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C^*-ALGEBRAS ARISING FROM INTERVAL EXCHANGE TRANSFORMATIONS

IAN F. PUTNAM

1. INTRODUCTION

An interval exchange transformation, or just interval exchange, is a bijective map of the half-open unit interval which is a piecewise translation with finitely many discontinuities. (In section 2, we will give a more elaborate definition. We will restrict our attention to interval exchanges satisfying the infinite distinct orbit condition – – see section 2.) There has been considerable interest in such transformations – see [6, 10, 11, 12, 14, 20, 27, 28].

At first glance, such maps do not fit into a "topological" context since they are not continuous. There are two approaches to repairing this difficulty which we outline in section 2 and refer the reader to [10] and [14] for further discussion. The first builds a homeomorphism (and thus an action of the group of integers, $\mathbf{Z}$) by modifying the unit interval resulting in a Cantor set. The second method is to mimic the flow under constant function construction. This yields a flow F (i.e., an action of the group of the reals, $\mathbf{R}$) on a compact, oriented 2-manifold (i.e., a surface) M with a closed transversal N homeomorphic to the circle, S^1, such that the Poincaré first return map on N is just the original interval exchange (identifying $[0,1)$ and S^1 in the obvious way and modulo the points of discontinuity). The flow F has a finite number of singularities and we let M_0 denote the complement of these in M. The flow F restricted to M_0 is minimal.

In section 3, we investigate the C^*-algebras associated to these two transformation groups. In general, if the locally compact group G acts on the locally compact Hausdorff space X, we denote the transformation group C^*-algebra or crossed product C^*-algebra [21, 16] by $C^*(G,X)$ or $C^*(G,X,\phi)$ in the case when we have some notation, ϕ, for the action. The case of the $\mathbb{Z}$-action on the Cantor set has been investigated in [18] and [19]. Here we will concentrate on $C^*(\mathbb{R}, M_0, F)$. First, we show that the two C^*-algebras are related (Theorem 3.2). The main result of this section (Theorems 3.3 and 3.4) is that $C^*(\mathbb{R}, M_0, F)$ is the inductive limit of a sequence of "tractable" C^*-algebras (which may be described as "non-commutative 1-complexes" [29]). As a consequence we derive results concerning the topological stable rank of these C^*-algebras. We also give an alternate description (up to strong Morita equivalence) of $C^*(\mathbb{R}, M_0, F)$ as operators on the Hilbert space $L^2(0,1)$.

The same dynamical properties of (M_0, F) which yield the construction of the C^*-subalgebras above also allow us to construct an embedding of $C^*(\mathbb{R}, M_0, F)$ into an AF-algebra [1] in section 4. Moreover, we show that this embedding induces an order isomorphism at the level of K_0-groups. This construction leads us to an investigation of asymptotics. Specifically, we show that a pair of asymptotic points for a dynamical system (see section 4) in a natural way gives rise to a pair of representations of the associated crossed product C^*-algebra whose difference is compact.

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2. INTERVAL EXCHANGES AND FLOWS ON SURFACES

An interval exchange is a bijection of the half-open unit interval defined in the following way. As initial data, we begin with an integer n (at least 2), σ in S_n, the permutation group of $\{1, 2, \ldots, n\}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\Delta^{n-1} = \{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \mid r_i \geqslant 0 \text{ for all } i \text{ and } \sum r_i = 1\}$, the standard $n-1$ simplex in $\mathbb{R}^n$. For $i = 0, \ldots, n$, we define

$$\beta(i) = \sum_{j \leqslant i} \alpha_j, \quad \beta'(i) = \sum_{\sigma(j) \leqslant i} \alpha_j.$$

Let $I(i) = [\beta(i-1), \beta(i))$ and $I'(i) = [\beta'(i-1), \beta'(i))$ and let

$$\tau(i) = \sum_{\sigma(j) < \sigma(i)} \alpha_j - \sum_{j < i} \alpha_j$$

for $i = 1, \ldots, n$. We define $T^{(\sigma,\alpha)}$ or just T to be the mapping of $[0,1)$ defined by

$$Tx = x + \tau(i), \; x \in I(i).$$

Roughly, T partitions $[0,1)$ into n subintervals of lenghts $\alpha_1, \ldots, \alpha_n$ and translates each so that they fit together to make up $[0,1)$ in a new order determined by σ. Note that $T(I(i)) = I'(\sigma(i))$. For the case $n = 2$ and $\sigma(1) = 2$, if we identify $[0,1)$ with the circle, S^1, in the obvious way, T is just rotation through angle $2\pi\alpha_2$.

We will assume throughout, as in most of the literature, that there is no $k < n$ such that $\sigma\{1, \ldots, k\} = \{1, \ldots, k\}$. The reader can quickly see why the existence of such a k would mean that our interval exchange is "reducible". It is worth remarking that we do not assume that there is no i such that $\sigma(i+1) = \sigma(i)+1$ as is frequently done. If there is such an i, one can rewrite the transformation using $n-1$ intervals. (One simply joins the i^{th} and $i+1^{\text{st}}$ together). However, for the topological construction we undertake in this section, this does make a difference; we refer the reader to the examples of section 7 of [18].

If $\alpha = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, then, of course, T is no more complicated than σ. We shall assume throughout that T (or (σ, α)) satisfies the infinite distinct orbit condition or IDOC. That is, the T-orbits of the points $\beta(1), \beta(2), \ldots, \beta(n-1)$ are all infinte and distinct. Recall that the orbit of a point x under T, or the T-orbit of x, is $\{T^n(x) \mid n \in \mathbf{Z}\}$. It is a non-trivial consequence of this hypothesis that T is minimal in the sense that the orbit of every point in $[0,1)$ is dense [6], [10]. We say that T (or (σ, α)) satisfies the integral independence condition or IIC if the set $\{\alpha_1, \ldots, \alpha_n\}$ is integrally independent or, equivalently, linearly independent over the rationals. It is straight-forward to see that the IIC implies the IDOC, but is not equivalent to it.

Of course, an interval exchange is not a homeomorphism of $[0,1)$. There are two resolutions to this problem. The first is to construct a Cantor set Σ and a homeomorphism ϕ of Σ so that $[0,1)$ is densely contained in Σ (in a natural way) and so that $\phi \mid [0,1) = T$ [10]. We proceed as follows. Let $D(T)$ denote the T-orbits of $\beta(1), \ldots, \beta(n-1)$, omitting the point 0. We want to consider the set $D(T) \times \{0,1\}$, but it will be more convenient to denote $(x,0)$ and $(x,1)$ by x^- and x^+, respectively. Let

$$\Sigma = [0,1] - D(T) \cup \{x^+, x^- \mid x \in D(T)\}.$$

There is an obvious linear order on Σ, using $x^- < x^+$, for all x in $D(T)$. Endowed with the order topology, Σ is a Cantor set since the minimality of T insures that $D(T)$ is dense. We include $[0,1)$ in Σ by mapping x in $D(T)$ to x^+. The definition of ϕ and the fact that it is a homeomorphism are both clear.

The second resolution of the discontinuities of T is to construct an action F of $\mathbf{R}$ (i.e. a flow) on a compact oriented surface M with a closed transversal N. The flow will have singularities, but the first return map on N is just the interval exchange T, except at the discontinuities of T, which flow directly into the singularities.

