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COMPACT PERTURBATION OF THE ALGEBRA OF A TENSOR PRODUCT OF NESTS

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1. INTRODUCTION

Let H be a complex, separable, infinite dimensional Hilbert space; $\mathcal{L}(H)$, $\mathcal{K}(H)$ denote the algebra of all bounded linear operators acting on H and, respectively, the ideal of all compact operators. A *nest* in H is a chain $\mathcal{N}$ of subspaces of H containing $\{0\}$ and H, which is closed under intersection and closed span. (By subspace we will always mean a closed linear manifold.)

It is well known that for a nest, there is a spectral measure $E(t)$ on $[0,1]$ such that $\mathcal{N} = \{E([0,t])H, t \in [0,1]\}$ and the compact subset supp$E$ of $[0,1]$ is order-isomorphic to and topologically homeomorphic to $\mathcal{N}$, when $\mathcal{N}$ is given the order topology (which is equivalent to the strong operator topology on $\mathcal{N}$). Here supp$E$ has the order and the related topology induced on it by the usual topology of the real line. Furthermore, E is uniquely determined by $\mathcal{N}$ up to order-preserving homeomorphisms of $[0,1]$ onto itself; that is, if F is an another spectral measure associated to $\mathcal{N}$ (as indicated above), then there exists a strictly increasing continuous function φ mapping $[0,1]$ onto itself such that $E = \varphi(F)$ and $F = \varphi^{-1}(E)$. In what follows, we will not distinguish a nest $\mathcal{N}$ from the support of its spectral measure. We will denote $M_{[c,d]} := E([c,d])H$ when $[c,d] \subset [0,1]$ and $M_t = M_{[0,t]}$.

For each $M \in \mathcal{N}$, let $M_- = \bigvee\{M' : M' \in \mathcal{N}, M' \subsetneq M\}$. If $M_- \neq M, M \ominus M_-$ is called an *atom* of $\mathcal{N}$ and the cardinal number dim$M \ominus M_-$ is called the *dimension of the atom*; $\lambda \in \mathcal{N}$ corresponds to an atom if and only if λ is the right end point of an interval in the complementary open set of $\mathcal{N}$ in $[0,1]$. A nest is called *continuous* if it has no atoms. (For more information about nest cf. [1], [2], [4].)

The nest algebra associated with $\mathcal{N}$ is the family of operarors defined by

$$\mathrm{alg}\mathcal{N} = \{T \in \mathcal{L}(H) : TM \subset M \text{ for all } M \in \mathcal{N}\}.$$

In [11], [12], D. A. Herrero solved a problem posed by W. B. Arveson by obtaining a spectral characterization of the sets

$$\mathcal{N}^\wedge = \{ UAU^* + K : U \text{ is unitary}, A \in \text{alg}\mathcal{N}, K \in \mathcal{K}(H) \}$$

and

$$\mathcal{N}_0^\wedge = \{ T \in \mathcal{L}(H) : \text{Given } \varepsilon > 0, \text{ there exist } U_\varepsilon \text{ unitary}, A_\varepsilon \in \text{alg}\mathcal{N} \text{ and } K_\varepsilon \in \mathcal{K}(H) \text{ such that } \|K_\varepsilon\| < \varepsilon \text{ and } T = U_\varepsilon A_\varepsilon U_\varepsilon^* + K_\varepsilon \}.$$

The results of [11] and [12] can be resumed as follows:

THEOREM 1.1. (Herrero).
(i) *If $\mathcal{N}$ is a well-ordered and all its atoms are finite dimensional, then*

$$\mathcal{N}^\wedge = \mathcal{N}_0^\wedge = (QT) := \{ T \in \mathcal{L}(H) : T \text{ is a quasitriangular operator} \};$$

(ii) *If $\mathcal{N}$ is well-ordered from above and all its atoms are finite dimensional, then*

$$\mathcal{N}^\wedge = \mathcal{N}_0^\wedge = (QT)^* = \{ T \in \mathcal{L}(H) \ : \ T^* \in (QT) \};$$

(iii) *In the remaining cases, $\mathcal{N}^\wedge = \mathcal{L}(H)$;*
(iv) *If $\mathcal{N}$ has only finitely many atoms $\{ M_i \ominus (M_i)_- \}_{i=1}^m$ and*

$$0 \leqslant d = \sum_{i=1}^m \dim M_i \ominus (M_i)_- < \infty,$$

then

$$\mathcal{N}_0^\wedge = \mathcal{L}(H)_d := \{ T \in \mathcal{L}(H) : \sum_{\lambda \in \sigma_0(T) - \sigma_e(T)^\wedge} \dim H(\lambda, T) \leqslant d \}.$$

where $\sigma_0(T)$ is the set of normal eigenvalues of T, $\sigma_e(T)$ is the essential spectrum of T, $\sigma_e(T)^\wedge$ is the polinomially convex hull of $\sigma_e(T)$, and $H(\lambda, T)$ is the Riesz spectral subspace of T associated with λ;
(v) *In the remaining cases, $\mathcal{N}_0^\wedge = \mathcal{L}(H)$.*

The purpose of this article is to consider the analogous problems for the tensor product of nests $\{ \mathcal{N}_i \}_{i=1}^k (2 \leqslant k < \infty)$.

DEFINITION 1.2. For $i = 1, 2, \ldots, k$, let $\mathcal{N}_i$ be a nest in the Hilbert space H_i,
(i) The tensor product $\bigotimes_{i=1}^k \mathcal{N}_i = \mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \cdots \otimes \mathcal{N}_k$ of the $\mathcal{N}_i$'s is the complete lattice generated by the family of subspaces of H,

$$\{ M^1 \otimes M^2 \otimes \cdots \otimes M^k : M^i \in \mathcal{N}_i (i = 1, 2, \ldots, k) \},$$

where $H = H_1 \otimes H_2 \otimes \cdots \otimes H_k$. (cf. K. Davidson [2, p. 381]);
(ii) The algebra of operators associated with $\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i$ is

$$\operatorname{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) = \left\{T \in \mathcal{L}(H) : \operatorname{Lat}T \supset \overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right\};$$

(iii) $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \left\{UTU^* + K : U \text{ is unitary}, T \in \operatorname{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \text{ and} K \in \mathcal{K}(H)\right\}$;
(iv) $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 = \{T \in \mathcal{L}(H)$: Given $\varepsilon > 0$, there exist U_ε unitary, $A_\varepsilon \in$ $\in \operatorname{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)$, $K_\varepsilon \in \mathcal{K}(H)$ such that $\|K_\varepsilon\| < \varepsilon$ and $T = U_\varepsilon A_\varepsilon U_\varepsilon^* + K_\varepsilon\}$.

The following two theorems are the main results of this article.

THEOREM 1.3. *For $i = 1, 2, \ldots, k$, let $\mathcal{N}_i$ be a nest in the Hilbert space H_i, and let $H = \overset{k}{\underset{i=1}{\otimes}} H_i$; then*
(i) *If, for all $i = 1, 2, \ldots, k$, $\mathcal{N}_i$ is well-ordered and all its atoms are finite dimensional, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 = (QT);$$

(ii) *If, for all $i = 1, 2, \ldots, k$, $\mathcal{N}_i$ is well-ordered from above and all its atoms are finite dimensional, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 = (QT)^*;$$

(iii) *If the family $\{\mathcal{N}_i\}_{i=1}^k$ does not satisfy (i), neither (ii), then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \mathcal{L}(H).$$

THEOREM 1.4. *For $i = 1, 2, \ldots, k$, let $\mathcal{N}_i$ be a nest in the Hilbert space H_i, and let $H = \overset{k}{\underset{i=1}{\otimes}} H_i$, and assume that neither all the $\mathcal{N}_i$'s are well-ordered nests with finite dimensional atoms, nor all of them are well-ordered from above with finite dimensional atoms; then*
(i) *If $\mathcal{N}_i$ has only finitely many atoms $\{M_j^i \ominus (M_j^i)_-\}_{j=1}^{m_i}$ and*

$$0 < d_i = \sum_{j=1}^{m_i} \dim\left[M_j^i \ominus (M_j^i)_-\right] < \infty\, (i = 1, 2, \ldots, k),$$

then

$$\mathcal{L}(H)_{\sum_{i=1}^k d_i - k + 1} \subset \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 \subset \mathcal{L}(H)_{\prod_{i=1}^k d_i}$$

and

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge \not\subset \mathcal{L}(H)_{\prod_{i=1}^{k} d_i - 1}.$$

Furthermore, if all $\mathcal{N}_i$'s are maximal nests then

$$\mathcal{L}(H)_{\sum_{i=1}^{k} d_i - k + 2} \not\subset \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge;$$

(ii) *If at least one of the nests is continuous, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}(H)_0;$$

(iii) *If $\mathcal{N}_i (i = 1, 2, \ldots, k)$ is not the type described in (i) or (ii), then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}(H).$$

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2. UNCOUNTBLE NESTS

LEMMA 2.1.(Voiculescu's corollary [17],[9, Chapter 4]). *Let $T \in \mathcal{L}(H)$ and ρ be a unital faithful *-representation of a separable C^*-subalgebra of the quotient Calkin algebra $\mathcal{A}(H) = \mathcal{L}(H)/\mathcal{K}(H)$ contaning the canonical image $\tilde{T} = \pi(T)$ of T, and $\tilde{I}$, on a separable space H_ρ. Let $A = \rho(\tilde{T})$. Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathcal{K}(H)$, with $\|K_\varepsilon\| < \varepsilon$, such that*

$$T - K_\varepsilon \simeq T \oplus A^{(\infty)}.$$

PROPOSITION 2.2. (Herrero [13]). *Let $T \in \mathcal{L}(H)$ and $\Gamma = \{\lambda_n\}_{n=-\infty}^{\infty}$ be a two-sided sequence of complex numbers such that*

(i) *all the limit points of Γ belong to $\sigma_e(T)^\wedge$; and*

(ii) card $\{n < 0 : \lambda_n \in \Omega\} =$ card$\{n > 0 : \lambda_n \in \Omega\} = \aleph_0$ *for each open set Ω such that $\Omega \cap \sigma_e(T)^\wedge \neq \emptyset$, but $\partial\Omega \cap \sigma_e(T)^\wedge = \emptyset$.*

Then there exist $K \in \mathcal{K}(H)$, U unitary and

$$A = \begin{pmatrix} \ddots & & & & & \\ & \lambda_{-2} & & & * & \\ & & \lambda_{-1} & & & \\ & & & \lambda_0 & & \\ & 0 & & & \lambda_1 & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e_{-2} \\ e_{-1} \\ e_0 \\ e_1 \\ e_2 \\ \vdots \end{matrix} \text{ such that } T = UAU^* + K,$$

where $\{ e_n \}_{n=-\infty}^{\infty}$ is an ONB of H.
Furthermore, if Γ also satisfies
(iii) *$\lambda_n \in \sigma(T)^\wedge$ for all n and*
(iv) $\operatorname{card}\{ n : \lambda_n = \lambda \} = \dim H(\lambda, T)$ *for each $\lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge$,*
then given $\varepsilon > 0$, K can be chosen so that $\|K\| < \varepsilon$.

An obvious consequence of Proposition 2.2 is that, up to a small compact perturbation, every operator T is unitarily equivalent to an operator in the nest $\mathcal{N}_{\mathcal{Z}}$ ($\mathcal{N}_{\mathcal{Z}}$ is a nest which is order-isomorphic to $\mathcal{Z}^* = \{-\infty\} \cup \mathcal{Z} \cup \{\infty\}$, $\mathcal{Z}$ is the set of all integers, and all its gaps are one-dimensional).

LEMMA 2.3. *Given T in $\mathcal{L}(H)$, $\varepsilon > 0$ and a sequence $\{\alpha_n\}_{n=-\infty}^{\infty}$ of complex numbers such that $\alpha_n \in \sigma_e(T)^\wedge$ for all n and, for each closen subset σ of $\sigma_e(T)^\wedge$,*

$$\operatorname{card}\{ n > 0 : \alpha_n \in \sigma \} = \operatorname{card}\{ n < 0 : \alpha_n \in \sigma \} = \aleph_0,$$

then there exists $K_\varepsilon \in \mathcal{K}(H)$, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K_\varepsilon \simeq \sum_{i=0}^{\infty} \oplus A_i$, where

$$A_0 = \begin{pmatrix} \ddots & & & & & \\ & \lambda_{-2} & & & * & \\ & & \lambda_{-1} & & & \\ & & & \lambda_0 & & \\ & 0 & & & \lambda_1 & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e^0_{-2} \\ e^0_{-1} \\ e^0_0 \\ e^0_1 \\ e^0_2 \\ \vdots \end{matrix},$$

$$A_i = \begin{pmatrix} \ddots & & & & & \\ & \alpha_{-2} & & & * & \\ & & \alpha_{-1} & & & \\ & & & \alpha_0 & & \\ & 0 & & & \alpha_1 & \\ & & & & & \alpha_2 \\ & & & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e^i_{-2} \\ e^i_{-1} \\ e^i_0 \\ e^i_1 \\ e^i_2 \\ \vdots \end{matrix}$$

$(i=1,2,\ldots)$, $\{e^i_j\}_{j=-\infty}^{\infty}$ $(i=0,1,2,\ldots)$ is a suitable ONB of the underlying space, $\operatorname{card}\{j : \lambda_j = \lambda\} = \dim H(\lambda, T)$ for each $\lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge$ and $\{\lambda_j : \lambda_j \notin \sigma_0(T) \setminus \sigma_e(T)^\wedge\}$ is order-isomorphic to $\{\alpha_n\}_{n=-\infty}^{\infty}$. (Roughly speaking, the finitely many or countable λ_j's corresponding to points in $\sigma_0(T) \setminus \sigma_e(T)^\wedge$ are "interpolated" in the two-sided sequence $\{\alpha_n\}_{n=-\infty}^{\infty}$.)

