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A MULTI-VARIABLE BERGER-SHAW THEOREM

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0. INTRODUCTION

A Hilbert space operator that commutes with its adjoint is said to be normal. On a finite dimensional space a normal operator can be diagonalized, while the spectral theorem provides multiplication as a model in the general case. Since such operators can be viewed as being reasonably well understood, it is natural to consider operators which are "almost normal". Many authors, including Livsitz ([12]), Carey and Pincus ([4]), Brown–Douglas–Fillmore ([3]) and Helton–Howe ([10]), sought a structure theory for almost normal operators. Berger and Shaw, however, obtained a surprising and unexpected result ([1]) which showed that a certain natural class of operators consisted of almost normal operators.

For operators S and T we let $[S,T]$ denote the commutator $[S,T] = ST - TS$ and $[T^*,T]$ the self-commutator of T. Since the operator T is normal if $[T^*,T] = 0$, one usually interprets "almost normal" to mean that $[T^*,T]$ is small in some sense. In the work referred to above, $[T^*,T]$ was finite rank, trace-class, or compact.

An operator T on $\mathcal{H}$ is said to be *hyponormal* if $[T^*,T]$ is positive and subnormal if $T = N|\mathcal{H}$, where N is a normal operator on space $\mathcal{K}$ which contains $\mathcal{H}$. Moreover, T is said to possess a finite rational cyclic set if there exist vectors $v_1, v_2, \ldots, v_n$ in $\mathcal{H}$ such that

$$\left\{ \sum_{i=1}^{n} r_i(T)v_i : r_i \in \mathrm{Rat}\,(\sigma(T)) \right\}$$

is dense in $\mathcal{H}$. Here $\mathrm{Rat}\,(X)$ denotes the rational functions with poles off X.

We can now state the theorem of Berger and Shaw.

THEOREM (Berger-Shaw). *An operator which is hyponormal and possesses a finite rational cyclic set has a trace-class self-commutator.*

Since much of the interest in almost normal operators concerned commuting N-tuples, it was natural to look for a generalization of this theorem to the multi-variable case. If we consider the N-tuple of coordinate multipliers on the Hardy space for the polydisk, we see that no straightforward generalization is possible. In this paper we show that the commutators are still trace-class for a commuting N-tuple if we assume that the joint spectrum lies on an algebraic curve. The proof requires techniques from commutative algebra. Before we can apply these, we must place the Berger-Shaw Theorem in the more algebraic framework of Hilbert modules (cf. [8]).

If T is an operator on the Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is a module over $\text{Rat}\,(\sigma(T))$ with the multiplication defined by

$$r(z) \times h = r(T)h.$$

Taking the norm closure of the operators defined by this module action we obtain a commutative Banach algebra A and $\mathcal{H}$ is a Hilbert module over A. Moreover, A contains a homomorphic image of $\text{Rat}\,(\sigma(T))$ and if T is hyponormal, then multiplication by z in $\text{Rat}\,(\sigma(T))$ corresponds to a hyponormal operator. We capture this situation in the following definition.

DEFINITION 1. Let A be a commutative Banach algebra containing a dense homomorphic image B of $\text{Rat}\,(X)$. A Hilbert module $\mathcal{H}$ over A is said to be *hyponormal* if the operator corresponding to z in $\text{Rat}\,(X)$ is hyponormal.

If an operator has a trace-class self-commutator, then commutators of rational functions in the operator will also be trace-class. Thus it is possible to express the conclusion of the Berger-Shaw Theorem more globally. To that end we introduce another definition.

DEFINITION 2. Let A be a commutative Banach algebra and B be a dense subalgebra. A Hilbert module $\mathcal{H}$ for A is said to be *p-reductive for B* if $[b_1^*, b_2]$ is in the Schatten p-class C_p for b_1, b_2 in B.

Thus we can restate the Berger-Shaw Theorem as follows:

THEOREM 1′. *Let A be a commutative Banach algebra containing a dense subalgebra B which is a homomorphic image of* $\text{Rat}\,(\Omega)$ *for the bounded planar domain Ω. A finitely generated hyponormal module over A is 1-reductive for B.*

Now we attempt analogous notions in the multivariate context. In the context of operator N-tuples, the following seems a natural generalization of hyponormality (cf. [6]).

DEFINITION 3. The operator N-tuple $T = (T_1, T_2, \dots, T_N)$ is *hyponormal* if the operator defined by the matrix $([T_j^*, T_i])$ is positive.

