J. OPERATOR THEORY
27(1992), 385–412

SIMILARITY AND QUASISIMILARITY OF QUASINORMAL OPERATORS

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1. INTRODUCTION

A bounded linear operator T on a complex separable Hilbert space is *quasinormal* if T and T^*T commute. In this paper, we study the problem when two quasinormal operators are similar or quasisimilar to each other.

Quasinormal operators were first studied by A. Brown [1]. Among other things, he obtained a structure theorem for such operators: T is quasinormal if and only if it is unitarily equivalent to an operator of the form $N \oplus (S \otimes A)$, where N is normal and A is a positive definite operator (denoted $A > 0$), that is, $(Ax, x) > 0$ for any nonzero vector x in the domain of A, and

$$S \otimes A = \begin{pmatrix} 0 & & & & \\ A & 0 & & & \\ 0 & A & 0 & & \\ 0 & 0 & A & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Moreover, in such a decomposition, N and A are uniquely determined by T. N and $S \otimes A$ are called the *normal* and *pure parts* of T, respectively ($S \otimes A$ is *pure* means that it has no nontrivial reducing subspace on which it is normal).

The question whether two similar quasinormal operators are unitarily equivalent has been in the air for many years (cf. [13]). It is easy to see that two quasinormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. To determine when two pure quasinormal operators $S \otimes A_1$ and $S \otimes A_2$ are similar seems quite elusive. In Section 2 below, we reduce this problem to that of the similarity of certain nests associated with A_1 and A_2 and completely solve it via the deep result of D. R. Larson and K. R. Davidson (cf. [6, Chapter 13]). Specifically, it is shown that $S\otimes A_1$ is similar to $S\otimes A_2$ if and only if A_1 and A_2 have equal spectra and their eigenvalues have equal multiplicities (Theorem 2.1). Thus similar quasinormal operators may not be unitarily equivalent answering negatively the question asked in [13].

The situation with quasisimilarity is much more complicated. Although the normal parts of quasisimilar quasinormal operators are still unitarily equivalent, their pure parts may not be quasisimilar as shown by an example of L. R. Williams (cf. [18, Example 1]). On the positive side, he showed that quasisimilar quasinormal operators must have equal essential spectra (cf. [18, Theorem 3]). (Since quasinormal operators are subnormal [10, Problem 195], this also follows from the recently proved result that quasisimilar subnormal operators have equal essential spectra [22]. That they have equal spectra follows from a more general result for quasisimilar hyponormal operators [3].) In the literature, quasisimilar quasinormal operators have often been constructed to illustrate certain properties of operators not preserved under quasisimilarity; see, for example, [19, Examples 2.2 and 2.3] and [11, p. 1445]. In Sections 3 and 4 below, we completely solve the problem when two quasinormal operators are quasisimilar. (Theorems 3.1 and 4.4). In particular, it follows from our main result that every pure quasinormal operator is quasisimilar to a very special one $S\otimes D$, where D is diagonal and positive definite (Corollary 3.19). We also determine when two pure quasinormal operators are injectively similar or densely similar (Theorem 3.18).

For operators T_1 and T_2 on Hilbert spaces H_1 and H_2, respectively, we say that T_1 is *densely* (resp. *injectively*) *intertwined to* T_2, denoted by $T_1 \overset{d}{\prec} T_2$ (resp. $T_1 \overset{i}{\prec} T_2$), if there is an operator $X : H_1 \to H_2$ with dense range (resp. trivial kernel) such that $XT_1 = T_2X$; T_1 is a *quasiaffine transform* of T_2, denoted by $T_1 \prec T_2$, if the intertwining operator X is a *quasiaffinity*, that is, it has trivial kernel and dense range. T_1 is *densely similar to* T_2 ($T_1 \overset{d}{\sim} T_2$) if $T_1 \overset{d}{\prec} T_2$ and $T_2 \overset{d}{\prec} T_1$; T_1 is *injectively similar to* T_2 ($T_1 \overset{i}{\sim} T_2$) if $T_1 \overset{i}{\prec} T_2$ and $T_2 \overset{i}{\prec} T_1$; T_1 is *quasisimilar to* T_2($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. T_1 is *similar to* T_2 ($T_1 \approx T_2$) if the intertwining X is invertible; they are *unitarily equivalent* ($T_1 \cong T_2$) if X is unitary.