Moreover, there also exists $K \in \mathcal{K}(H)$, with $\|K\| < \varepsilon + \frac{1}{2}\max\{\operatorname{dist}[\lambda, \sigma_e(T)], \lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge\}$, such that $T - K \simeq \sum_{j=1}^{\infty} \oplus T_j$, where

$$T_j = \begin{pmatrix} \ddots & & & & & & \\ & \alpha_{-2} & T^j_{-2-1} & T^j_{-20} & T^j_{-21} & T^j_{-22} & \\ & & \alpha_{-1} & T^j_{-10} & T^j_{-11} & T^j_{-12} & \\ & & & \alpha_0 & T^j_{01} & T^j_{02} & \\ & 0 & & & \alpha_1 & T^j_{12} & \\ & & & & & \alpha_2 & \\ & & & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ H^j_{-2} \\ H^j_{-1} \\ H^j_0 \\ H^j_1 \\ H^j_2 \\ \vdots \end{matrix},$$

H^j_l is infinite dimensional $(j = 1,2,\ldots; l = 0, \pm1, \pm2, \ldots)$, and the operators T^j_{lm} $(j = 1,2,\ldots; l,m = 0,\pm1,\pm2,\ldots)$ are simultaneously diagonal with respect to the same ONB (up to a suitable identification of the underlying spaces).

Proof. (i) From Lemma 2.1, there exist $K_1 \in \mathcal{K}(H)$ and U unitary such that $\|K_1\| < \frac{\varepsilon}{2}$ and $U(T-K_1)U^* = T \oplus \sum_{i=1}^{\infty} \oplus R_i$ where $R_i \simeq A^{(\infty)} \simeq R_i^{(\infty)}$ $(i = 1,2,\ldots)$. Here $A^{(\infty)} = A \oplus A \oplus A \oplus \cdots$. By Proposition 2.2, there exist C_i compact and U_i unitary such that $\|C_i\| < \frac{\varepsilon}{2^{i+1}}$,

$$A_0 = U_0(T - C_0)U_0^* = \begin{pmatrix} \ddots & & & & \\ & \lambda_{-1} & & & * \\ & & \lambda_0 & & \\ & 0 & & \lambda_1 & \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e^0_{-1} \\ e^0_0 \\ e^0_1 \\ \vdots \end{matrix},$$

and

$$A_i = U_i(R_i - C_i)U_i^* = \begin{pmatrix} \ddots & & & & \\ & \alpha_{-1} & & & * \\ & & \alpha_0 & & \\ & 0 & & \alpha_1 & \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e^i_{-1} \\ e^i_0 \\ e^i_1 \\ \vdots \end{matrix},$$

$(i = 1, 2, 3, \ldots)$, where $\{\lambda_j\}_{j=-\infty}^{\infty}$ is a sequence of complex numbers which satisfies the conditions mentionated in the lemma.

Let $C = \sum_{i=0}^{\infty} \oplus C_i$ and $V = \sum_{i=0}^{\infty} \oplus U_i$, then we have $W(T - K_\varepsilon)W^* = \sum_{i=0}^{\infty} \oplus A_i$, where $W = VU$ is unitary, $K_\varepsilon = K_1 + U^*CU$ is compact, and $\|K_\varepsilon\| < \varepsilon$.

(ii) From Corollary 2.11 of [10], there exists $K' \in \mathcal{K}(H)$ such that

$$\|K'\| < \frac{\varepsilon}{2} + \frac{1}{2}\max\{\,\mathrm{dist}\,[\lambda, \sigma_e(T)] : \lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge\,\},$$

and $\sigma_0(T - K') = \sigma_0(T) \cap \sigma_e(T)^\wedge$. By applying (i) to $T - K'$ and $\frac{\varepsilon}{2}$, we have

$$T - K' - K'' \simeq \sum_{i=0}^{\infty} \oplus A_i,$$

where $\|K''\| < \frac{\varepsilon}{2}$ and A_0 has the same form as the A_i's $(i > 0)$.

If we regroup $\{A_i\}_{i=0}^{\infty} = \{A_{j,k} : j, k = 1, 2, \ldots\}$, then

$$T - K' - K'' \simeq \sum_{j=1}^{\infty} \oplus \left(\sum_{k=1}^{\infty} \oplus A_{j,k} \right).$$

Assume

$$A_{j,k} = \begin{pmatrix} \ddots & & & & \\ & \alpha_{-1} & t^{jk}_{-10} & t^{jk}_{-11} & \\ & & \alpha_0 & t^{jk}_{01} & \\ 0 & & & \alpha_1 & \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ e^{jk}_{-1} \\ e^{jk}_0 \\ e^{jk}_1 \\ \vdots \end{matrix},$$

then

$$T_j = \sum_{k=1}^{\infty} \oplus A_{j,k} \simeq \begin{pmatrix} \ddots & & & & \\ & \alpha_{-1} & T^j_{-10} & T^j_{-11} & \\ & & \alpha_0 & T^j_{01} & \\ 0 & & & \alpha_1 & \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ H^j_{-1} \\ H^j_0 \\ H^j_1 \\ \vdots \end{matrix},$$

where

$$T^j_{lm} = \begin{pmatrix} t^{j1}_{lm} & & \\ & t^{j2}_{lm} & 0 \\ 0 & & \ddots \end{pmatrix} \in \mathcal{L}(H^j_m, H^j_l) \qquad (-\infty < l < m < \infty)$$

and $H^j_l = \bigvee_{k=1}^{\infty} \{e^{jk}_l\}$. Thus $T - (K' + K'') \simeq \sum_{j=1}^{\infty} \oplus T_j$ and $\|K' + K''\| < \varepsilon +$ $+\frac{1}{2}\max\{\mathrm{dist}\,[\lambda, \sigma_e(T)],\, \lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge\,\}$. ■

REMARK 2.4. (i) Clearly, $\sum_{j=1}^{\infty} \oplus T_j$ can be replaced by $\sum_{j\in\Lambda} \oplus T_j$, where Λ is any finite or countable index set, not necessarily the set of all natural numbers; for instance, we can set $\Lambda = \mathcal{Z}^k$, the set of all k-tuples of integers, etc.
(ii) In the second part of the lemma, $\|K\|$ can be chosen arbitrarily small if and only if $\sigma_0(T) \subset \sigma_e(T)^\wedge$.

PROPOSITION 2.5. *For $i = 1, 2, \ldots, k$, let $\mathcal{N}_i$ be a perfect nest (i.e., the support of spectral measure of $\mathcal{N}_i$ is a perfect set) in the Hilbert space H_i; then $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge =$ $= \mathcal{L}(H)$ and $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge_0 \supset \mathcal{L}(H)_0$.*

Proof. Since $\mathcal{N}_i$ is perfect, $E_i(\{0\}) = E_i(\{1\}) = 0$ (E_i is the spectral measure associated with $\mathcal{N}_i$, $i = 1, 2, \ldots, k$)) and we can choose $\{C_n^i\}_{n\in\mathcal{Z}} \subset [0, 1]$ strictly increasing such that $C_n^i \nearrow 1 (n \to \infty)$ and $C_n^i \searrow 0 (n \to -\infty)$, in such a way that $C_n^i (n \in \mathcal{Z})$ is a limit point of $\mathcal{N}_i$ both from the right as well as from the left.

Now we can find an order-preserving homeomorphism $\varphi_i : (0, 1) \to \mathcal{R}$ such that $\varphi_i(C_n^i) = n$. Thus, up to a change of variable, we can assume $\mathcal{N}_i$ has spectral measure E_i supported on $\mathcal{R}$, every $n \in \mathcal{Z}$ is a limit point of $\mathcal{N}_i$ both from the right as well as from the left, and

$$\mathcal{N}_i = \{\,0; E_i\left((-\infty, t]\right) H_i,\ (t \in \mathcal{R}); H_i\,\}\ (i = 1, 2, \ldots, k).$$

Given $T \in \mathcal{L}(H)$, it follows from Lemma 2.3 and Remark 2.4 (i) that there exist $K \in \mathcal{K}(H)$ and W unitary such that

$$W(T - K)W^* = \sum_{\beta\in\mathcal{Z}^{k-1}} \oplus T_\beta,$$

where

$$T_\beta = \begin{pmatrix} \ddots & & & \\ & \alpha_{-1} & & * \\ & & \alpha_0 & \\ 0 & & & \alpha_1 \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ H_{-1}^\beta \\ H_0^\beta \\ H_1^\beta \\ \vdots \end{matrix}, \quad \dim H_j^\beta = \infty$$

$(\beta \in \mathcal{Z}^{k-1}, -\infty < j < \infty)$.

Consider the decomposition $H = \overset{k}{\underset{i=1}{\otimes}} H_i = \sum_{\gamma\in\mathcal{Z}^k} \oplus H_\gamma$. (If $\gamma = (n_1, \ldots, n_k)$, then $H_\gamma = E\left(\prod_{i=1}^{k} [n_i, n_{i+1}]\right)$ is infinite dimensional because n_i and $n_i + 1$ are limit points of $\mathcal{N}_i$ from both sides, where $E = E_1 \times E_2 \times \cdots \times E_k$.)

Define unitary mappings

$$U_\beta : \sum_{n=-\infty}^{\infty} \oplus H_n^\beta \to \sum_{n=-\infty}^{\infty} \oplus H_{(n,\beta+n\overline{u})} \qquad (\beta \in \mathcal{Z}^{k-1})$$

such that $U_\beta H_n^\beta = H_{(n,\beta+n\overline{u})}\,(n = 0, \pm 1, \ldots)$, where $\overline{u} = (1, 1, \ldots, 1) \in \mathcal{Z}^{k-1}$ is the "unit" of translation along the main diagonal and

$$(n, \beta + n\overline{u}) = (n, n_1 + n, n_2 + n, \ldots, n_{k-1} + n) \in \mathcal{Z}^k,$$

if $\beta = (n_1, \ldots, n_{k-1})$.

Thus,

$$U_\beta T_\beta U_\beta^* = \begin{pmatrix} \ddots & & & \\ & \alpha_{-1} & & * \\ & & \alpha_0 & \\ 0 & & & \alpha_1 \\ & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ H_{(-1,\beta-\overline{u})} \\ H_{(0,\beta)} \\ H_{(1,\beta+\overline{u})} \\ \vdots \end{matrix} .$$

Clearly, for each $\beta \in \mathcal{Z}^{k-1}$ and each $q(\beta) \in \mathcal{Z}$,

$$\left[\sum_{n=-\infty}^{q(\beta)} \oplus H_{(n,\beta+n\overline{u})}\right] \oplus R_{(q(\beta)+1,\beta+(q(\beta)+1)\overline{u})} \in \operatorname{Lat} U_\beta T_\beta U_\beta^*,$$

for each subspace $R_{(q(\beta)+1,\beta+(q(\beta)+1)\overline{u})}$ of $H_{(q(\beta)+1,\beta+(q(\beta)+1)\overline{u})}$. Hence, if $U = \sum_{\beta \in \mathcal{Z}^{k-1}} \oplus U_\beta$ then

$$E\left(\prod_{i=1}^{k}(-\infty, x_i)\right) \in \operatorname{Lat}[UW(T-K)W^*U^*]$$

for all $(x_1, x_2, \ldots, x_k) \in \mathcal{R}^k$; that is, $UW(T-K)W^*U^* \in \operatorname{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)$ and $T \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge$. Thus, we conclude that $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge = \mathcal{L}(H)$.

If $T \in \mathcal{L}(H)_0$, given $\varepsilon > 0$,it follows from Remark 2.4 (ii) that K can be chosen so that $\|K\| < \varepsilon$. Thus $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge \supset \mathcal{L}(H)_0$. ■

COROLLARY 2.6. *If $\mathcal{N}_i$ is a continuous nest in a Hilbert space $H_i\,(i = 1, 2, \ldots, k)$, then* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge = \mathcal{L}(H)$.

LEMMA 2.7. *Let $\mathcal{N}_\ell$ be a nest in the Hilbert space $H_\ell(\ell = 1, 2, \ldots, k)$, and assume that $H = \overset{k}{\underset{\ell=1}{\otimes}} H_\ell$ can be decomposed as $H = \sum_{i=1}^{\infty} \oplus G_i$ in such a way that for every*

$x = (x_1, \ldots, x_k) \in \prod_{\ell=1}^{k}[0,1]_\ell$, $\overset{k}{\underset{\ell=1}{\otimes}} M^\ell_{x_\ell} = \sum_{i=1}^{\infty} \oplus M_{i,x}$ *and* $M_{i,x} \in \overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}$, *where* $M_{i,x} \subset \subset G_i$ *and* $\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}$ *is a tensor product of* $m_i (m_i < \infty)$ *nests in Hilbert space* G_i, $(i = = 1, 2, \ldots)$.

If $T \in \mathcal{L}(H)$ *can be decomposed as* $T = \sum_{i=1}^{\infty} \oplus T_i \quad (T_i \in \mathcal{L}(G_i))$, *and if for some finite number* p, $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge \quad (i = 1, 2, \ldots, p)$ *and* $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge_0 \quad (i = = p+1, \ldots)$; *then* $T \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge$. *If* $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge_0$ *for all* i, *then* $T \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge_0$.

Proof. Since $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge$, $M_{ix} \in \mathrm{Lat} U_i(T_i - K_i)U_i^*$ for some compact K_i and unitary $U_i \quad (i = 1, 2, \ldots, p)$. Given $\varepsilon > 0$, since $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge_0 \quad (i = p+1, \ldots)$, $M_{ix} \in \mathrm{Lat}\, U_i(T_i - K_i)U_i^*$ for some compact K_i with $\|K_i\| < \frac{\varepsilon}{2^i}$ and unitary U_i. Thus,

$$\overset{k}{\underset{\ell=1}{\otimes}} M^\ell_{x_\ell} = \sum_{i=1}^{\infty} \oplus M_{ix} \in \mathrm{Lat}[U(T-K)U^*]$$

where $U = \sum_{i=1}^{\infty} \oplus U_i$ and $K = \sum_{i=1}^{\infty} \oplus K_i$. Therefore, $T \in \left(\overset{k}{\underset{\ell=1}{\otimes}} \mathcal{N}_\ell\right)^\wedge$.

It is obvious that if $T_i \in \left(\overset{m_i}{\underset{j=1}{\otimes}} \mathcal{N}_{ij}\right)^\wedge_0$ for all i, then $\|K\| < \varepsilon$. Thus, $T \in \in \left(\overset{k}{\underset{\ell=1}{\otimes}} \mathcal{N}_\ell\right)^\wedge_0$. ■

REMARK 2.8. Lemma 2.7 remains true (with the obvious notational changes) if $T = \sum_{i=1}^{m} \oplus T_i \,(1 \leqslant m < \infty)$, or $T = \sum_{\gamma \in \Lambda} \oplus T_\gamma$ (Λ is a countable index set).