We can now generalize the notion of hyponormal module.

DEFINITION 4. Let $\mathcal{H}$ be a Hilbert module over the Banach algebra A. Assume there is a dense subalgebra B of A, such that B is Noetherian and that there exists a set of generators $(b_1, b_2, \ldots, b_N)$ for B which forms a hyponormal (joint subnormal) N-tuple. Then $\mathcal{H}$ is said to be a *hyponormal* (*subnormal*) module over A.

As we stated earlier, a generalization of the Berger-Shaw Theorem to arbitrary hyponormal modules is not possible. The example of the Hardy module $H^2(\mathbb{D}^N)$ over $A(\mathbb{D}^N)$ with dense subalgebra $\mathbb{C}[z_1, z_2, \ldots, z_N]$ shows that the validity of the Berger-Shaw Theorem depends on the algebraic properties of the Noetherian ring B. This is the place where techniques from algebraic geometry and commutative ring theory come in. For complete reference we refer to [11], [14].

We begin by recalling some of the definitions that will be used in stating our generalization of the Berger-Shaw Theorem.

DEFINITON 5. For R a commutative ring, the Krull dimension of R, denoted $\dim R$, is defined to be the length of a maximal chain of prime ideals,

$$\dim R = \max\{\ell \mid p_0 \subset p_1 \subset \cdots \subset p_\ell \subset R,\ p_i \text{ prime ideals}\}.$$

DEFINITION 6. A subset V of $\mathbb{C}^n$ is said to be an algebraic closed set if V is the set of common zeroes of a set of polynomials. If $I(V)$ denotes the set of polynomials that vanish on V, then V is said to be an *algebraic curve* if $\dim \mathbb{C}[z_1, \ldots, z_n]/I(V) = 1$.

Noether's normalization theorem enables one to treat an N-dimensional ring as a polynomial ring in N-variables via a finite to one covering map.

In case $\dim B = 1$, the normalization theorem enables us to "reduce" B to $\mathbb{C}[z]$, where the Berger-Shaw Theorem is known to be true. In order to make this reduction work, we have to modify and extend Noether's theorem as follows.

THEOREM (Noether's Normalization Theorem). *Let R be a Noetherian ring over the complex number field $\mathbb{C}$ having $\dim R = n$ with generators $\{x_1, \ldots, x_m\}$.*

1) *Then there exists a complex $n \times m$ matrix $A = (a_{ij})$ such that R is integral over* $\mathbb{C}\left[\sum_{j=1}^{m} a_{1j}x_j, \ldots, \sum_{j=1}^{m} a_{nj}x_j\right]$.

2) *If E is the linear span in $\mathbb{C}^m$ of the row vectors of those $n \times m$ matrix $A = (a_{ij})$ such that R is integral over* $\mathbb{C}\left[\sum_{j=1}^{m} a_{1j}x_j, \ldots, \sum_{j=1}^{m} a_{nj}x_j\right]$, *then $\dim E = m$.*

Proof. To prove this version of Noether's theorem, we need to re-examine the proof of Noether's original theorem as given in ([11], p.18), noting that the ground field $\mathbb{C}$ in our case is infinite.

Let F be any nonzero polynomial in $\mathbb{C}[z_1,\dots,z_n]$. The crucial step in the proof of Noether's normalization theorem ([11], Lemma 1.6, p. 18) is to prove for any F in $\mathbb{C}[z_1,\dots,z_n]$ one can find polynomials $Y_2,\dots,Y_n$ such that $\mathbb{C}[z_1,\dots,z_n]$ is integral over $\mathbb{C}[F,Y_1,\dots,Y_n]$. When this is established, the normalization theorem follows by induction on the number of generators of R over $\mathbb{C}$ and by the observation that such an R is a quotient ring of a polynomial ring over $\mathbb{C}$. In our case, the ground field is $\mathbb{C}$, the polynomials $Y_2,\dots,Y_n$ mentioned above can be taken to be linear expressions of $X_1,\dots,X_m$ and then the same proof in [11] will yield the above theorem. For completeness we show below how the Y_i's can be replaced by linear expressions. If we let $Y_2 = X_2 - \alpha_2 X_1,\dots,Y_n = X_n - \alpha_n X_1$, then the polynomial

$$H(T) = F(T, Y_2 + \alpha_2 T,\dots,Y_n + \alpha_n T) - F \text{ is in } \mathbb{C}[Y_2,\dots,Y_n][T]$$