Begin with $P = [0,1] \times [0,1]$. Let $V'(0) = V(0) = \{0\} \times [0,1]$ and $V'(n) = V(n) = \{1\} \times [0,1]$. For $i = 1, \ldots, n-1$, let $V(i) = (\beta(i), 1)$ and $V'(i) = (\beta'(i), 0)$.

We define M to be the quotient of P obtained by collapsing $V(0)$ and $V(n)$ to single points and by identifying $I(i)^- \times \{1\}$ with $I'(\sigma(i))^- \times \{0\}$ via T, for $i = 1, 2, \ldots, n$. That is, M is approximately the mapping cylinder for T. We remark that we will frequently use co-ordinates in P for the image point under the quotient map in M. We define σ_0, a permutation of $\{0, 1, \ldots, n\}$ by

$$\sigma_0(j) = \begin{cases} \sigma^{-1}(1) - 1 & \text{if } j = 0 \\ n & \text{if } j = \sigma^{-1}(n) \\ \sigma^{-1}(\sigma(j) + 1) - 1 & \text{otherwise.} \end{cases}$$

We let $N(\sigma)$ denote the number of cyclic components of σ_0 and we let $X(\sigma)$ denote the image of $\{V(0), \ldots, V(n), V'(0), \ldots, V'(n)\}$ in M. Then it is easy to check that if $\sim$ denotes the equivalence relation on $\{V(0), \ldots, V(n)\}$ generated by $V(i) \sim \sim V(\sigma_0(i))$, $i = 0, \ldots, n$, then $V(i) \sim V(j)$ if and only if they have the same image in M. It is clear that each $V'(i)$ has the same image as some $V(j)$. Thus, we may identify $X(\sigma)$ with $\{V(0), \ldots, V(n)\}/\sim$ and $X(\sigma)$ is just $N(\sigma)$ points in M. In fact, M is a compact oriented surface with genus $(n - N(\sigma) + 1)/2$. Let $M_0 = M - X(\sigma)$.

To construct a flow F on M, $(F : \mathbb{R} \times M \longrightarrow M)$ we proceed exactly as in the flow under constant function construction (Chap. 11 [6]), except that we have a problem at each $V(i)$ (in P) which is identified with several $V'(j)$'s by the quotient map. To resolve this we define F to be the identity on $X(\sigma)$ (and slowed suitably nearby). In other words, F is obtained by integrating the vector field $\mathcal{X}(p) = \omega(p) \cdot (0, 1)$, for p in M, where $\omega(p)$ is a certain positive scalar function on M vanishing exactly on $X(\sigma)$ and $(0, 1)$ is in $\mathbb{R}^2$ which we may identify with T_pM, the tangent space at p, via P in an obvious way (at least for p in M_0). The transversal N is the image in M of $[0, 1] \times \left\{\frac{1}{2}\right\}$.

REMARK. For a more geometric description of M, including a complex structure with singularities, see Masur [14].

REMARK. In the case $n = 2$ mentioned earlier, M is the two-torus. However, F is not the Kronecker flow, but the Kronecker flow stopped at one point.

3. THE C^*-ALGEBRAS

In this section, we examine the C^*-algebras associated with the dynamical systems of section 2, especially $C^*(\mathbb{R}, M_0, F)$.

We begin by observing that we have the following F-invariant short exact sequence

$$0 \longrightarrow C_0(M_0) \longrightarrow C(M) \longrightarrow C(X(\sigma)) \longrightarrow 0 \tag{3.1}$$

and taking crossed products we obtain [26] the short exact sequence

$$(3.2) \qquad 0 \longrightarrow C^*(\mathbf{R}, M_0, F) \longrightarrow C^*(\mathbf{R}, M, F) \longrightarrow \bigoplus_{X(\sigma)} C_0(\mathbf{R}) \longrightarrow 0,$$

also using the fact

$$C^*(\mathbf{R}, X(\sigma), F) \cong \bigoplus_{x \in X(\sigma)} C^*(\mathbf{R}, \{x\}, F) \cong \oplus C_0(\mathbf{R}).$$

We also note that if $N(\sigma) = 1$, the sequence has an obvious splitting. Since the flow F on M_0 is minimal, $C^*(\mathbf{R}, M_0, F)$ is simple and so we will concentrate our attention on this C^*-algebra.

To begin with, however, we will consider any flow F on a locally compact space Y, with no periodic points; i.e. $F(t, x) = x$ only if $t = 0$. Suppose U is any non-empty open subset of Y. We wish to consider a new flow, which we denote F^U, on Y which is obtained from F by stopping F on $Y - U$. We let $Y - U = Z$. That is, we have a continuous map $\gamma : Y \times \mathbf{R} \to \mathbf{R}$ such that, for x in U, $\gamma(x, \cdot)$ is a strictly increasing function and for x in Z, $\gamma(x, t) = 0$ for all t in $\mathbf{R}$. The flow F^U is given by

$$F^U(t, x) = F(\gamma(x, t), x)$$

for all x in Y and t in $\mathbf{R}$. The fact that F^U is a flow is equivalent to the condition

$$\gamma(x, t) - \gamma(x, s) = \gamma(F(\gamma(x, s), x), t - s),$$

for all x in Y, t, s in $\mathbf{R}$. For fixed U, there are many choices for F^U but they will all have the same orbit structure and give rise to $*$-isomorphic C^*-algebras. We note that the orbit of x in U under F^U is just the connected component of the F-orbit of x intersected with U which contains x. From this point of view, we see that the foliation of U given by our flow F^U is just the restriction to U of the foliation of Y determined by F, as in Connes [4]. We want to show that there is a natural inclusion, which we will denote by ρ^U, of $C^*(\mathbf{R}, U, F^U)$ in $C^*(\mathbf{R}, Y, F)$ [4]. The underlying groupoids for the transformation group $(\mathbf{R}, U, F^U)$ and $(\mathbf{R}, Y, F)$ are $\mathbf{R} \times U$ and $\mathbf{R} \times Y$, respectively [21]. The map defined by

$$\overline{\gamma}(t, x) = (\gamma(x, t), x)$$

is an isomorphism between $\mathbf{R} \times U$ and an open sub-groupoid of $\mathbf{R} \times Y$, as in 4.4.1 of Wang [29]. There arises a slight subtlety regarding Haar systems [21]. As a transformation groupoid $\mathbf{R} \times U$ has its own Haar system [21] which will differ from that of its image under $\overline{\gamma}$ in $\mathbf{R} \times Y$. There are two resolutions to this problem. First, one can appeal to II.2.11 of [21] which asserts that the two systems will give C^*-algebras which are strongly Morita equivalent, hence stably isomorphic [3]. By [9], both of these C^*-algebras are already stable and we conclude that they are actually $*$-isomorphic. A more concrete procedure is possible if we assume that the function $\gamma(x,t)$ above is actually differentiable in t and that the derivate $\gamma'(x,t)$ is continous on $Y \times \mathbb{R}$. In this case, we can define

$$\rho^U : C_c(\mathbb{R} \times U) \to C_c(\mathbb{R} \times Y)$$

by

$$\rho^U(f)(t,x) = \begin{cases} 0 & \text{if } t \neq \gamma(x,s) \text{ for any } s \text{ in } \mathbb{R} \\ \gamma'(x,s)^{-1} f(s,x) & \text{if } t = \gamma(x,s) \end{cases}$$

Using the property of γ above, one can show directly that ρ^U is a $*$-homomorphism and extends to a map from $C^*(\mathbb{R}, U, F^U)$ to $C^*(\mathbb{R}, Y, F)$.

