounds on width of resonances and resonant states. These results are deduced on studying a suitable Dirichlet problem and constructing a precise parametrix of the Grushin problem. The details are often the same as in [6] and [12], therefore will only be sketched. Finally we show in Section 6 that in a large domain in $\mathbb{C}$ below the bottom of the essential spectrum of the N-body operator with zero homogeneous field, the resonances of the generalized Stark Hamiltonian are all generated by the discrete eigenvalues of the corresponding N-body operator without Stark effect, hence are all exponentially close to the real axis. This gives in particular an estimate on the number of resonances in above-mentioned domain.

2. GENERALIZED WEINBERG-VAN WINTER EQUATION

The Weinberg-van Winter equation is a fundamental tool in the spectral analysis of N-body Schrödinger operators. For regular N-body operators, see [10] for the references and the result. Since we are not able to find an exact reference for the same equation for generalized N-body operators, we give in this Section a formulation of the generalized Weinberg-van Winter equation, which will be used in Section 4.

Let us first introduce the generalized N-body Stark Hamiltonians. Recall the definition of a generalized N-body operator. Let X be a real vector space with $\dim X = d$, and q a positively definite quadratic form over X. Let $\{X_a, a \in A\}$ be a family of linear subspaces in X, satisfying some axioms. See for example [4]. Denote by X^a the orthogonal complement of X_a in X (with respect to the scalar product induced by q) and by π^a the orthogonal projection from X onto X^a. For $x \in X$, we write $x^a = \pi^a x$ and $x = x^a + x_a$. For $a \in A$, let $N(a)$ ($\#a$, resp.) denote the number of particles (clusters, resp.) in a. $N(a)$ ($\#a$, resp.) is by definition the maximal number n such that there are $a_j \in A$, $j = 1, \ldots, n$, with $a_j \neq a_k$, if $j \neq k$ and $a_1 = a_{\min} \subset a_2 \subset \ldots \subset a_n = a$ ($a_1 = a \subset \ldots \subset a_n = a_{\max}$). Here $a_{\min}$ and $a_{\max}$ are respectively the minimal and the maximal element in A. Put $N = N(a_{\max})$. Then the generalized N-body Schrödinger operators of the form:

$$P = -\Delta + \sum_{a \in A} V_a(x^a). \tag{2.1}$$

Here $-\Delta$ is the Laplace-Beltrami operator over (X, q) and $\{V_a, a \in A\}$ is a family of interaction potentials. Throughout this work we assume that V_a is $-\Delta^a$-compact in $\mathcal{L}^2(X^a)$, where $-\Delta^a$ is the Laplace-Beltrami operator over X^a. Let X_1 be a real linear form on X. The generalized Stark Hamiltonians studied in this paper are of the form:

$$P(\beta) = P + \beta X_1, \tag{2.2}$$

where P is given by (2.1) and $\beta > 0$ is a small parameter. In physical situations the linear form βX_1 arises from a weak homogeneous electric field (i.e., the Stark effect) and β is proportional to the field strength (cf. [7]). Clearly operators $P(\beta)$ contain regular N-body Schrödinger operators with Stark effect as particular examples (cf. [4]). By rescaling $\beta > 0$, we can assume that

$$X_1 = \langle e, x \rangle, \quad x \in X, \quad \text{for some } e \in S^{d-1} \equiv \{x \in X, |x| = 1\}. \tag{2.3}$$

Here $|x| = q(x)^{\frac{1}{2}}$.

To formulate the Weinberg-van Winter equation in a general setting, let P_0 be a closed operator in $\mathcal{L}^2(X)$ and $\{V_a,\ a \in A\}$ a family of P_0-bounded perturbation with relative bound 0. For $a \in A$, put:

$$A_a = \{b \in A; b \subseteq a\}, \quad B_a = \{b \in A; b \notin A_a\},$$

$$C_a = \{c \in A; c \neq a_{\max} \text{ and } c = b \cup a, \text{ for some } b \in B_a\}.$$

A string of cluster decompositions S is a collection $(a_1, a_2, \ldots, a_k)$ with $a_{j+1} \in C_{a_j} \cup \cup\{a_{\max}\}$ and $a_j \neq a_{j+1}$. For any given string of cluster decompositions $S = (a_1, \ldots, a_k)$, we define the lenght of S to be $|S| = k$ and order of S to be $N(S) = N(a_k)$. For $a \in A$, $b \in C_a$, define

$$I_b^a = \sum_{c \in B_a,\ c \cup a = b} V_c. \tag{2.4}$$

For a given string $S = (a_1, \ldots, a_k)$, we put:

$$I_1^S = V_{a_2} \text{ and } I_j^S = I_{a_{j+1}}^{a_j}, \text{ for } j = 2, \ldots, k-1.$$

Now consider the operator $P = P_0 + \sum_{a \in A} V_a$. The spectral properties of P have much to do with the subhamiltonians P_a, $a \in A$, $a \neq a_{\max}$, defined by

$$P_a = P_0 + \sum_{b \in A_a} V_b.$$

For $z \notin \sigma(P_a)$, put $R_a(z) = (P_a - z)^{-1}$. If $z \notin \bigcup \sigma(P_a)$, we define:

$$\begin{aligned} D(z) &= \sum_{N(S) \leqslant N-1} (-1)^{k-1} R_{a_k}(z) I_{k-1}^S R_{a_{k-1}}(z) \ldots I_1^S R_{a_1}(z), \text{ and} \\ I(z) &= \sum_{N(S)=N} (-1)^{k-1} I_{k-1}^S R_{a_{k-1}}(z) I_{k-2}^S \ldots I_1^S R_{a_1}(z). \end{aligned} \tag{2.5}$$

PROPOSITION 2.1. *For $z \notin \sigma(P)$, put $R(z) = (P - z)^{-1}$. Then the following Weinberg-van Winter equation holds:*

$$R(z) = D(z) + R(z) I(z), \text{ for } z \notin \sigma(P) \cup \left(\bigcup_{a \neq a_{\max}} \sigma(P_a) \right). \tag{2.6}$$

Proof. We successively expand $R(z)$ in terms of $R_a(z)$, making use of the second resolvent equation. For the simplicity of notations we omit the dependence on z, which is asssumed to be in the resolvent set of all operators. Put $R_0 = R_{a_{\min}}$. Then

$$R = R_0 - \sum_a R V_a R_0 = R_0 + \sum_a (R I_a R_a V_a R_0 - R_a V_a R_0).$$

Here $I_a = \sum_{b \in B_a} V_b$. For each $a \in A$, $a \neq a_{\max}$, we split B_a into two parts: B'_a and B''_a, according that $a \subset b$ or $a \not\subset b$. Let $\sum'$ $\left(\sum'', \text{ resp.}\right)$ denote the sum over $b \in B'_a$

(B''_a, resp.). Then for $a \neq a_{\max}$, we have

$$\begin{aligned} RI_aR_aV_aR_0 &= \sum{}' (R_bV_bR_aV_aR_0 - RI_bR_bV_bR_aV_aR_0) + \\ &+ \sum_{c=a\cup b}{}'' (R_cV_bR_aV_aR_0 - RI_cR_cV_bR_aV_aR_0) = \\ &= \sum_{b\in C_a} (R_bI_b^aR_aV_aR_0 - RI_bR_bI_b^aR_aV_aR_0). \end{aligned}$$

Here I_b^a is defined in (2.4). Consequently,

$$\begin{aligned} R = R_0 - RV_{a_{\max}}R_0 - \sum_{a\neq a_{\max}} &(R_aV_aR_0 - RV_{a_{\max}}R_aV_aR_0 - \\ &- \sum_{b\in C_a}(R_bI_b^aR_aV_aR_0 - RI_bR_bI_b^aR_aV_aR_0)). \end{aligned} \tag{2.7}$$

If $N(b) = N-1$, then $I_b = I^b_{a_{\max}}$. If $N(b) \leqslant N-2$, repeating the above processes, we obtain:

$$RI_bR_bI_b^aR_aV_aR_0 = (RV_{a_{\max}}R_b + \sum_{c\in C_b}(R_cI_c^bR_b - RI_cR_cI_c^bR_b))I_b^aR_aV_aR_0.$$