and $H(X_1) = 0$. If we expand $H(T)$ in powers of T, then

$$H(T) = p(\alpha_2,\dots,\alpha_n)T^{\delta} + \text{lower order terms}$$

where p is a nonzero polynomial. Since $\mathbb{C}$ is an infinite field, $V(p)$ is an exceptional set in $\mathbb{C}^{n-1}$. So we conclude that except for an exceptional set of $(\alpha_2,\dots,\alpha_n)$ in $\mathbb{C}^{n-1}$, there exists a monic polynomial in $\mathbb{C}[Y_2,\dots,Y_n][T]$ having X_1 as its root.

In case $\dim R = 1$, there exist a finite set of vectors ($1 \times m$ matrices), denoted by $(a_1^1,\dots,a_m^1),\dots,(a_1^k,\dots,a_m^k)$, such that

1) R is integral over $\mathbb{C}[a_1^i x_1 + \cdots + a_m^i x_m]$ for each i and $1 \leqslant i \leqslant k$, and

2) each x_j is a linear combination of $\left\{\sum a_j^i x_j\right\}_{i=1}^{k}$.

If R is the coordinate ring $\mathbb{C}[z_1,\dots,z_n]/I$, where $\sqrt{I} = I(U)$, for U an algebraic curve, then there exists $(a_{ij})_{i=1,j=1}^{km}$ such that if $g^i = \sum_{j=1}^{m} a_{ij} z_j$, $i = 1,\dots,k$ then

1) $\mathbb{C}[z_1,\dots,z_n]$ is integral over $\mathbb{C}[I, g^i]$, and

2) each z_j is a linear combination of the g^i's.

1. MAIN RESULT

THEOREM 1. *Let $\mathcal{H}$ be a finitely generated hyponormal Hilbert module over A with the dense subalgera B. If the Krull dimension $\dim B = 1$, then $\mathcal{H}$ is 1-reductive for B.*

This theorem has many corollaries.

THEOREM 2. *Let $T = (T_1, \ldots, T_n)$ be a joint hyponormal N-tuple of operators on the Hilbert space $\mathcal{H}$, and I be the ideal of $\mathbb{C}[z_1, \ldots, z_n]$ defined by*

$$I = \{p \mid p(T_1, \ldots, T_n) = 0\}.$$

If $\mathcal{H}$ is finitely generated over T and $\mathbb{C}[z_1, \ldots, z_n]/I$ has Krull dimension 1, then $[T_i^*, T_j]$ is trace class for all i, j.

If we assume in the above theorem that T is jointly subnormal, then the condition on the vanishing ideal I is precisely that the Taylor spectrum of T is contained in an algebraic curve. This is because if T is subnormal, then $p(T)$ is subnormal for all polynomials p. If the spectrum of T is contained in an algebraic curve U, then for any p in $I(U)$, $\sigma(p(T)) = p(\sigma(T)) = \{0\}$, and hence $p(T) = 0$. So the vanishing ideal I of T contains $I(U)$ and $\dim \mathbb{C}[z_1, \ldots, z_n]/I \leqslant \dim \mathbb{C}[z_1, \ldots, z_n]/I(U) = 1$. On the other hand, if $\dim \mathbb{C}[z_1, \ldots, z_n]/I \leqslant 1$ since $\sqrt{I} = I$, it follows that there is an algebraic closed set U, $\dim U \leqslant 1$, such that $I = I(U)$. For any λ in $\mathrm{Sp}(T)$ and p in $I(U)$, we have $p(\lambda)$ in $\sigma(p(T)) = \{0\}$. Hence by the Hilbert Nullstellen Satz we have λ in U, and hence $\mathrm{Sp}(T) \subset U$. Thus we obtain the following:

THEOREM 3. *Let $S = (S_1, \ldots, S_n)$ be a joint subnormal n-tuple on $\mathcal{H}$. If $\mathcal{H}$ is finitely generated over S and the Taylor spectrum of S is contained in an algebraic curve, then $[S_i^*, S_j]$ is trace class for all i, j.*

To prove Theorem 1, we must make several reductions. We begin by reducing the proof of the main theorem to that of Theorem 2.