LEMMA 2.9. *If* $\mathcal{N}_i \,(i = 1, 2, \ldots, p)$ *is a perfect nest and* $\mathcal{N}_i \,(i = p+1, \ldots, k)$ *is an at most denumerable nest, then* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge = \mathcal{L}(H)$ *and* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge_0 \supset \mathcal{L}(H)_0$.

Proof. If $L_c = M^i_c \ominus (M^i_c)_-$ is an atom of $\mathcal{N}_i$ for some i with $\dim L_c = s_c > 1$, let $\{e_j\}_{j=1}^{s_c}$ be an ONB of L_c. Consider the nest

$$\mathcal{N}_i' = \mathcal{N}_i \cup \{(M^i_c \oplus \vee\{e_1, \ldots, e_\ell\} : \ell = 1, 2, \ldots, s_c\};$$

since $\mathcal{N}_i' \supset \mathcal{N}_i$, $\mathrm{alg}(\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_{i-1} \otimes \mathcal{N}_i' \otimes \mathcal{N}_{i+1} \otimes \cdots \otimes \mathcal{N}_k) \subset \mathrm{alg}\left(\overset{k}{\underset{j=1}{\otimes}} \mathcal{N}_j\right)$. Thus, $\left(\overset{k}{\underset{j=1}{\otimes}} \mathcal{N}_j\right)^\wedge = \mathcal{L}(H)$ whenever $\left[\left(\overset{i-1}{\underset{j=1}{\otimes}} \mathcal{N}_j\right) \otimes \mathcal{N}_i' \otimes \left(\overset{k}{\underset{q=i+1}{\otimes}} \mathcal{N}_q\right)\right]^\wedge = \mathcal{L}(H)$, and

$\left(\overset{k}{\underset{j=1}{\otimes}} \mathcal{N}_j\right)^{\wedge}_0 \supset \mathcal{L}(H)_0$ whenever $\left[\left(\overset{i-1}{\underset{j=1}{\otimes}} \mathcal{N}_j\right) \otimes \mathcal{N}'_i \otimes \left(\overset{k}{\underset{q=i+1}{\otimes}} \mathcal{N}_q\right)\right]^{\wedge}_0 \supset L(H)_0$. Therefore, we can assume without loss generality that all atoms of $\mathcal{N}_i$ $(i = p+1, \dots, k)$ are one-dimensional.

Denote these atoms by $\{[e^i_j]\}_{j=1}^{\infty}$, where $[e^i_j] = M^i_{c^i_j} \ominus (M^i_{c^i_j})_-$ and $\overset{\infty}{\underset{j=1}{\bigvee}} [e^i_j] = H_i$. Given $x_i \in [0,1]\,(i = 1, 2, \dots, k)$, we can write

$$\begin{aligned}\overset{k}{\underset{i=1}{\otimes}} M^i_{x_i} &= \sum_{\substack{c^\ell_{j_\ell} \leqslant x_\ell \\ (\ell=p+1,\dots,k)}} \oplus \left[\left(\overset{p}{\underset{i=1}{\otimes}} M^i_{x_i}\right) \otimes \left(\overset{k}{\underset{\ell=p+1}{\otimes}} [e^\ell_{j_\ell}]\right)\right] = \\ &= \sum_{\gamma \in \prod_{\ell=p+1}^{k} [0,1]_\ell} \oplus \left[\left(\overset{p-1}{\underset{i=1}{\otimes}} M^i_{x_i}\right) \otimes M(\gamma)_{x_p}\right],\end{aligned}$$

where $\gamma = \left(c^{p+1}_{p+1}, \dots, c^k_{j_k}\right)$, $M(\gamma)_{x_p} = M^p_{x_p} \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right] \in \mathcal{N}_p(\gamma)$, and $\mathcal{N}_p(\gamma)$ is a nest in the Hilbert space $H_p \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right]$, unitarily equivalent to $\mathcal{N}_p$.

By proposition 2.5, $\left[\left(\overset{p-1}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \otimes \mathcal{N}_p(\gamma)\right]^{\wedge} = \mathcal{L}(H_\gamma)$ and $\left[\left(\overset{p-1}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \otimes \mathcal{N}_p(\gamma)\right]^{\wedge}_0 \supset$ $\supset \mathcal{L}(H_\gamma)_0$, where $H_\gamma = \left(\overset{p}{\underset{i=1}{\otimes}} H_i\right) \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right]$.

Given $T \in \mathcal{L}(H)$ and $\varepsilon > 0$, it follows from Lemma 2.1 that there exist U unitary and $K \in \mathcal{K}(H)$, with $\|K\| < \varepsilon$, such that

$$U(T-K)U^* = \sum_{\gamma \in \prod_{\ell=p+1}^{k} [0,1]_\ell} \oplus A_\gamma,$$

where $A_{\gamma_0} = T$ and $A_\gamma \simeq A \simeq A^{(\infty)}$ if $\gamma \neq \gamma_0$. (Here $\gamma_0 = (1, 1, \dots, 1)$.)

By proposition 2.5, $T \in \left[\left(\overset{p-1}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \otimes \mathcal{N}_p(\gamma_0)\right]^{\wedge}$ and $A_\gamma \in \left[\left(\overset{p-1}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \otimes \mathcal{N}_p(\gamma)\right]^{\wedge}_0$ $(\gamma \neq \gamma_0)$. By Lemma 2.7, $U(T-K)U^* \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}$; therefore, $T \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}$.

If $T \in \mathcal{L}(H)_0$, then $U^*(T-K)U^* \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$. Since K can be chosen of arbitrarily small norm, $T \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$. ■

PROPOSITION 2.10. *Let $\mathcal{N}_i$ be an uncountable nest in the Hilbert space H_i, $(i = 1, 2, \dots, k)$; then $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \mathcal{L}(H)$ and $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 \supset \mathcal{L}(H)_0$.*

Proof. We will proceed by induction on k. If $k = 1$, then the result is true by Theorem 1.1.

Suppose the statement is true for the tensor product of $k \leqslant n-1$ nests, and let $k = n$. As in the proof of Lemma 2.9, we can assume that all the atoms of $\mathcal{N}_i$ ($i = 1, 2, \ldots, n$) are one-dimensional.

Let $F_i = \operatorname{supp} E_i$, where E_i is the spectral measure of $\mathcal{N}_i$ ($i = 1, 2, \ldots, n$).

Set

$$\mathcal{N}_i^1 = \{ E_i([0,t]) \cap G_i \,:\, 0 \leqslant t \leqslant 1 \}$$

and

$$\mathcal{N}_i^2 = \{ E_i([0,t] \cap (F_i \setminus G_i)) \,:\, 0 \leqslant t \leqslant 1 \},$$

where G_i is the perfect part of F_i. Then $\mathcal{N}_i^1$ is a perfect nest and $\mathcal{N}_i^2$ is an at most denumerable nest ($\mathcal{N}_i^2$ can be absent).

Let

$$M_t^i = E_i([0,t]),$$
$$M_t^{i,1} = E_i([0,t]) \cap G_i),$$
$$M_t^{i,2} = E_i([0,t] \cap (F_i \setminus G_i)) \quad (i = 1, 2, \ldots, n;\ 0 \leqslant t \leqslant 1).$$

Given $x_i \in [0,1]$ ($i = 1, 2, \ldots, n$), it is not difficult to see that

$$\bigotimes_{i=1}^{n} M_{x_i}^i = \left(\bigotimes_{i=1}^{n} M_{x_i}^{i,1}\right) \oplus \left[\left(\bigotimes_{i=1}^{n-1} M_{x_i}^i\right) \otimes M_{x_n}^{n,2}\right] \oplus \left[\sum_{\substack{s_i = 1 \text{ or } 2 \\ (i=1,2,\ldots,n-1) \\ s_n = 1 \\ n < \sum_{i=1}^{n} s_i < 2n}} \oplus \left(\bigotimes_{i=1}^{n} M_{x_i}^{i,s_i}\right) \right].$$

Observe that

$$\left(\bigotimes_{i=1}^{n-1} M_{x_i}^i\right) \otimes M_{x_n}^{n,2} = \sum_{c_j^n \leqslant x_n} \oplus \left[\left(\bigotimes_{i=1}^{n-2} M_{x_i}^i\right) \otimes M^{n-1}(j)_{x_{n-1}}\right],$$

where $M^{n-1}(j)_{x_{n-1}} = M_{x_{n-1}}^{n-1} \otimes [e_j^n]$, $[e_j^n]$ is an atom of $\mathcal{N}_n^2$, $[e_j^n] = M_{c_j^n}^{n,2} \ominus \left(M_{c_j^n}^{n,2}\right)_-$, and $\left(\bigotimes_{i=1}^{n-2} M_{x_i}^i\right) \otimes M^{n-1}(j)_{x_{n-1}} \in \left(\bigotimes_{i=1}^{n-2} \mathcal{N}_i\right) \otimes \mathcal{N}_{n-1}(j)$. Here $\mathcal{N}_{n-1}(j) = \{ M^{n-1}(j)_t \otimes [e_j^n] \,:\, 0 \leqslant t \leqslant 1 \}$ is uncountable and unitarily equivalent to $\mathcal{N}_{n-1}$. By the inductive assumption,

$$\left[\left(\bigotimes_{i=1}^{n-2} \mathcal{N}_i\right) \otimes \mathcal{N}(j)_{n-1}\right]_0^{\wedge} \supset \mathcal{L}\left(\bigotimes_{i=1}^{n-1} H_i \otimes [e_j^n]\right)_0.$$

Observe that $\overset{n}{\underset{i=1}{\otimes}} M_{x_i}^{i,s_i} \in \overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i^{s_i}$, by Lemma 2.9, $\left(\overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i^{s_i}\right)_0^{\wedge} \supset \mathcal{L}\left(\overset{n}{\underset{i=1}{\otimes}} H_i^{s_i}\right)_0$, where

$$H_i^{s_i} = \begin{cases} E_i([0,1] \cap G_i) & s_i = 1 \\ H_i \ominus E_i([0,1] \cap G_i) & s_i = 2 \end{cases}$$

Observe that $\overset{n}{\underset{i=1}{\otimes}} M_{x_i}^{i,1} \in \overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i^1$. By Proposition 2.5, $\left(\overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i^1\right)^{\wedge} = \mathcal{L}\left(\overset{n}{\underset{i=1}{\otimes}} H_i^1\right)$ and $\left(\overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i^1\right)_0^{\wedge} \supset \mathcal{L}\left(\overset{n}{\underset{i=1}{\otimes}} H_i^1\right)_0$. By Lemmas 2.1 and 2.7, $\left(\overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \mathcal{L}(H)$ and $\left(\overset{n}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^{\wedge} \supset \mathcal{L}(H)_0$. ■

COROLLARY 2.11. *If at least one of the nests $\mathcal{N}_i$'s is uncountable, then* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \mathcal{L}(H)$ *and* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^{\wedge} \supset \mathcal{L}(H)_0$.

Proof. This follows immediately from Lemma 2.7 and Proposition 2.10. ■

3. COUNTABLE NESTS

LEMMA 3.1. *If $\mathcal{N}_i$ $(i = 1, 2)$ is a well-ordered nest in the Hilbert space H_i and all its atoms are finite dimensional, then there exists a nest $\mathcal{N}$ in $H = H_1 \otimes H_2$ such that $\mathcal{N} \subset \mathcal{N}_1 \otimes \mathcal{N}_2$, $\mathcal{N}$ is well-ordered and all its atoms are finite dimensional.*

Proof. Assume that the order type of $\mathcal{N}_i$ is $\alpha_i + m_i$ where α_i is a countable limit ordinal and m_i is finite $(i = 1, 2)$; i.e., $\mathcal{N}_i = \{ c_j^i \}_{j=0}^{\alpha_i+m_i}$, $c_0^i = 0, c_{\alpha_i+m_i}^i = 1$ and $c_j^i < c_{j+1}^i$. Assume $L_j^i = M_{c_j^i}^i \ominus \left(M_{c_j^i}^i\right)_-$ is the atom of $\mathcal{N}_i$ at c_j^i, then $\dim L_j^i < \infty$ and $\sum_{j=1}^{\alpha_i+m_i} \oplus L_j^i = H_i \,(i = 1, 2)$.

Define $\mathcal{N}$ as the ordinal sum $\mathcal{N} = \sum_{1 \leqslant j \leqslant \alpha_2+m_2} \mathcal{N}_{1,j}$, where $\mathcal{N}_{1,j} = \{ M_t^1 \otimes L_j^2 : 0 \leqslant t \leqslant 1 \}$. (If $m_2 = 0$, then $\mathcal{N} = \sum_{1 \leqslant j \leqslant \alpha_2} \mathcal{N}_{1,j} \cup \{H\}$.) It is immediate that $\mathcal{N}$ has the required properties. ■

PROPOSITION 3.2.
(i) *If $\mathcal{N}_i$ $(i = 1, 2, \ldots, k)$ is well-ordered, and all its atoms are finite dimensional, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^{\wedge} = (QT);$$

(ii) *If $\mathcal{N}_i$ $(i = 1, 2, \ldots, k)$ is well-ordered from above, and all its atoms are finite dimensional, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge} = \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^{\wedge} = (QT)^*.$$

Proof. (i) By Lemma 3.1, we have a nest $\mathcal{N}^2$ in $H_1 \otimes H_2$, such that $\mathcal{N}^2$ is well-ordered, all its atoms are finite dimensional and $\mathcal{N}_1 \otimes \mathcal{N}_2 \supset \mathcal{N}^2$. By applying Lemma 3.1 to $\mathcal{N}^2$ and $\mathcal{N}_3$, we can find a nest $\mathcal{N}^3$ in $\bigotimes_{i=1}^{3} H_i$, such that $\mathcal{N}^3$ is well-ordered, all its atoms are finite dimensional and

$$\mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \mathcal{N}_3 \supset \mathcal{N}^2 \otimes \mathcal{N}_3 \supset \mathcal{N}^3.$$

After $k-1$ steps, we will have a nest $\mathcal{N}^k$ in $\bigotimes_{i=1}^{k} H_i$ such that $\mathcal{N}^k$ is well-ordered, all its atoms are finite dimensional and $\bigotimes_{i=1}^{k} \mathcal{N}_i \supset \mathcal{N}^k$. Thus, by Theorem 1.1,

$$\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge \subset \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge \subset (\mathcal{N}^k)^\wedge = (QT).$$

Assume the set of atoms of $\mathcal{N}_i$ is $\{ L_j^i \}_{j=1}^{\infty} (i = 1, 2, \ldots, k)$. For a multi-index $\gamma = (j_2, \ldots, j_k)$, consider the nest $\mathcal{N}(\gamma) = \left\{ M_t^1 \otimes \left[\bigotimes_{i=2}^{k} L_{j_i}^i\right] : 0 \leqslant t \leqslant 1 \right\}$ in the Hilbert space $H(\gamma) = H_1 \otimes \left(\bigotimes_{i=2}^{k} L_{j_i}^i\right)$. It is obvious that $\mathcal{N}(\gamma)$ is well-ordered, and all its atoms are finite dimensional. Thus, by Theorem 1.1, $\mathcal{N}(\gamma)^\wedge = (QT)$.