Since for $c \in C_b$, $b \in C_a$, one has $N(c) \geqslant N(b)+1 \geqslant N(a)+2$, after iterating the process at most $(N-1)$-times, we arrive at the expression:

$$\begin{aligned} R = R_0 - RV_{a_{\max}}R_0 - \sum_{a\neq a_{\max}} &(R_aV_aR_0 - RV_{a_{\max}}R_aV_aR_0 - \\ &- \sum_{b\in C_a}(R_bI_b^aR_bV_aR_0 - RV_{a_{\max}}R_bI_b^aR_aV_aR_0 - \\ &\qquad - \sum_{c\in C_b}(\ldots \\ &\qquad\qquad \cdots\cdots\cdots \\ - \sum_{u\in C_v}(R_uI_u^vR_v\ldots R_bI_b^aR_aV_aR_0 &- RI_uR_uI_u^vR_v\ldots R_aV_aR_0)\ldots))). \end{aligned} \tag{2.8}$$

Here C_v, $v \in A$, is such that $\forall u \in C_v$, $N(u) = N-1$. Remark that all strings of lenght k appear after k-times of iteration. Since $I_u = I^u_{a_{\max}}$ for $N(u) = N-1$, resumming (2.8) according to the order of strings, we obtain (2.6). ■

In a way similar to (2.5), we can also construct $D'(z)$ and $I'(z)$, having the same properties as $D(z)$ and $I(z)$ such that

$$R(z) = D'(z) + I'(z)R(z). \tag{2.9}$$

(For selfadjoint operators we need just take $D'(z)$ and $I'(z)$ to be the adjoint of $D(\overline{z})$ and $I(\overline{z})$, respectively). (2.6) and (2.9) are useful in the spectral analysis of generalized N-body operators. For example the following result can be proven in a usual way by utilizing the identity:

$$(P-z)D(z) = I - I(z), \quad \text{for } z \notin \bigcup_{a \neq a_{\max}} \sigma(P_a)$$

and the holomorphic Fredholm alternative theorem (see [10]).

COROLLARY 2.2. *Assume that there exists a smooth function $f : \mathbf{R} \to [1,+\infty[$ such that $\lim_{t\to\infty} f(t) = \infty$ and that $f(|x^a|)V_a$ is P_0-bounded with relative bound zero. Let H^t_{loc} denote the local Sobolev spaces on X of order $t \in \mathbf{R}$. Assume that $D(P_0) \subset \subset H^s_{\mathrm{loc}}$ for some $s > 0$ and $f(|x^a|)V_a$, $a \in A$, maps continuously H^s_{loc} to H^s_{loc} and that for all $b \in A$, $[P_b, f(|x^a|)]R_b(z)f(|x^a|)^{-1}$, $z \notin \sigma(P_b)$, extends to a bounded operator on $\mathcal{L}^2(X)$. Let Ω be a connected component of $\mathbf{C} \setminus \bigcup_{a\neq a_{\max}} \sigma(P_a)$ such that there is at least one $z \in \Omega$ with $z \notin \sigma(P)$. Then $\sigma_{\mathrm{ess}}(P) \cap \Omega = \emptyset$.*

Note that if $c = a \cup b$, then $X_c = X_a \cap X_b$ and we can show that $|x^a| + |x^b| \geqslant \delta |x^c|$, for some $\delta > 0$ and for any $x \in X$. Consequently, the assumptions in Corollary 2.2 imply that $I(z)$ defined in (2.5) is compact and holomorphic in Ω. Finally let us single out that in Proposition 2.1, P_0 and V_a, $a \in A$, are not assumed to be symmetric, so we can apply it to complex distorted N-body operators. Corollary 2.2 is a version of the HVZ Theorem. See [3] for other proofs of the HVZ Theorem in selfadjoint case.

3. ANALYTICAL DISTORTION

Let $P(\beta)$ be a generalized N-body Stark hamiltonian (cf. (2.2)), $P = P(0)$ the N-body operator. Let the potentials $V_a = V_a(x^a)$ satisfy the conditions stated at the beginning of Section 2. Under additional analyticity on V_a, we shall define the analytical distortion of $P(\beta)$. Let S^{d-1} be the unit spfere in X. For any $\omega \in S^{d-1}$, P^ω define by

$$P^\omega = -\Delta + \sum_{\pi^a\omega = 0} V_a(x^a)$$

is a subhamiltonian of P and under the assumptions on V_a, one can prove as in [1]:

$$\sigma_{\mathrm{ess}}(P) = [\Sigma, +\infty[, \quad \text{with } \Sigma = \min \Sigma_\omega \text{ and } \Sigma_\omega = \inf \sigma(P^\omega). \tag{3.1}$$

Let $\Sigma(x)$ be defined by:

$$\Sigma(x) = \begin{cases} \Sigma_\omega, & \text{if } x \neq 0 \text{ and } x/|x| = \omega \ ; \\ \Sigma, & \text{if } x = 0. \end{cases}$$

(See Agmon [1].) For $\lambda_0 < \Sigma$, let $S(\beta)$ denote the distance from $0 \in X$ to the set $\Gamma = \{x; \beta X_1 + \Sigma(x) = \lambda_0\}$ in the Agmon metric $(\beta X_1 + \Sigma(x) - \lambda_0)_+ \mathrm{d}x^2$ (cf. [1]). Here $\mathrm{d}x^2$ is the metric on (X, q). A direct estimate gives the lower bound for $S(\beta)$:

$$S(\beta) \geqslant \frac{2(\Sigma - \lambda_0)^{\frac{3}{2}}}{(3\beta)}. \tag{3.2}$$

For $r > 0$, put $\rho(r) = \inf_{\omega \in S^{d-1}} (\beta r \langle e, \omega \rangle + \Sigma_\omega - \lambda_0)_+$. Let $S_1(\beta)$ denote the distance from 0 to Γ in the metric $\rho(r)\mathrm{d}x^2$. Let $\mathrm{d}(\cdot\,; \beta)$ and $\mathrm{d}'(\cdot\,; \beta)$ denote the distance from $x \in X$ to 0 in the metric $(\beta X_1 + \Sigma(x) - \lambda_0)_+ \mathrm{d}x^2$ and $\rho(r)\mathrm{d}x^2$, respectively. Clearly $\mathrm{d}(x; \beta) \geqslant \mathrm{d}'(x; \beta)$, $\forall x \in X$ and $\mathrm{d}'(x; \beta)$ is spherically symmetric (i.e., only depending on $|x|$). The following result shows that we can choose a spherically symmetrical distortion, without losing signifiant information on the tunneling estimates.

LEMMA 3.1. *There exists $C > 0$, independent of $\beta > 0$, such that* $\mathrm{d}(x; \beta) \geqslant S(\beta) - C$ *for all x with* $\mathrm{d}'(x; \beta) = S_1(\beta)$.

Proof. Since $\mathrm{d}(x; \beta) \geqslant S_1(\beta)$ on the sphere $\{\mathrm{d}'(x; \beta) = S_1(\beta)\}$, we only need to prove

$$S(\beta) \geqslant S_1(\beta) - C, \quad \text{for some } C > 0 \text{ independent of } \beta. \tag{3.3}$$

Let $S_a = \{\omega \in S^{d-1}; \omega \in X_a \text{ and for any } b \in B_a,\ \omega \notin X_b\}$. $\{S_a\}$ form a covering of S^{d-1} and on S_a, $\Sigma(x)$ is constant:

$$\Sigma(x) = \Sigma^a \equiv \inf \sigma(P_a).$$

Note that $\Sigma(-x) = \Sigma(x)$. We have:

$$\rho(r) = \min_a (\beta r \mu_a + \Sigma^a - \lambda_0)_+, \quad \text{with } \mu_a = \inf\{\langle e, \omega \rangle; \omega \in S_a, \langle e, \omega \rangle < 0\}. \tag{3.4}$$

This shows that there exist k ($k \leqslant |A|$) points: $r_0 = 0 < r_1 \ldots < r_k$ with r_k the radius (in the metric $\mathrm{d}x^2$) of the sphere $\{x; \mathrm{d}'(x; \beta) = S_1(\beta)\}$ such that

$$\rho(r) = (\beta r \mu_{a_j} + \Sigma^{a_j} - \lambda_0)_+, \quad \text{for } r \in [r_j, r_{j+1}],\ j = 0, 1, \ldots, k-1. \tag{3.5}$$