We end this section with some additional notation and terminology. If H_1 and H_2 are two Hilbert spaces, we denote by $\mathcal{B}(H_1, H_2)$ the space of all (bounded linear) operators from H_1 to H_2; we write $\mathcal{B}(H)$ in place of $\mathcal{B}(H, H)$. If T is an operator on H, we denote $T^{(\infty)}$ for the direct sum of $\aleph_0$ copies of T (acting on the direct sum $H^{(\infty)}$ of $\aleph_0$ copies of H). If $T \in \mathcal{B}(H), \sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$ denote its spectrum, point spectrum and approximate point spectrum, respectively, and $m(T)$ its *minimum* *modulus*, that is, $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$. If H is infinite-dimensional, let $\sigma_e(T)$ and $\|T\|_e$ denote the *essential spectrum* and the *essential norm* of T, that is, the spectrum and the norm of the image of T in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ under the natural quotient map, where $\mathcal{K}(H)$ denotes the ideal of compact operators in $\mathcal{B}(H)$. If H is finite dimensional, we let $\|T\|_e = 0$ for convenience. For any normal operator $N, E_N(\cdot)$ denotes its spectral measure.

We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$. H^2 is the Hardy space of analytic functions on $\mathbb{D}$ with square integrable boundary values and H^∞ the Banach algebra of all bounded analytic functions on $\mathbb{D}$ with supremum norm. We denote by S the (simple) unilateral shift on H^2 defined by $(Sf)(z) = zf(z), z \in \mathbb{D}, f \in H^2$. For $1 \leqslant n \leqslant \infty$, S_n will denote the direct sum of n copies of S (acting on the direct sum H_n^2 of n copies of H^2).

We would like to express our appreciation to Man-Duen Choi, John B. Conway and David R. Larson for helpful discussions either orally or in correspondence concerning the similarity problem for quasinormal operators.

2. SIMILARITY

In this section, we completely solve the similarity problem for quasinormal operators. Our main result is

THEOREM 2.1. *For $j = 1, 2$, let $T_j = N_j \oplus (S \otimes A_j)$ be a quasinormal operator, where N_j is normal and $A_j > 0$. Then T_1 is similar to T_2 if and only if N_1 is unitarily equivalent to $N_2, \sigma(A_1) = \sigma(A_2)$ and* $\dim\ker(A_1 - \lambda) = \dim\ker(A_2 - \lambda)$ *for any $\lambda \in \sigma(A_1)$.*

We start the proof with the following lemma which gives the spectrum and approximate point spectrum of a pure quasinormal operator.

LEMMA 2.2. *If $T = S \otimes A$ is a pure quasinormal operator, where $A > 0$, then $\sigma(T) = \{z : |z| \leqslant \|A\|\}$ and $\sigma_{\rm ap}(T) = \{z : |z| \in \sigma(A)\}$.*

Proof. The assertion on $\sigma(T)$ was obtained in [18, Corollary 2] by a direct method. Here we use instead the more general result on spectra of tensor products of operators: $\sigma(T) = \sigma(S)\sigma(A)$ and $\sigma_{\rm ap}(T) \supseteq \sigma_{\rm ap}(S)\sigma_{\rm ap}(A)$ (cf. [2]); here $\sigma(S)\sigma(A) = \{uv : u \in \sigma(S) \text{ and } v \in \sigma(A)\}$ and similarly for $\sigma_{\rm ap}(S)\sigma_{\rm ap}(A)$. Since $\sigma(S) = \overline{\mathbb{D}}$ and $\sigma_{\rm ap}(S) = \partial\mathbb{D}$, we infer that $\sigma(T) = \{z : |z| \leqslant \|A\|\}$ and $\sigma_{\rm ap}(T) \supseteq \{z : |z| \in \sigma(A)\}$. On the other hand, let U be the bilateral shift: $(Uf)(e^{it}) = e^{it}f(e^{it})$ for $f \in \mathcal{L}^2(\partial\mathbb{D})$ and $W = U \otimes A$. Then W is a normal extension of T. Hence $\sigma_{\rm ap}(T) \subseteq \sigma_{\rm ap}(W) = \sigma(W)$. From [2], we have $\sigma(W) = \sigma(U)\sigma(A)$. Since $\sigma(U) = \partial\mathbb{D}$, it follows that $\sigma(W) = \{z :$