Given $T \in (QT)$ and $\varepsilon > 0$, it follows from Lemma 2.1 that there exist K compact, with $\|K\| < \varepsilon$, and U unitary such that

$$U(T-K)U^* = \sum_{\gamma \in N^{k-1}} \oplus A_\gamma$$

where N denotes the set of natural numbers, $A_{(1,\ldots,1)} = T$ and $A_\gamma \simeq A \simeq A^{(\infty)}$ for $\gamma \neq (1, \ldots, 1)$.

If $\lambda \in \rho_{s-F}(T) = \rho_{s-F}(A)$, then

$$\operatorname{ind}(\lambda - A) = \begin{cases} 0, & \text{if } \operatorname{ind}(\lambda - T) < \infty \\ \infty, & \text{if } \operatorname{ind}(\lambda - T) = \infty \end{cases}$$

Thus, $A_\gamma \in (QT)$. By Theorem 1.1, $A_\gamma \in \mathcal{N}(\gamma)^\wedge = \mathcal{N}(\gamma)_0^\wedge$. Since $\sum_\gamma \oplus H(\gamma) = H$, by Lemma 2.7, $U(T-K)U^* \in \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge$. Since K can be chosen of arbitrarily small norm,

$$T \in \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge.$$

Hence, $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge = \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge = (QT)$.

(ii) This follows immediately from (i) by taking adjoints. ∎

If $\{\mathcal{N}_i\}_{i=1}^k$ is a finite family of countable nests not of the forms described in Proposition 3.2, then either

(1) At least one nest has both an increasing and a decreasing sequence of atoms; or

(2) At least one nest has an infinite dimensional atom; or

(3) Some nests (at least one) have the form of nests described in Proposition 3.2(i) and the others (at least one) have the form of nests described in Proposition 3.2(ii).

We need the following

LEMMA 3.3. *If* $\left(\overset{p}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}\left(\overset{p}{\underset{i=1}{\otimes}} H_i\right)$, *and* $\mathcal{N}_{p+1}, \ldots, \mathcal{N}_k$ *are countable nests, then*

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}(H).$$

Proof. We can assume, without loss of generality, that all atoms of $\mathcal{N}_i$ ($i = p+1, \ldots, k$) are one-dimensional. Denote them by $\{e^i_{j_i}\}_{j_i=1}^\infty$. For a multi-index $\gamma = (j_{p+1}, \ldots, j_k) \in N^{k-p}$, consider the nest $\mathcal{N}(\gamma)$ in the Hilbert space $H_p \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right]$:

$$\mathcal{N}(\gamma) = \left\{ M^p_t \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right] : 0 \leqslant t \leqslant 1 \right\}.$$

Since $\mathcal{N}(\gamma)$ is unitarily equivalent to $\mathcal{N}_p$, $\left[\left(\overset{p-1}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \otimes \mathcal{N}(\gamma)\right]_0^\wedge = \mathcal{L}(H(\gamma))$, where $H(\gamma) \left(\overset{p}{\underset{i=1}{\otimes}} H_i\right) \otimes \left[\overset{k}{\underset{\ell=p+1}{\otimes}} e^\ell_{j_\ell}\right]$. Since $\sum_{\gamma \in N^{k-p}} \oplus H(\gamma) = H$, by Lemmas 2.1 and 2.7 we have

$$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}(H). \qquad \blacksquare$$

From Theorem 1.1 and Lemma 3.3, we obtain the following.

COROLLARY 3.4. *Let* $\{\mathcal{N}_i\}_{i=1}^k$ *be a finite family of countable nests. If at least one of them has both an increasing and a decreasing sequence of atoms, or at least one of them has an infinite dimensional atom, then* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^\wedge = \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)_0^\wedge = \mathcal{L}(H)$.

LEMMA 3.5. $(\mathcal{N}_\omega \otimes \mathcal{N}_{\omega^*})^\wedge = (\mathcal{N}_\omega \otimes \mathcal{N}_{\omega^*})_0^\wedge = \mathcal{L}(H)$.
(Here $\mathcal{N}_\omega = \left\{ 0;\ \bigvee_{j=1}^m [e^1_j],\ (m = 1, 2, \ldots);\ H_1 \right\}$, $\mathcal{N}_{\omega^*} = \left\{ 0;\ \bigvee_{j=m}^\infty [e^2_j],\ (m = 1, 2, \ldots) \right\}$, $\{e^1_j\}_{j=1}^\infty$ *and* $\{e^2_j\}_{j=1}^\infty$ *are ONB of* H_1 *and* H_2*, respectively, and* $H = H_1 \otimes H_2$.*)*

Proof. Let $f_{i,j} = e_i^1 \otimes e_j^2\ (i, j = 1, 2, \ldots)$. Consider the nest $\mathcal{N}(\ell)$

$$\mathcal{N}(\ell) = \left\{ 0;\ \bigvee_{j=m}^{\infty} \{f_{\ell,j}\},\ (m \geqslant \ell);\ \left[\bigvee_{j=\ell}^{\infty}\{f_{\ell,j}\}\right] \oplus \left[\bigvee_{j=\ell}^{m}\{f_{j,\ell}\}\right],\right.$$
$$\left.(m > \ell);\ \left[\bigvee_{j=\ell}^{\infty}\{f_{\ell,j}\}\right] \oplus \left[\bigvee_{j=\ell}^{\infty}\{f_{j,\ell}\}\right]\right\} \quad (\ell = 1, 2, \ldots).$$

It is obvious that $\mathcal{N}(\ell)$ is unitary equivalent to $\mathcal{N}_Z$ and, by Theorem 1.1, $(\mathcal{N}(\ell))^{\wedge} = (\mathcal{N}(\ell))_0^{\wedge} = \mathcal{L}[H(\ell)]$, where $H(\ell) = \left[\bigvee_{j=\ell}^{\infty}\{f_{\ell,j}\}\right] \oplus \left[\bigvee_{j=\ell}^{\infty}\{f_{j,\ell}\}\right]$. Since $\sum_{\ell=1}^{\infty} \oplus \oplus H(\ell) = H$, it follows from Lemma 2.7,

$$(\mathcal{N}_\omega \otimes \mathcal{N}_{\omega^*})^{\wedge} = (\mathcal{N}_\omega \otimes \mathcal{N}_{\omega^*})_0^{\wedge} = \mathcal{L}(H). \qquad \blacksquare$$

PROPOSITION 3.6. *Let $\mathcal{N}_i$ be a countable nest with finite dimensional atoms in the Hilbert space $H_i\ (i = 1, 2, \ldots, k)$. If at least one of the $\mathcal{N}_i$'s is well-ordered and at least one of the $\mathcal{N}_i$'s is well-ordered from above, then*

$$\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^{\wedge} = \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^{\wedge} = \mathcal{L}(H), \qquad \text{where } H = \bigotimes_{i=1}^{k} H_i.$$

Proof. By using Lemma 3.3, we can directly assume that $k = 2$, $\mathcal{N}_1$ is well-ordered and $\mathcal{N}_2$ is well-ordered from above. Suppose the ordinal types of $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\alpha + m$ and $\beta + n$, respectively, where α and β are limit ordinals, m and n are finite $(m, n \geqslant 0)$. Without loss of generality, we can assume that all the atoms of $\mathcal{N}_1$ and $\mathcal{N}_2$ are one-dimensional. Assume that $m > n \geqslant 1$ (the order cases follow by similar argument); then

$$\mathcal{N}_1 = \left\{ 0;\ \bigvee_{0 \leqslant j < \ell} \{e_j^1\}, 0 < \ell \leqslant \alpha + m \right\}$$

and

$$\mathcal{N}_2 = \left\{ 0;\ \bigvee_{\ell \leqslant j < \beta + n} \{e_j^2\}, 0 \leqslant \ell < \beta + n \right\},$$

where $\{ e_j^1 : 1 \leqslant j < \alpha + m \}$ and $\{ e_j^2 : 1 \leqslant j < \beta + n \}$ are orthonormal basis of H_1 and H_2, respectively.

Consider the nests:

$$\mathcal{N}(\gamma)_1 = \left\{ 0;\ \bigvee_{\gamma \leqslant j < \gamma + \ell} \{e_j^1\}, (\ell = 1, 2, \ldots);\ \bigvee_{\gamma \leqslant j < \gamma + \omega} \{e_j^1\} \right\}$$

in the Hilbert space $H(\gamma)_1$, where $\gamma(\omega \leqslant \gamma < \alpha)$ is a limit ordinal;

$$\mathcal{N}(\delta)_2 = \left\{ 0; \bigvee_{\delta+\ell \leqslant j < \delta+\omega} \{e_j^2\}, (\ell = 0, 1, 2, \ldots) \right\}$$

in the Hilbert space $H(\delta)_2$, where δ is either 0 or a limit ordinal strictly smaller than β;

$$\mathcal{N}(0)_1 = \left\{ 0; \bigvee_{0 \leqslant j < \ell} \{e_j^1\}, (\ell = 1, 2, \ldots); \bigvee_{0 \leqslant j < \omega} \{e_j^1\}; \right.$$
$$\left. \left[\bigvee_{0 \leqslant j < \omega} \{e_j^1\} \right] \oplus \left[\bigvee_{\alpha \leqslant j < \alpha+\ell} \{e_j^1\} \right], \quad (\ell = 1, 2, \ldots, m-n) \right\}$$

in the Hilbert space $H(0)_1$;

$$\mathcal{N}_h^* = \left\{ 0; \bigvee_{0 \leqslant i < \ell} \{f_{i,\beta+h}\}, \quad (1 \leqslant \ell \leqslant \alpha + m - n + h + 1); \right.$$
$$\left. \left[\bigvee_{0 \leqslant i \leqslant \alpha+m-n+h} \{f_{i,\beta+h}\} \right] \oplus \left[\bigvee_{\ell \leqslant j < \beta+h} \{f_{\alpha+m-n+h,j}\} \right], \ (0 \leqslant \ell < \beta + h) \right\}$$

in the Hilbert space H_h^* $(h = 0, 1, 2, \ldots, n-1)$, where $f_{i,j} = e_i^1 \otimes e_j^2$.

It is easy to check that $\mathcal{N}(\gamma)_1 \simeq \mathcal{N}_\omega$ $(\omega \leqslant \gamma < \alpha)$ and $\mathcal{N}(\delta)_2 \simeq \mathcal{N}_{\omega^*}$ $(\omega \leqslant \delta < \beta)$. Thus, by Lemma 3.5,

$$[\mathcal{N}(\gamma)_1 \otimes \mathcal{N}(\delta)_2]^\wedge = [\mathcal{N}(\gamma)_1 \otimes \mathcal{N}(\delta)_2]_0^\wedge = \mathcal{L}[H(\gamma)_1 \otimes H(\delta)_2].$$

By a similar argument used in the proof of Lemma 3.5,

$$[\mathcal{N}(0)_1 \otimes \mathcal{N}(\delta)_2]^\wedge = [\mathcal{N}(0)_1 \otimes \mathcal{N}(\delta)_2]_0^\wedge = \mathcal{L}[H(0)_1 \otimes H(\delta)_2], \quad (\omega \leqslant \delta < \beta).$$

Note that $\mathcal{N}_h^*(h = 1, 2, \ldots, n)$ has both an increasing and a decreasing sequence of atoms. By Theorem 1.1,

$$(\mathcal{N}_h^*)^\wedge = (\mathcal{N}_h^*)_0^\wedge = \mathcal{L}(H_h^*)$$

Thus by Lemma 2.7, it follows that

$$\left(\bigotimes_{i=1}^{k} \mathcal{N}_i \right)^\wedge = \left(\bigotimes_{i=1}^{k} \mathcal{N}_i \right)_0^\wedge = \mathcal{L}(H). \qquad \blacksquare$$

The proofs of Theorems 1.3 are now complete. Indeed, Theorem 1.4(ii) is also proved.

4. THE ANALYSIS OF $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge$

In this section we will analyze the set $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge$ and complete the proof of Theorem 1.4.

DEFINITION 4.1. Suppose that for each $i = 1, 2, \ldots, k$, the nest $\mathcal{N}_i$ has an atom $E^i = M^i \ominus M_-^i$, then the subspace $E = \bigotimes_{i=1}^{k} E^i$ of $H = \bigotimes_{i=1}^{k} H_i$ is called a *quark* of $\bigotimes_{i=1}^{k} \mathcal{N}_i$.

PROPOSITION 4.2.
(i) *If* $T \in \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge$ *and* $\sigma_0(T) \setminus \sigma_e(T)^\wedge \neq \emptyset$, *then* $\mathcal{N}_i$ $(i = 1, 2, \ldots, k)$ *has at least one atom.*
(ii) *If for each* $i = 1, 2, \ldots, k, \mathcal{N}_i$ *has only finitely many atoms* $L_1^i, \ldots, L_{m_i}^i$, *and*

$$0 < d_i = \sum_{j=1}^{m_i} \dim L_j^i < \infty,$$

then

$$\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^\wedge \subset \mathcal{L}(H)_d,$$

where $d = \prod_{i=1}^{k} d_i$.

