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MODEL THEORY ON THE UNIT BALL IN $\mathbb{C}^m$

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0. INTRODUCTION

In this paper we continue the investigations started in [4]. In [1] and [2] Jim Agler provided a powerful approach to the model theory of operators by a clever synthesis of the Stinespring Representation Theorem, the Arveson Extension Theorem [3, Theorem 1.2.3] and some basic properties of the reproducing kernels of a special class of functional Hilbert spaces associated with the open unit disk $\mathbb{C}$. Some several variables ramifications of this approach were explored in [4], where it was shown that the program initiated in [1] and [2] can be carried out with some success for the model theory of operator tuples of contractions by considering a special class of functional Hilbert spaces associated with the unit polydisk. The purpose of this paper is to explore in the same spirit the model theory of operator tuples whose Taylor spectrum is contained in the closed unit ball in $\mathbb{C}^m$. The arguments here parallel those in [1], [2] and [4]; one must, however, avoid the danger of reading too much into the results in [4], since the results in [4] reflect some features unique to the unit polydisk. The development of the present paper is geared toward obtaining an intrinsic characterization of those subnormal tuples whose Taylor spectrum is contained in the closed unit ball in $\mathbb{C}^m$.

Section 1 fixes the notation and executes some preliminaries. Section 2 deals with some extension results as interpreted in terms of the positive definite kernels associated with the unit ball. Section 3 constructs a special class of functional Hilbert spaces $\mathcal{M}$ of which the classical Hardy space and the Bergman space of the ball are prototypes. Most of Section 3 deals with the discussion of the spectral and Fredholm properties of the multiplication tuples M_z on the spaces $\mathcal{M}$. Section 4 interprets the general extension results of Section 2 with reference to the particular models constructed in Section 3. Section 5 provides an intrinsic characterization of those subnormal tuples whose Taylor spectrum is contained in the closed unit ball.

The author has done his best to avoid the repetitions of arguments present in the earlier works; consequently the reader is required to have a thorough familiarity with the work in [1], [2] and [4]. For the definition of the Taylor spectrum, the essential Taylor spectrum and the Fredholmness of a commuting tuple of operators, the reader is referred to [16], [17] and [7].

1. PRELIMINARIES

The set of bounded linear operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}^{(n)}$ will denote the direct sum of $\mathcal{H}$ with itself n times. An m-tuple $(T_1,\ldots,T_m)$ of commuting operators in $\mathcal{B}(\mathcal{H})$ will be denoted by T; while T^* will stand for $(T_1^*,\ldots,T_m^*)$. Similarly, any m-tuple $(z_1,\ldots,z_m)$ of complex numbers will be abbreviated to z with z^* having the obvious meaning. For any m-tuple $(s_1,\ldots,s_m)$ of non-negative integers, T^s will denote $T_1^{s_1}T_2^{s_2}\ldots T_m^{s_m}$, while z^s will stand for $z_1^{s_1}z_2^{s_2}\ldots z_m^{s_m}$. For any open subset G of $\mathbb{C}^m$, $H(G)$ will be the set of holomorphic functions on G and $L^2(\bar{G},\mu)$ will be the set of square-integrable functions with respect to the measure μ on the closure $\bar{G}$ of G. The open unit ball and the closed unit ball in $\mathbb{C}^m$ will respectively be denoted by $\mathbb{B}^{2m}$ and $\bar{\mathbb{B}}^{2m}$; and S^{2m-1} will represent the unit sphere, which is the topological boundary $\partial\mathbb{B}^{2m}$ of $\mathbb{B}^{2m}$. Most of the notation will be clear from the context in which it appears. All the Hilbert spaces occurring below are separable.

We begin by introducing a functional calculus for (T,T^*), where the Taylor spectrum $\sigma(T)$ of T is contained in some open ball $\Omega = \{z \in \mathbb{C}^m : |z_1|^2 + \cdots + |z_m|^2 < r^2\}$, $r \geqslant 1$. (The expression functional calculus is used in a somewhat liberal sense; it certainly is not meant to imply any algebra homomorphism.) It follows from Corollary 3.14 in [7] that $\sigma(T^*)$ is contained in Ω as well. If $f(z,w)$ is a holomorphic function on the Cartesian product $\Omega\times\Omega$, then for any element u in $\mathcal{H}$, interpret $f(T,T^*)u$ as

$$(1)\qquad f(T,T^*)u = \frac{1}{(2\pi i)^{2m}} \int_{\partial\Omega_1} M_{T^*}(w)\left[\int_{\partial\Omega_1} M_T(z)\, f(z,w)u \,\mathrm{d}z\right]\mathrm{d}w\ ,$$

where Ω_1 is some open ball containing $\sigma(T)$ and with the closure $\bar{\Omega}_1$ in Ω; $\partial\Omega_1$ is the surface of Ω_1; and $M_T(z)$ and $M_{T^*}(w)$ denote the Martinelli kernels corresponding to the commuting tuples T and T^* respectively (see [18], or [19], Chap. III, Prop. 11.1). We note in particular the following consequences of (1).