$: |z| \in \sigma(A)\}$. Thus $\sigma_{\mathrm{ap}}(T) \subseteq \{z : |z| \in \sigma(A)\}$ completing the proof. ■

The idea of the proof of Theorem 2.1 is to reduce the similarity of pure quasi-normal operators to that of associated nests. Here are the basic definitions and some of the results we need from the nest algebra theory; the main reference is [6]. A collection $\mathcal{N}$ of (closed) subspaces of a Hilbert space H is a *nest* if

(1) $\{0\}$ and H belong to $\mathcal{N}$,

(2) $\mathcal{N}$ is a chain, that is, for any M, N in $\mathcal{N}$, either $M \subseteq N$ or $N \subseteq M$,

and

(3) $\mathcal{N}$ is closed under intersection and span, that is, for any $\{N_\alpha\} \subseteq \mathcal{N}, \bigcap_\alpha N_\alpha$ and $\bigvee_\alpha N_\alpha$ belong to $\mathcal{N}$.

The *nest algebra* $\tau(\mathcal{N})$ associated with a nest $\mathcal{N}$ is the set of all operators T such that $TN \subseteq N$ for every N in $\mathcal{N}$. For any positive definite operator A, there is a naturally associated nest $\mathcal{N}_A$, the one generated by the subspaces $\{E_A[0,\lambda]H : \lambda \geqslant 0\}$. More precisely, $\mathcal{N}_A$ consists of subspaces of the form $E_A[0,\lambda]H$ for $\lambda \in \sigma(A)$ or $E_A[0,\lambda)H$ for $\lambda \in \sigma_{\mathrm{p}}(A)$. As proved in [7] or [14], $\tau(\mathcal{N}_A)$ is the set of those operators T for which $\sup_{n\geqslant 0} \|A^n T A^{-n}\| < \infty$. Two nests $\mathcal{N}$ and $\mathcal{M}$ are *similar* if there is an invertible operator X such that $\mathcal{M} = \{XN : N \in \mathcal{N}\}$. The Similarity Theorem of Larson and Davidson says that $\mathcal{N}$ and $\mathcal{M}$ are similar if and only if there is an order isomorphism θ of $\mathcal{N}$ onto $\mathcal{M}$ which preserves dimension, that is, satisfies $\dim \theta(N_2) \ominus \theta(N_1) = \dim N_2 \ominus N_1$ for any subspaces N_1, N_2 in $\mathcal{N}$ with $N_1 \subseteq N_2$. (cf. [6, Corollary 13.21]).

Now we are ready for the proof of the main result in this section.

Proof of Theorem 2.1. As the arguments for the proof of [4, Proposition 2.6] show, T_1 is similar to T_2 if and only if N_1 is unitarily equivalent to N_2 and $S \otimes A_1$ is similar to $S \otimes A_2$. Thus we may concentrate on the pure parts.