Suppose $\mathcal{H}$, A and B are as in Theorem 1. Since B is a Noetherian ring, by definition we can select a set of generators $(b_1, \ldots, b_n)$ acting as a joint hyponormal operator n-tuple on $\mathcal{H}$. It is easily seen that $\mathcal{H}$ is still finitely generated over $(b_1, \ldots, b_n)$. Let X be the homomorphism:

$$X : \mathbb{C}[z_1, \ldots, z_n] \to B$$

defined by $X(z_i) = b_i$, $i = 1, \ldots, n$. Then X is surjective and if we denote $\ker X$ by I, then I is just the vanishing ideal of $(b_1, \ldots, b_n)$ and

$$\mathbb{C}[z_1, \ldots, z_n]/I \cong B.$$

By assumption, $\dim B = \dim \mathbb{C}[z_1, \ldots, z_n]/I = 1$ and thus we have reduced the proof of the main theorem to that of Theorem 2. To accomplish this we need some elementary lemmas:

LEMMA 1. *If $T = (T_1, \ldots, T_n)$ is jointly and $g_i(T) = \sum_{j=1}^{n} a_{ij} T_j$, $i = 1, \ldots, m$, where the a_{ij}'s are complex scalars, then $(g_1(T), \ldots, g_m(T))$ is also a jointly hyponormal m-tuple.*

Proof. We first determine a new product on $B(\mathcal{H})$. For A, B in $B(\mathcal{H})$, define $A \circ B = [B, A] = BA - AB$. It is easy to see that the product $\circ$ is bilinear,

$$(X+Y)\circ Z = X\circ Z + Y\circ Z$$
$$(\alpha X)\circ Y = \alpha(X\circ Y)$$
$$X\circ Y = -Y\circ X.$$

We can extend this binary operation to the matrix algebra over $B(\mathcal{H})$:

$$\circ : M_{m\times k}(B(\mathcal{H})) \times M_{k\times n}(B(\mathcal{H})) \to M_{m\times n}(B(\mathcal{H})).$$

Then the condition for $T = (T_1, \ldots, T_n)$ to be hyponormal is simply

$$\begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \circ (T_1^*, \ldots, T_n^*) \geqslant 0.$$

We now prove that $(g_1(T), \ldots, g_m(T))$ is also a jointly hyponormal m-tuple. Since

$$\begin{pmatrix} g_1(T) \\ \vdots \\ g_m(T) \end{pmatrix} = (a_{ij}) \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$$
$$(g_1(T)^*, \ldots, g_m^*(T)) = (T_1^*, \ldots, T_m^*)(a_{ij})^*,$$

it follows that

$$\begin{pmatrix} g_1(T) \\ \vdots \\ g_m(T) \end{pmatrix} \circ (g_1(T)^*, \ldots, g_m(T)^*) = (a_{ij}) \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \circ (T_1^*, \ldots, T_m^*)(a_{ij})^* \geqslant 0,$$

by the hyponormality of T. This proves that $(g_1(T), \ldots, g_m(T))$ is a joint hyponormal m-tuple. ∎

LEMMA 2. *If $T = (T_1, \ldots, T_n)$ is a jointly hyponormal n-tuple and $[T_i^*, T_i]$ is in $\mathcal{C}_p$ for some p and $i = 1, 2, \ldots, n$, then $[T_i^*, T_j]$ is in $\mathcal{C}_p$ for all i, j.*

Proof. By the hyponormality of $T = (T_1, \ldots, T_n)$,

$$([T_j^*, T_i]) \geqslant 0.$$

In the C^*-algebra $M_{n\times n}(B(\mathcal{H}))$, any nonnegative element can always be written in the form X^*X for some $X = (x_{ij})$ in $M_{n\times n}(B(\mathcal{H}))$, or

$$X^*X = ([T_j^*, T_i]),$$

that is,

$$\sum_{k=1}^{n} X_{ki}^{*} X_{kj} = [T_j^*, T_i] \text{ for all } i, j. \tag{1.1}$$

When $i = j$, we have

$$\sum_{k=1}^{n} X_{ki}^{*} X_{kj} = [T_i^*, T_i] \text{ is in } \mathcal{C}_p,$$

by assumption, so X_{ki} is in $\mathcal{C}_{2p}$ for all i and k. Then we conclude from (1.1) that all $[T_j^*, T_i]$'s are $\mathcal{C}_p$ elements. ∎

Proof of Theorem 2. We will first prove for any linear homogeneous polynomial $g(z_1, \ldots, z_n) = a_1 z_1 + \cdots + a_n z_n$, if $\mathbb{C}[z_1, \ldots, z_n]$ is integral over $\mathbb{C}[I, g]$, then $[g(T)^*, g(T)]$ must be in $\mathcal{C}_1$.