(a) If $p(z,w)=\sum_{s,t} c_{st} z^s w^t$ is a polynomial in $2m$ complex variables (z,w), then

$$p(T,T^*)=\sum_{s,t} c_{st} T^{*t} T^s \ . \tag{2}$$

(b) If $f_1, f_2 \in H(\Omega)$ and $f \in H(\Omega \times \Omega)$, then

$$f_2(T^*) f(T,T^*) f_1(T) = g(T,T^*) \ , \tag{3}$$

where $g(z,w)=f_1(z)f_2(w)f(z,w)$.

The interpretation of $f_1(T)$ and $f_2(T^*)$ in (3) is well-known and (3) itself can be deduced by verifying it first for powers of z and w and then noting that any f in $H(\Omega \times \Omega)$ can be expressed as

$$f(z,w)=\sum_{s,t} a_{st} z^s w^t \ ,$$

where the series on the right converges uniformly to f on compact subsets of $\Omega \times \Omega$ [14].

DEFINITION 1.1. An $n \times n$ matrix $[g_{ij}]$ of functions defined on $\mathbb{B}^{2m} \times \mathbb{B}^{2m}$ is positive definite if for all positive integers p, all vectors C_r in $\mathbb{C}^{(n)}$, and all points $\lambda^{(r)}$ in $\mathbb{B}^{2m} (1 \leqslant r \leqslant p)$,

$$\sum_{1 \leqslant r,s \leqslant p} < G_{rs} C_r, \ C_s >_{\mathbb{C}^{(n)}} \geqslant 0 \ ,$$

where G_{rs} is the $n \times n$ matrix $[g_{ij}(\lambda^{(r)}, \lambda^{(s)})]$ and where $< ., . >_{\mathbb{C}^{(n)}}$ denotes the inner product in $\mathbb{C}^{(n)}$.

LEMMA 1.2. *Let $[g_{ij}]$ be positive definite as in Definition 1.1. Also, let $g_{ij}(z^*, w)$ be holomorphic on $\mathbb{B}^{2m} \times \mathbb{B}^{2m}$ for $1 \leqslant i,j \leqslant n$. Then for any r such that $0 < r < 1$, there exist functions f_{il}, $1 \leqslant i \leqslant n$, $l \geqslant 1$, defined and holomorphic on $r\mathbb{B}^{2m}$ such that*

$$g_{ij}(z,w)=\sum_{l=1}^{\infty} f_{jl}^*(z) f_{il}(w) \text{ for } 1 \leqslant i \ , j \leqslant n \ ,$$

and where the series on the right converges uniformly on compact subsets of $(r\mathbb{B}^{2m}) \times \times (r\mathbb{B}^{2m})$.

Proof. Refer to the proof of Lemma 1.2 in [4] and do some trivial modifications. ∎

For the sake of easy reference, we reproduce Definition 1.3 in [4] and present a specialized version of Theorem 1.4 in [4].

DEFINITION 1.3. Let T be a tuple of m commuting operators on a Hilbert space $\mathcal{H}$. We say that T extends to S if there exist a Hilbert space $\mathcal{K}$, $S_i \in \mathcal{B}(\mathcal{K})$ $(1 \leqslant i \leqslant m)$, and an isometry V from $\mathcal{H}$ into $\mathcal{K}$ such that Range V is invariant for each S_i and $T_i = V^* S_i V$ for each i. (It then follows that $p(T, T^*) = V^* p(S, S^*) V$ for any $p(T, T^*)$, $p(S, S^*)$ as in (2)).

THEOREM 1.4. *Let T be a tuple of m commuting operators in $\mathcal{B}(\mathcal{H})$ and S a tuple of m commuting operators in $\mathcal{B}(\mathcal{H}_1)$, where $\mathcal{H}$ and $\mathcal{H}_1$ are some Hilbert spaces. Statements (i) and (ii) below are equivalent.*

(i) There exist a Hilbert space $\mathcal{K}$ and a $*$-representation π: $\mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{K})$ such that $\pi(1) = 1$, (here 1 represents the identity operator) and T extends to $\pi(S) = (\pi(S_1), \ldots, \pi(S_m))$.

(ii) For any positive integer n and n^2 polynomials p_{ij} in $2m$ complex variables, $[p_{ij}(S, S^*)] \geqslant 0$ in $\mathcal{B}(\mathcal{H}_1^{(n)})$ implies $[p_{ij}(T, T^*)] \geqslant 0$ in $\mathcal{B}(\mathcal{H}^{(n)})$.