Assume that A_1 and A_2 act on the spaces H_1 and H_2, respectively. Note that the similarity of $S \otimes A_1$ and $S \otimes A_2$ implies that $\sigma_{\mathrm{ap}}(S \otimes A_1) = \sigma_{\mathrm{ap}}(S \otimes A_2)$. Hence $\sigma(A_1) = \sigma(A_2)$ by Lemma 2.2. Next let $X = [X_{ij}]_{i,j=0}^{\infty}$ be an invertible operator such that $X(S\otimes A_1) = (S\otimes A_2)X$. Carrying out the matrix multiplications and comparing the entries in the strict upper triangular parts of the resulting matrices yield

$$X_{ij}A_1 = \begin{cases} 0 & \text{if } j > i = 0 \\ A_2 X_{i-1,j-1} & \text{if } j > i \geqslant 1 \end{cases}.$$

Since A_1 has dense range, we infer, for the case $i = 0$, that $X_{0j} = 0$ for all $j \geqslant 1$. By induction on i, we obtain $X_{ij} = 0$ for any $j > i \geqslant 0$. This shows that X is lower triangular. The same arguments applied to X^{-1} show that the same holds for X^{-1}.

It follows that X_{ii} is invertible for any i. On the other hand, we also have $X_{ii}A_1 = A_2X_{i-1,i-1}$ for $i \geqslant 1$ whence $X_{ii}A_1^i = A_2^iX_{00}$ or $X_{ii} = A_2^iX_{00}A_1^{-i}$ for all i. Therefore, $\sup_{n\geqslant 0} \|A_2^nX_{00}A_1^{-n}\| = \sup_{n\geqslant 0}\|X_{nn}\| \leqslant \|X\|$. Similarly, $\sup_{n\geqslant 0}\|A_1^nX_{00}^{-1}A_2^{-n}\| \leqslant \|X^{-1}\|$. Let

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & X_{00}^{-1} \\ X_{00} & 0 \end{pmatrix}.$$

It is easily seen that

$$A^nYA^{-n} = \begin{pmatrix} 0 & A_1^nX_{00}^{-1}A_2^{-n} \\ A_2^nX_{00}A_1^{-n} & 0 \end{pmatrix}$$

with $\sup_{n\geqslant 0}\|A^nYA^{-n}\| \leqslant \max\{\|X\|, \|X^{-1}\|\} < \infty$. By the remarks preceding the proof, Y belongs to the nest algebra $\tau(\mathcal{N}_A)$, where $\mathcal{N}_A$ is the nest generated by the subspaces $E_A[0,\lambda](H_1 \oplus H_2), \lambda \geqslant 0$. Since $\mathcal{N}_A$ consists of subspaces of the form

$$E_{A_1}[0,\lambda]H_1 \oplus E_{A_2}[0,\lambda]H_2 \quad \text{for } \lambda \in \sigma(A_1)$$

or

$$E_{A_1}[0,\lambda)H_1 \oplus E_{A_2}[0,\lambda)H_2 \quad \text{for } \lambda \in \sigma_p(A_1) \cup \sigma_p(A_2),$$

we deduce that

$$X_{00}E_{A_1}[0,\lambda]H_1 = E_{A_2}[0,\lambda]H_2 \quad \text{for all } \lambda \in \sigma(A_1)$$

and

$$X_{00}E_{A_1}[0,\lambda)H_1 = E_{A_2}[0,\lambda)H_2 \quad \text{for all } \lambda \in \sigma_p(A_1) \cup \sigma_p(A_2).$$

In particular, for any $\lambda \in \sigma_p(A_1)$, the compression of X_{00} to $E_{A_1}(\{\lambda\})H_1$ is an invertible operator from $E_{A_1}(\{\lambda\})H_1$ onto $E_{A_2}(\{\lambda\})H_2$. Thus $\dim E_{A_1}(\{\lambda\})H_1 = \dim E_{A_2}(\{\lambda\})H_2$ or, equivalently, $\dim\ker(A_1 - \lambda) = \dim\ker(A_2 - \lambda)$ for any $\lambda \in \sigma_p(A_1)$. By symmetry, the same holds for $\lambda \in \sigma_p(A_2)$. This proves the necessity condition.