Remark that $\rho(r_k) = 0$ and the distance (in $\mathrm{d}x^2$) between Γ and the set $\{r_k\omega; \omega \in S_{a_k}\}$ is zero. For any given $\varepsilon > 0$, take $\omega_j \in S_{a_j}$ such that

$$\rho(r) \leqslant (\beta r \langle e, \omega_j \rangle + \Sigma_{\omega_j} - \lambda_0)_+ \leqslant \rho(r) + \varepsilon\beta, \quad \text{for } r \in [r_j, r_{j+1}]. \tag{3.6}$$

Note that we can also choose $\omega_k \in S_{a_k}$ such that the distance between $x_0 \equiv r_k\omega_k$ and Γ is small: $\mathrm{dist}(x_0, \Gamma) < \varepsilon$. Let $\gamma : [0, 1] \to X$ be a minimal geodesic in the metric $\rho(r)\mathrm{d}x^2$, joining 0 to x_0. γ is Lipschitz continuous and is radial: $\gamma(t) = r(t)\omega_k$, with $r(0) = 0$, $r(1) = r_k$ and $r'(t) \geqslant 0$ a.e. Since $\mathrm{d}'(x;\beta)$ only depends on $|x|$, the length $|\gamma|_\rho$ of γ in $\rho(|x|)\mathrm{d}x^2$ is equal to $S_1(\beta)$. Making use of γ, we can construct an approximate minimal geodesic γ' in the metric $(\beta X_1 + \Sigma(x) - \lambda_0)_+\mathrm{d}x^2$, joining 0 to x_0, such that

$$\gamma'(t) = r(t)\omega_j, \quad \text{if } r(t) \in [0, r_1 - \delta[\cup \left(\bigcup_{j=2}[r_j + \delta, r_{j+1} - \delta]\right) \cup [r_{k-1} + \delta, r_k].$$

and that the derivate of γ' when t is such that $r(t) \in [r_j - \delta, r_j + \delta]$ is bounded. Here $\delta > 0$ is a fixed small constant. Since the number $k \leqslant |A|$ and the metrics are uniformly bounded in $\{x; |X_1| \leqslant C/\beta\}$ for any $C > 0$, we can estimate the lenght $|\gamma'|$ of γ' in the metric $(\beta X_1 + \Sigma(x) - \lambda_0)_+\mathrm{d}x^2$:

$$|\gamma'| \leqslant |\gamma|_\rho + C_1 = S_1(\beta) + C_1 \text{ and } |\gamma'| \geqslant \mathrm{d}(x_0;\beta) \geqslant S(\beta) - C_2,$$

(since x_0 is near Γ) for some $C_1, C_2 > 0$ independent of β. The lemma is proven. ■

Let χ be a smooth function on $\mathbf{R}$, such that $0 \leqslant \chi \leqslant 1$, $\chi' \geqslant 0$ and $\operatorname{supp}\chi \subset \subset]1 - \eta, \infty[$, $\chi = 1$ on $[1-\frac{\eta}{2}, \infty[$. Here $\eta > 0$ is sufficiently small but fixed. We denote by the same letter the function on X depending on β defined by: $\chi(x;\beta) = \chi\left(\frac{|x|}{r_k}\right)$, with r_k given in (3.5). Clearly, $|\partial^\alpha\chi(x;\beta)| \leqslant C_\alpha\beta^{|\alpha|}$, for all $\alpha \in \mathbf{N}^d$. Put $M = \{x; |x| < (1-\eta)r_k\}$. On $M, \chi(x;\beta) = 0$. By Lemma 3.1, one has:

$$\mathrm{d}(x;\beta) \geqslant S(\beta) - \frac{\varepsilon(\eta)}{\beta},$$

for $x \in \partial M$. Here and in the following, $\varepsilon(\eta) > 0$ is some function of $\eta > 0$ and $\varepsilon(\eta)$ tends to zero as $\eta \to 0$. For $\theta \in R$, $|\theta|$ small, let $U(\theta)$ denote the unitary operator on $\mathcal{L}^2(X)$, induced by the diffeomorphism on $X : x \to \mathrm{e}^{\theta\chi(x;\beta)}x$, for $\beta > 0$ small. Put:

$$P(\beta;\theta) = U(\theta)P(\beta)U(\theta)^{-1}, \quad P(\theta) = U(\theta)PU(\theta)^{-1}.$$

To study the analytical extension of $P(\beta;\theta)$ in θ into a small complex neighbourhood of zero, we make the following assumption on V_a:

(3.7) There exists a smooth weight function $f : \mathbf{R} \to [1, \infty[$, $f(t) \to \infty$ as $t \to \infty$ and a small neighbourhood Ω of $0 \in \mathbf{C}$ such that as operators in $\mathcal{L}^2(X)$, the map $\theta \to F_a(\theta) \equiv f(|x^a|)V_a(x^a(\theta))$, $a \in A$, defined for θ real, extends to a $-\Delta$-bounded operator-valued holomorphic function of $\theta \in \Omega$, and that $F_a(\theta)$ and $\partial_\theta F_a(\theta)$, $\theta \in \Omega$, have relative bound zero and for each fixed x_a, as operators in

$\mathcal{L}^2(X^a)$, $F(\cdot, x_a; \theta)$ is $-\Delta^a$ compact. Here $x^a(\theta) = e^{\theta \chi} x^a$, which also depends on x_a.

REMARKS. a. The assumptions on V_a allow singularities and slow decay of the potentials. For example, the potentials of the form:

$$V_a(x^a) = \frac{c_a}{|x^a|^{r^a}},\ 0 < r^a < \min\left(2, \frac{n_a}{2}\right),\ c_a \in \mathbf{R},$$

(here $n_a = \dim X^a \geqslant 3$), and

$$V_a(x^a) = \frac{c_a(|x^a| - 1)^2}{((1 + |x^a|^2)\log|x^a|)},\ c_a \in \mathbf{R},$$

satisfy (3.7). If $a = a_{\max}$, the singularities of V_a local in X are also allowed. In particular, the N-body Schrödinger operator with Stark effect obtained by removing the mass center from the regular N-body Scrödinger operator with Coulomb potentials is included.

b. If $a \neq a_{\max}$, $x^a(\theta) = e^{\theta} x^a$ for all $x^a \in X^a$, when $|x_a|$ is large enough. The assumption (3.7) implies that V_a is dilation analytical in the sense of Combes. We shall use this point in the proofs.

By (3.7), $P(\beta, \theta)$ can naturally be defined for $\theta \in \Omega$. Put $B_+ = \{z; |z| < \varepsilon_0$ and $\operatorname{Im} z > 0\}$ for $\varepsilon_0 < \eta$ small enough so that $B_+ \subset \Omega$.

LEMMA 3.2. *Let $P_0(\beta, \theta)$ be the distorted free Stark Hamiltonian. Then the following holds:*

(a) *$P_0(\beta, \theta)$, $\theta \in B_+$, defined on $D \equiv D(-\Delta) \cap D(X_1)$ is closed and is a holomorphic family of type (A);*

(b) *Let $\varepsilon_0 < \eta$, where $\eta > 0$ is the small constant used in the definition of χ. There exist $C > 0$ and $\beta_0 > 0$ such that for $0 < \beta < \beta_0$ and $\theta \in B_+$, the numerical range of $P_0(\beta, \theta)$ is contained in $\left\{z; \operatorname{Re} ie^{-\theta} z > \lambda' + \frac{\eta}{C}\right\}$. Here $\lambda' = \inf\{\beta X_1; |x| \leqslant r_k\}$;*

(c) *The spectrum of $P_0(\beta, \theta)$, $\theta \in B_+$, is empty: $\sigma(P_0(\beta, \theta)) = \emptyset$ and for any $E \in \mathbf{R}$,*

$$\sup_{\operatorname{Re} ie^{-\theta} z < E} \|P_0(\beta, \theta - z)^{-1}\| < +\infty. \tag{3.8}$$

Proof. It is the same as the usual case. See Section 2 in [12]. Note that the spherical choice of χ is used in estimating the imaginary part of the symbol and the numerical range of $P_0(\beta, \theta)$. The details are omitted. See [12]. ∎

Remark that the bound in (3.8) may depend on $\beta > 0$, but (b) of Lemma 3.2 assures that if $E < \lambda'$, then the sup in (3.8) is independent of β.