Theorem 1.4 is indeed the essence of the complete positivity considerations underlying the statements of extension results in Section 2.

2. HOLOMORPHIC KERNELS AND EXTENSION RESULTS

DEFINITION 2.1. An analytic model atom $\mathcal{M}$ over $\mathbb{B}^{2m}$ is a Hilbert space of holomorphic functions on $\mathbb{B}^{2m}$ satisfying the following properties.

(i) For any $\lambda \in \mathbb{B}^{2m}$; $f, g \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{C}$, $(\alpha f + \beta g)(\lambda) = \alpha f(\lambda) + \beta g(\lambda)$.

(ii) For any λ in $\mathbb{B}^{2m}$, there exists a constant c_λ such that $|f(\lambda)| \leqslant c_\lambda \|f\|_{\mathcal{M}}$ for every f in $\mathcal{M}$. Here $\| \cdot \|_{\mathcal{M}}$ denotes the norm induced by the inner product $< \cdot, \cdot >_{\mathcal{M}}$ on $\mathcal{M}$.

(iii) The maps $(M_{z_i} f)(z) = z_i f(z)$ define bounded operators on $\mathcal{M}$.

(iv) $\mathcal{M}$ contains constant functions and the Taylor spectrum $\sigma(M_z)$ of $M_z = (M_{z_1}, \ldots, M_{z_m})$ is contained in $\bar{\mathbb{B}}^{2m}$.

Note that by virtue of (ii) above, there exists for any λ in $\mathbb{B}^{2m}$, a function κ_λ in $\mathcal{M}$ such that $f(\lambda) =< f, \kappa_\lambda >_{\mathcal{M}}$ for every f in $\mathcal{M}$. The kernel functions κ_λ give rise to the kernel $\kappa(\lambda, \mu)$ of $\mathcal{M}$ defined by $\kappa(\lambda, \mu) = \kappa_{\mu^*}(\lambda)$. Note that $\kappa(\lambda, \mu) \in H(\mathbb{B}^{2m} \times \mathbb{B}^{2m})$; κ will be referred to as the holomorphic kernel of $\mathcal{M}$.

DEFINITION 2.2. An analytic model atom $\mathcal{M}$ over $\mathbb{B}^{2m}$ is called regular if it satisfies the following properties.

(i) Polynomials in z are dense in $\mathcal{M}$.

(ii) The kernel $\kappa(\lambda, \mu)$ does not vanish anywhere on $\mathbb{B}^{2m} \times \mathbb{B}^{2m}$ and is symmetric; that is, $\kappa(\lambda, \mu) = \kappa(\mu, \lambda)$.

(iii) The operator tuple $\lambda - M_z = (\lambda_1 - M_{z_1}, \ldots, \lambda_m - M_{z_m})$ is Fredholm for every λ in $\mathbb{B}^{2m}$.

The Hardy space and the Bergman space of the ball are well-known examples of regular analytic model atoms over $\mathbb{B}^{2m}$ ([7], [9]).

LEMMA 2.3. *Let $\mathcal{M}$ be a regular analytic model atom over $\mathbb{B}^{2m}$. Then for any positive integer n and n^2 polynomials p_{ij}, $[p_{ij}(M_z^*, M_z)] \geqslant 0$ in $\mathcal{B}(\mathcal{M}^{(n)})$ if and only if $[p_{ij}(z^*, w)\kappa(z^*, w)]$ is positive definite.*

Proof. Argue as in proposition 2.5 in [1]. ∎

Using Lemma 1.2, 2.3 and Theorem 1.4, one can prove Theorem 2.4 below. For details, consult the proof of Theorem 2.5 in [4].

THEOREM 2.4. *Let $\mathcal{M}$ be a regular analytic model atom over $\mathbb{B}^{2m}$ with the holomorphic kernel κ. Let $\mathcal{H}$ be a Hilbert space and let T be a tuple of m commuting operators in $\mathcal{B}(\mathcal{H})$ such that the Taylor spectrum $\sigma(T)$ of T is contained in the open unit ball $\mathbb{B}^{2m}$. If $\frac{1}{\kappa}(T, T^*) \geqslant 0$, then there exist a Hilbert space $\mathcal{K}$ and a $*$–representation $\pi: \mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{K})$ with $\pi(1) = 1$ such that T extends to $\pi(M_z^*)$.*

THEOREM 2.5. *Let $\mathcal{M}$, κ, T be as in Theorem 2.4. Suppose that $\frac{1}{\kappa}$ extends to a holomorphic function on a neighborhood of $\bar{\mathbb{B}}^{2m} \times \bar{\mathbb{B}}^{2m}$. Then statements (i) and (ii) below are equivalent.*

(i) $\frac{1}{\kappa}(T, T^*) \geqslant 0$

(ii) There exist a Hilbert space $\mathcal{K}$ and a $*$–representation π: $\mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{K})$ with $\pi(1) = 1$ such that T extends to $\pi(M_z^*)$.