To prove the converse, let $\mathcal{N}_j$ be the nest generated by $\{E_{A_j}[0,\lambda]H_j : \lambda \geqslant 0\}$, $j = 1,2$. Define $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ by

$$\theta(E_{A_1}[0,\lambda]H_1) = E_{A_2}[0,\lambda]H_2 \quad \text{for } \lambda \in \sigma(A_1) = \sigma(A_2)$$

and

$$\theta(E_{A_1}[0,\lambda)H_1) = E_{A_2}[0,\lambda)H_2 \quad \text{for } \lambda \in \sigma_p(A_1) = \sigma_p(A_2).$$

By our assumption, θ is a dimension-preserving order isomorphism from $\mathcal{N}_1$ onto $\mathcal{N}_2$. The Similarity Theorem says that there exists an invertible operator X from H_1 onto H_2 which implements θ. Let

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & X^{-1} \\ X & 0 \end{pmatrix}.$$

From above, we have $Y \in \tau(\mathcal{N}_A)$ whence $\sup_{n\geqslant 0}\|A^nYA^{-n}\| < \infty$. Since

$$A^nYA^{-n} = \begin{pmatrix} 0 & A_1^nX^{-1}A_2^{-n} \\ A_2^nXA_1^{-n} & 0 \end{pmatrix},$$

this implies that $\sup_{n\geqslant 0}\|A_2^nXA_1^{-n}\| < \infty$ and $\sup_{n\geqslant 0}\|A_1^nX^{-1}A_2^{-n}\| < \infty$. Let $Z_n = A_2^nXA_1^{-n}$ for $n \geqslant 0$ and

$$Z = \begin{pmatrix} Z_0 & & & 0 \\ & Z_1 & & \\ 0 & & & \ddots \end{pmatrix}.$$

It is easily seen that Z is an invertible operator satisfying $Z(S \otimes A_1) = (S \otimes A_2)Z$. This completes the proof. ■

We remark that the necessity condition of the preceding theorem can also be proved using the results in Section 3, viz., Lemmas 3.2 and 3.3 and Corollary 3.10.

It follows from Theorem 2.1 that similar quasinormal operators may not be unitarily equivalent. However, as the next corollary shows, they are the same in special cases.

COROLLARY 2.3. *For $j = 1, 2$, let $T_j = N_j \oplus (S \otimes A_j)$ be quasinormal, where N_j is normal and $A_j > 0$. Assume further that both A_j's are diagonal. Then T_1 is similar to T_2 if and only if they are unitarily equivalent.*

3. QUASISIMILARITY OF PURE PARTS

In this section, we consider the quasisimilarity of pure quasinormal operators. Our main result is the following

THEOREM 3.1. *Two pure quasinormal operators $S \otimes A$ and $S \otimes B$ with $A, B > 0$ are quasisimilar if and only if the following hold:*

(a) $m(A) = m(B)$ *and* $\dim\ker(A - m(A)) = \dim\ker(B - m(B))$,

(b) $\|A\|_e = \|B\|_e$ *and* $\dim\ker(A - \lambda) = \dim\ker(B - \lambda)$ *for any* $\lambda > \|A\|_e$,

and , in case $\Gamma(A)$ is finite,

(c) $\dim\ker(A - \|A\|_e) = \dim\ker(B - \|B\|_e)$.

Here for any positive definite A, we let $\Gamma(A) = \{\lambda \in \sigma(A) : \lambda > \|A\|_e\}$. The proof of Theorem 3.1 splitted into a series of lemmas and propositions. We start with some conditions on operators T_1 and T_2 for which $XT_1 = T_2X$ has no nontrivial solution.

LEMMA 3.2. *Let T_1 and T_2 be operators on spaces H_1 and H_2, respectively, and X an operator from H_1 to H_2 such that $XT_1 = T_2X$. If either $m(T_2) > \|T_1\|$ or $m(T_1^*) > \|T_2\|$, then $X = 0$.*

Proof. Since $XT_1 = T_2X$ is equivalent to $X^*T_2^* = T_1^*X^*$, we need only consider $m(T_2) > \|T_1\|$. If W is the left inverse of T_2 on H_2 with $\|W\| = m(T_2)^{-1}$, then $WXT_1 = X$ and hence $\|W\|\,\|X\|\,\|T_1\| \geqslant \|X\|$. Therefore $\|X\|\,\|T_1\| \geqslant m(T_2)\|X\|$. Consequently, $X = 0$. ∎