Let $v_1, \ldots, v_s$ be the vectors in $\mathcal{H}$ such that

$$\{p_1(T)v_1 + \cdots + p_s(T)v_s \mid p_i \in \mathbb{C}[z_1, \ldots, z_n]\}$$

is dense in $\mathcal{H}$. Since $\mathbb{C}[z_1, \ldots, z_n]$ is integral over $\mathbb{C}[I, g]$, $\mathbb{C}[z_1, \ldots, z_n]$ is a $\mathbb{C}[I, g]$-module of finite dimension. Thus there are $f_1, \ldots, f_t$ in $\mathbb{C}[z_1, \ldots, z_n]$ such that for any polynomial p in $\mathbb{C}[z_1, \ldots, z_n]$, there are elements $q_1, \ldots, q_t$ from $\mathbb{C}[I, g]$ such that

$$p = q_1 f_1 + \cdots + q_t f_t. \tag{1.2}$$

Let $\mathbb{C}[g(T)]$ be the ring of operators which are polynomials of $g(T)$. For any q in $\mathbb{C}[I, g]$, it is not hard to see $q(T)$ is in $\mathbb{C}[g(T)]$, because the ideal I annihilates the operator n-tuple T.

If we can prove that the hyponormal operator $g(T)$ is finitely cyclic on $\mathcal{H}$, then by the Berger-Shaw theorem we conclude that $[g^*(T), g(T)]$ is in $\mathcal{C}_1$. To see this, we define $v_{ij} = f_j(T)v_i$ for $i = 1, \ldots, s$, $j = 1, \ldots, t$ and claim that

$$\left\{ \sum_{i=1,\ j=1}^{s,t} \mathbb{C}[g(T)]v_{ij} \right\} \text{ is dense in } \mathcal{H}.$$

For polynomials $p_1, \ldots, p_s$ in $\mathbb{C}[z_1, \ldots, z_n]$ by (1.2) there are q_{ij}'s in $\mathbb{C}[I, g]$, $i = 1, \ldots, s$, $j = 1, \ldots, t$ such that

$$p_i = \sum_{j=1}^{t} q_{ij} f_j, \ i = 1, \ldots, s.$$

Then

$$p_i(T) = \sum_{j=1}^{t} q_{ij}(T) f_j(T), \ i = 1, \ldots, s$$

and

$$\sum_{i=1}^{s} p_i(T)v_i = \sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}(T)f_j(T)v_i =$$
$$= \sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}(T)v_{ij} \in \left\{ \sum_{i,j} \mathbb{C}[g(T)]v_{ij} \right\}.$$

The last inclusion is because $q_{ij}(T)$ is in $\mathbb{C}[g(T)]$. Thus we have

$$\left\{ \sum_{i=1}^{s} p_i(T)v_i \mid p_i \in \mathbb{C}[z_1, \ldots, z_n] \right\} \subset \left\{ \sum_{i,j} \mathbb{C}[g(T)]v_{ij} \right\}$$

and this completes the proof that $[g(T)^*, g(T)]$ is in $\mathcal{C}_1$.

Since, by the discussion at the end of introduction, we can find linear homogeneous polynomials $g_1, \ldots, g_m$ such that the z_i's are linear combinations of $g_1, \ldots, g_m$, and $\mathbb{C}[z_1, \ldots, z_n]$ is integral over $\mathbb{C}[I, g_i]$ for $i = 1, \ldots, m$, there exist (a_{ij}) such that

$$z_i = \sum_{j=1}^{m} a_{ij} g_j, \ i = 1, \ldots, n.$$

Applying Lemma 1 to the operator m-tuple $(g_1(T), \ldots, g_m(T))$, it follows that $(g_1(T), \ldots, g_m(T))$ is a jointly hyponormal m-tuple. By the first part of this proof we know that $[g_i(T)^*, g_i(T)]$ is in $\mathcal{C}_1$ for all $i = 1, \ldots, m$, and then by Lemma 2, the $[g_i(T)^*, g_j(T)]$'s are in $\mathcal{C}_1$ for all i, j.