Proof. (i) Given $0 < \varepsilon \ll 1$, there exist $A \in \operatorname{alg}\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)$ and U unitary such that $T - UAU^* \in \mathcal{K}(H)$ and $\|T - UAU^*\| < \varepsilon$.

If Γ is a Jordan curve containing p points of $\sigma_0(T) \setminus \sigma_e(T)^\wedge$ inside (counting multiplicity), and $\Gamma \cap \sigma(T) = \emptyset$, then Γ contains inside p points of $\sigma_0(A) \setminus \sigma_e(A)^\wedge$ (counting multiplicity) and $\Gamma \cap \sigma(A) = \emptyset$, provided ε is small enough.

Set $F = \frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} d\lambda$, then $F \in \operatorname{alg}\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)$ and $\operatorname{rank}(F) = p$. (observe that F is a uniform limit of polynomials in A.) For each $q, 1 \leqslant q \leqslant k$, consider the nest

$$\mathcal{N}_q^* = \{H_1 \otimes \cdots \otimes H_{q-1} \otimes M^q \otimes H_{q+1} \otimes \cdots \otimes H_k \,:\, M^q \in \mathcal{N}_q\}.$$

It is obvious that $F \in \operatorname{alg}\mathcal{N}_q^*$. From Lemma 3.5 of [2], there is a partition $E_1^q, \ldots, E_s^q$ of $\mathcal{N}_q^*$ so that for each $1 \leqslant i \leqslant s$, either E_i^q is an atom or $\|F_{ii}\| = \|E_i^q F E_i^q\| < \varepsilon$. Assume the matrix of the p-rank operator F with respect to the decomposition $H =$

$= E_1^q \oplus \cdots \oplus E_s^q$ has the form

$$F = \begin{pmatrix} F_{11} & & & & \\ & \ddots & & * & \\ & & F_{ii} & & \\ & 0 & & \ddots & \\ & & & & F_{ss} \end{pmatrix} \begin{matrix} E_1^q \\ \vdots \\ E_i^q \\ \vdots \\ E_s^q \end{matrix}$$

From [8], $\sigma(F) = \{0,1\} = \bigcup_{i=1}^{s} \sigma(F_{ii})$. If none of the E_i^q's is an atom, then $\|F_{ii}\| < \varepsilon < 1$ and $\sigma(F_{ii}) = \{0\}$ $(i = 1,2,\ldots,s)$, contradicting $\bigcup_{i=1}^{s} \sigma(F_{ii}) = \{0,1\}$. Thus at least one of E_i^q's is an atom of $\mathcal{N}_q^*$. Therefore, $\mathcal{N}_q$ has at least one atom $(q = 1,2,\ldots,k)$.

(ii) We assume $\mathcal{N}_i \simeq \bigcup_{j=1}^{m_i} [a_j^i, b_{j+1}^i]$, where $0 = a_0^i \leqslant b_1^i < a_1^i \leqslant b_2^i < a_2^i \leqslant \cdots \leqslant b_{m_i}^i < a_{m_i}^i \leqslant b_{m_i+1}^i = 1$ and $M_{a_j}^i \ominus M_{b_j}^i = L_j^i$ $\quad (i = 1,2,\ldots,k; j = 1,2,\ldots,m_i)$. For the subspace $M_\gamma = \otimes M^i_{(a^i_{\ell_i - 1}, a^i_{\ell_i}]}$ $\quad (\gamma = (\ell_1,\ldots,\ell_k) \in N^k, 1 \leqslant \ell_i \leqslant m_i, i = 1,2,\ldots,k)$, we construct the nest $\mathcal{N}_\gamma$:

$$\mathcal{N}_\gamma = \Bigg\{ 0;\ M^1_{(a^1_{\ell_1-1},t]} \otimes \left(\bigotimes_{i=2}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right),\ (a^1_{\ell_1-1} < t \leqslant b^1_{\ell_1});$$
$$\left[M^1_{(a^1_{\ell_1-1},b^1_{\ell_1}]} \otimes \left(\bigotimes_{i=2}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right) \right] \oplus \left[L^1_{\ell_1} \otimes M^2_{(a^2_{\ell_2-1},t]} \otimes \left(\bigotimes_{i=3}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right) \right],$$
$$(a^2_{\ell_2-1} < t \leqslant b^2_{\ell_2});\ \left[M^1_{(a^1_{\ell_1-1},b^1_{\ell_1}]} \otimes \left(\bigotimes_{i=2}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right) \right] \oplus \left[L^1_{\ell_1} \otimes M^2_{(a^2_{\ell_2-1},b^2_{\ell_2}]} \otimes \right.$$
$$\left. \otimes \left(\bigotimes_{i=3}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right) \right] \oplus \left[L^1_{\ell_1} \otimes L^2_{\ell_2} \otimes M^3_{(a^3_{\ell_3-1},t]} \otimes \left(\bigotimes_{i=4}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]} \right) \right],$$
$$(a^3_{\ell_3-1} < t \leqslant b^3_{\ell_3});\ldots; M_\gamma \ominus \left(\bigotimes_{i=1}^{k} L^i_{\ell_i} \right); M_\gamma \Bigg\}.$$

This nest has exactly one atom $L_\gamma = \bigotimes_{i=1}^{k} L^i_{\ell_i}$ with dimension $d_\gamma = \prod_{i=1}^{k} d^i_{\ell_i}$, where $d^i_{\ell_i} = \dim L^i_{\ell_i}$.

In a similar way we construct the nest $\mathcal{N}_\delta^0$ in the space $M_\gamma^0 = \bigotimes_{i=1}^{k} M^i_{(a^i_{\ell_i-1},a^i_{\ell_i}]}$ $(\gamma = (\ell_1,\ldots,\ell_k) \in N^k,\ 1 \leqslant \ell_1 \leqslant m_i + 1,\ i = 1,2,\ldots,k$, and some $\ell_i = m_i + 1$; we define $a^i_{m_i+1} = 1)$. It is not difficult to see $\mathcal{N}_\delta^0$ is a continuous nest.

Set the ordinal sum

$$\mathcal{N} = \sum_\gamma \mathcal{N}_\gamma + \sum_\gamma \mathcal{N}_\gamma^0$$

according to the order:

1. $\mathcal{N}_\gamma \prec \mathcal{N}_{\gamma'}$ or $\mathcal{N}_\gamma^0 \prec \mathcal{N}_{\gamma'}^0$, if $\ell_k < \ell'_k$ or $\ell_j = \ell'_j$ $(j = h, h+1,\ldots,k)$ and $\ell_{h-1} < \ell'_{h-1}$ for some h $(1 < h \leqslant k), (\gamma = (\ell_1,\ldots,\ell_k), \gamma' = (\ell'_1,\ldots,\ell'_k))$;

2. $\mathcal{N}_\gamma \prec \mathcal{N}^0_{\gamma'}$ for all γ and γ'.

Thus $\mathcal{N}$ is a nest in the Hilbert space H with $\prod_{i=1}^{k} m_i$ atoms $\{L_\gamma\}$ where $\gamma = (\ell_1, \dots, \ell_k) \in N^k, 1 \leqslant \ell_j \leqslant m_j (j = 1, 2, \dots, k)$, and $\sum \dim L_\gamma = \prod_{i=1}^{k} d_i = d$. (In other words, the atoms of $\mathcal{N}$ are the quarks of $\bigotimes_{i=1}^{k} \mathcal{N}_i$ in a suitable order.)

Consider the operator F in (i). By Lemma 3.5 [2] there is a partition $E_1, E_2, \dots, E_p$ of $\mathcal{N}$ such that for each $1 \leqslant i \leqslant p$, either E_i is an atom of $\mathcal{N}$ or $\|F_i\| = \|E_i F E_i\| < \varepsilon$. Let the matrix of F with respect to the decomposition $H = \sum_{i=1}^{p} \oplus E_i$ be

$$F = \begin{pmatrix} F_1 & & & & \\ & \ddots & & * & \\ & & F_i & & \\ & 0 & & \ddots & \\ & & & & F_p \end{pmatrix} \begin{matrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_p \end{matrix}$$

Then $\bigcup_{i=1}^{p} \sigma(F_i) = \sigma(F) = \{0, 1\}$. If E_i is not an atom, since $\|F_i\| < \varepsilon < 1$, $\sigma(F_i) = \{0\}$. Thus, the $p1$'s, the eigenvalues of F, are contained in the set $\cup\{d(F_i) : E_i$ is an atom of $\mathcal{N}\}$, where $d(F_i)$ is the diagonal of F_i with respect to some upper triangular matrix representation in the corresponding space. Since $\{ E_i : E_i$ is an atom of $\mathcal{N}\} \subset \{L_\gamma\}, p \leqslant \sum_{E_i \text{ is an atom}} \dim E_i \leqslant \sum \dim L_\gamma = \prod_{i=1}^{k} d_i = d$. Therefore, $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0 \subset \mathcal{L}(H)_d$. ■

From Corollary 2.11 and Proposition 4.3, we obtain the following

COROLLARY 4.3. *If at least one of the nests $\{\mathcal{N}_i\}_{i=1}^{k}$ is continuous, then* $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0 = \mathcal{L}(H)_0$.

(This proves Theorem 1.4(ii).)

From definition 4.4 to Proposition 4.9, we will assume that $\mathcal{N}_i \simeq \bigcup_{j=1}^{m_i} [a^i_j, b^i_{j+1}]$, where $0 = a^i_0 \leqslant b^i_1 < a^i_1 \leqslant b^i_2 < a^i_2 \leqslant \dots \leqslant b^i_{m_i} < a^i_{m_i} \leqslant b^i_{m_i+1} = 1$ and $L^i_j = M^i_{a_j} \ominus M^i_{b_j}$, $1 \leqslant d^i_j = \dim L^i_j < \infty (j = 1, 2, \dots, m_i; 1 \leqslant m_i < \infty; i = 1, 2, \dots, k)$.

DEFINITION 4.4. An s-tuple of quarks $L = (L_{\gamma_1}, L_{\gamma_2}, \dots, L_{\gamma_s}), (1 \leqslant s \leqslant \sum_{i=1}^{k} m_i - k + 1)$, will be called a *route* of the tensor product $\bigotimes_{i=1}^{k} \mathcal{N}_i$ if for each pair of consecutive quarks L_{γ_t} and $L_{\gamma_{t+1}}, \ell^{t+1}_i = \ell^t_i$ for all $i (1 \leqslant i \leqslant k)$ except for one index q, and

$\ell_q^{t+1} = \ell_q^t + 1$. The cardinal number $d_L = \sum_{i=1}^{s} d_{\gamma_i}$ will be called the *dimension of the route* L, where $\gamma_j = (\ell_1^j, \ldots, \ell_k^j), 1 \leqslant \ell_i^j \leqslant m_i, 1 \leqslant i \leqslant k$, $L_{\gamma_j} = \bigotimes_{i=1}^{k} L^i_{\ell_i^j}, d_{\gamma_j} = \dim L_{\gamma_j}$ and $1 \leqslant j \leqslant s$.

LEMMA 4.5. *Let $L = (L_{\gamma_1}, \ldots, L_{\gamma_s})$ be a route of $\bigotimes_{i=1}^{k} \mathcal{N}_i$, then $d_L \geqslant \sum_{h=1}^{k} \sum_{i=\ell_h^1}^{\ell_h^s} d_i^h - k + 1$, where $d_i^k = \dim L_i^h$.*

Proof. We prove the lemma by induction on the number s of steps.

When $s = 1, L = (L_{\gamma_1}), \gamma_1 = (\ell_1^1, \ldots, \ell_k^1)$ and $d_L = \prod_{i=1}^{k} d^i_{\ell_i^1}$. Since $\prod_{i=1}^{k} d^i_{\ell_i^1} - \sum_{h=1}^{k} d^h_{\ell_h^1} + (k-1) = (d^1_{\ell_1^1 - 1}) d^2_{\ell_2^1} \ldots d^k_{\ell_k^1} + (d^2_{\ell_2^1} - 1) d^3_{\ell_3^1} \ldots d^k_{\ell_k^1} + \cdots + (d^{k-1}_{\ell_{k-1}^1} - 1) d^k_{\ell_k^1} - \sum_{h=1}^{k-1} (d^h_{\ell_h^1} - 1) \geqslant 0$, the conclusion is true.

Suppose the conclusion is true for a s steps route and consider a $s+1$ steps route $L = (L_{\gamma_1}, \ldots, L_{\gamma_{s+1}})$. Assume $\ell_i^{s+1} = \ell_i^s (i \neq q)$ and $\ell_q^{s+1} = \ell_q^s + 1$. By inductive assumption,

$$d_L = \sum_{i=1}^{s} d_{\gamma_i} + d_{\gamma_{s+1}} \geqslant \sum_{h=1}^{k} \sum_{i=\ell_h^1}^{\ell_h^s} d_i^h - k + 1 + \prod_{i=1}^{k} d^i_{\ell_i^{s+1}} \geqslant$$

$$\geqslant \sum_{h=1}^{k} \sum_{i=\ell_h^1}^{\ell_h^s} d_i^h + d^q_{\ell_q^s+1} - k + 1 = \sum_{h=1}^{k} \sum_{i=\ell_h^1}^{\ell_h^{s+1}} d_i^h - k + 1.$$

Thus the proof of the Lemma is complete. ■

PROPOSITION 4.6. *Let L be a route of the tensor product $\bigotimes_{i=1}^{k} \mathcal{N}_i$, then $\mathcal{L}(H)_{d_L} \subset \subset \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^{\wedge}$.*

Proof. Case 1. First consider $k = 2$ case. Let $L = (L_{\gamma_1}, \ldots, L_{\gamma_s})$ and $\gamma_j = (\ell_1^j, \ell_2^j)$, $1 \leqslant j \leqslant s$.

Note that each quark L_{γ_j} corresponds to a point $P_{\gamma_j} = (a^1_{\ell_1^j}, a^2_{\ell_2^j}) \in [0,1] \times [0,1]$. Set $P_{\gamma_0} = (a^1_{\ell_1^1}, 0)$. Then the broken line $P_{\gamma_0} P_{\gamma_1} \ldots P_{\gamma_s}$, with the spectral measure induced from $E = E_1 \times E_2$, determines a nest $\mathcal{N}_L$ in some subspace H_L of $H = H_1 \otimes H_2$. It is obvious that $\mathcal{N}_L$ has at least s atoms $\{L_{\gamma_j}\}_{j=1}^{s}$ and the sum of dimensions of atoms is at least d_L.