We first want to show that this inclusion, at the level of K-theory, is compatible with Connes' analogue of the Thom isomorphism [5].

Proposition 3.1. *Given $(\mathbb{R}, Y, F)$ with no periodic orbits and $U \subseteq Y$ non-empty and open, let $(\mathbb{R}, U, F^U)$ be as above. Then, for $i = 0$, 1, the following diagram is commutative*

$$\begin{array}{ccc} K_i(C^*(\mathbb{R}, U, F^U)) & \xrightarrow{\rho^U{}_*} & K_i(C^*(\mathbb{R}, Y, F)) \\ \downarrow & & \downarrow \\ K_{i+1}(C_0(U)) & \xrightarrow{j_*} & K_{i+1}(C_0(Y)) \end{array}$$

where the vertical arrows are Connes' analogue of the Thom isomorphism and j denotes the natural inclusion of $C_0(U)$ in $C_0(Y)$.

Proof. Let $CX = (0,1] \times Y$ and define a flow $CF = \mathrm{id} \times F$ on CX. Let $V = (0,1) \times \times Y \cup \{1\} \times U$. Then $SY = (0,1) \times Y$ is open and invariant under $(CF)^V$. Moreover, $(CF)^V \mid SY$ is topologically equivalent to $SF = \mathrm{id} \times F$. Clearly $(CF)^V \mid \{1\} \times U$ is just $F^U \mid U$. So we obtain the $(CF)^V$ invariant short exact sequence

$$0 \longrightarrow C_0(SY) \longrightarrow C_0(V) \longrightarrow C_0(U) \longrightarrow 0$$

and taking crossed-products, using the observations above, we also have the short exact sequence

$$0 \longrightarrow C^*(\mathbb{R}, SY, SF) \longrightarrow C^*(\mathbb{R}, V, (CF)^V) \longrightarrow C^*(\mathbb{R}, U, F^U) \longrightarrow 0.$$

Next, $C^*(\mathbb{R}, SY, SF) \cong C_0(0,1) \otimes C^*(\mathbb{R}, Y, F)$. The two sequences above are, in fact, the mapping cone sequences for the maps j and ρ^U, respectively. So by standard mapping cone arguments [25] and the fact that Connes' map is natural, we obtain the desired result. ∎

We digress for a moment, returning to interval exchanges. Specifically, we want to show that there is indeed a relationship between the C^*-algebras $C^*(\mathbb{Z}, \Sigma, \phi)$ and $C^*(\mathbb{R}, M_0, F)$.

THEOREM 3.2. *Let* $(\mathbb{Z}, \Sigma, \phi)$ *and* $(\mathbb{R}, M_0, F)$ *arise from an interval exchange* $T^{(\sigma,\alpha)}$ *as in section 2. Then*

(i) $K_0(C^*(\mathbb{Z}, \Sigma, \phi)) \cong \mathbb{Z}^n$
$K_1(C^*(\mathbb{Z}, \Sigma, \phi)) \cong \mathbb{Z}$
$K_0(C^*(\mathbb{R}, M_0, F)) \cong \mathbb{Z}^n$
$K_1(C^*(\mathbb{R}, M_0, F)) \cong \mathbb{Z}$,

(ii) *there is an injective* $*$*-homomorphism*
$\rho : C^*(\mathbb{R}, M_0, F) \to C^*(\mathbb{Z}, \Sigma, \phi) \otimes \mathcal{K}$,
where $\mathcal{K}$ *denotes the* C^**-algebra of compact operators on a separable, infinite dimensional Hilbert space.*

(iii) *using the isomorphisms of* (i),
$\rho * : K_0(C^*(\mathbb{R}, M_0, F)) \to K_0(C^*(\mathbb{Z}, \Sigma, \phi))$
is given by multiplication by L^σ, *an* $n \times n$ *matrix,*

$$L^\sigma_{ij} = \begin{cases} 1 & \text{if } i > j \text{ and } \sigma(i) < \sigma(j) \\ -1 & \text{if } i < j \text{ and } \sigma(i) > \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\rho * : K_1(C^*(\mathbb{R}, M_0, F)) \to K_1(C^*(\mathbb{Z}, \Sigma, \phi))$$

is the zero map.

Proof. (i) The first two isomorphisms were shown in [18]. In fact, it was shown that $[\chi_{[\beta(i-1)^+, \beta(i)^-]}]$, $i = 1, \ldots, n$, where χ denotes the characteristic function, are generators for the K_0-group. The second pair of isomorphisms are computed via Connes' analogue of the Thom isomorphism and the short exact sequence

$$0 \longrightarrow C_0((0,1) \times (0,1)) \longrightarrow C_0(M_0) \longrightarrow \bigoplus_{i=1}^{n} C_0(I(i)^0 \times \{1\}) \longrightarrow 0.$$

(ii) Let $(\mathbb{R}, \Sigma^\sim, \phi^\sim)$ be the flow obtained from $(\mathbb{Z}, \Sigma, \phi)$ by the flow under constant function construction. There is an obvious surjection of Σ onto $[0, 1]$ which gives us a surjection $\pi : \Sigma^\sim \to M$. Let $U = \pi^{-1}(M_0)$. The flow $(\phi^\sim)^U$ may be chosen so that $\pi \mid U$ is equivariant between $(\mathbb{R}, U, (\phi^\sim)^U)$ and $(\mathbb{R}, M_0, F)$. Therefore, we have an embedding

$$\pi^* : C^*(\mathbb{R}, M_0, F) \to C^*(\mathbb{R}, U, (\phi^\sim)^U)$$

which we follow by ρ^U and finally a $*$-isomorphism from $C^*(\mathbb{R}, \Sigma^{\sim}, \phi^{\sim})$ to $C^*(\mathbb{Z}, \Sigma, \phi) \otimes \mathcal{K}$ [3,22] to obtain ρ.
(iii) Basically, Proposition 3.1 and the description of the generators of the K-groups from part (i) allows everything to be computed at the level of spaces. The computation is routine and we omit it. ∎

We state our next result in a general context.

THEOREM 3.3. *Suppose that F is a flow on a non-compact, locally compact space Y whose one point compactification is metrizable. Suppose that for each y in Y the orbit of y under F has infinity as a limit point (i.e., its closure is not compact in Y). Then there is an increasing sequence of C^*-subalgebras of $C^*(\mathbb{R}, Y, F)$, $A_1 \subseteq A_2 \subseteq \cdots$, whose union is dense in $C^*(\mathbb{R}, Y, F)$ and such that each A_j is CCR* [16].

REMARK. As will be evident in the proof the analogous result will be true for $\mathbb{Z}$-actions. The basic idea here is the same as in the construction of A_Y in [18]. Here, Y is just the point at infinity and so A_Y is the entire crossed product algebra.

Before beginning the proof of 3.3, we wish to apply the result to our case of the flow arising from an interval exchange. It would seem that in many specific examples, the above result may be improved.