Since $T_i = \sum_{j=1}^{m} a_{ij} g_j(T)$, $i = 1, \ldots, m$, we have

$$[T_i^*, T_j] = \left[\left(\sum_{k=1}^{n} a_{ik} g_k(T) \right)^*, \sum_{\ell=1}^{n} a_{j\ell} g_\ell(T) \right] =$$
$$= \sum_{k=1}^{n}\sum_{\ell=1}^{n} \bar{a}_{ik} a_{j\ell} [g_k(T)^*, g_\ell(T)] \text{ is in } \mathcal{C}_1.$$

This completes the proof. ■

Our result can also be stated in the language of Helton-Howe [10] as follows.

THEOREM 4. *Let $\mathcal{H}$ be a hyponormal finitely generated Hilbert module over A having the dense subalgebra B with $\dim B = 1$. If $C^\infty(B) = \{T \mid T$ is a C^∞ function of elements in $B\}$, then $C^\infty(B)$ is a one-dimensional crypto integral algebra.*

A consequence of this is that one can define naturally a cyclic cocycle ([5]) on $C^\infty(B)/\mathcal{C}_1$ using the module $\mathcal{H}$. We plan to study this cyclic cocycle in another paper.

2. TRANSCENDENTAL CASE

The theorems we proved in the last section are very algebraic in nature. Besides the hyponormality and finite rank assumptions, the essential geometric condition is that the spectrum of the module be contained in an algebraic curve. A natural question is what about analytic curves or even two-dimensional spaces? We begin this section with an example.

Consider the pair of analytic Toeplitz operators T_z and T_θ acting on $H^2(D)$, where θ is the singular inner function with unit mass at 1. We make $H^2(D)$ into a contractive Hilbert module over $A(D^2)$ using (T_z, T_θ). An easy calculation shows that the module spectrum is the closure in C^2 of the graph

$$\{(z, \theta(z)) : z \in D\},$$

which has Hausdorff dimension two and is analytic except for a singularity at $(1,1)$. Finally, one can show that the self-commutator $[T_\theta, T_\theta^*]$ is not even compact.

This example shows that no generalization of the Berger-Shaw theorem is possible under the hypothesis that the module spectrum has two-dimensional Hausdorff measure. Such a result might hold, however, if one assumes that the module spectrum is contained in a compact two-dimensional manifold. The example above shows also that the appropriate notion of analyticity for the theorem to hold might be subtle. In what follows we obtain a generalization of the Berger-Shaw theorem by assuming that the module spectrum is a compact subset of an analytic curve.

DEFINITION 7 ([9]). An analytic space is a topological ringed space $(X, \mathcal{O})$, that is, a structure sheaf $\mathcal{O}$ on a topological space X, such that locally it is isomorphic to the ringed space for an analytic set in a polydisk.

A one-dimensional analytic space is called an *analytic* curve.

Let $(X, \mathcal{O})$ be an analytic space. We can naturally construct a sheaf $\tilde{\mathcal{O}}$ on X so that each stalk $\tilde{\mathcal{O}}_z$ at a point z is the quotient field on $\mathcal{O}_z$. A function on X is called meromorphic if it is a global cross section of $\tilde{\mathcal{O}}$, or an element of $\tilde{\mathcal{O}}(X)$. It is easy to define poles and zeroes of a meromorphic function by checking the valuation in the local rings $\tilde{\mathcal{O}}_z$.

One important theorem about analytic spaces is:

PROPOSITION ([9]). *Let $(X, \mathcal{O})$ be a compact analytic space of dimension n and $M(X)$ be the field of all meromorphic functions on X. Then $M(X)$ is a field of algebraic functions and the transcendental degree of $M(X)$ is less than or equal to n.*

It is known that any compact analytic curve is the analytic space of an algebraic curve. If X is a compact analytic curve, then $M(X)$ is a field of transcendental degree

1 over $\mathbb{C}$. That is, for any nonalgebraic element z in $M(X)$, $\mathbb{C}(z)$ is a subfield of $M(X)$ which is isomorphic to the field of rational functions in one variable, and $M(X)$ is a finite dimensional vector space over $\mathbb{C}(z)$. $M(X)$ is an algebraic extension of $\mathbb{C}(z)$ and by the primitive element theorem, there exists an element f in $M(X)$ such that

$$M(X) = \mathbb{C}(z)(f)$$

where f is algebraic over $\mathbb{C}(z)$. The point of using an analytic curve is that the definition of these spaces is coordinate-free.