Proof. Consult the proof of Theorem 2.6 in [4]. ∎

3. A CLASS OF REGULAR ANALYTIC MODEL ATOMS OVER $\mathbb{B}^{2m}$

The holomorphic kernels of the Hardy space and the Bergman space of the unit ball are respectively

$$\frac{(m-1)!}{2\pi^m} \frac{1}{(1 - z_1 w_1 - \cdots - z_m w_m)^m} \quad \text{and}$$

$$\frac{m!}{\pi^m} \frac{1}{(1 - z_1 w_1 - \cdots - z_m w_m)^{m+1}} .$$

This suggests that we consider the functional Hilbert spaces $\mathcal{M}_{m+p}$ whose holomorphic kernels are given by $\frac{1}{(1 - z_1 w_1 - \cdots - z_m w_m)^{m+p}} (p \geqslant 0)$. (The constant factors are of no particular significance for the discussion of the extension results to follow.)

Note that for z, w in $\mathbb{B}^{2m}$, $\frac{1}{(1-z_1w_1-\cdots-z_mw_m)^{m+p}} = 1 + \sum_{k_1\geqslant 0,\ldots,k_m\geqslant 0,k\neq 0} \frac{(m+p)(m+p+1)\cdots(m+p+|k|-1)}{k_1!k_2!\cdots k_m!} z^k w^k$, where $|k| = k_1 + \cdots + k_m$.

Letting $e_{m+p,0} = 1$ and $e_{m+p,k} = \frac{k_1!\cdots k_m!}{(m+p)\cdots(m+p+|k|-1)}$ otherwise, we can define $\mathcal{M}_{m+p}$ to be the set of all those power series $f(z) = \sum_k a_k z^k$ for which $\|f\|^2 = \sum_k |a_k|^2 e_{m+p,k} < \infty$. (The definition of the inner product on $\mathcal{M}_{m+p}$ is obvious.) We plan to show that the multiplication tuple $M_{m+p,z}$ on $\mathcal{M}_{m+p,z}$ is subnormal (see Definition 5.1 below). This will be accomplished by realizing $M_{m+p,z}$ as the multiplication tuple on $H^2(\eta_{m+p})$, where $H^2(\eta_{m+p})$ denotes the completion of the set of polynomials in $L^2(\bar{\mathbb{B}}^{2m}, \eta_{m+p})$ and where η_{m+p} are some suitably chosen measures supported on $\bar{\mathbb{B}}^{2m}$. Noting that under the transform $(z_1,\ldots,z_m)\mapsto(|z_1|^2,\ldots,|z_m|^2)$, $\bar{\mathbb{B}}^{2m}$ gets mapped onto the closed simplex $P = \{x\in\mathbb{R}^m\colon x_1\geqslant 0,\ldots,x_m\geqslant 0, 1-x_1-\cdots-x_m\geqslant 0\}$, we start by solving the following moment problem: Find measures ν_{m+p} upported on the closed simplex P such that

$$e_{m+p,k} = \int_P x^k \mathrm{d}\nu_{m+p}(x) \ .$$

We illustrate the argument for $m = 2$; it easily generalizes to higher dimensions. Choose ν_{2+0} to be the linear measure supported on the edge $E = \{x\in\mathbf{R}^m\colon x_1\geqslant 0,\ x_2\geqslant 0,\ 1 = x_1+x_2\}$ of the simplex P. It easy to see that

$$\int_P x_1^{k_1}x_2^{k_2}\mathrm{d}\nu_{2+0}(x_1,x_2) = \int_0^1 x_1^{k_1}(1-x_1)^{k_2}\mathrm{d}x_1 =$$

$$= \frac{k_1!\ k_2!}{(k_1+k_2+1)!} = e_{2+0,k} \ .$$

For $p\geqslant 1$, choose $\mathrm{d}\nu_{2+p}(x) = p(p+1)(1-x_1-x_2)^{p-1}\mathrm{d}x_1\mathrm{d}x_2$, so that

$$\int_P x_1^{k_1}x_2^{k_2}\mathrm{d}\nu_{2+p}(x_1,x_2) =$$

$$= \int_0^1 x_1^{k_1}\int_0^{1-x_1} x_2^{k_2}(1-x_1-x_2)^{p-1}p(p+1)\mathrm{d}x_1\mathrm{d}x_2 =$$

$$= \int_0^1 x_1^{k_1}(1-x_1)^{p+k_2}\beta(k_2+1,p)p(p+1)\mathrm{d}x_1 \ ,$$

where β denotes the standard beta function. This further reduces to

$$\int_P x_1^{k_1}x_2^{k_2}\mathrm{d}\nu_{2+p}(x_1,x_2) =$$

$$= \beta(k_1+1, p+k_2+1)\beta(k_2+1, p)p(p+1) =$$
$$= \frac{k_1!(p+k_2)!}{(p+|k|+1)!} \cdot \frac{k_2!(p-1)!}{(p+k_2)!} \cdot p(p+1) = e_{2+p,k} .$$