THEOREM 3.4. *Let (M, F) be the flow arising from an interval exchange $T^{(\sigma,\alpha)}$ (satisfying the IDOC). Then there is an increasing sequence of C^*-subalgebras of $C^*(\mathbb{R}, M_0, F)$, $A_1 \subseteq A_2 \subseteq \cdots$, whose union is dense in $C^*(\mathbb{R}, M_0, F)$ with each A_j CCR. Moreover, for each j, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{n} C_0(0,1) \otimes \mathcal{K} \longrightarrow A_j \longrightarrow \bigoplus_{i=1}^{2n-1} \mathcal{K} \longrightarrow 0.$$

Proof of 3.3. Let $U_1 \subseteq U_2 \subseteq \cdots$ be any increasing sequence of open subsets of Y whose union is all of Y and such that for each j, U_j^- is compact (here we use our hypothesis on Y). Let $A_j = C^*(\mathbb{R}, U_j, F^{U_j})$. (For convenience will denote F^{U_j} by F_j). The containments $A_1 \subseteq A_2 \subseteq \cdots$ are clear, as is the fact that the union is dense in $C^*(\mathbb{R}, Y, F)$. We will show that the flow (U_j, F_j) has closed orbits, for $j = 1, 2, \ldots$. It is clear that this flow has no periodic orbits so, by a result of Williams [32], A_j is CCR.

Let y be in U_j. We claim that there is $t \geqslant 0$ such that $F(t, y) \in Y - U_j^-$. If $\{F(t,y) \mid t \geqslant 0\}$ has no accumulation point in Y then the claim is clearly true. Otherwise, let z be such an accumulation point. By hypothesis, there is r in $\mathbb{R}$ so that $F(r, z) \in Y - U_j^-$. Then $\{F(t,y) \mid t \geqslant r\}$ has an accumulation point in $Y - U_j^-$ and as $\{F(t,y) \mid 0 \geqslant t \geqslant r\}$ is compact, $\{F(t,y) \mid t \geqslant 0\}$ must have an accumulation point in $Y - U_j^-$. The claim follows since $Y - U_j^-$ is open.

Therefore,

$$t_0 = \inf\{t \geqslant 0 \mid F(t,y) \in Y - U_j^-\}$$

exists, as does

$$t_1 = \sup\{t \leqslant 0 \mid F(t,y) \in Y - U_j^-\}.$$

Moreover, the F_j-orbit of y is just $\{F(t,y) \mid t_0 < t < t_1\} = U_j \cap \{F(t,y) \mid t_0 \leqslant t \leqslant t_1\}$ since U_j is open. The set $\{F(t,y) \mid t_0 \leqslant t \leqslant t_1\}$ is compact in Y so the F_j-orbit of y is closed in U_j. This completes the proof of 3.3. ∎

Proof of 3.4. We expand on the idea of the last proof, choosing $U_1 \subseteq U_2 \subseteq \cdots$ with sufficient care so that the A_j's have the desired property. The set U_1 is chosen first. Then we describe an iterative procedure for obtaining U_{j+1} from U_j.

Choose an integer K sufficiently large so that, for each $i = 1, \ldots, n$

$$\{T^k(0) \mid 1 \leqslant k \leqslant K\} \cap I(i)$$

has at least two points. For $i = 0, 1, \ldots, n-1$ let

$$x(1,i) = \inf(\{T^k(0) \mid 1 \leqslant k \leqslant K\} \cap I(i+1))$$

and for $i = 1, \ldots, n$ let

$$x(0,i) = \sup(\{T^k(0) \mid 1 \leqslant k \leqslant K\} \cap I(i)).$$

For convienence, let $\Omega = \{0,1\} \times \{1, \ldots, n-1\} \cup \{(1,0),(0,n)\}$. For each ω in Ω, let $k(\omega)$ be the positive integer such that $x(\omega) = T^{k(\omega)}(0)$. We also define

$$x'(1,i) = \inf(\{T^k(0) \mid 2 \leqslant k \leqslant K+1\} \cap I'(i+1))$$

$$x'(0,i) = \sup(\{T^k(0) \mid 2 \leqslant k \leqslant K+1\} \cap I'(i))$$

for appropriate i and $k'(\omega)$ is such that $T^{k'(\omega)}(0) = x'(\omega)$ for ω in Ω. Note that $T\{x(\omega) \mid \omega \in \Omega\} = \{x'(\omega) \mid \omega \in \Omega\}$. We define closed sets $Z(i), Z'(i)$ in $P = [0,1] \times [0,1]$ by

$$Z(0) = Z'(0) = [0, x(1,0)] \times [0,1]$$
$$Z(n) = Z'(n) = [x(0,n), 1] \times [0,1]$$
$$Z(i) = [x(0,i), x(1,i)] \times \left[\frac{3}{4}, 1\right]$$
$$Z'(i) = [x'(0,i), x'(1,i)] \times \left[0, \frac{1}{4}\right]$$

for $i = 1, \ldots, n-1$. We let $U(= U_1)$ be the complement of the union of the images of the $Z(i), Z'(i)$ under the quotient map $\pi : P \to M$.

We make the following claim: The F^U-orbits of the points of $\left\{\left(T(0), \frac{1}{2}\right), \left(x'(\delta, i), \frac{1}{2}\right) \middle| \delta = 0, 1 \quad i = 1, \ldots, n-1\right\} \subseteq U$, are pairwise disjoint. Moreover, if we denote by Γ their union, which is closed (in U) and F^U-invariant, then

(i) $(\mathbb{R}, \Gamma, F^U) \cong (\mathbb{R}, \bigcup_1^{2n-1} \mathbb{R}, \tau)$

(ii) $(\mathbb{R}, U - \Gamma, F^U) \cong (\mathbb{R}, \bigcup_1^n (0,1) \times \mathbb{R}, \text{ id} \times \tau)$

where $\cong$ denotes topological conjugacy of flows [2] and τ denotes the canonical flow on $\mathbb{R}$ (i.e. translation).

Notice that once we establish the claim, then we have the following short exact sequence for $A_1 = C^*(\mathbb{R}, U, F^U)$

$$0 \longrightarrow C^*(\mathbb{R}, U - \Gamma, F^U) \longrightarrow C^*(\mathbb{R}, U, F^U) \longrightarrow C^*(\mathbb{R}, \Gamma, F^U) \longrightarrow 0$$

and using the conjugacies of (i) and (ii)

$$C^*(\mathbb{R}, U - \Gamma, F^U) \cong \bigoplus_1^n C_0(0,1) \otimes \mathcal{K}$$
$$C^*(\mathbb{R}, \Gamma, F^U) \cong \bigoplus_1^{2n-1} \mathcal{K}.$$

As for the claim, let y be one of the points of the set above; say $y = \left(T^k(0), \frac{1}{2}\right)$, for some k. Define $y^+ = \lim_{t \to \infty} F^U(t, y)$ and $y^- = \lim_{t \to -\infty} F^U(t, y)$. It's clear that $y^- = \left(T^k(0), \frac{1}{4}\right)$, in fact; from which, it follows at once that these points all have disjoint F^U-orbits. Let us consider the point $\left(x(\delta, i), \frac{3}{4}\right)$, for some (δ, i). From the definitions, $x(\delta, i) = T^{k(\delta, i)}(0)$ with $1 \leqslant k(\delta, i) \leqslant K$. Choose k' from $\{k'(\delta' i') \mid \delta' = 0, 1 \quad i' = 1, \ldots, n-1\} \cup \{1\}$ to be the largest element of this set less than or equal to $k(\delta, i)$. Also let $y = \left(T^{k'}(0), \frac{1}{2}\right)$. It follows from the choice of k' and the definition of $x(\delta, i)$ that $y^+ = \left(x(\delta, i), \frac{3}{4}\right)$. What we have shown is that each of the corners of the $Z(i)$'s is a y^+ for some y in our set. There are $2n-2$ such corners and the conclusion of the claim follows easily from these observations.