For each $\gamma = (\ell_1, \ell_2)$, $0 \leqslant \ell_1 \leqslant m_i$, $i = 1,2$, consider the nest $\mathcal{N}^i_{\ell_i} = \{0; M^1_{(a^1_{\ell_1}, t]}, (a^1_{\ell_1} < t \leqslant b^1_{\ell_1})\}$ and the subspace $G_\gamma = M^1_{(a^1_{\ell_1}, b^1_{\ell_1+1}]} \otimes M^2_{(a^2_{\ell_2}, b^2_{\ell_2+1}]}$ (if $G_\gamma = (0)$ for some γ, omit it). It is obvious that $\mathcal{N}^1_{\ell_1} \otimes \mathcal{N}^2_{\ell_2}$ is the tensor product in G_γ. Let S_1 be the set of those $\mathcal{N}^1_{\ell_1} \otimes \mathcal{N}^2_{\ell_2}$, i.e. $S_1 = \{\mathcal{N}^1_{\ell_1} \otimes \mathcal{N}^2_{\ell_2} : \gamma = (\ell_1, \ell_2) \in \Lambda_1\}$, where Λ_1 is some index set.

Consider the vertical segment $V_\gamma(\gamma = (j,h), 1 \leqslant j \leqslant m_1, 0 \leqslant h \leqslant m_2)$ determined by the points $(a^1_j, a^2_h), (a^1_j, b^2_{h+1})$, and satisfying $V_\gamma \not\subset P_{\gamma_0}P_{\gamma_1} \ldots P_{\gamma_s}$. If the point $(a^1_j, a^2_h) \notin P_{\gamma_0}P_{\gamma_1} \ldots P_{\gamma_s}$, construct the nest $\mathcal{N}_\gamma = \{0; L^1_j \otimes M^2_{(a^2_h, t]}, (a^2_h \leqslant t \leqslant b^2_{h+1})\}$; if $(a^1_j, a^2_h) \in P_{\gamma_0}P_{\gamma_1} \cdots P_{\gamma_s}$, construct the nest $\mathcal{N}_\gamma = \{0; L^1_j \otimes M^2_{(a^2_h, t]}, (a^2_h < t \leqslant b^2_{h+1})\}$ Let S_2 be the set of the $\mathcal{N}_\gamma$'s, i.e. $S_2 = \{\mathcal{N}_\gamma : \gamma \in \Lambda_2\}$, where Λ_2 is some index set.

Consider the horizontal segment $Z_\gamma(\gamma = (j,h), 0 \leqslant j \leqslant m_1, 1 \leqslant h \leqslant m_2)$ determined by (a^1_j, a^2_h) and (b^1_{j+1}, a^2_h), and satisfying $Z_\gamma \not\subset P_{\gamma_0}P_{\gamma_1} \cdots P_{\gamma_s}$. Construct the nest $\mathcal{N}^*_\gamma = \{0; M^1_{(a^1_j, t]} \otimes L^2_h, (a^1_j < t \leqslant b^1_{j+1})\}$. Let S_3 be the set of the $\mathcal{N}^*_\gamma$'s, that is, $S_3 = \{\mathcal{N}^*_\gamma : \gamma \in \Lambda_3\}$, where Λ_3 is some index set.

Given $T \in \mathcal{L}(H)_{d_L}$ and $\varepsilon > 0$, by using Lemma 2.1 we can find K compact, with $\|K\| < \varepsilon$, such that $T - K \simeq T \oplus (\bigoplus_{\gamma \in \Lambda_1} A_\gamma) \oplus (\bigoplus_{\gamma' \in \Lambda_2} B_{\gamma'}) \oplus (\bigoplus_{\gamma'' \in \Lambda_3} C_{\gamma''})$, where $A_\gamma \simeq B_{\gamma'} \simeq C_{\gamma''} \simeq A \simeq A^{(\infty)}$ for $\gamma \in \Lambda_1, \gamma' \in \Lambda_2, \gamma'' \in \Lambda_3$. By Theorem 1.1, $T \in (\mathcal{N}_L)^\wedge_0$ and for all $\gamma' \in \Lambda_2, \gamma'' \in \Lambda_3, B_{\gamma'} \in (\mathcal{N}_{\gamma'})^\wedge_0$, and $C_{\gamma''} \in (\mathcal{N}_{\gamma''})^\wedge_0$. By Proposition 2.5, $A_\gamma \in (\mathcal{N}^1_{\ell_1} \otimes \mathcal{N}^2_{\ell_2})^\wedge_0$ for all $\gamma = (\ell_1, \ell_2) \in \Lambda_1$. Thus from Lemma 2.7, $T \in (\mathcal{N}_1 \otimes \mathcal{N}_2)^\wedge_0$.

Case 2. $k > 2$. Let $L = (L_{\gamma_1}, \ldots, L_{\gamma_s})$ and $\gamma_j = (\ell^j_1, \ldots, \ell^j_k), (1 \leqslant j \leqslant s)$. Note that each quark L_{γ_j} corresponds to a point $P_{\gamma_j} = (a^1_{\ell^j_1}, \ldots, a^k_{\ell^j_k})$ in the k-dimensional unit cube $\prod_{i=1}^k [0,1]_i$. Set $P_{\gamma_0} = (a^1_{\ell^1_1}, \ldots, a^1_{\ell^1_{k-1}}, 0)$. Then the broken line $P_{\gamma_0}P_{\gamma_1} \cdots P_{\gamma_s}$, with the spectral measure induced by $E = \prod_{i=1}^k E_i$, determines a nest $\mathcal{N}_L$ having at least s atoms $\{L_{\gamma_i}\}^s_{i=1}$ (the sum of the dimensions is at least d_L). Cut the unit cube $\prod_{i=1}^k [0,1]_i$ with the hyperplane $x_i = a^i_{j_i} (j_i = 1, 2, \ldots, m_i; i = 1, 2, \ldots, k)$. In addition to the broken line $P_{\gamma_0}P_{\gamma_1} \cdots P_{\gamma_s}$, there is a finite number of small u-dimensional pieces $(1 \leqslant u \leqslant k)$: edges, faces,.... Each piece determines a subspace of H and a nest (when $u = 1$) or a tensor product of nests; then, by combining Lemma 2.1, Theorem 1.1, Proposition 2.5 and Lemma 2.7, we deduce that $T \in \left(\bigotimes_{i=1}^k \mathcal{N}_i\right)^\wedge_0$. ■

From Proposition 4.6 and Lemma 4.5 we obtain the following

COROLLARY 4.7. *$\mathcal{L}(H)_p \subset \left(\bigotimes_{i=1}^k \mathcal{N}_i\right)^\wedge_0$, where $p = \sum_{i=1}^k d_i - k + 1, d_i = \sum_{j=1}^{m_i} d^i_j$ and $d^i_j = \dim L^i_j (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, k)$.*

LEMMA 4.8. *For each $m \geqslant 2$, there exists $T \in \mathcal{L}(\mathcal{R}^m)$ such that $\sigma(T) = \{1, 2, \ldots, m\}$ and for every permutation π of this set, T admits an upper triangular* *matrix representation*

$$T = \begin{pmatrix} \pi(1) & t^{\pi}_{12} & \dots & t^{\pi}_{1m} \\ & \pi(2) & \dots & t^{\pi}_{2m} \\ & 0 & \ddots & \vdots \\ & & & \pi(m) \end{pmatrix} \begin{matrix} e^{\pi}_1 \\ e^{\pi}_2 \\ \vdots \\ e^{\pi}_m \end{matrix}$$

with respect to an ONB $\{e^{\pi}_h\}_{h=1}^m$, *such that* $t^{\pi}_{ij} \neq 0$ *for* $1 \leqslant i < j \leqslant m$.

Proof. Let

$$T = \begin{pmatrix} 1 & t_{12} & \dots & t_{1m} \\ & 2 & \dots & t_{2m} \\ & 0 & \ddots & \\ & & & m \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{matrix}$$

with respect to the canonical ONB $\{e_i\}_{i=1}^m$. The real numbers $t_{ij} (1 \leqslant i < j \leqslant m)$ will be chosen later.

Simple computations show that $\ker(T-1)$ is generated by the unit vector $g_1 = = e_1, \ker(T-2)$ is generated by the unit vector $g_2 = \frac{1}{\sqrt{t_{12}^2+1}}(t_{12}e_1 + e_2)$, in general, $\ker(T-n)(1 \leqslant n \leqslant m)$ is generated by a unit vector of the form $g_n = (\sqrt{q_n})^{-1}(p_{n1}e_1 + +p_{n2}e_2 + \cdots + p_{n,n-1}e_{n-1} + e_n)$, where q_n and $p_{n1}, p_{n2}, \dots, p_{n,n-1}$ are polynomials in the t_{ij}'s.

Given a permutation π, we define the ONB $\{e^{\pi}_n\}_{n=1}^m$ by means of the Gram-Schmidt orthonormalization process, as follows

$$e^{\pi}_1 = g_1$$

and, inductively,

$$e^{\pi}_n = \|g_{\pi(n)} - \sum_{j=1}^{n-1}(g_{\pi(n)}, e^{\pi}_j)e^{\pi}_j\|^{-1} \left[g_{\pi(n)} - \sum_{j=1}^{n-1}(g_{\pi(n)}, e^{\pi}_j)e^{\pi}_j \right], \quad (n = 2, 3, \dots, m).$$

Observe that

$$g_{\pi(n)} - \sum_{j=1}^{n-1}(g_{\pi(n)}, e^{\pi}_j)e^{\pi}_j = (\sqrt{q_{\pi(n)}})^{-1} \left(\sum_{j=1}^{m} p^{\pi}_{ni}e_i \right),$$

where $p^{\pi}_{ni}(1 \leqslant n, i \leqslant m)$ is a polynomial in the t_{ij}'s.

Thus, $e^{\pi}_n = (\sqrt{b_{\pi(n)}})^{-1} \left(\sum_{i=1}^{m} d^{\pi}_{ni}e_i \right)$, where $b_{\pi(n)}, d^{\pi}_{ni}(1 \leqslant n, i \leqslant m)$ are polynomials in the t_{ij}'s. It is not difficult to check that $t^{\pi}_{ij} = (Te^{\pi}_j, e^{\pi}_i) = (q^{\pi}_{ij})^{-\frac{1}{2}}h^{\pi}_{ij}$ and

$h^{\pi}_{ij}, q^{\pi}_{ij}$ are polynomials in the $\frac{m(m-1)}{2}$ real variables $t_{uv}(1 \leqslant u < v \leqslant m)$. Note that for

$$T = \begin{pmatrix} \pi(1) & t^{\pi}_{12} & \dots & t^{\pi}_{1m} \\ & \pi(2) & \dots & t^{\pi}_{2m} \\ & 0 & \ddots & \vdots \\ & & & \pi(m) \end{pmatrix} \begin{matrix} e^{\pi}_1 \\ e^{\pi}_2 \\ \vdots \\ e^{\pi}_m \end{matrix}$$

and π^{-1}, if we use the same process, the original ONB $\{e_i\}$ and the original matrix

$$T = \begin{pmatrix} 1 & t_{12} & \dots & t_{1m} \\ & 2 & \dots & t_{2m} \\ & & \ddots & \vdots \\ & 0 & & m \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{matrix}$$

can be recovered.

We will prove that $h^{\pi}_{ij}(1 \leqslant i < j \leqslant m)$ is a non-identically zero polynomial. If $h^{\pi}_{i_0j_0} \equiv 0$ for some i_0, j_0, we consider the operator

$$A = \begin{pmatrix} \pi(1) & 1 & \dots & 1 \\ & \pi(2) & \dots & 1 \\ & 0 & \ddots & \vdots \\ & & & \pi(m) \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{matrix};$$

then the Gram-Schmidt process yields

$$A = \begin{pmatrix} 1 & x_{12} & \dots & x_{1m} \\ & 2 & \dots & x_{2m} \\ & 0 & \ddots & \vdots \\ & & & m \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{matrix}$$

for a suitable ONB $\{f_i\}_{i=1}^m$ and constants $x_{ij}(1 \leqslant i < j \leqslant m)$. Thus, $|t^{\pi}_{ij}(x_{12}, x_{13}, \dots, \dots, x_{m-1m})| = 1$; that is, if $t_{ij} = x_{ij}$, then $|t^{\pi}_{ij}| = 1(1 \leqslant i < j \leqslant m)$. This contradicts $t^{\pi}_{i_0j_0} \equiv 0$, therefore h^{π}_{ij} is always a non-identically zero polynomial $(1 \leqslant i < j \leqslant m)$.

Set $P = \prod\{h^{\pi}_{ij} : 1 \leqslant i < j \leqslant m, \pi \text{ runs all permutations}\}$. It is not difficult to prove that $N(P) = \{t = (t_{12}, t_{13}, \dots, t_{m-1,m}) \in \mathcal{R}^{\frac{m(m-1)}{2}} : P(t) = 0\} \neq \mathcal{R}^{\frac{m(m-1)}{2}}$. Thus, every T with $t \in \mathcal{R}^{\frac{m(m-1)}{2}} \setminus N(P)$ satisfies all our requirements. ■

PROPOSITION 4.9. (i) *$\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 \not\subset \mathcal{L}(H)_{d-1}$ where $d = \prod_{i=1}^{k} d_i, d_i = \sum_{j=1}^{m_i} d^i_j$, $i = 1, 2, \dots, k$.*

(ii) *If $\mathcal{N}_i$ $(i = 1, 2, \dots, k)$ is maximal, then*

$$\mathcal{L}(H)_{p+1} \not\subset \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0. \quad (p = \sum_{i=1}^{k} d_i - k + 1).$$

Proof. (i) Let E_a be the sum of all quarks of $\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i$. It is obvious that $E_a \in \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$ (in fact, $E_a \in \mathrm{alg}(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i)$). Since the rank of E_a is d, $E_a \in \mathcal{L}(H)_d \setminus \mathcal{L}(H)_{d-1}$. Therefore, $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 \not\subset \mathcal{L}(H)_{d-1}$.