PROPOSITION 3.3. *Let (3.7) be satisfied. Then $P(\beta,\theta)$, $\theta \in B_+$, defined on D is closed and is a holomorphic family of type (A). For any open set $O \subset B_+$, there is $R \gg 1$ such that for $0 < \beta < \beta_0$, $P(\beta,\theta) - z$ is invertible for $\theta \in O$, $\operatorname{Re} z < \lambda' + \frac{2\eta}{C}$, $\operatorname{Im} z > R$ and the resolvent is jointly analytical there.*

Proof. The first part of the Proposition is clear by the assumptions on V_a. To show the second part, note that the (b) in Lemma 3.2 implies:

$$\|P_0(\beta,\theta - z)^{-1}\| \leqslant \frac{1}{\mathrm{d}(z)}, \quad \mathrm{d}(z) = \operatorname{dist}\left(z; \left\{ie^{-\theta} w; \operatorname{Re} w > \lambda' + \frac{\eta}{C}\right\}\right). \tag{3.9}$$

For $\operatorname{Re} z < \lambda' + \frac{\eta}{C}$, $\operatorname{Im} z > 1$, one has: $\mathrm{d}(z) \geqslant \operatorname{Im} \frac{z}{C'}$ and by constructing a parametrix for $P_0(\beta,\theta) - z$, we can show that $\|\Delta(P_0(\beta,\theta) - z_0)^{-1}\| \leqslant C(\operatorname{Im}\theta) \leqslant C$, if $\theta \in O$. Since each $V_a(\theta) \equiv V_a(x^a(\theta))$ is $-\Delta$-Bounded with relative bound zero, we have

$$\begin{aligned} \|V(\theta)R_0(z;\beta,\theta)\| &\leqslant \varepsilon\|\Delta R_0(z;\beta,\theta)\| + C_\varepsilon\|R_0(z;\beta,\theta)\| \leqslant \\ &\leqslant \varepsilon\left(C + \frac{C|z - z_0|}{\mathrm{d}(z)}\right) + \frac{C_\varepsilon}{\mathrm{d}(z)}, \quad \text{for } \theta \in O. \end{aligned} \tag{3.10}$$

Here $V(\theta) = \sum V_a(\theta)$ and $R_0(z;\beta,\theta) = (P_0(\beta,\theta) - z)^{-1}$. Remark that in the region $\left\{\operatorname{Re} z < \lambda' + \frac{\eta}{C}, \operatorname{Im} z > 0\right\}$, $\frac{|z - z_0|}{\mathrm{d}(z)}$ is bounded (z_0 being fixed). So when $\operatorname{Im} z > R$, we can make the right hand side of (3:10) be bounded by $\frac{1}{2}$, by choosing $\varepsilon > 0$ small and $R \gg 1$. This means $P(\beta,\theta) - z$ is invertible in the above region. The second part of the Proposition follows. ∎

In this work, the resonance of $P(\beta)$ shall be defined as the discrete spectrum of $P(\beta,\theta)$, for some $\theta \in B_+$. See the remark after Theorem 4.2. By Proposition 3.3, we can show that the discrete spectrum of $P(\beta,\theta)$ in a proper region is essentially independent of $\theta \in B_+$ and of the choice of χ. See [5] and [9], for example.

4. QUASI-INVERTIBILITY OF DISTORTED STARK HAMILTONIANS

The purpose of this section is to construct a good approximation of the resolvent $R(z;\beta,\theta) = (P(\beta,\theta) - z)^{-1}$, outside some compact set in X. This approximation is important in the construction of a parametrix of the Grushin problem for $P(\beta,\theta)$ studied in Section 5. We will prove that $P(\beta,\theta)$ is quasi-invertible, i.e., modifying $P(\beta,\theta)$ in a compact domain independent of β yields an operator invertible in a large complex domain containing λ_0. To prove this, let φ be a smooth function on X, $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 0$ when $|x| \leqslant 1$; 1, when $|x| \geqslant 2$. For $R > 1$, put $\rho_R(x) = \varphi\left(\frac{x}{R}\right)^2$. Let $P' = P_0 + V'$, $V' = \rho_R V$.

LEMMA 4.1. *Let V_a satisfy the assumption (3.7) (with $\theta = 0$). Let $\Sigma(\cdot)$ be defined as in Section 2. Then for any $\varepsilon > 0$ small, there exists $R_0 > 1$ s.t. for $R > R_0$,*

$$\langle P'u, u\rangle \geqslant \langle(\Sigma(x) - \varepsilon^2)u, u\rangle, \quad u \in C_0^\infty(X). \tag{4.1}$$

Proof. As in [1], we have for $R_0 \gg 1, \langle Pu, u\rangle \geqslant \left\langle\left(\Sigma(x) - \frac{\varepsilon^2}{2}\right)u, u\right\rangle$ for any $u \in C_0^\infty$, $\operatorname{supp} u \cap \{|x| < R_0\} = \emptyset$. For $R > R_0$, put $u' = \varphi_R u$. Then

$$\begin{aligned}\langle P'u, u\rangle &= \langle Pu', u'\rangle + \langle P_0u, u\rangle - \langle P_0u', u'\rangle \geqslant \\ &\geqslant \left\langle\left(\Sigma(x) - \frac{\varepsilon^2}{2}\right)u', u'\right\rangle + \|\nabla u\|^2 - \|\varphi_R\nabla u\|^2 - \\ &- \|(\nabla\varphi_R)u\|^2 - 2\mathrm{Re}\langle[\nabla, \varphi_R u]u, \varphi_R\nabla u\rangle \geqslant \\ &\geqslant \langle(\Sigma(x) - \frac{\varepsilon^2}{2})u', u'\rangle - O\left(\frac{1}{R}\right)(\|u\|^2 + \|\nabla u\|^2).\end{aligned} \tag{4.2}$$

Note that $\Sigma(x) \leqslant 0$, so $\left\langle\left(\Sigma(x) - \frac{\varepsilon^2}{2}\right)u', u'\right\rangle \geqslant \left\langle\left(\Sigma(x) - \frac{\varepsilon^2}{2}\right)u, u\right\rangle$ and by the assumptions on V, we can derive that $\|\nabla u\|^2 \leqslant 2\langle P'u, u\rangle + C\langle u, u\rangle$, for some C independent of u. (4.1) follows from by choosing $R > R_0$ and $R_0 \gg 1$. ■

By an argument of density, we can derive from Lemma 4.1

$$\sigma(P') \subset [\Sigma - \varepsilon^2, +\infty[, \quad \Sigma \equiv \inf \sigma_{\mathrm{ess}}(P). \tag{4.3}$$

In the following, we shall take $\varepsilon = \eta$ and $R > 1$ is fixed such that Lemma 4.1 is valid. Since P and P' differ only in a compact set independent of β, the distorted operator $P'(\beta, \theta)$ of $P'(\beta) = P' + \beta X_1$ can also be defined for $\theta \in B_+$ and $\beta > 0$ sufficiently small, and has the similar properties as $P(\beta, \theta)$. The following result is important in this paper.

THEOREM 4.2. *For $a, b \in \mathbf{R}, \theta \in B_+$, define $E(\theta, a, b) = \{z, \operatorname{Re} z < a, \operatorname{Re} ie^{-(1-\eta)\theta} \cdot z < b\}$. Let $P'(\beta, \theta)$ be defined as above. Then for $\theta \in B_+$ and β_0 small enough, there is $C > 0$ such that*

$$\sigma(P'(\beta, \theta)) \cap E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta) = \emptyset, \quad 0 < \beta < \beta_0, \tag{4.3}$$

and

$$\|(|\beta X_1|^{\frac{1}{2}} + (1 - \Delta)^{\frac{1}{2}})R'(z; \beta, \theta)\| \leqslant C \tag{4.4}$$

for $z \in E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta)$ and $0 < \beta < \beta_0$. Here $R'(z; \beta, \theta)$ is the resolvent of $P'(\beta, \theta)$.

REMARKS. (a) Since $P(\beta,\theta)$ differs from $P'(\beta,\theta)$ in a compact in X, it follows from Theorem 4.2 that the spectrum of $P(\beta,\theta)$ in $E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta)$ is discrete. The resonances of $P(\beta)$ in $E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta)$ is defined to be the eigenvalues of $P(\beta,\theta)$ in this region. Concerning the relation between different definitions of resonances, we refer to [5].