Define

$$d\eta_{m+p}(r_1 e^{i\theta_1}, \dots, r_m e^{i\theta_m}) =$$
$$= d\nu_{m+p}(r_1^2, \dots, r_m^2) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \cdots \frac{d\theta_m}{2\pi} .$$

It is clear that for $p \geqslant 1$, η_{m+p} has $\bar{\mathbb{B}}^{2m}$ for its support and $M_{m+p,z}$ can be identified with M_z on $H^2(\eta_{m+p})$. The case $p = 0$ is the well-known case of the Szegö tuple on the Hardy space of the unit ball. Our observations above can now be summarized in Proposition 3.1 below.

PROPOSITION 3.1. *For $p \geqslant 0$, $M_{m+p,z}$ on $\mathcal{M}_{m+p}$ is a (cyclic) subnormal tuple with $\sigma(M_{m+p,z}) = \bar{\mathbb{B}}^{2m}$.*

PROPOSITION 3.2. *For $p \geqslant 0$, $M_{m+p,z}$ on $\mathcal{M}_{m+p}$ is essentially normal; that is, $M^*_{m+p,z_j} M_{m+p,z_i} - M_{m+p,z_i} M^*_{m+p,z_j}$ is compact for all i and j.*

Proof. Again we illustrate the argument for $m = 2$. Suppressing the notation $e_{m+p,k}$ to e_k, we see that for $k_1 \geqslant 1$, as $|k| \to \infty$,

$$\frac{e_{(k_1+1,k_2)}}{e_{(k_1,k_2)}} - \frac{e_{(k_1,k_2)}}{e_{(k_1-1,k_2)}} = \frac{p+k_2+1}{(p+|k|+1)(p+|k|+2)} \leqslant \frac{1}{p+|k|+2} \to 0 ;$$

and as $k_2 \to \infty$, $\dfrac{e_{(1,k_2)}}{e_{(0,k_2)}} = \dfrac{1}{p+k_2+2} \to 0$. By the obvious relationship of the weights of M_{z_1} to e_k, this shows that $M^*_{z_1} M_{z_1} - M_{z_1} M^*_{z_1}$ is compact. Other commutators may be checked directly or one may appeal to the symmetry of the situation. ∎

PROPOSITION 3.3. *For $p \geqslant 0$, $M_{m+p,z}$ on $\mathcal{M}_{m+p}$ is irreducible; that is, M_{m+p,z_i} do not possess a common non-trivial reducing subspace.*

Proof. This follows directly from Corollary 13 in [11]. ∎

For $p \geqslant 0$, let P_{m+p} denote the projection of $L^2(\bar{\mathbb{B}}^{2m}, \eta_{m+p})$ onto $H^2(\eta_{m+p})$. For any continuous function χ defined on $\bar{\mathbb{B}}^{2m}$, indicate for $p \geqslant 1$ the Toeplitz operator $P_{m+p}(\chi f)$ $(f \in H^2(\eta_{m+p}))$ by $T_{m+p,\chi}$. For any continuous function φ on S^{2m-1}, similarly interpret $T_{m+0,\varphi}$.

LEMMA 3.4. *For $p \geqslant 1$, $T_{m+p,\chi}$ compact implies $\chi / S^{2m-1} = 0$. Also, $T_{m+0,\varphi}$ compact implies $\varphi = 0$.*

Proof. The cases $p = 0$ and $p = 1$ are dealt with in Lemma 2 in [6]. For $p \geqslant 2$, argue as in Lemma 2 in [6] with the volumetric measure η_{m+1} there replaced by η_{m+p}. ∎

LEMMA 3.5. *Let $\{f_{m+p,k}\}_k$ denote the orthonormal basis in $H^2(\eta_{m+p})$ and $U_{p,q}$ denote the unitary transformation defined by*

$$U_{p,q} f_{m+q,k} = f_{m+p,k} .$$

Then for any $p,\ q \geqslant 1$,

$$U_{p,q}^* T_{m+p,\chi} U_{p,q} = T_{m+q,\chi} + K_{p,q}(\chi) ,$$

where $K_{p,q}(\chi)$ is a compact operator. Also, for any $p \geqslant 1$,

$$U_{p,0}^* T_{m+p,\chi} U_{p,0} = T_{m+0,\varphi} + K_{p,0}(\chi) ,$$

where $K_{p,0}(\chi)$ is a compact operator and where $\varphi = \chi/S^{2m-1}$.