(ii) Let $F_1 \in \mathcal{L}(\mathcal{R}^{p+1})$ satisfy the requirements of Lemma 4.8.:

$$F_1 = \begin{pmatrix} 1 & & & \\ & 2 & & * \\ & 0 & \ddots & \\ & & & p+1 \end{pmatrix}, \quad \text{and let } F = F_1 \oplus 0^{(\infty)};$$

then $F \in \mathcal{L}(H)_{p+1}$. Suppose $F \in \left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$; then given $\varepsilon, 0 < \varepsilon \ll 1$, there exist $A \in \mathrm{alg}(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i)$ and U unitary such that $F - UAU^* \in \mathcal{K}(H)$ and $\|F - UAU^*\| < \varepsilon$. By replacing (if necessary) F by $F' = U^*FU$, we can directly assume that $U = I$. If ε is small enough, then the uppersemicontinuity properties of the spectrum imply that $\sigma(A) = \{\lambda_i\}_{i=1}^{p+1} \cup \sigma_0$, where $|\lambda_i - i| = 0(\varepsilon)(\ll 1; i = 1, 2, \ldots, p+1)$ and σ_0 is included in a closed disk of radius less then $1/2$ centered at the origin; moreover, each λ_i is a normal eigenvalue of A with dimension 1 [9, Chapter 1].

Let Γ be the (positively oriented) circle of radius $p+3/2$ centered at $p+2$. Clearly, $\Gamma \cap \sigma(A) = \emptyset$, so that we can define

$$A' = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \lambda(\lambda - A)^{-1} \mathrm{d}\lambda.$$

It follows that $A' \in \mathrm{alg}(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i), \sigma(A') = \{0; \lambda_i\}_{i=1}^{p+1}, \mathrm{rank} A' = \mathrm{rank} F = p+1$ and $\|A - A'\| = 0(\varepsilon)$.

Let $q(\lambda)$ be the only polynomial of degree $p+1$ satisfying $q(0) = 0, q(\lambda_n) = n$, $(n = 1, 2, \ldots, p+1)$; then $\|q(n) - n\| = 0(\varepsilon)$ for $n = 0, 1, 2, \ldots, p+1$. Thus, the operator $A'' = q(A') \in \mathrm{alg}(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i)$ satisfies $\sigma(A'') = \sigma_0(A'') = \{n\}_{n=0}^{p+1}, \mathrm{rank} A'' = p+1$ and $\|A - A''\| \leqslant \|A - A'\| + \|A' - A''\| = 0(\varepsilon)$. If

$$E = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda - F)^{-1} \mathrm{d}\lambda \quad \text{and} \quad E' = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda - A'')^{-1} \mathrm{d}\lambda,$$

then $\|E - E'\| = 0(\varepsilon), \|E' - E'^*\| \leqslant \|E' - E\| + \|E^* - E'^*\| = 0(\varepsilon)$. Thus, $\mathrm{rank} E = = \mathrm{rank} E' = p+1$ and $\|A'' - E'A'E'\| = 0(\varepsilon)$. Therefore, for practical purposes, we can directly assume that $A = A'', \sigma(A) = \sigma(F) = \{0, 1, 2, \ldots, p+1\}$ and $\mathrm{rank} F = = \mathrm{rank} A = p+1$.

Consider the nest $\mathcal{N}$ constructed in the proof of proposition 4.2. Since $\mathcal{N}_i(i = 1,2,\dots,k)$ is a maximal nest, $\mathcal{N}$ is a nest in the Hilbert space H with d atoms $\{L_\gamma\}$ (corresponding to the quarks of $\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i$ suitably ordered), where $\gamma = (\ell_1,\dots,\ell_k) \in N^k, 1 \leqslant \ell_i \leqslant d_i \quad i = 1,2,\dots,k$. Assume $L_\gamma = [f_\gamma], f_\gamma = \overset{k}{\underset{i=1}{\otimes}} e^i_{\ell_i}$ and $L^i_{\ell_i} = [e^i_{\ell_i}]$. Since $\mathrm{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) \subset \mathrm{alg}\mathcal{N}, A \in \mathrm{alg}\mathcal{N}$. By Ringrose's theorem ([2] Theorem 3.4) the matrix of the $p+1$ rank operator A, with respect to the decomposition $H = (M_{(1,1,\dots,1)} \ominus [f_{(1,1,\dots,1)}]) \oplus [f_{(1,1,\dots,1)}] \oplus \cdots \oplus (M_\gamma \ominus [f_\gamma]) \oplus [f_\gamma] \oplus \cdots \oplus (M_{(d_1,\dots,d_k)} \ominus [f_{(d_1,\dots,d_k)}]) \oplus [f_{(d_1,\dots,d_k)}] \oplus (\sum \oplus M^0_\gamma)$, has the form

$$A = \begin{pmatrix} 0 & & & & & & & \\ & \lambda_{(1,1,\dots,1)} & & & & & & \\ & & \ddots & & * & & & \\ & & & 0 & & & & \\ & & & & \lambda_\gamma & & & \\ & & & & & \ddots & & \\ & & 0 & & & 0 & & \\ & & & & & & \lambda_{(d_1,\dots,d_k)} & \\ & & & & & & & 0 \end{pmatrix} \begin{matrix} M_{(1,1,\dots,1)} \oplus [f_{(1,1,\dots,1)}] \\ [f_{(1,1,\dots,1)}] \\ \vdots \\ M_\gamma \ominus [f_\gamma] \\ [f_\gamma] \\ \vdots \\ M_{(d_1,\dots,d_k)} \ominus [f_{(d_1,\dots,d_k)}] \\ [f_{(d_1,\dots,d_k)}] \\ \sum \oplus M^0_\gamma \end{matrix},$$

where for each $j (1 \leqslant j \leqslant p+1)$ exactly one λ_γ is equal to j, and the remaining ones are equal to 0.

Denote by $\lambda_{\gamma(j)}$ the j-th non-zero entry in the diagonal A. Our assumptions about F indicate that there exists an ONB $\{e_i\}_{i=1}^{p+1}$ of $\mathcal{R}^{p+1}$ such that

$$F = \begin{pmatrix} \lambda_{\gamma(1)} & t_{12} & \cdots & t_{1,p+1} \\ & \lambda_{\gamma(2)} & \cdots & t_{2,p+1} \\ & 0 & \ddots & \vdots \\ & & & \lambda_{\gamma(p+1)} \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{p+1} \end{matrix} \oplus 0^{(\infty)}$$

and $t_{ij} \neq 0$ for $1 \leqslant i < j \leqslant p+1$. Since $\|E' - E'^*\| = 0(\varepsilon)$, E' is "almost"orthogonal. Thus, A has the form

$$A = \begin{pmatrix} \lambda_{\gamma(1)} & \alpha_{1,2} & \cdots & \alpha_{1,p+1} \\ & \lambda_{\gamma(2)} & \cdots & \alpha_{2,p+1} \\ & 0 & \ddots & \vdots \\ & & & \lambda_{\gamma(p+1)} \end{pmatrix} \begin{matrix} f_{\gamma(1)} \\ f_{\gamma(2)} \\ \vdots \\ f_{\gamma(p+1)} \end{matrix} \oplus \underset{H \ominus \bigvee_{j=1}^{p+1} [f_{\gamma(j)}]}{0} + 0(\varepsilon).$$

Since $\|F - A\| < \varepsilon$, we can find $p+1$ complex number $\beta_j (j = 1,2,\dots,p+1)$ such that $|\beta_j| = 1$ and $\|f_{\gamma(j)} - \beta_j e_j\| = 0(\varepsilon)$. By replacing $f_{\gamma(j)}$ by $\beta_j^{-1} f_{\gamma(j)}$, we can directly assume that $\|f_{\gamma(j)} - e_j\| = 0(\varepsilon)$ $(j = 1, 2, \ldots, p+1)$. Let $\alpha_{s,u} = (Af_{\gamma(u)}, f_{\gamma(s)})$. By using Lemma 4.8, $\mathrm{Re}\alpha_{s,u} = (Fe_u, e_s) + 0(\varepsilon) = t_{s,u} + 0(\varepsilon)$ $(1 \leqslant s < u \leqslant p+1)$, and therefore $\alpha_{s,u} \neq 0$ provided ε is small enough. Since $(Af_{\gamma(u+1)}, f_{\gamma(u)}) \neq 0, Af_{\gamma(u+1)}$ cannot be orthogonal to $f_{\gamma(u)}$, and therefore $G_{\gamma(u+1)} \supset G_{\gamma(u)}$ $(u = 1, 2, \ldots, p)$, where $G_\gamma = \bigotimes_{i=1}^{k} M^i_{a_{\ell_i}}$ $(\gamma = (\ell_1, \ldots, \ell_k))$.

Thus, we have a strictly increasing chain of length $p+1$: $G_{\gamma(1)} \subset G_{\gamma(2)} \subset \cdots \subset \subset G_{\gamma(p+1)}$. But partially ordered (by inclusion) family $\{G_\gamma\}$ does not include any chain of length $p+1$. This contradiction indicates that $F \notin \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0$, that is,

$$\mathcal{L}(H)_{p+1} \not\subset \left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0 . \qquad \blacksquare$$

REMARK 4.10. The following example shows that the condition of maximal nests in proposition 4.9(ii) cannot be removed.

EXAMPLE. Let $\mathcal{N}_1 \simeq \mathcal{N}_2 \simeq [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $d^i = \dim(M^i_{\frac{2}{3}} \ominus M^i_{\frac{1}{3}}) = 2$ $(i = = 1, 2)$. It is not difficult to prove that $\mathcal{L}(H)_{p+1} = \mathcal{L}(H)_4 \subset (\mathcal{N}_1 \otimes \mathcal{N}_2)^\wedge_0$.

PROPOSITION 4.11. *Let $\mathcal{N}_i$ be a nest in the Hilbert space H_i with at least one atom L^i $(i = 1, 2, \ldots, k)$. Assume, moreover, that $\left(\bigotimes_{i=1}^{p} \mathcal{N}_i\right)^\wedge_0 = \mathcal{L}\left(\bigotimes_{i=1}^{p} H_i\right)$ for some p $(1 \leqslant p < k)$; then $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0 = \mathcal{L}(H)\,(H = \bigotimes_{i=1}^{k} H_i)$.*

Proof. We proceed by induction on k. If $k = p$, the result follows from our hypothesis. Assume that $\left(\bigotimes_{i=1}^{p+s-1} \mathcal{N}_i\right)^\wedge_0 = \mathcal{L}\left(\bigotimes_{i=1}^{p+s-1} H_i\right)$. By replacing (if necessary) $\mathcal{N}_{p+s}$ by a maximal nest, we can directly assume that $\dim L^{p+s} = 1$.

Let $\mathcal{N}'_{p+s}$ be the nest in the Hilbert space $H'_{p+s} = H_{p+s} \ominus L^{p+s}$ defined by "collapsing" L^{p+s} to a point: $\mathcal{N}'_{p+s} = \{M \in \mathcal{N}_{p+s} : M \not\supset L^{p+s}\} \cup \{M \ominus L^{p+s} : : M \in \mathcal{N}_{p+s}$ and $M \supset L^{p+s}\}$. Given $T \in \mathcal{L}(H)$ and $\varepsilon > 0$, by Lemma 2.1, there exists $K_\varepsilon \in \mathcal{K}(H)$, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K_\varepsilon \simeq T \oplus A$ and $A \simeq A^{(\infty)}$. It is obvious that we can assume that $\mathcal{N}'_{p+s}$ is uncountable. Thus, since $\sigma_0(A) = \emptyset, A \in \in \left[\left(\bigotimes_{i=1}^{p+s-1} \mathcal{N}_i\right) \otimes \mathcal{N}'_{p+s}\right]^\wedge_0$. On the other hand, by our inductive hypothesis, $T \in \in \left(\bigotimes_{i=1}^{p+s-1} \mathcal{N}_i\right)^\wedge_0$. But $\bigotimes_{i=1}^{p+s-1} \mathcal{N}_i$ is unitarily equivalent to $\left(\bigotimes_{i=1}^{p+s-1} \mathcal{N}_i\right) \otimes L^{p+s}$ because $\dim L^{p+s} = 1$. By Lemma 2.7, $T - K_\varepsilon \in \left(\bigotimes_{i=1}^{p+s} \mathcal{N}_i\right)^\wedge_0$. Since ε can be chosen arbitrarily small and $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^\wedge_0$ is invariant under unitary equivalence, we conclude that

$\left(\overset{p+s}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_{0} = \mathcal{L}\left(\overset{p+s}{\underset{i=1}{\otimes}} H_i\right)$. The proof of Proposition 4.11 is now complete. ∎

PROPOSITION 4.12. *Let $\mathcal{N}_i$ be a nest in the Hilbert space H_i with at least one atom ($i = 2, 3, \ldots, k$) and $\mathcal{N}_1$ is well-ordered (or well-ordered from above) with finite dimensional atoms. Assume, moreover, $\{\mathcal{N}_i\}_{i=1}^{k}$ do not satisfy the condition of Proposition 3.2; then* $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_{0} = \mathcal{L}(H)$.

Proof. Let $\mathcal{N}_1$ be a well-ordered nest. From the results in section 3 and Proposition 4.11, we can assume that $k = 2$ and that $\mathcal{N}_2$ is an uncountable nest with a finite number of atoms.

Let L^2 be an atom of $\mathcal{N}_2$, and let $L^2 = M_d^2 \ominus M_c^2$. Consider two nests:

$$\mathcal{N}_2^1 = \{ M^2 \in \mathcal{N}_2 : M^2 \not\supset L^2 \};$$

$$\mathcal{N}_2^2 = \{ M^2 \ominus M_d^2 : M^2 \in \mathcal{N}_2 \text{ and } M^2 \supset L^2 \}.$$

There are three possibilities:

1. $\mathcal{N}_2^1, \mathcal{N}_2^2$ are uncountale nests;
2. $\mathcal{N}_2^1$ is a finite nest;
3. $\mathcal{N}_2^2$ is a finite nest.