We have produced $U_1 = U$ satisfying conditions which imply $A_1 = C^*(\mathbb{R}, U_1, F_1)$ is of the desired form. We now show how to construct U_2 with the same conditions. As noted above, $2n-2$ points of the set considered earlier flow directly into corners of the $Z(i)$'s under F_1. One $y = \left(x'(\delta, i), \frac{1}{2}\right)$ has y^+ contained in the interior of the line segment $[x(0, j), x(1, j)] \times \left\{\frac{3}{4}\right\}$, for some j. (In fact, (δ, i) is such that

$k'(\delta,i) = \sup\{k'(\delta',i') \mid \delta' = 0,1\ i' = 1, \ldots, n-1\}$.) Let $y^+ = \left(T^\ell(0), \frac{3}{4}\right)$, for some $\ell \geqslant 1$. Notice that $T^\ell(0) = \beta(j)$ is not possible by the IDOC. Redefine the $x(\delta,i)$, etc by replacing K by ℓ. (In the case $T^\ell(0) < \beta(j)$ and $\sigma(j) = n$ and in the case $T^\ell(0) > \beta(j)$ and $\sigma(j+1) = 1$, replace K by $\ell + 1$.) Then we obtain $x(\delta,i), x'(\delta,i), k(\delta,i), k'(\delta,i)$ as before. Now, define $Z(i) = [x(0,i), x(1,i)] \times \left[\frac{7}{8}, 1\right]$ and $Z'(i) = [x'(0,i), x'(1,i)] \times \left[0, \frac{1}{8}\right]$. It is clear that U_2, defined as before, has the same properties as U_1 and $A_2 = C^*(\mathbb{R}, U_2, F_2)$ will satisfy the conclusion of the theorem. It is clear how to continue this process inductively to obtain U_3, U_4, etc. From the fact that the orbit of 0 under T is dense in $[0,1)$ it follows that the union of all U_j will be M_0. ∎

REMARK 3.5. The primitive ideal space of one of the C^*-subalgebras A_j of 3.4 consists of open line segments and points. However, it is not Hausdorff. (The C^*--algebra A_j might be called a "non-commutative 1-complex".) One may instead form the "dual graph" of A_j, using a vertex for each copy of $C_0(0,1) \otimes \mathcal{K}$ and an edge for each $\mathcal{K}$ in the quotient algebra. This is indeed a graph and if one considers how it arises dynamically, there is a natural embedding of this graph in M_0 which is transverse to the flow. This is the same construction as in Wang [29], who considered foliations of the plane. Unlike the situation there, our graphs are not trees – indeed, they are "transverse" cell decompositions of M_0. Of course, what is substantially different here from the situation in [29] is that $C^*(\mathbb{R}, M_0, F)$ is not associated to such a decomposition (as in [29]) but is the inductive limit of C^*-subalgebras of this type. We also refer the reader to Wang [30, 31].

As an immediate corollary, we have the following result concerning the topological stable ranks of the C^*-algebras we are considering. We refer the reader to Rieffel [23] for the relevant definitions.

THEOREM 3.6. *Let (M,F) be the flow arising from an interval exchange $T^{(\sigma,\alpha)}$. Then*

(i) $\operatorname{tsr}(C^*(\mathbb{R}, M_0, F)) = 1$

(ii) $\operatorname{tsr}(C^*(\mathbb{R}, M, F)) = \begin{cases} 1 & \text{if } N(\sigma) = 1 \\ 2 & \text{if } N(\sigma) \geqslant 2 \end{cases}$

Proof. For the first part, from [23]

$$\operatorname{tsr}(C^*(\mathbb{R}, M_0, F)) \leqslant \liminf \operatorname{tsr}(A_j).$$

The results of [23] also imply $\operatorname{tsr}(C_0(0,1) \otimes \mathcal{K}) = \operatorname{tsr}(\mathcal{K}) = 1$. A straight-forward argument (see Theorem 3.1 of [19]) using the fact that each invertible element in

$K^{\sim}$ (the C^*-algebra obtained by adding a unit to the compacts) is homotopic to the identity then gives the result that $\operatorname{tsr}(A_j) = 1$, for all j.

As for part (ii), $\operatorname{tsr}(C^*(\mathbf{R}, M, F)) \leqslant 2$ follows from the result of [23], part (i) above and the short exact sequence (3.2). In the case $N(\sigma) = 1$, then $X(\sigma)$ is a single point and sequences (3.1) and (3.2) both split and $\operatorname{tsr}(C^*(\mathbf{R}, M, F)) = 1$ follows. If $N(\sigma) > 1$, then one considers the six-term exact sequence for K-groups obtained from (3.2). One can show directly that the index map is non-zero and, by arguments similar to those in the proof of Theorem 3.1 of [19], one concludes $\operatorname{tsr}(C^*(\mathbf{R}, M, F)) \geqslant 2$. This completes the proof. ■

We remark that, just as in Theorem 3.1 of [19], if $\operatorname{tsr}(C^*(\mathbf{R}, M, F)) = 2$, then $C^*(\mathbf{R}, M, F)$ does not satisfy the cancellation property [23].

We conclude this section by giving a different description of $C^*(\mathbf{R}, M_0, F)$ which may be compared to the description of $C^*(\mathbf{Z}, \Sigma, \phi)$ given in [18]. We let U_T denote the unitary operator on $L^2(0,1)$ defined by

$$U_T\xi = \xi \circ T^{-1},$$

for ξ in $L^2(0,1)$, where T is an interval exchange transformation. We consider functions on $(0,1)$ as multiplication operators on $L^2(0,1)$.

THEOREM 3.7. *Let T be an interval exchange and let (M_0, F) be the flow associated with T as before. Let B denote the C^*-algebra generated by $C_0(0,1)$ and $U_T C_0((0,1) - \{\beta(1), \ldots, \beta(n-1)\})$. Then $C^*(\mathbf{R}, M_0, F)$ and B are strongly Morita equivalent* [22].

Proof. Let G denote the groupoid associated with the flow (M_0, F). Let $N = (0,1) \times \left\{\frac{1}{2}\right\} \subseteq M_0$ which is a closed transversal. We use the results (specifically, Theorem 2.8) and notations of Muhly et al. [15]. Letting G_N^N be the groupoid associated with N (see [15]), $C^*(G_N^N)$ and $C^*(G) = C^*(\mathbf{R}, M_0, F)$ are strongly Morita equivalent. Moreover, it is readily seen that B is the image of $C^*(G_N^N)$ under an obvious representation on $L^2(0,1)$. Since we assume T satisfies the IDOC, T is minimal. Then by II.4.6 of Renault [21], $C^*(G_N^N)$ is simple and so the representation on $L^2(0,1)$ is faithful and hence $C^*(G_N^N) \cong B$. ■

REMARK 3.8. Again for the special case $n = 2$, one may choose a different closed transversal N (homeomorphic to S^1 and not containing the singularity) and follow the same proof obtaining a C^*-algebra B which may be described as follows. Let u and v be unitary operators satisfying $uv = e^{2\pi i\alpha_2}vu$. The C^*-algebra they generate is $*$-isomorphic to the irrational rotation C^*-algebra A_{α_2} [26]. Then B is the C^*-subalgebra generated by v and $u(v-1)$.