Proof. Argue as in Lemma 3 in [6] using the weights obtained from $e_{m+p,k}$. ∎

PROPOSITION 3.6. *For $p \geqslant 0$, the essential Taylor spectrum $\sigma_e(M_{m+p,z})$ of $M_{m+p,z}$ on $\mathcal{M}_{m+p}$ is the unit sphere S^{2m-1}.*

Proof. The assertion is well-known for $p = 0, 1$ ([7], [9]). For $p \geqslant 1$, using Proposition 3.2, 3.3, Lemma 3.4, 3.5 and arguing as in Theorem 1 in [6], it follows that the C^*-algebra $C^*(M_{m+p,z})$ generated by M_{m+p,z_i} contains the algebra of compact operators $K(\mathcal{M}_{m+p})$ and $\frac{C^*(M_{m+p,z})}{K(\mathcal{M}_{m+p})}$ is isomorphic to $C(S^{2m-1})$, the space of continuous functions on S^{2m-1}. In particular $\sigma_e(M_{m+p,z}) = S^{2m-1}$. ∎

All of our observations above are summarized in Theorem 3.7 below.

THEOREM 3.7. *For $p \geqslant 0$, let $\mathcal{M}_{m+p}$ be the functional Hilbert space whose holomorphic kernel is given by $1/(1 - z_1 w_1 - \cdots - z_m w_m)^{m+p}$; $z, w \in \mathbb{B}^{2m}$. The multiplication tuple $M_{m+p,z}$ on $\mathcal{M}_{m+p}$ is an essentially normal irreducible (cyclic) subnormal tuple with $\sigma(M_{m+p,z}) = \bar{\mathbb{B}}^{2m}$ and $\sigma_e(M_{m+p,z}) = S^{2m-1}$. In particular, for any $p \geqslant 0$, $\mathcal{M}_{m+p}$ is a regular analytic model atom over $\mathbb{B}^{2m}$.*

REMARK 3.8. It was shown in [10] that each M_{m+0,z_i} can be identified with the direct sum $\bigoplus_k M_{k,z}$, where each $M_{k,z}$ denotes multiplication by the single variable z on a weighted Bergman space of the unit disk, indexed by k. It is left to the reader to verify that for any $p \geqslant 1$, a similar statement is true for M_{m+p,z_i}.

REMARK 3.9. For $m = 2$, suppressing the dependence on $2 + p$, $\inf_{|k|\geqslant 0} \left\{ \frac{e_{(k_1+1,k_2)}}{e_{(k_1,k_2)}} + \frac{e_{(k_1,k_2+1)}}{e_{(k_1,k_2)}} \right\}^{1/2} = \sqrt{\frac{2}{p+2}} > 0$. Also, agreeing to interpret expressions like $\frac{e_{(o,k_2)}}{e_{(0-1,k_2)}}$ to be zero, $\inf_{|k|>0} \left\{ \frac{e_{(k_1,k_2)}}{e_{(k_1-1,k_2)}} + \frac{e_{(k_1,k_2)}}{e_{(k_1,k_2-1)}} \right\}^{1/2} = \frac{1}{\sqrt{p+2}} > 0$.

It follows from Corollary 4.3 in [9] that $M_{2+p,z}$ has Fredholm index equal to -1. (Here the Fredholm index of a commuting operator tuple T is understood to be the negative of the Euler characteristic of a certain koszul complex associated with T (see [7]).)

4. APPLICATIONS

Using the regular analytic model atoms constructed in Section 3, we can now derive some interesting extension results for commuting operator tuples.

THEOREM 4.1. *Let T be a tuple of m commuting operators on $\mathcal{H}$ such that $\sigma(T) \subset \mathbf{B}^{2m}$. Then $(1 - z_1w_1 - \cdots - z_mw_m)^{m+p}(T,T^*) \geqslant 0$ if and only if there exist a Hilbert space $\mathcal{K}$ and a $*$-representation π: $\mathcal{B}(\mathcal{M}_{m+p}) \to \mathcal{B}(\mathcal{K})$ such that $\pi(1) = 1$ and T extends to $\pi(M^*_{m+p,z})$.*

Proof. This is a direct consequence of Theorem 2.5. ∎

The reader may want to decipher the positivity conditions in Theorem 4.1 for $p = 0$ and $p = 1$.

THEOREM 4.2. *Let T be a tuple of m commuting operators on $\mathcal{H}$ such that $\sigma(T) \subseteq \bar{\mathbf{B}}^{2m}$. The statements (i) and (ii) below are equivalent.*

(i) $(1 - z_1w_1 - \cdots - z_mw_m)^k(T,T^*) \geqslant 0$ for all k such that $1 \leqslant k \leqslant m+p$.