Case 1. Construct the nests

$$\mathcal{N} = \{ M^1 \otimes L^2 : M^1 \in \mathcal{N}_1\} \cup \{ H_1 \otimes M^2_{[d,t]} : d \leqslant t \leqslant 1\};$$
$$\mathcal{N}_\lambda^1 = L_\lambda^1 \otimes \mathcal{N}_2^1 = \{ L_\lambda^1 \otimes M : M \in \mathcal{N}_2^1 \};$$
$$\mathcal{N}_\lambda^2 = L_\lambda^1 \otimes \mathcal{N}_2^2 = \{ L_\lambda^1 \otimes M : M \in \mathcal{N}_2^2 \} \quad (\lambda \in \Lambda),$$

where $\{ L_\lambda^1 \}_{\lambda \in \Lambda}$ is the set of atoms of $\mathcal{N}_1$ and Λ is a countable index set.

Given $T \in \mathcal{L}(H)$ and $\varepsilon > 0$, by Lemma 2.1 there exists K_ε compact, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K \simeq T \oplus \left(\underset{\lambda \in \Lambda}{\oplus} A_\lambda\right) \oplus \left(\underset{\lambda \in \Lambda}{\oplus} B_\lambda\right)$, where $A_\lambda \simeq B_\lambda \simeq A_\lambda^{(\infty)}$. By Theorem 1.1, $T \in (\mathcal{N})^{\wedge}_{0}$. Since $\sigma_0(A_\lambda) = \sigma_0(B_\lambda) = \emptyset, A_\lambda \in (\mathcal{N}_\lambda^1)^{\wedge}_{0}$ and $B_\lambda \in (\mathcal{N}_\lambda^2)^{\wedge}_{0}$ (Use Corollary 2.11). Thus by Lemma 2.7, $T \in (\mathcal{N}_1 \otimes \mathcal{N}_2)^{\wedge}_{0}$.

Case 2. Suppose the (finitely many) atoms of $\mathcal{N}_2^1$ are $\{ L_i^2 \}_{i=1}^{s}$. Since $\mathcal{N}_2^2$ is uncountable with a finite number of atoms, $\mathcal{N}_2^2$ contains some interval $[a, b]$. Choose real numbers $\{ t_i \}_{i=1}^{s}$ such that $a < t_1 < t_2 < \cdots < t_s < b$. Construct the nests

$$\mathcal{N}^1 = \{ M^1 \otimes L_1^2 : M^1 \in \mathcal{N}_1 \} \cup \{ H_1 \otimes M^2_{(d,t]} : d < t < t_1 \};$$
$$\mathcal{N}^2 = \{ M^1 \otimes L_2^2 : M^1 \in \mathcal{N}_1 \} \cup \{ H_1 \otimes M^2_{[t_1,t]} : t_1 \leqslant t < t_2 \};$$
$$\vdots$$
$$\mathcal{N}^s = \{ M^1 \otimes L_s^2 : M^1 \in \mathcal{N}_1 \} \cup \{ H_1 \otimes M^2_{[t_{s-1},t]} : t_{s-1} \leqslant t < t_s \};$$
$$\mathcal{N}^{s+1} = \{ M^1 \otimes L^2 : M^1 \in \mathcal{N}^1 \} \cup \{ H_1 \otimes M^2_{[t_s,t]} : t_s \leqslant t \leqslant 1 \}$$

Given $T \in \mathcal{L}(H)$ and $\varepsilon > 0$, by Lemma 2.1, we have $T - K_\varepsilon \simeq T \oplus \left(\overset{s}{\underset{i=1}{\oplus}} A_i\right) \oplus$ $\oplus \left(\underset{\lambda\in\Lambda}{\oplus} B_\lambda\right)$; where $K_\varepsilon \in K(H), \|K_\varepsilon\| < \varepsilon$ and $A_i \simeq B_\lambda \simeq B_\lambda^{(\infty)}$ $(i = 1, 2, \ldots, s; \lambda \in$ $\in \Lambda)$. By Theorem 1.1, $T \in (\mathcal{N}^{s+1})_0^\wedge$. Since $\sigma_0(A_i) = \sigma_0(B_\lambda) = \emptyset, A_i \in (\mathcal{N}^i)_0^\wedge$ and $B_\lambda \in (\mathcal{N}_\lambda^2)_0^\wedge (i = 1, 2, \ldots, s; \lambda \in \Lambda)$. Thus, by Lemma 2.7, $T \in (\mathcal{N}_1 \otimes \mathcal{N}_2)_0^\wedge$.

Case 3 follows by the same argument as in case 2.

Thus, when $\mathcal{N}_1$ is well-ordered, the conclusion is true. By using same argument, we can arrive to the same conclusion when $\mathcal{N}_1$ is well-ordered from above. The proof of Theorem 1.4 is now complete. ∎

5. FINAL REMARKS

I. As in [14], we say that a subspace $\mathcal{L}$ of $\mathcal{L}(H)$ is a *model* for $\mathcal{L}(H)$ if every T in $\mathcal{L}(H)$ is unitarily equivalent to some operator in $\mathcal{L}$.

In [5], T. Fall, W. B. Arveson and P. S. Muhly proved that $\text{alg}\mathcal{N} + \mathcal{K}(H)$ is norm closed in $\mathcal{L}(H)$ for each nest $\mathcal{N}$ in Hilbert space H. Thus, by Theorem 1.1, if $\mathcal{N}$ is neither well-ordered nor well-ordered from above, with finite dimensional atoms, $\text{alg}\mathcal{N} + \mathcal{K}(H)$ is a model for $\mathcal{L}(H)$. This conclusion is still true in the tensor product of nests case.

LEMMA 5.1 (K. R. Davidson, personal communication). *For $i = 1, 2, \ldots, k$, let $\mathcal{N}_i$ be a nest in the Hilbert space H_i and let $H = \overset{k}{\underset{i=1}{\otimes}} H_i$; then $\text{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) + \mathcal{K}(H)$ is a norm-closed subalgebra of $\mathcal{L}(H)$.*

Proof. Since every nest is a completely distributive commutative subspace lattice, from corollary 23.10 of [2] we have

$$\text{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) = \overset{k}{\underset{i=1}{\otimes}} (\text{alg}\mathcal{N}_i)$$

By Theorem 23.7 of [2], $(\text{alg}\mathcal{N}_i) \cap \mathcal{K}(H_i)$ is weak* dense in $\text{alg}\mathcal{N}_i (i = 1, 2, \ldots, k)$. Therefore, $\overset{k}{\underset{i=1}{\otimes}} [(\text{alg}\mathcal{N}_i) \cap \mathcal{K}(H_i)]$ is weak* dense in $\text{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)$. By Theorem 1.1 of Fall-Arveson-Muhly [5], $\text{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) + \mathcal{K}(H)$ is norm-closed in $\mathcal{L}(H)$. ∎

Thus, with this lemma at hand, Theorem 1.3 can be rewritten as follows.

THEOREM 1.3'. *Let $\mathcal{N}_i$ be a nest in the Hilbert space $H_i (i = 1, 2, \ldots, k)$. Suppose that neither all $\mathcal{N}_i$'s nor all $\mathcal{N}_i^\perp$'s ($\mathcal{N}_i^\perp = \{ M^\perp : M \in \mathcal{N}_i \}$) are well-ordered with finite dimensional atoms; then the algebra $\text{alg}\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right) + \mathcal{K}(H)$ is a model for $\mathcal{L}(H)$,* where $H = \bigotimes_{i=1}^{k} H_i$.

II. Instead of finite tensor products, we can consider the tensor product of a countable family of nests.

DEFINITION 5.2. Let $\mathcal{N}_i$ be a nest in the Hilbert space $H_i (i = 1, 2, \ldots)$ and let $H = \bigotimes_{i=1}^{\infty} H_i$.

(i) $\bigotimes_{i=1}^{\infty} \mathcal{N}_i$ is the complete lattice generated by the family of subspaces of H : $\{ \bigotimes_{i=1}^{\infty} M^i : M^i \in \mathcal{N}_i (i = 1, 2, \ldots)$ and $M^i = H_i$ except for a finite number of indices $i \}$;

(ii) $\mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right) = \{ T \in \mathcal{L}(H) : \mathrm{Lat}T \supset \bigotimes_{i=1}^{\infty} \mathcal{N}_i \}$;

(iii) $(\otimes \mathcal{N}_i)^{\wedge} = \{ UAU^* + K : U$ is unitary $, A \in \mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)$ and $K \in \mathcal{K}(H) \}$;

(iv) $\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)_0^{\wedge} = \{ T \in \mathcal{L}(H) :$ Given $\varepsilon > 0$, there exists U_ε unitary, $A_\varepsilon \in \in \mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)$ and $K_\varepsilon \in \mathcal{K}(H)$ such that $\|K_\varepsilon\| < \varepsilon$ and $T = U_\varepsilon A_\varepsilon U_\varepsilon^* + K_\varepsilon \}$.

PROPOSITION 5.4. *For $i = 1, 2, \ldots$, let $\mathcal{N}_i$ be Volterra nest in the Hilbert space $H_i = L^2([0,1])$ with Lebesgue measure (c.f.[2] p.23), and let $H = \bigotimes_{i=1}^{\infty} H_i$; then*

$$\left[\mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)\right] \cap \mathcal{K}(H) = \{0\} \quad \textit{and} \quad \left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)_0^{\wedge} \cap \mathcal{K}(H) = \{0\}.$$

Proof. Set $P_n = \bigotimes_{i=1}^{\infty} M_i$ where $M_i = H_i (i \neq n), M_n = \{ f \in L^2([0,1]) : f(x) = = 0$ a.e. on $[\frac{1}{2}, 1] \}$ $(n = 1, 2, \ldots)$. Then it is not difficult to check that for each subsequence $\{ P_{n_i} \}_{i=1}^{\infty}$, we have

$$\bigvee_{i=1}^{\infty} P_{n_i} = I \text{ and } \bigwedge_{i=1}^{\infty} P_{n_i} = 0.$$

Thus by Theorem 1.1.4 of [7], $\left[\mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)\right] \cap \mathcal{K}(H) = \{0\}$.

Assume $K \in \left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)_0^{\wedge} \cap \mathcal{K}(H)$; that is, given $\varepsilon > 0$, there exist $K_\varepsilon \in \mathcal{K}(H)$ and U_ε unitary such that $\|K_\varepsilon\| < \varepsilon$ and $U_\varepsilon(K - K_\varepsilon)U_\varepsilon^* \in \mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right)$. Since $\mathrm{alg}\left(\bigotimes_{i=1}^{\infty} \mathcal{N}_i\right) \cap \mathcal{K}(H) = \{0\}, U_\varepsilon(K - K_\varepsilon)U_\varepsilon^* = 0$ or $K = K_\varepsilon$. Since ε can be arbitrary small, $K = 0$. ∎

We close this section with some open problems. It is immediate that $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)^{\wedge}$ is always a closed subset of $\mathcal{L}(H)$; moreover, so is $\left(\bigotimes_{i=1}^{k} \mathcal{N}_i\right)_0^{\wedge}$ in the cases when this set coincides with $\mathcal{L}(H), \mathcal{L}(H)_d$ (for some $d \geqslant 0$), (QT) or $(QT)^*$. As we have seen, in the remaining cases, $\mathcal{L}(H)_{d_1} \subset \left(\overset{\infty}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0 \subset \mathcal{L}(H)_{d_2}$ for some d_1, d_2 satisfying $1 \leqslant d_1 < d_2 < \infty$.

PROBLEM 1. Is $\left(\overset{k}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$ always a closed subset of $\mathcal{L}(H)$?

PROBLEM 2. Suppose $\mathcal{N}_i$ is the Volterra nest for all $i = 1, 2, \ldots$. Is $\left(\overset{\infty}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}$ a norm-closed subset of $\mathcal{L}(H)$? What about $\left(\overset{\infty}{\underset{i=1}{\otimes}} \mathcal{N}_i\right)^{\wedge}_0$?

The author conjectures that the first problem has an affirmative answer, but the two questions of problem 2 have negative answers.

REFERENCES

1. ARVESON, W. B., Interpolation problems in nest algebras, *J. Functional Analysis* **20**(1975), 208–233.
2. DAVIDSON, K. R., *Nest algebra*, Research Notes in Mathematics **191**(*Pitman* 1988).
3. DAVIDSON, K. R.; HERRERO, D. A., Quasisimilarity of nests, *Pac. J. Math.* **138**(1989), 243–254.
4. ERDÖS, J., Unitary invariants for nests, *Pac. J. Math.* **23**(1967), 229–256.
5. FALL, T.; ARVESON, W. B.; MUHLY, P., Perturbations of nest algebras, *J. Operator Theory* **1**(1979), 137–150.
6. FROELICH, J., Compact operators in the algebra of a partially ordered measure space, *J. Operator Theory* **10**(1983), 353–355.
7. FROELICH, J., Compact operators, invariant subspaces, and spectral synthesis, *J. Functional Analysis* **81**(1988), 1–37.
8. HERRERO, D. A., On the spectra of the restrictions of an operator, *Trans. Amer. Math. Soc.* **233**(1977), 45–58.
9. HERRERO, D. A., Approximation of Hilbert space operators I, *Research Notes in Mathematics* **72**(Pitman 1982).
10. HERRERO, D. A., Economical compact perturbations I: Erasing normal eigenvalues, *J. Operator Theory* **10**(1983), 293–306.
11. HERRERO, D. A., Compact perturbations of continuous nest algebras, *J. London Math. Soc.* **(2)**, **27**(1983), 339–344.
12. HERRERO, D. A., Compact perturbations of nest algebra, index obstructions, and a problem of Arveson, *J. Functional Analysis*, **55**(1984), 78–109.
13. HERRERO, D. A., The Diagonal entries in the formula 'Quasitriangular-compact= triangular', and restrictions of quasitriangularity, *Trans. Amer. Math. Soc.* **298** (1986), 6–42.
14. HERRERO, D. A.; LARSON, D. R., Ideals of nest algebras and model for operators, preprint, 1987.
15. LARSON, D. R., Nest algebra and similarity transformations, *Ann. Math.* **121**(1985), 409–427.

16. RINGROSE, J. R., Super-Diagonal forms for compact linear operators, *Proc. London Math. Soc.* (**3**) **12**(1962), 367–384.
17. VOICULESCU, D., A non-commutative Weyl-Von Neumann theorem, *Rev. Roum. Math. Pures et Appl.* **21**(1976), 97–113.

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