4. ASYMPTOTICS AND *AF*-EMBEDDING

In this section we examine the question of embedding $C^*(\mathbb{R}, M_0, F)$ into an approximately finite dimensional or AF-algebra. There has been considerable interest in such questions (see [13, 17, 18]). Here we show that there is an embedding of $C^*(\mathbb{R}, M_0, F)$ into an AF-algebra which induces an order-isomorphism at the level of K_0-groups. In fact, we show much more, namely that there is a locally compact space $\tilde{M}_0$ with flow $\tilde{F}$ and an equivariant surjection $\pi : \tilde{M}_0 \to M_0$, so that $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$ is AF. In particular, as our embedding is really obtained at the level of flows it allows us to avoid Berg's technique (see [13]). This is due to the same dynamical properties which yield Theorems 3.3 and 3.4.

Finaly, we show how the notion of asymptotic points for a dynamical system relates to pairs of representations of a C^*-algebra with compact difference. We examine the case of $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$ and show that there is a pair of representations whose equalizer is $*$-isomorphic to $C^*(\mathbb{R}, M_0, F)$.

Let $T = T^{(\sigma,\alpha)}$ be an interval exchange (with the IDOC). Basically, to obtain $\tilde{M}_0$ from M_0, we split the F-orbit of $\left(T(0), \frac{1}{2}\right)$ into 2 parallel orbits. Note that the F-orbit of $\left(T(0), \frac{1}{2}\right)$ is

$$\{(T^k(0), s) \mid k \geqslant 1 \ s \in (0,1]\}.$$

since as t approaches $-\infty$, $F\left(\left(T(0), \frac{1}{2}\right), t\right)$ converges to

$$(T(0), 0) = (\beta'(\sigma(1) - 1), 0).$$

Let $D_0(T) = \{T^k(0) \mid k \geqslant 1\}$. As in section 2, let

$$\Sigma_0 = [0,1] - D_0(T) \cup \{x^+, x^- \mid x \in D_0(T)\}$$

Let $\tilde{M}$ be the compact set obtained from $\Sigma_0 \times [0,1]$, as follows. First, identify the points $(T(0)^+, 0)$ and $(T(0)^-, 0)$ and denote the resulting point by $(T(0), 0)$. Identify each of $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ to a point. Finally for each $i = 1, \ldots, n$ identify $[\beta(i-1), \beta(i)] \times \{1\}$ with $[\beta'(\sigma(i)-1), \beta'(\sigma(i))] \times \{0\}$ via T-meaning that for x in D_0, we identify $(x^+, 1)$ and $(x^-, 1)$ with $(T(x)^+, 0)$ and $(T(x)^-, 0)$, respectively. Let $\tilde{F}$ be the obvious vertical flow stopped at the images of $(\beta(i), 1)$, $i = 0, ,1 \ldots, n$, in $\tilde{M}$. There is an obvious surjection $\pi : \tilde{M} \to M$ and we may choose $\tilde{F}$ so that π is equivariant; i.e. $F \circ \mathrm{id} \times \pi = \pi \circ \tilde{F}$ as maps from $\mathbb{R} \times \tilde{M}$ to M. Let $\tilde{M}_0 = \pi^{-1}(M_0)$. Any such equivariant map induces a map at the level of C^*-algebras which, in our case, will be denoted by $\pi : C^*(\mathbb{R}, M_0, F) \to C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$.

THEOREM 4.1.
(i) $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$ *is an AF-algebra.*
(ii) $\pi* : K_0(C^*(\mathbf{R}, M_0, F)) \to K_0(C^*(\mathbf{R}, \tilde{M}_0, \tilde{F}))$ *is an order isomorphism.*

Proof. (i) We construct a sequence $\tilde{U}_1 \subsetneq \tilde{U}_2 \subsetneq \cdots$ as in the proof of 3.4 for $(\tilde{M}_0, \tilde{F})$. Using the notation and definitions as in 3.4, we let

$$\tilde{Z}(i) = [x(0,i)^+, x(1,i)^-] \times \left[\frac{3}{4}, 1\right]$$

using the fact that each $x(0,i), x(1,i)$ is in $D_0(T)$. We define $\tilde{Z}'(i)$ analogously. We then define $\tilde{U}_1$ as in 3.4 and it is straight-forward to see that

$$C^*(\mathbf{R}, \tilde{U}_1, \tilde{F}^{\tilde{U}_1}) \cong \bigoplus_{i=1}^{n} C(\Sigma_i) \otimes \mathcal{K} \tag{4.1}$$

where each $\Sigma_i \subsetneq \Sigma_0$ is a Cantor set. We inductively define $\tilde{U}_j$, for $j = 2, 3, \ldots$ as in 3.4 and we see that $C^*(\mathbf{R}, \tilde{M}_0, \tilde{F})$ is the inductive limit of AF-algebras and so is AF.

(ii) We will first show that $\pi*$ maps the positive cone onto the positive cone. Let k and ℓ be natural numbers and define

$$E_0 = [T^k(0)^+, T^\ell(0)^-] \times \left\{\frac{1}{2}\right\} \subseteq \Sigma_0 \times \left\{\frac{1}{2}\right\} \subseteq \tilde{M}.$$

This set is a closed transversal to the flow; i.e. it is closed and, for some $\varepsilon > 0$,

$$\tilde{F} : E_0 \times (-\varepsilon, \varepsilon) \to \tilde{M}_0$$

is a homeomorphism onto an open subset, W, of $\tilde{M}_0$. As indicated in [4], this set gives rise to an element in $K_0(C^*(\mathbf{R}, \tilde{M}_0, \tilde{F}))$ in the following way. By our earlier results, we have an inclusion ρ^W of $C^*(\mathbf{R}, W, \tilde{F}^W)$ in $C^*(\mathbf{R}, \tilde{M}_0, \tilde{F})$. Moreover, it is clear that the transformation group $(\mathbf{R}, W, \tilde{F}^W)$ is conjugate to $(\mathbf{R}, E_0 \times \mathbf{R}, \mathrm{id} \times \tau)$, so that $C^*(\mathbb{R}, W, \tilde{F}^W) \simeq C(E_0) \otimes \mathcal{K}$. Letting e_0 be any rank one projection in $\mathcal{K}$, we consider the class of the image of $I \otimes e_0$ under ρ^W. It is a consequence of the proof of (i) that every positive element of $K_0(C^*(\mathbf{R}, \tilde{M}_0, \tilde{F}))$ is the sum of elements arising in this way. So to show surjectivity it suffices to consider such elements. We also suppose that k, ℓ are such that there is no i between k and ℓ with $T^i(0) \in [T^k(0), T^\ell(0)]$. Define a map ζ from E_0 to $[0,1]$ by setting $\zeta\left(x, \frac{1}{2}\right), \zeta\left(x^+, \frac{1}{2}\right)$ or $\zeta\left(x^-, \frac{1}{2}\right)$ equal to $(x - T^k(0))(T^\ell(0) - T^k(0))^{-1}$. Since the points $\left(T^\ell(0)^-, \frac{1}{2}\right)$ and $\left(T^k(0)^-, \frac{1}{2}\right)$ lie in the same $\tilde{F}$ orbit, there is r in $\mathbb{R}$ so that $\tilde{F}\left(r, \left(T^\ell(0)^-, \frac{1}{2}\right)\right) = \left(T^k(0)^-, \frac{1}{2}\right)$. For each t in $[0,1]$, we let

$$E_t = \left\{ \tilde{F}\left(\mathrm{tr}\zeta\left(x, \frac{1}{2}\right), \left(x, \frac{1}{2}\right)\right) \,\middle|\, \left(x, \frac{1}{2}\right) \in E_0 \right\}$$