(ii) There exist a Hilbert space $\mathcal{K}$ and a $*$-representation $\pi: \mathcal{B}(\mathcal{M}_{m+p}) \to \mathcal{B}(\mathcal{K})$ such that $\pi(1) = 1$ and T extends to $\pi(M^*_{m+p,z})$.

Proof. We first show that (ii) implies (i). Provided (ii) is true, appealing to Lemma 2.3 for the case $n = 1$, $\kappa(z,w) = 1/(1 - z_1w_1 - \cdots - z_mw_m)^{m+p}$ and $p_{ij}(z,w) = (1 - z_1w_1 - \cdots - z_mw_m)^k (1 \leqslant k \leqslant m+p)$, we see that $(1 - z_1w_1 - \cdots - z_mw_m)^k(\pi(M^*_{m+p,z}), \pi(M_{m+p,z})) \geqslant 0$ for $1 \leqslant k \leqslant m+p$. Since T extends to $\pi(M^*_{m+p,z})$, however, it follows that $(1 - z_1w_1 - \cdots - z_mw_m)^k(T,T^*) \geqslant 0$ for $1 \leqslant k \leqslant m+p$.

Conversely, suppose (ii) is true. By the spectral mapping property of the Taylor spectrum, $\sigma(rT) \subset \mathbf{B}^{2m}$ if $0 < r < 1$. Note that

$$(1 - z_1w_1 - \cdots - z_mw_m)^{m+p}(rT, rT^*) =$$
$$= (1 - r^2z_1w_1 - \cdots - r^2z_mw_m)^{m+p}(T,T^*) =$$
$$= ((1 - z_1w_1 - \cdots - z_mw_m) + (1-r^2)z_1w_1 + \cdots + (1-r^2)z_mw_m)^{m+p}(T,T^*) .$$

Now appeal to (3) and argue as in Theorem 3.1 in [4]. ∎

For an arbitrary tuple T of m commuting operators, we agree to interpret $p(z,w)(T,T^*)$ as in (2).

THEOREM 4.3. *If* $(1 - z_1 w_1 - \cdots - z_m w_m)(T, T^*) \geqslant 0$, *then* T *extends to* $\pi(M_z^*)$, *where* M_z *denotes the multiplication tuple on the Hardy space of the unit polydisk in* $\mathbb{C}^m$; *and* π *has the usual meaning.*

Proof. Note that $(1 - z_1 w_1 - \cdots - z_m w_m)(T, T^*) = 1 - T_1^* T_1 - \cdots - T_m^* T_m$. It is an exercise in elementary algebra that the positivity of the last expression guarantees the positivity of $(1 - z_1 w_1)^{k_1}(1 - z_2 w_2)^{k_2} \cdots (1 - z_m w_m)^{k_m}(T, T^*)$ for $0 \leqslant k_j \leqslant 1$ $(1 \leqslant \leqslant j \leqslant m)$. One can then invoke Theorem 3.1 in [4]. ■

REMARK 4.4. It was observed in [4] that the extension of T to $\pi(M_z^*)$ where M_z is as in Theorem 4.3 is equivalent to the concept of regular unitary dilation of T (see [13]). The reader is urged to compare Theorem 4.3 to Proposition 9.2, (iii) in [13].

QUESTION 4.5. For $m > 1$, can one obtain any interesting extension result analagous to the one in Theorem 4.3, corresponding to the conditions $(1 - z_1 w_1 - \cdots - z_m w_m)^k (T, T^*) \geqslant 0$, $1 \leqslant k \leqslant p$; where p is some integer satisfying $2 \leqslant p \leqslant m - 1$?

REMARK 4.6. We illustrate a partial answer to the above question for $m = 2$. Supposing $\sigma(T_1)$, $\sigma(T_2) \subset D$ and $(1 - z_1 w_1 - z_2 w_2)^k (T, T^*) \geqslant 0$ for all k such that $1 \leqslant k \leqslant p$; one can easily check the posivity of

$$((1 - z_1 w_1)^p (1 - z_2 w_2)^p)(T, T^*) =$$

$$= ((1 - z_1 w_1 - z_2 w_2) + z_1 w_1 z_2 w_2)^p (T, T^*)$$

and then appealing to Theorem 2.5 in [4] conclude that (T_1, T_2) extends to $(\pi(M_{z_1}^*)$, $\pi(M_{z_1}^*))$ on $H^2(\mu_p) \otimes H^2(\mu_p)$ (see [4] for the notation). (Note that the condition $\sigma(T_1, T_2) \subset \mathbf{B}^4$ will certainly guarantee $\sigma(T_1)$, $\sigma(T_2) \subset D$ by the projection property of the Taylor spectrum.)