(b) The resolvent estimates (4.4) is useful in establishing the exponential decay of resonant states (i.e., the eigenfunctions of $P(\beta,\theta)$ and the later will be important for estimating the number of resonances of $P(\beta)$ in $E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta)$. See Section 6.

PROPOSITION 4.3. *Let* $M_1=\left\{x;\mathrm{d}'(x;\beta)<\left(1-\frac{\eta}{3}\right)r_k\right\}$ *and* $M_2=\left\{x;\mathrm{d}'(x;\beta)>\right.$ $\left.>\left(1-\frac{\eta}{2}\right)r_k\right\}$. *(See Section 3 for the definition of* r_k*). Let* L *denote the square root of the selfadjoint realization of* $-\Delta+\langle\beta X_1\rangle$. *Then we have:*

(a) *There exists* $c>0$ *such that* $\|(P'(\beta,\theta)-z)u\|\geqslant c\|Lu\|$, *for* $u\in C_0^\infty(X)$, $\operatorname{supp}u\subset M_1$ *and for* $\operatorname{Re}z<\lambda_0+\frac{\eta}{C}$ *and* $\theta\in B_+$.

(b) *For any* $\theta\in B_+$, *there is* c' *such that* $\|(P'(\beta,\theta)-z)u\|\geqslant c'\|Lu\|$, *for* u *in* $C_0^\infty(X)$, $\operatorname{supp}u\subset M_2$ *and for* $z\in E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta)$. *Here* c *and* c' *are independent of* β *when* $\beta>0$ *is small enough.*

Admitting for the moment Proposition 4.3, we first give the proof for Theorem 4.32.

Proof of Theorem 4.2. Let ρ_j, $j=1,2,3$, be smooth functions on X, such that

(i) $\partial^\alpha\rho_j(x)=0(\beta^{|\alpha|})$;

(ii) $\operatorname{supp}\rho_j\subset M_j$, $0\leqslant\rho_j\leqslant1$ for $j=1,2$, and $\rho_1^2+\rho_2^2\equiv1$ on X and

(iii) $\operatorname{supp}\rho_3\subset M_1$, $\rho_3\equiv1$ on $\operatorname{supp}\rho_1(1-\rho_1)$.

Put $u_j=\rho_ju$, $u\in C_0^\infty$, $j=1,2,3$. By Proposition 4.3, there exists $c>0$,

$$\|(P'(\beta,\theta)-z)u_j\|^2\geqslant c\|Lu_j\|^2,\quad j=1,2,\tag{4.5}$$

uniformly in $z\in E$ $(\equiv E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta))$. We compute:

$$\begin{aligned}\sum_{j=1}^{2}\|P'(\beta,\theta)-z)u_j\|^2&=\|(P(\beta,\theta)-z)u\|^2+\\&+\sum_j(2\mathrm{Re}\langle[P_0(\beta,\theta),\rho_j]u,\rho_j(P'(\beta,\theta)-z)u\rangle+\|[P_0(\beta,\theta),\rho_j]u\|^2).\end{aligned}\tag{4.6}$$

By the choice of ρ_j, $j=1,2,$:

$$\sum_j\rho_j[P_0(\beta,\theta),\rho_j]=\mathrm{e}^{-2\theta}\sum_j|\nabla\rho_j|^2=0(\beta^2)$$

and

$$\sum_j \|[P_0(\beta,\theta),\rho_j]u\|^2 \leqslant 0(\beta)(\|u_3\|^2 + \|\nabla u_3\|^2).$$

As in the proof of (a) of proposition 4.3, we have:

$$\begin{aligned}\|\nabla u_3\|^2 = \langle -\Delta u_3, u_3\rangle &\leqslant c\mathrm{Re}\langle (P'(\beta,\theta)-z)u_3, u_3\rangle, \quad z\in E,\\ &\leqslant C\{|P'(\beta,\theta)-z)u\|\,\|u\| + 0(\beta)(\|\nabla u_3\|^2 + \|u\|^2)\}.\end{aligned}$$

Therefore for $\beta > 0$ small, we obtain:

$$(4.7)\qquad \left|\sum_j (2\mathrm{Re}\langle [P_0(\beta,\theta),\rho_j]u, \rho_j(P'(\beta,\theta)-z)u\rangle + \|[P_0(\beta,\theta),\rho_j]u\|^2)\right| \leqslant$$
$$\leqslant 0(\beta)(\|\nabla\|^2 + \|u\|).$$

From (4.5), (4.6) and (4.7), it follows that

$$(4.8)\qquad \|(P'(\beta,\theta)-z)u\|^2 \geqslant c\|Lu\|^2,$$

for some $c > 0$ independent of $z \in E$, $u \in C_0^\infty(X)$ and $\beta > 0$ small. (4.8) shows that $P'(\beta,\theta) : D \to \mathcal{L}^2(X)$ is injective. Note that $P'(\beta,\theta)^*$ differs from $P'(\beta,\overline{\theta})$ by a first order differential operator with smooth coefficients of order $O(\beta)$. By studying $P'(\beta,\theta)$ for $\mathrm{Im}\,\theta < 0$, we can prove that (4.8) is also valid for $P'(\beta,\theta)^* - \overline{z}$ with $\mathrm{Im}\,\theta > 0$ and $z \in E$. This proves $P'(\beta,\theta) - z$ is invertible for $z \in E$ and Theorem 4.2 follows from (4.8). ■

Remark that the constants appeared in the above results may depend on $\mathrm{Im}\,\theta$, but it is easy to prove that the dependence is locally uniform in $\theta \in B_+$. Now we want to give the proof for Proposition 4.3.

Proof of Proposition 4.3. (a) Remark first that $V_a' \equiv \rho_R V_a$ still satisfies (3.7). Applying Lemma 4.1, we obtain for $\delta > 0$ small

$$(4.9)\qquad \begin{aligned}\langle P'u,u\rangle &= (1-\delta)\langle P'u,u\rangle + \delta\langle P'u,u\rangle \geqslant\\ &\geqslant (1+c(\delta))\langle (\Sigma(x)-\eta^2)u,u\rangle + \delta\langle -\Delta u,u\rangle,\end{aligned}$$

for $u \in D$. Here $c(\delta) \downarrow 0$ when $\delta \to 0$. By (3.7), $\partial_\theta V'(\theta)$ is $-\Delta$-bounded with relative bound zero. Comparing $P'(\theta)$ with p', we obtain

$$(4.10)\qquad |\mathrm{Re}\langle (P'(\theta)-P')u,u\rangle| \leqslant C|\theta|(\langle -\Delta u,u\rangle + \|u\|^2),$$

for $\theta \in B_+$ ($\equiv \{z; \mathrm{Im}\,z > 0$ and $|z| < \varepsilon_0\}$. If δ and $\varepsilon_0 \ll \delta$ are chosen to be sufficiently small, from (4.9) and (4.10), it results

$$\mathrm{Re}\,\langle P'(\theta)u,u\rangle \geqslant \langle (\Sigma(x)-2\eta^2)u,u\rangle + \frac{\delta}{2}\langle -\Delta u,u\rangle.$$

Now if $\operatorname{supp} u \subset M_1$, we have

$$\text{(4.11)}\quad \begin{aligned} \operatorname{Re}\langle (P(\beta,\theta)-z)u,u\rangle &\geqslant \operatorname{Re}\langle (\beta X_1 e^{\theta \chi} + \Sigma(x) - 2\eta^2 - z)u,u\rangle + \frac{\delta}{2}\langle -\Delta u,u\rangle \geqslant \\ &\geqslant \delta^2\langle (-\Delta+1)u,u\rangle, \end{aligned}$$

if $\operatorname{Re} z < \lambda_0 + \frac{\eta}{C}$ for $C > 0$ large enough. We have used $\inf\{\beta X_1 + \Sigma(x); x \in M_1\} \geqslant \geqslant \lambda_0 + C'\eta$ for some $C' > 0$. (See the choice of M_1). This proves (a) of Proposition 4.3.