LEMMA 3.3. *Suppose T and $A > 0$ are operators on H and K, respectively, and X is an operator from H to $K^{(\infty)}$ such that $XT = (S \otimes A)X$. If $m(A) = \|T\|$ and $m(A) \notin \sigma_p(A)$, then $X = 0$.*

Proof. For $n = 1, 2, \ldots$, let

$$K_n = E_A([m(A) + \frac{1}{n}, \infty))K, \quad A_n = A|K_n \tag{3.3.1}$$

and P_n be the orthogonal projection from $K^{(\infty)}$ onto $K_n^{(\infty)}$. A simple calculation shows that

$$P_nXT = (S \otimes A_n)P_nX. \tag{3.3.2}$$

Since $m(A_n) \geqslant m(A) + \frac{1}{n} > \|T\|$, from (3.3.2) and Lemma 3.2, we infer that $P_nX = 0$. It is obvious from (3.3.1) and the assumption $m(A) \notin \sigma_p(A)$ that P_n converges to $I_{K^{(\infty)}}$ in the strong operator topology. Thus we have $X = 0$. ∎

Using a similar method, we have the following.

LEMMA 3.4. *Let T and $A > 0$ be operators on H and K, respectively. Suppose $X \in \mathcal{B}(K^{(\infty)}, H)$ and $X(S \otimes A) = TX$. If $\|A\| = m(T)$ and $\|A\| \notin \sigma_p(A)$, then $X = 0$.*

Lemmas 3.2, 3.3 and 3.4 lead us to

COROLLARY 3.5. *Suppose T and V are operators and $A > 0$.*
(a) *Suppose $T \overset{i}{\prec} (S \otimes A) \oplus V$. If $A > \|T\|$, then $T \overset{i}{\prec} V$.*
(b) *Suppose $(S \otimes A) \oplus V \overset{d}{\prec} T$. If $A < m(T)$, then $V \overset{d}{\prec} T$.*

We first consider necessary conditions for injectively similar pure quasinormal operators.

LEMMA 3.6. *If* $S \otimes A \overset{\mathrm{i}}{\prec} S \otimes B$ *where* $A, B > 0$, *then* $m(A) \geqslant m(B)$ *and* $\dim \ker(A - m(B)) \leqslant \dim \ker(B - m(B))$.

Proof. Suppose A acts on Hilbert space H. We first show that $m(A) \geqslant m(B)$. Assume $m(B) > m(A)$. Choose δ such that $m(B) > \delta > m(A)$. Let

$$H_\delta = E_A([m(A), \delta))H \quad \text{and} \quad A_\delta = A|H_\delta.$$

Then $S \otimes A_\delta \overset{\mathrm{i}}{\prec} S \otimes A \overset{\mathrm{i}}{\prec} S \otimes B$. From Corollary 3.5, we infer that $\|A_\delta\| \geqslant m(B)$ which is a contradiction.

Next, to prove that $\dim \ker(A - m(B)) \leqslant \dim \ker(B - m(B))$, it suffices to consider the case $m(A) = m(B) \equiv m > 0$. Let

$$H_1 = \ker(A - m), \ K_1 = \ker(B - m) \text{ and } B_2 = B|K_1^{\perp}.$$

From Corollary 3.5 and $S \otimes mI_{H_1} \overset{\mathrm{i}}{\prec} (S \otimes mI_{K_1}) \oplus (S \otimes B_2)$, it follows that $S \otimes I_{H_1} \overset{\mathrm{i}}{\prec} S \otimes$ $\otimes I_{K_1}$, and hence $\dim H_1 \leqslant \dim K_1$ by [20, Lemma 2]. This completes the proof. ∎

COROLLARY 3.7. *If* $S \otimes A \overset{\mathrm{i}}{\sim} S \otimes B$, *where* $A, B > 0$, *then* $m(A) = m(B)$ *and* $\dim \ker(A - m(A)) = \dim \ker(B - m(B))$.