Now we will define modules with spectrum contained in an analytic curve. There is a coordinate-free definition of the spectrum of a Hilbert module over A, [8], but the inclusion of this module spectrum in an analytic curve is not simply a topological embedding. Rather it must be consistent with the ringed structure of the spaces. We will see that it is the Hilbert module language that makes possible this coordinate-free version.

DEFINITION 8. Let $\mathcal{H}$ be a Hilbert module over A. If there exists an analytic space $(X, \mathcal{O})$, a compact subset $Y \subset X$, and an injection i:

$$i : \tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y) \to A$$

such that $B = i(\tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y))$ is dense in A, then we say that $\mathcal{H}$ has spectrum contained in X.

The algebra $\tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y)$ in the above definition is the ring of meromorphic functions which are holomorphic on Y. When X is an analytic curve, $\tilde{\mathcal{O}}(X)$ is a field of algebraic functions and is of transcendental degree 1, but the subring $\tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y)$ of $\tilde{\mathcal{O}}(X)$ may not be a ring of Krull dimension 1. An example of this is the following: if X is the projective plane $\mathbb{CP}^1$ and Y be an annulus on $\mathbb{C}$ considered as a subset of $\mathbb{CP}^1$, then $\tilde{\mathcal{O}}(X) \simeq \mathcal{O}(z)$, $\tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y) = \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[z], g \mid Y \neq 0 \right\}$. Thus the hypothesis in Theorem 1 is not satisfied in this transcendental case, but we still have the following:

THEOREM 5. *Let $\mathcal{H}$ be a Hilbert module over A which is finitely generated and subnormal. If the spectrum of $\mathcal{H}$ is contained in a compact analytic curve, then $[b_1^*, b_2]$ is trace class for b_1, b_2 in B, where B is given in Definition 8.*

Proof. Since for any pair $b = (b_1, b_2)$, b is a jointly subnormal pair, to prove that $[b_i^*, b_j]$ is in trace class, by Lemma 2, we only need to prove that

$$[b_1^*, b_1] \text{ is trace class for each } b \text{ in } B.$$

Let b be an element of B. If b is an algebraic element, that is, $p(b) = 0$ for some polynomial, then b must be normal. So we consider only the case when b is not algebraic. In that case, let z in $\tilde{B} = \tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y)$ be such that $i(z) = b$. Then z is not algebraic in $\tilde{B}$ so $M(X) = \tilde{\mathcal{O}}(\tilde{X})$ is algebraic over $\mathbb{C}(z)$. Consequently the quotient field $\tilde{\tilde{B}}$ of $\tilde{B}$ is also an algebraic extension of $\mathbb{C}(z)$, and by the primitive element theorem [13], there is an element g in $\tilde{B}$ such that:

$$\tilde{\tilde{B}} = \mathbb{C}(z)(g).$$

If p is the minimal polynomial of g in $\mathbb{C}(z)[x]$ with $\deg p = k$, then

$$\tilde{\tilde{B}} \cong \mathbb{C}(z)(g)/(p(x)) \simeq \left\{ \sum_{i=0}^{k-1} a_i g^i \mid a_i \in \mathbb{C}(z) \right\}.$$

Since $\tilde{B} \subset \tilde{\tilde{B}}$, then for any element w in $\tilde{B}$ there are polynomials p_i, q_i, $i = 1, \ldots, k-1$, in $\mathbb{C}[z]$ such that

$$w = \sum_{i=0}^{k-1} \frac{p_i}{q_i} g^i.$$

Thus we can conclude that for any w in $\tilde{B}$ there are polynomials $p, q_0, \ldots, q_{k-1}$ in $\mathbb{C}[z]$ such that

$$p(z)w = \sum_{i=0}^{k-1} q_i(z) g^i.$$

By applying i to both sides of the above equation, we have that for any f in B there exist polynomials $p, q_0, \ldots, q_{k-1}$ such that

$$p(b)f = q_0(b)g_0 + \cdots + q_{k-1}(b)g_{k-1},$$

where $g_j = i(g^j)$ for $j = 0, \ldots, k-1$.

With the above preparations in algebra, we can prove that $[b^*, b]$ is trace class by proving that the subnormal operator b on $\mathcal{H}$ is effectually finite cyclic.