To prove (b), let $P_a''(\beta,\theta) = e^{-2\theta}\Delta + \beta e^{\theta} X_1 + \sum_{b \in A_a} V_b(e^{\theta} x^b)$, $a \neq a_{\max}$. $P_a''(\beta,\theta)$ is well defined for $\theta \in B_+$ (see Remark (b) after (3.7)). If $u \in D$, $\operatorname{supp} u \subset M_2$, one has:

$$\text{(4.12)}\quad P_a''(\beta,\theta)u = P_a'(\beta,\theta)u.$$

Let $\{\chi_a, a \in A\}$ be a partition of the unity such that

(i) χ_a is smooth on X, $0 \leqslant \chi_a \leqslant 1$ and $\operatorname{supp}\chi_a \subset J_a \cap \left\{|x| > \frac{\varepsilon}{\beta}\right\}$, where $J_a = = \{x; |x^b| > \delta|x|, \forall b \in B_a\}$, $\delta > 0$ is a fixed small constant.;

(ii) $\sum_a \chi_a(x)^2 \equiv 1$ on M_2;

(iii) $|\partial^\alpha \chi_a| = 0(\beta^\alpha)$. Put $u_a = \chi_a u$. By IMS formula (cf. [3]), one has:

$$\text{(4.13)}\quad \begin{aligned} \|(P'(\beta,\theta)-z)u\|^2 = \sum_a (&\|(P'(\beta,\theta)-z)u_a\|^2 - \|[\chi_a, P_0'(\theta)]u\|^2 - \\ &-4\operatorname{Re}\langle e^{-2\theta}|\nabla\chi_a|^2 u, (P'(\beta,\theta)-z)u\rangle). \end{aligned}$$

We write $P'(\beta,\theta) = P_a'(\beta,\theta) + I_a(\theta)$. By (4.12) and the assumption (3.7), we have:

$$\|(P'(\beta,\theta) - P_a''(\beta,\theta))u_a\| = \|I_a(\theta)u_a\| \leqslant \varepsilon(\beta)(\|\Delta u_a\| + \|u_a\|).$$

From (4.13), it results:

$$\text{(4.14)}\quad \|(P'(\beta,\theta)-z)u\|^2 \geqslant \sum_a ((1-\varepsilon(\beta))\|(P_a''(\beta,\theta)-z)u_a\|^2 - \varepsilon(\beta)\|(1-\Delta)u_a\|^2).$$

Here $\varepsilon(\beta)$ is some positive function of β, tending to zero with β. The spectrum properties of the dilated Stark Hamiltonians $P''(\beta,\theta)$ can be analyzed as in [6]. In particular, making use of the generalized Weinberg – van Winter equation and the methods of [7] and [6], we can prove by an induction on $N(a)$, $a \neq a_{max}$, that for $\theta \in B_+$,

$$\text{(4.15)}\quad \sigma(P_a''(\beta,\theta)) \subset \{z; \operatorname{Re} ie^{-\theta} z > \Sigma_a - C\eta\},$$

and

$$\|R_a''(z;\beta,\theta)\| \leqslant \frac{C}{|\mathrm{Re}\,\mathrm{ie}^{-\theta}z - \Sigma_a + C\eta|}. \tag{4.16}$$

for $\beta > 0$ small enough and for some $C > 0$ independent of z with $\mathrm{Re}\,\mathrm{ie}^{-\theta}z < \Sigma_a - C\eta$ and $\beta \in]0,\beta_0]$. Here $R_a''(z;\beta,\theta)$ is the resolvent of $P_a''(\beta,\theta)$ and $\Sigma_a = \inf\sigma(P_a)$. (Compare with Theorem 3. in [8]). The upper bound in (4.16) follows from the semi-group arguments used in [7]. Note that $\frac{|z|}{|\mathrm{Re}\,\mathrm{ie}^{-\theta} - \Sigma_a + C\eta|}$ is bounded for $z \in E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta)$. From (4.16) we obtain:

$$\|(|\beta X_1| + 1 - \Delta)R_a''(z;\beta,\theta)\| \leqslant C, \quad \text{for } z \in E \text{ and } \beta > 0 \text{ small.}$$

This means:

$$\|(P_a''(\beta,\theta) - z)u_a\| \geqslant C'^{-1}\|(|\beta X_1| + 1 - \Delta)u_a\|, \quad \text{for } z \in E.$$

(b) of Proposition 4.3 follows from (4.14) by taking $\beta \in]0,\beta_0]$ with $\beta_0 > 0$ sufficiently small. ∎

5. EXPONENTIAL ESTIMATES ON RESONANCES AND RESONANT STATES

Seeing the results in Section 4, we can proceed as in [6] and [12] to obtain precise estimates on the resonances of $P(\beta)$ generated by the discrete spectrum of $P(0)$. We only give a sketch for the proofs. Let $\Sigma = \inf\sigma_{\mathrm{ess}}(P)$ and $\lambda_0 < \Sigma$. Let $P(\beta,\theta)$ be defined in Section 3. By Theorem 4.2, for $\theta \in B_+$ and for $\beta > 0$ small enough, $\sigma(P(\beta,\theta)) \cap E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta) = \emptyset$, for some $C > 0$. Take $\eta > 0$ small enough so that $\lambda_0 + \frac{\eta}{C} < \Sigma - C\eta$. Then the spectrum of $P(\beta,\theta)$ near λ_0 is discrete, or in other words, the resonances of $P(\beta)$ near λ_0 is well defined. We shall study in detail these resonances.

Let $M = \{x; |x| < (1-\eta)r_k\}$. Let $P^D(\beta)$ denote the Dirichlet realization of $P(\beta,\theta)$ over M. As in [1], for any $\varepsilon > 0$, there exists $R > 0$ s.t. for any $u \in C_0^\infty(M)$ with $\mathrm{supp}\, u \cap \{|x| < R\} = \emptyset$,

$$\begin{aligned}\langle P^D(\beta)u, u\rangle = \langle P^D(0)u, u\rangle + \langle \beta X_1 u, u\rangle = \\ \langle P(0)u, u\rangle + \langle \beta X_1 u, u\rangle \geqslant \langle (\Sigma(x) + \beta X_1 - \varepsilon)u, u\rangle \geqslant (\Lambda - \varepsilon)\langle u, u\rangle.\end{aligned} \tag{5.1}$$

Here $\Lambda \equiv \inf_{x\in M} \Sigma(x) + \beta X_1 \geqslant \lambda_0 + \frac{\eta}{C}$ for some $C > 0$. Since $\varepsilon > 0$ is arbitrary, by an argument of density, we derive from (5.1) that $\sigma_{\mathrm{ess}}(P^D(\beta)) \subset [\Lambda, +\infty[$ and the spectrum of $P^D(\beta)$ below Λ is discrete. The following result on the stability of the eigenvalues of P under the perturbation of the homogeneous field can be proven as in [12].

THEOREM 5.1. *Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of $P^D(\beta)$ in $I \equiv \left]-\infty, \lambda_0 + \frac{\eta}{C}\right[$ repeated according to the multiplicity. Then for $\beta > 0$ small enough, there are exactly m eigenvalues of $P^D(\beta)$ in I. Let $\mu_1(\beta), \dots, \mu_m(\beta)$ denote these eigenvalues of $P^D(\beta)$ and $u_1(\beta), \dots, u_m(\beta)$ the associated orthonormalized eigenfunctions. Then after a suitable rearrangement, we have:*

$$\mu_j(\beta) = \lambda_j + 0(\beta) \quad \text{and for any } \varepsilon > 0,$$

$$\sup_{0<\beta\leqslant\beta_0} \left\| \mathrm{e}^{(1-\varepsilon)d_j} u_j(\beta) \right\|_{H^1(M)} < +\infty, \quad j = 1, \dots, m.$$

Here $\mathrm{d}_j(x)$ denotes the distance from x to 0 in the metric $(\beta X_1 + \Sigma(x) - \lambda_j)_+ \mathrm{d}x^2$ on M.