5. COMMUTING NORMAL EXTENSION

DEFINITION 5.1 Let T be a tuple of m commuting operators in $\mathcal{B}(\mathcal{H})$. If there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and m commuting normal operators $N_1, \ldots, N_m$ in $\mathcal{B}(\mathcal{K})$ such that T extends to $N = (N_1, \ldots, N_m)$, then T is said to be subnormal and N is said to be a commuting normal extension of S.

An intrinsic characterization of subnormal tuples of contractions was provided in [4]. The following is the analog of Theorem 4.1 in [4].

THEOREM 5.2. *Let* T *be a tuple of* m *commuting operators in* $\mathcal{B}(\mathcal{H})$. *Statements (i) and (ii) below are equivalent.*

(i) T *is subnormal with* $\sigma(T) \subseteq \overline{\mathbf{B}}^{2m}$.

(ii) $(1 - z_1 w_1 - \cdots - z_m w_m)^k (T, T^*) \geqslant 0$ *for all positive integers* k.

Proof. We first show that (i) implies (ii). Thus let T be subnormal with $\sigma(T) \subseteq \subseteq \bar{\mathbb{B}}^{2m}$. Let N on $\mathcal{K}$ be the minimal normal extension of T so that $\mathcal{K}$ is the smallest reducing subspace for N containing $\mathcal{H}$. It follows from the spectral inclusion property (see [15]) that $\sigma(N) \subseteq \sigma(T)$.

If $P_{\mathcal{H}}$ denotes the projection of $\mathcal{K}$ onto $\mathcal{H}$, then

$$(1 - z_1 w_1 - \cdots - z_m w_m)^k (T, T^*) = P_{\mathcal{H}}(1 - N_1^* N_1 - \cdots - N_m^* N_m)^k / \mathcal{H} \geqslant 0 \ .$$

(Note we used the fact that $N_i N_j^* = N_j^* N_i$ for all i and j.)

Conversely, suppose $(1 - z_1 w_1 - \cdots - z_m w_m)^k (T, T^*) \geqslant 0$ for all $k \geqslant 1$. For any integers $k_i \geqslant 0$ $(1 \leqslant i \leqslant m)$, and α equal to 0 or 1,

$$((1 - z_1 w_1 - \cdots - z_m w_m)^{\alpha} \prod_{i=1}^{m} (1 - z_i w_i)^{k_i})(T, T^*) =$$

$$= ((1 - z_1 w_1 - \cdots - z_m w_m)^{\alpha} \prod_{i=1}^{m} ((1 - z_1 w_1 - \cdots - z_m w_m) +$$

$$+ z_1 w_1 + \cdots + \hat{z_i w_i} + \cdots + z_m w_m)^{k_i})(T, T^*) \geqslant 0$$

where ˆ denotes omission.

By Proposition 5 in [5], T is a subnormal tuple of contractions satisfying the integral representation $T^{*k} T^k = \int_P x^k \mathrm{d}\rho_{_T}(x)$, where k is any arbitrary tuple of non-negative integers, and $\rho_{_T}$ is a unique operator-valued probability measure supported on the simplex $P = \{x \in \mathbb{R}^m \colon x_1 \geqslant 0, \ldots, x_m \geqslant 0, 1 - x_1 - x_2 - \cdots - x_m \geqslant 0\}$. If N on $\mathcal{K}$ is the minimal normal extension of T, then denote by $E(z)$ the spectral measure of N. For any Borel set A in $\mathbb{R}^m$, define $E'(A) = E(\{z \in \mathbb{C}^m \colon (|z_1|^2, \ldots, |z_m|^2) \in A\})$. Referring to the proof of Theorem 3.2 in [12], it is clear that $\rho_{_T}$ may be identified with $P_{\mathcal{H}} E'(.)/\mathcal{H}$ where $P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Since E' (support of $\rho_{_T}$) $\mathcal{K}$ is reducing for $N_1, \ldots, N_m$ and contains $\mathcal{H}$, it follows from the minimality of N that E' (support of $\rho_{_T}$) $\mathcal{K} = \mathcal{K}$ and hence the support of $\rho_{_T}$ is the same as the support of E'. But this clearly means that $\sigma(N)$ is contained in the closed unit ball $\bar{\mathbb{B}}^{2m}$. It was shown in [8] that the Taylor spectrum of a subnormal tuple must be contained in the polynomial convex hull of the spectrum of its minimal normal extension. Since $\bar{\mathbb{B}}^{2m}$ is polynomially convex, however, it follows that $\sigma(T)$ must be contained in $\bar{\mathbb{B}}^{2m}$. ■

It is pointed out that the discussion of Section 4 attempts to provide interpretations of conditions (ii) in Theorem 5.2 as truncated.

To recapitulate, the work in [4] and the present paper constitutes an effort to understand some aspects of the model theory of operator tuples with reference to two important Reinhardt domains in $\mathbb{C}^m$, the unit polydisk and the unit ball.

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