Since $\mathcal{H}$ is finitely cyclic over A, so is $\mathcal{H}$ over B. Let $w_1, \ldots, w_s$ be the generators. Then

$$\mathcal{H}' = \left\{ \sum_{i=1}^{s} f_i w_i \mid f_i \in B \right\}$$

is dense in $\mathcal{H}$. If we define $w_{ij} = g_j w_i$ for $i = 1, \ldots, s$, $j = 0, \ldots, k-1$, then for any element $\sum_{i=1}^{s} f_i w_i$ in the dense subspace $\mathcal{H}'$, there exist polynomials p, q_{ij}, $i = 1, \ldots, s$, $j = 0, \ldots, k-1$ such that

$$p(b) \left(\sum_{i=1}^{s} f_i w_i \right) = \sum_{ij} q_{ij}(b) w_{ij}.$$

This shows that b is effectually finite cyclic and we can conclude our result by applying the following theorem of Berger and Shaw.

THEOREM (Berger-Shaw [2]). *Let A be a hyponormal operator such that A is effectually rational cyclic. That is, such that there exist $w_1, \ldots, w_n$ in $\mathcal{H}$ for which*

$$\mathcal{L} = \left\{ f \in \mathcal{H} \mid \text{there exists } r, r_1, \ldots, r_n \in \operatorname{Rat}(\sigma(A)), r(A)f = \sum_{i=1}^{n} r_i(A) w_i \right\}$$

is dense in $\mathcal{H}$. Then

$$\operatorname{tr}[A^*, A] \leqslant \frac{n}{\pi} m_2(\sigma(A)).$$

3. EXAMPLES

The most natural examples arise from multiplication operators acting on the Hardy or Bergman space of an algebraic or analytic curve.

EXAMPLE 1. Let V be an algebraic curve in $\mathbb{C}^n$, E be an algebraic vector bundle over V and let

$$P(E, V) = \{\text{all algebraic cross sections of } E\}.$$

It is easy to see that $P(E, V)$ is a finitely generated module over $\mathbb{C}[z_1, \ldots, z_n]$. If, in addition, we choose a Riemannian metric on E and a compactly supported finite measure μ on V, then there is a naturally defined $L^2(E, \mathrm{d}\mu)$ space and we define the Hardy space

$$P^2(E\mathrm{d}\mu) = L^2\text{-closure of } P(E, V) \text{ in } L^2(E, \mathrm{d}\mu).$$

Obviously, $L^2(E, \mathrm{d}\mu)$ becomes a reductive module over $\mathbb{C}[z_1, \ldots, z_n]$ and $P^2(E, \mathrm{d}\mu)$ becomes a subnormal finitely cyclic module over $C[z_1, \ldots, z_n]$. If $M_z = (M_{z_1}, \ldots, M_{z_n})$ on $P^2(E, \mathrm{d}\mu)$ where $(M_{z_i} f)(z) = z_i f(z)$, $i = 1, \ldots, n$ for f in $P^2(E, \mathrm{d}\mu)$, then our result shows that $[M_{z_j}^*, M_{z_i}]$ is a trace class operator for all i, j.

EXAMPLE 2. Let X be a compact analytic curve and μ be a probability measure on a proper closed subset Y of X. If $\mathcal{H}$ is the closure of $\tilde{\mathcal{O}}(X) \cap \mathcal{O}(Y)$ in $L^2(\mathrm{d}\mu)$, then for meromorphic functions f, g on X with poles off Y, the multiplication operators T_f, T_g on $\mathcal{H}$ have the property that $[T_f^*, T_g]$ is trace class.

For a special case of Example 2, let X be a compact Riemann surface, Y be $X \backslash \mathbb{D}$ (X with a unit disk deleted) and $\mathrm{d}\mu$ be the area measure on Y. If $\mathcal{H}$ is defined to be as above, we obtain a generalized Bergman space on the Riemann surface. If $\mathcal{I}^\infty(X \backslash D)$ is the operator algebra generated by the C^∞-functional calculus of the T_g's with g a meromorphic function having poles outside of Y, then there is an exact sequence

$$O \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{I}^\infty(X \backslash D) \longrightarrow C^\infty(\partial \mathbb{D}) \longrightarrow 0.$$

This exact sequence will determine an element in $\mathrm{Ext}(\partial\mathbb{D}) \simeq \mathbb{Z}$. A careful computation of the index of Toeplitz operators will show that this element in $\mathrm{Ext}(\partial\mathbb{D})$ is a generator of $\mathrm{Ext}(\partial\mathbb{D}) \simeq \mathbb{Z}$.

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