The proof of Theorem 5.1 is based on the exponential decay of the eigenfunctions of $P^D(\beta)$ and $P(0)$. See [1]. The details are omitted (cf. [12] for similar results). From now on, we assume that $\lambda_0 < \Sigma$ is an eigenvalue of P and we want to study the resonances of $P(\beta)$ near λ_0. The main result is the following

THEOREM 5.2. *Let $\{V_a, a \in A\}$ satisfy the assumption (3.7). Let $\lambda_0 < 0$ be an eigenvalue of P with multiplicity m. Let $\mu_j(\beta)$, $j = 1, \dots, m$ be the eigenvalues of $P^D(\beta)$ such that $\mu_j(\beta) = \lambda_0 + 0(\beta)$. For $\theta \in B_+$, let Ω be a small complex neighbourhood of λ_0, contained in $E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta)$. Then for $\beta > 0$ small enough, there are exactly m resonances (counted according to the algebraic multiplicity) of $P(\beta)$ in Ω. Let $z_j(\beta)$, $j = 1, \dots, m$ denote these resonances. Then after a suitable rearrangement, we have for some $C > 0$ independent of β,*

$$|z_j(\beta) - \mu_j(\beta)| \leqslant C\mathrm{e}^{-2S(\beta) + \frac{\varepsilon(\eta)}{\beta}}, \quad j = 1, \dots, m,$$

for $\beta > 0$ small enough. Here $\varepsilon(\eta)$ is some positive constant tending to zero with η. In particular we have the bound on the width of the resonances:

$$|\mathrm{Im}\, z_j(\beta)| = 0_\varepsilon\left(\mathrm{e}^{-2S(\beta) + \frac{\varepsilon}{\beta}}\right), \quad \text{for any } \varepsilon > 0.$$

$S(\beta)$ is the tunneling factor defined through the Agmon metric $(\beta X_1 + \Sigma(x) - \lambda_0)_+ \mathrm{d}x^2$ in Section 3.

Proof. The result of Theorem 5.2 follows from a parametrix for a Grushin problem of $P(\beta, \theta)$, $\theta \in B_+$. The details are similar to [6] and [12]. We only give a sketch.

Let $u_j(\beta)$, $j=1,\dots,m$ be the orthonormalized eigenfunctions of $P^D(\beta)$ associated to $\mu_j(\beta)$, $j=1,\dots,m$. For $\theta\in B_+$ fixed, let Ω be a small neighbourhood of λ_0, $\Omega\subset E(\theta,\lambda_0+\frac{\eta}{C},\Sigma-C\eta)$ and $\Omega\cap\sigma(P)=\{\lambda_0\}$. For $z\in\Omega$, consider the Grushin problem for $P(\beta,\theta)$:

$$\mathcal{P}(z)=\begin{pmatrix} P(\beta,\theta)-z & R^- \\ R^+ & 0\end{pmatrix} : D\oplus C^m\to\mathcal{L}^2(X)\oplus C^m,$$

Here $R^+:\mathcal{L}^2\to C^m$ is defined by $:(R^+u)_j=\langle\rho u_j,u\rangle$, $j=1,\dots,m$, $u\in\mathcal{L}^2$ and ρ is a cutt-off function with compact support in M and $\rho(x)=1$ if $\mathrm{d}(x;\beta)\leqslant S(\beta)-\frac{2\eta}{\beta}$. $R^-:C^m\to D$ is the adjoint of R^+. Applying Theorem 4.2, we can construct a precise parametrix for $\mathcal{P}(z)$. Let ψ be a cut-off function such that :

$$\psi(x)=\begin{cases}1, & \text{if } \mathrm{d}(x;\beta)<\dfrac{\left(S(\beta)-\frac{\eta}{\beta}\right)}{2};\\[2ex] 0, & \text{if } \mathrm{d}(x;\beta)>\dfrac{\left(S(\beta)+\frac{\eta}{\beta}\right)}{2}.\end{cases}$$

Define $\mathcal{F}(z):\mathcal{L}^2\oplus C^m\to D\oplus C^m$ by:

$$\mathcal{F}(z)=\begin{pmatrix}\rho(x)(P'^D(\beta)-z)^{-1}\psi+(P'(\beta,\theta)-z)^{-1}(1-\psi) & R^- \\ R^+\psi & \mathrm{Diag}(z-\mu_j(\beta))\end{pmatrix}$$

where $P'^D(\beta)=\Pi P^D(\beta)\Pi$, Π being the orthogonal projection in $\mathcal{L}^2$ onto the orthogonal complement of the span $\{u_1(\beta),\dots,u_m(\beta)\}$, and $P'(\beta,\theta)$ is the operator constructed in Section 4. In particular $P'(\beta,\theta)$ differs from $P(\beta,\theta)$ only in a compact set independent of β. Note that $(P'^D(\beta)-z)^{-1}$ is holomorphic in $z\in\Omega$, so is $\mathcal{F}(z)$. Making use of Theorem 4.2, we can prove as in [12] that

$$\mathcal{P}(z)\mathcal{F}(z)=I+O\left(\mathrm{e}^{\frac{-S(\beta)}{2}+\frac{\varepsilon(\eta)}{\beta}}\right)\tag{5.2}$$

in the norm of bounded operators on $\mathcal{L}^2(X)\oplus C^m$. This proves the right inverse of $\mathcal{P}(z)$ exists when $\beta>0$ is small enough. Similarly we can prove that the left inverse of $\mathcal{P}(z)$ also exists. So $\mathcal{P}(z)$, $z\in\Omega$, is invertible. Let $\mathcal{S}(z)$ denote the inverse of $\mathcal{P}(z)$. Utilizing (5.2) and the Neumann series for the inverse, we can approximate $\mathcal{S}(z)$ up to any exponentially small order. In particular, if $\mathcal{S}(z)$ is written in the matrix form,

$$\mathcal{S}(z)=\begin{pmatrix}E(z) & E^+(z)\\ E^-(z) & E^{-+}(z)\end{pmatrix}$$

then by a detailed calculus as done in [6] and [12], we can show

$$E^{-+}(z) = \mathrm{Diag}(z - \mu_j(\beta)) + O\left(\mathrm{e}^{-2S(\beta) + \frac{\varepsilon(\eta)}{\beta}}\right). \tag{5.3}$$

As in [6], the resonances of $P(\beta)$ (i.e., the eigenvalues of $P(\beta,\theta)$ in Ω are precisely the zeros of $\det E^{-+}(z)$ in Ω. (5.3) yields the results of Theorem 5.2. ■

Our method for proving Theorem 5.2 also gives precise information on the resonant states (i.e., the eigenfunctions of $P(\beta,\theta)$). For $\varepsilon > 0$ small, define Π by

$$\Pi = -\frac{1}{(2\pi \mathrm{i})} \int_{|z-\lambda_0|=\varepsilon} (P(\beta,\theta) - z)^{-1} \mathrm{d}z.$$

By Theorem 5.2, Rank $\Pi = m$ for $\beta > 0$ small enough. Let ρ be the cut-off function used in the proof of Theorem 5.2. Put $u_j'(\beta,\theta) = \Pi(\rho u_j(\beta))$, $j = 1,\dots,m$. By the construction of the parametrix for $\mathcal{P}(z)$, we have:

$$(P(\beta,\theta) - z)^{-1} = E(z) - E^{+}(z)E^{-+}(z)^{-1}E^{-}(z), \quad \text{(see [6], Appendix)}.$$

By Theorem 4.2, we see that $\|(P(\beta,\theta) - z)^{-1}\| = 0(1)$, for $|z - \lambda_0| = \varepsilon$ and $\beta > 0$ small. From the estimate $(P(\beta,\theta) - z_j(\beta))(\rho u_j(\beta)) = 0\left(\mathrm{e}^{-S(\beta) + \frac{\varepsilon(\eta)}{\beta}}\right)$, we obtain the following

THEOREM 5.3. *For $\beta > 0$ small, $\{u_j(\beta,\theta);\ j = 1,\dots,m\}$ is a basis of* Ran Π *and*

$$\langle u_j(\beta,\theta), u_k(\beta,\theta)\rangle = \delta_{jk} + O\left(\mathrm{e}^{-S(\beta) + \frac{\varepsilon(\eta)}{\beta}}\right) \tag{5.4}$$

$$\|u_k(\beta,\theta) - u_k(\beta)\| = O\left(\mathrm{e}^{-S(\beta) + \frac{\varepsilon(\eta)}{\beta}}\right), \quad k = 1,\dots,m. \tag{5.5}$$

Remark that if λ_0 is simple ($m = 1$), then $u_1(\beta,\theta)$ is an eigenfunction of $P(\beta,\theta)$ and (5.5) says that the eigenfunction of $P^D(\beta)$ gives good approximation for the resonant state.