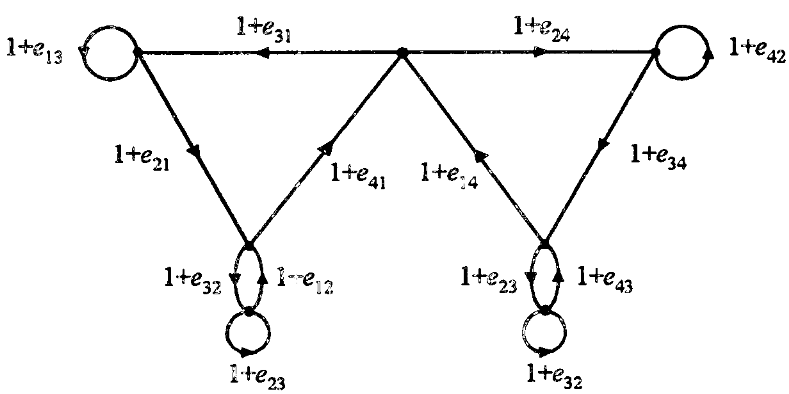
where the x should have a + or − for appropriate points between $T^k(0)$ and $T^\ell(0)$, a + for $x = T^k(0)$ and a − for $x = T^\ell(0)^-$. From the hypotheses above, each E_t is a homeomorphic image of E_0. The projections in $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$ constructed from these transversals will be homotopic and thus determine the same class in K-theory. Finally, it is easily seen that $\pi(E_1)$ is actually a closed transversal to F in M_0 and that $\pi^{-1}(\pi(E_1)) = E_1$. Thus the K_0-class determined by E_0 is in the image of $K_0(C^*(\mathbb{R}, M_0, F))$ under $\pi*$.

As for the injectivity of $\pi*$, let C_j be the C^*-subalgebra of $C^*(\mathbb{R}, \tilde{U}_j, \tilde{F}^{\tilde{U}_j})$ which is mapped to $\bigoplus_{i=1}^{n} \mathbb{C} \otimes \mathcal{K}$ under the isomorphism of (4.1), for $j = 1, 2, \ldots$. It is not hard to check that $C_j \subseteq C_{j+1}$ for all j, and that the union of the C_j's is dense in $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$. It is clear that $K_0(C_j) \cong \mathbb{Z}^n$, for all j, and that the embeddings induce maps $K_0(C_j) \to K_0(C_{j+1})$ which are all isomorphisms (of abelian groups). In fact, using $K_0(C_j) \cong \mathbb{Z}^n$, each embedding is given by multiplication by a matrix of 1's on the diagonal, one off-diagonal 1 and the rest of the entries 0. Therefore, we see that $K_0(C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})) \cong \mathbb{Z}^n$ and since the map from $K_0(C^*(\mathbb{R}, M_0, F)) \cong \mathbb{Z}^n$ (by 3.2) is known to be surjective (from part(i)) it must be injective as well. ■

REMARK. One of the central questions concerning interval exchanges has been the so-called Keane conjecture. In [10], Keane asked if every minimal interval exchange is uniquely ergodic (i.e. Lebesgue measure is the only invariant probability measure on [0,1)). For $n = 2$ and 3 this is true, but Keynes and Newton [12] and subsequently Keane [11] found counter-examples. Finally Veech [27] and Masur [14] showed that, for fixed σ, the set of α in Δ^{n-1} such that $T^{(\sigma,\alpha)}$ is uniquely ergodic has full Lebesgue measure in Δ^{n-1} (i.e., almost all interval exchanges are uniquely ergodic). Using Theorem 4.1 and a result of Blackadar (pg. 58 of [1]), the computation of the $T^{(\sigma,\alpha)}$-invariant measures may be reduced to the computation of the state space of the simple dimension group $K_0(C^*(\mathbb{R}, \tilde{M}_0, \tilde{F}))$ (see[1]). The Bratteli diagram [2] for the C_j's may be viewed as the continued fraction expansion of (σ, α). In the case $n = 2$, it is essentially the continued fraction expansion of α_2 as interpreted by Effros and Shen [8]. It is not the same expansion as defined by Rauzy [20] and used by Veech [27] in his proof of the Keane conjecture.

I will try to be a little more specific without going into great detail. I will only discuss the case $n = 4$, comparing tehniques with those of Veech [28]. Consider the iterative procedure in the proof of 3.4. At the first stage, we have K, $x(\delta, i)$, etc. Let $i_0 = \sigma(1) - 1$ and consider the following four intervals $[x'(0, i_0), \beta'(i_0)]$, $[\beta(i_0), x'(1, i_0)]$ and $[x'(0, i), x'(1, i)]$ for $i \in \{1, 2, 3\} - \{i_0\}$. These form the top edges of the boxes $Z'(j)$'s. Under F, each will flow into one of the boxes $Z(j)$; in this way we obtain a map from $\{i_0^-, i_0^+\} \cup (\{1, 2, 3\} - \{i_0\})$ to $\{1, 2, 3\}$. We call such a map a configuration.

It turns out that there are only seven configurations. Along with the configuration, we obtain a point in Δ^3 by scaling the lengths of the four intervals so they sum to one. For a fixed configuration, the points which arise from a sub-simplex. This space of pairs, (configuration, point in Δ^3), is our analogue of the space of interval exchanges. The iteration of the procedure in 3.4 then becomes a two-to-one map of this space into itself. This is our analogue of the Rauzy transform. One can draw a schematic diagram as follows: each configuration is represented by a vertex. The points on Δ^3 for a fixed configuration go to one of two new configurations (depending on whether the last iterate of 0 under T which we are considering is to the right or left of the appropriate $\beta(i)$) and we draw two corresponding directed edges between the appropriate vertices. As in [28], we have a 4×4 matrix attached to each edge in the graph so that the following holds: if (ν, α) is a point in our space mapped to (ν', α'), then there is an edge ν to ν' and α' is equal to $|A^{-1}\alpha|^{-1} \cdot A^{-1}\alpha$, where A is the matrix associated with this edge and $|A\alpha|$ denotes the sum of the entries of $A\alpha$. Our graph looks like



using e_{ij} to denote standard matrix units. (I have no explanation why the underlying graph is the same as on page 135 of [28].) In our situation it is easy to write down the appropriate Gauss measure for the transformation [28]; in fact, on each subsimplex it's just $(x_1x_2x_3x_4)^{-1}d\lambda$, where λ denotes Lebesgue measure on Δ^3. To complete the proof of the Keane conjecture, one must still show that this transformation is ergodic. With this view, the proof of 3.4 takes an interval exchange and provides an infinite sequence of points (in fact an orbit) in this space. In fact, if we ignore the point in Δ^3 at each stage, we get an infinite path in our graph. The matrices on the traversed edges give us the Bratteli diagram for the C_j's and information regarding invariant measures for the original interval exchange. (This also appears in [28]). Of course, this AF-algebra must be simple and so the Bratteli diagram must have certain properties [2]. However, if we take an arbitrary infinite path in this graph, it turns out to be relatively easy to see if the associated AF-algebra is simple. (Such a statement seems imposible for the Rauzy transform.) It is then reasonable to ask if every such path is the expansion of some interval exchange. If this is so, one can presumably use ideas like those in [8] to produce a path which yields a simple AF-algebra having more than one trace. Then one would have shown the existence of a non-uniquely ergodic IDOC interval exchange.

We will now give a different description of $C^*(\mathbb{R}, M_0, F)$ as a C^*-subalgebra of $C^*(\mathbb{R}, \tilde{M}_0, \tilde{F})$. This will rely on observing the connection between asymptotics of a general dynamical system and pairs of representations of the associated C^*-algebra whose difference is compact.

We begin with the general situation of a locally compact group G acting on a compact metrizable space X. We say two points x and y in X are asymptotic if, for every open set U in $X \times X$ containing the diagonal $\Delta = \{(x,x) \in X \times X \mid x \in X\}$, the set $\{g \in G \mid (g \cdot x, g \cdot y) \notin U\}$ is pre-compact in G. In the case $G = \mathbb{R}$ or $G = \mathbb{Z}$, this is equivalent to the condition

$$\lim_{|t| \to \infty} d(t \cdot x, t \cdot y) = 0,$$

where d is any metric on X. In the case X is locally compact, we say x and y are asymptotic if they are asymptotic in X^+, the one point compactification of X, letting G act trivially on the point at infinity.

For each point x in X, we obtain a representation ρ_x of $C^*(G, X)$ on the Hilbert space $L^2(G)$ defined by

$$[\rho_x(f)\xi](t) = \int_G f(s^{-1}t, s \cdot x)\xi(s)\mathrm{d}s$$

where $f : G \times X \to \mathbb{C}$ is continuous and of compact support and considered as an element of $C^*(G, X)$, ξ is in $L^2(G)$ and t in G [26].

THEOREM 4.2. *In the transformation group (G, X) the points x and y are asymptotic if and only if $\rho_x(a) - \rho_y(a)$ is a compact operator for every a in $C^*(G, X)$.*

Proof. ($\Rightarrow$) It suffices to consider $f(t, z) = f_1(t)f_2(z)$, (t, z) in $G \times X$, where $f_1 : G \to \mathbb{C}$ and $f_2 : X \to \mathbb{C}$ are each continuous and of compact support. If we define g_2 in $C_0(G)$ by $g_2(s) = f_2(s \cdot x) - f_2(s \cdot y)$ then $g_x(f) - g_y(f) = \lambda(g)$ where $g : G \times G \to \mathbb{C}$ is defined by $g(s,t) = f_1(s)g_2(t)$ and is regarded as an element of $C^*(G, G)$ (natural action of G on itself) and λ denotes the left regular representantion. It is well-known that λ is an isomorphism between $C^*(G, G)$ and $\mathcal{K}(L^2(G))$ [24].

($\Leftarrow$) We suppose that x and y are not asymptotic; so we may find a sequence $\{t_n\}_1^\infty$ in G tending to infinity such that the sequence $\{(t_n \cdot x,\ t_n \cdot y)\}_1^\infty$ has no limit points in Δ. Since X^+ is compact and by passing to a subsequence we may assume that $t_n \cdot x$ converges to x_0 and $t_n \cdot y$ converges to a distinct point y_0. Since x_0 and y_0 are distinct, one of them, say x_0, is not the point at infinity. We may find f in $C_0(X)$ such that f is identically one on V_0 a neighbourhood of x_0 and zero on W_0 a neighbourhood of y_0. We may then find Z a compact neighbourhood of the identity in G and neighbourhoods V_1 and W_1 of x_0 and y_0 so that $Z \cdot V_1 \subseteq V_0$ and $Z \cdot W_1 \subseteq W_0$. Since Z is compact and the t_n's tend to infinity, we may again pass to a subsequence so that $t_n \cdot Z$ is disjoint from $t_m \cdot Z$, for $n \neq m$. Let ξ be any unit vector in $L^2(G)$ with support in Z. Let ξ_n be ξ translated by t_n, for each n. The sequence $\{\xi_n\}_1^\infty$ is an orthonormal family in $L^2(G)$. If we choose $g : G \to \mathbb{C}$ continuous with compact support contained in Z and so that $\|g * \xi - \xi\|_2$ is small, and letting $a(t, z) = f(z) \cdot g(t)$ in $C^*(G, X)$ the sequence

$$\|(\rho_x(a) - \rho_y(a))\xi_n\|$$

will be bounded below and so $\rho_x(a) - \rho_y(a)$ is not compact. ∎

THEOREM 4.3. *Let $(\mathbb{R}, \tilde{M}_0, \tilde{F})$ be the flow constructed from the interval exchange $T^{(\sigma,\alpha)}$, as before. Let $x = \left(T(0)^-, \frac{1}{2}\right)$ and $y = \left(T(0)^+, \frac{1}{2}\right)$ in M_0. Then x and y are asymptotic under $\tilde{F}$. Moreover, they are the unique such pair in the sense that if x' and y' are any other pair of asymptotic points, then either* (a) $x' = y'$, (b) $x' = \tilde{F}(r, x)$ *and $y' = \tilde{F}(r, y)$ for some r in $\mathbb{R}$ or* (c) $x' = \tilde{F}(r, y)$ *and $y' = \tilde{F}(r, x)$ for some r in $\mathbb{R}$. The equalizer of ρ_x and ρ_y,*

$$\{a \in C^*(\mathbb{R}, \tilde{M}_0, \tilde{F}) | \rho_x(a) = \rho_y(a)\}$$

is precisely the image of $C^*(\mathbb{R}, M_0, F)$ under π.

Proof. Everything is very straight-forward except the uniqueness statement. We will give a sketch of the proof of that part only. It is easy to see that (a), (b) or (c) is equivalent to the condition $\pi(x') \neq \pi(y')$ in M_0. So it suffices to show that if x_1 and x_2 in M_0 are asymptotic for F, then $x_1 = x_2$. This may be deduced by examing the interval exchange T. If y_1 and y_2 are in the same interval $I(i)$, then $|T(y_1) - T(y_2)| = |y_1 - y_2|$. If y_1 is in $I(i)$ and y_2 is in $I(i+1)$ and if $\sigma(i+1) = \sigma(i)+1$, then again $|T(y_1) - t(y_2)| = |y_1 - y_2|$. However, if $\sigma(i+1) \neq \sigma(i)+1$ there are positive constants δ_1 and δ_2 such that $|y_1 - y_2| < \delta_1$ implies that $|T(y_1) - T(y_2)| > \delta_2$. Using the minimality of T and the facts above one may deduce that T has no non-trivial asymptotic pairs and then neither will F. ∎

REMARK. For any C^*-algebra A, a pair of representations of A on the same Hilbert space, $\rho_0, \rho_1 : A \to \mathcal{L}(H)$, with the property that $\rho_0(a) - \rho_1(a) \in \mathcal{K}(H)$ for all a in A determines an element of the Kasparov group $KK(A, \mathbb{C})$, see [7] or [1]. It is interesting to note that using the universal coefficient theorem [1], Theorem 4.1 (ii) and Theorem 4.2, the class of (ρ_x, ρ_y) of 4.2 is trivial in $KK(C^*(\mathbb{R}, \tilde{M}_0, \tilde{F}), \mathbb{C})$, while its equalizer is considerably more interesting.

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IAN F. PUTNAM
Department of Mathematics, Statistics
and Computing Science,
Dalhousie University,
Halifax, N.S.,
Canada B3H 4H8.

current address:
Department of Mathematics
and Statistics,
University of Victoria,
P.O. Box 3045,
Victoria, B.C.,
Canada V8W 3P4

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