6. AN ESTIMATE ON THE NUMBER OF RESONANCES

Theorem 4.2 is global in the sense that it implies the spectrum of $P(\beta,\theta)$ is discrete in a large complex domain $E \equiv E(\theta, \lambda_0 + \frac{\eta}{C}, \Sigma - C\eta)$, for some $C > 0$ large enough. In this Section, we propose to study the number of eigenvalues of $P(\beta,\theta)$ in E. Let $\lambda_0 < \Sigma$ and let $\eta > 0$ be small enough so that $\lambda_0 + \frac{\eta}{C} < \Sigma - C\eta$. We assume: $\lambda_0 + \frac{\eta}{C} \notin \sigma(P)$. The number of eigenvalues of P below $\lambda_0 + \frac{\eta}{C}$ is finite.

THEOREM 6.1. *Let N denote the number of eigenvalues of P in $\left]-\infty, \lambda_0 + \frac{\eta}{C}\right[$. Under the assumption (3.7), for $\theta \in B_+$ and for $\beta > 0$ small enough, there are exactly N resonances (counted with their algebraic multiplicity) of $P(\beta)$ inside E.*

REMARK. According to Theorem 5.2, there are at least N resonances of $P(\beta)$ in E, which are exponentially close to the real axis. Theorem 6.1 says that these are the only resonances of $P(\beta)$ inside E.

To prove Theorem 6.1, we shall need the following a priori energy estimate which implies the exponential decay of the resonant states.

TEOREM 6.2. *Put $b = \beta^{\frac{1}{2}}$. For $\theta \in B_+$, there exists $C > 0$ and $\beta_0 > 0$ such that*

$$\|e^{b\langle x\rangle}u\|_{H^1(X)} \leqslant C(\|e^{b\langle x\rangle}(P(\beta,\theta) - z)u\| + \|\chi V u\|) \tag{6.1}$$

for $u \in C_0^\infty(X)$, $0 < \beta \leqslant \beta_0$ and $z \in E$. Here χ is a cut-off function with support contained in a compact set independent of z and β.

Proof. Let $P'(\beta,\theta)$ be the operator introduced in Section 4. Then

$$\|u\|_{H^1(X)} \leqslant C\|(P'(\beta,\theta) - z)u\|, \quad \text{for all } u \in D \text{ and } z \in E. \tag{6.2}$$

Put $P'_b(\beta,\theta) = e^{b\langle x\rangle}P'(\beta,\theta)e^{-b\langle x\rangle}$. Then $P'_b(\beta,\theta) = P'(\beta,\theta) + Q$, where Q is a first order operator with coefficients of the order $0(b)$. So (6.2) is still valid for $P'_b(\beta,\theta)$, so long as $b > 0$ is small enough:

$$\|u\|_{H^1(X)} \leqslant C'\|(P'_b(\beta,\theta) - z)u\|, \quad \text{for all } u \in D \text{ and } z \in E. \tag{6.3}$$

Now for $u \in C_0^\infty(X)$, replacing u by $e^{b\langle x\rangle}u$ in (6.3), we obtain

$$\|e^{b\langle x\rangle}u\|_{H^1(X)} \leqslant C'\|e^{b\langle x\rangle}(P'(\beta,\theta) - z)u\| \leqslant$$

$$\leqslant C'(\|e^{b\langle x\rangle}(P(\beta,\theta) - z)u\| + \|(1-\rho_R)Vu\|)$$

where ρ_R is the cut-off function used in Section 4. Theorem 6.2 is proven. ■

By an argument of density, we see that Theorem 6.2 remains true for $u \in D$ such that $e^{b(x)}(P(\beta,\theta)-z)u \in \mathcal{L}^2$. This will be used in the proof of the following corollary:

COROLLARY 6.3. *There exists $C > 0$ such that if u is a normalized eigenfunction of $P(\beta,\theta)$: $P(\beta,\theta)u = zu$, $z \in E$, then one has for $\beta > 0$ small enough,*

$$\|e^{b(x)}u\|_{H^1} \leqslant C\langle z\rangle.$$

Proof. By the assumptions on V, one has for any $\varepsilon > 0$,

$$\|Vw\| \leqslant \varepsilon\|\Delta w\| + C_\varepsilon\|w\| \leqslant 2\varepsilon\|(P_0(\beta,\theta) - z)w\| + C'(\|\beta X_1 w\| + |z|\,\|w\|).$$

The above estimates yield:

$$\|Vw\| \leqslant 4\varepsilon\|(P(\beta,\theta) - z)w\| + C\|(\langle X_1\rangle + \langle z\rangle)w\|, \quad \text{for all } w \in D \text{ and } z \in E. \tag{6.4}$$

Now let u be a normalized eigenfunction of $P(\beta,\theta)$ associated with eigenvalue $z \in E$. Applying Theorem 6.2 and (6.4) $w = (1 - \rho_R)u$, we obtain:

$$\|e^{b(x)}u\|_{H^1} \leqslant 0(\varepsilon)\|[P_0(\beta,\theta),\rho_R]u\| + C\langle z\rangle \leqslant 0\left(\frac{\varepsilon}{R}\right)\|u\|_{H^1} + C\langle z\rangle.$$

Corolarry 6.3 follows if we take $\varepsilon > 0$ sufficiently small (or $R > 1$ sufficiently large). ■

Remark that the result in Corollary 6.3 is not optimal, but it is sufficient for obtaining Theorem 6.1. In two-body case more precise results on the decay of resonant states have been obtained in [13].

Proof of Theorem 6.1. By computing the numerical range of $P(\beta,\theta)$, we can show as in [12] that $\sigma(P(\beta,\theta)) \subset \{z \in \mathbf{C}; z = e^\theta t + s,\ t \in \mathbf{R},\ s > -C\}$, for some $C > 0$. See Proposition 3.1 in [12]. Note that in [12], some particular structure of $P_0(\beta,\theta)$ was used in order to simplify the computation of the numerical range of distorted Stark Hamiltonian. In the present work, $P_0(\beta,\theta)$ has the similar structure, due to the choice of χ (see Section 3). So we only have to estimate the number of the eigenvalues in region $E_1 = E \cap (\mathbf{C} \setminus \{z = e^\theta t + s;\ t \in \mathbf{R},\ s > -C\})$, which is a compact independent of $\beta > 0$, due to the choice of E. Let $B(\varepsilon)$ denote an ε-neighbourhood of $\sigma(P) \cap \left]-\infty, \lambda_0 + \frac{\eta}{C}\right[$ in $\mathbf{C}$. By Theorem 5.2, if $\varepsilon > 0$ is small enough and $0 < \beta < \beta_\varepsilon$, there are exactly N resonances of $P(\beta)$ in $B(\varepsilon)$. We want to show that in $E_2 \equiv E_1 \setminus B(\varepsilon)$, there is no resonance of $P(\beta)$, provided $\beta > 0$ is small enough. Assume there were $z_0 \in E_2$ and $u \in D$, $\|u\| = 1$, such that $P(\beta,\theta)u = z_0 u$.

Take a cut-off function ψ such that $\operatorname{supp}\psi \subset M$, $0 \leqslant \psi \leqslant 1$, $\psi(x) = 1$ for $|x| < \frac{\varepsilon_0}{\beta}$, for some $\varepsilon_0 > 0$ and the derivatives of ψ are of order $0(\beta)$. Put $w = \psi u$. By Corollary 5.3,

$$\|w\| = 1 + 0(e^{\frac{-c}{b}}), \quad \text{and since } P(\beta,\theta) = P(\beta) \text{ on } M,$$

$$(P - z_0)w = 0(b), \quad \text{where } b = \beta^{\frac{1}{2}},$$

for some $c > 0$. This means there exists $C' > 0$ such that $\sigma(P) \cap \{z; \exists z_0 \in E_2, \text{s.t.} |z - z_0| < C'b\}$ is nonvoid. This is impossible if $\beta_0 > 0$ is small enough s.t. $C'b < \varepsilon$ for $\beta \leqslant \beta_0$, because $\operatorname{dist}(\sigma(P), E_2) \geqslant \varepsilon$. This proves Theorem 6.1. ■

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