Now we consider densely similar pure quasinormal operators. The following lemma (whose proof is left to the reader) will be used in the proof of Lemma 3.9.

LEMMA 3.8. *Suppose* $A, B > 0$ *act on* H, K, *respectively. If* $S \otimes A \overset{\mathrm{d}}{\prec} S \otimes B$, *then* $\dim H \geqslant \dim K$.

Recall that if A is a positive definite operator and $\lambda \in \Gamma(A)$, then λ is an isolated eigenvalue with finite multiplicity. Thus if $\Gamma(A) \neq \emptyset$, then we can list its elements, counting multiplicities, in descending order $\{\lambda_j(A)\}_{j=1}^{n(A)}$, $1 \leqslant n(A) \leqslant \aleph_0$.

LEMMA 3.9. *If* $S \otimes A \overset{\mathrm{d}}{\prec} S \otimes B$, *where* $A, B > 0$, *then the following hold:*

(a) $\|A\|_e \geqslant \|B\|_e$.

(b) *If* $\{\lambda_j(B)\}_{j=1}^{m}$ *are the eigenvalues of* B *in* $(\|A\|_e, \infty)$ *in descending order, then*

(b1) $m \leqslant n(A)$ *and* $\lambda_j(A) \geqslant \lambda_j(B)$ *for finite* $j \leqslant m$,

and, in case $n(A) < \infty$,

(b2) $\dim \ker(A - \|A\|_e) \geqslant \dim \ker(B - \|A\|_e) - n(A) + m$.

Proof. Let A and B act on Hilbert spaces H and K, respectively. For any $\varepsilon > 0$, let $H_\varepsilon = E_A([\varepsilon, \infty))H$, $K_\varepsilon = E_B([\varepsilon, \infty))K$, $A_\varepsilon = A|H_\varepsilon$ and $B_\varepsilon = B|K_\varepsilon$. From Corollary 3.5, we infer that $S \otimes A_\varepsilon \overset{\mathrm{d}}{\prec} S \otimes B_\varepsilon$. Thus, by Lemma 3.8,

$$\dim K_\varepsilon \leqslant \dim H_\varepsilon. \tag{3.9.1}$$

(a) Assume $\|B\|_e > \|A\|_e$. Choose λ such that $\|B\|_e > \lambda > \|A\|_e$. Letting $\varepsilon = \lambda$ in (3.9.1), we have $\dim K_\lambda \leqslant \dim H_\lambda$ which contradicts the fact that $\dim K_\lambda = \infty$ and $\dim H_\lambda < \infty$.

(b1) It is obvious from (3.9.1) that $m \leqslant n(A)$. Suppose $\lambda_j(B) > \lambda_j(A)$ for some j. Let $\varepsilon = \lambda_j(B)$. Then $\dim H_\varepsilon \leqslant j-1$ and $\dim K_\varepsilon \geqslant j$, contradicting (3.9.1).

(b2) Let $\varepsilon = \|A\|_e$. Observe that $\dim \ker(A - \|A\|_e) = \dim H_\varepsilon - n(A)$ and $\dim \ker(B - \|A\|_e) = \dim K_\varepsilon - m$. Our assertion follows from (3.9.1). ∎

COROLLARY 3.10. *If $S \otimes A \overset{d}{\sim} S \otimes B$, where $A, B > 0$, then the following hold:*

(a) $\|A\|_e = \|B\|_e$,

(b) $\dim \ker(A - \lambda) = \dim \ker(B - \lambda)$ *for all* $\lambda > \|A\|_e$,

(c) $\sigma_e(S \otimes A) = \sigma_e(S \otimes B)$, and in case $\Gamma(A)$ is finite,

(d) $\dim \ker(A - \|A\|_e) = \dim \ker(B - \|B\|_e)$.

Proof. (a), (b) and (d) follow easily from Lemma 3.9, and (c) is a consequence of (a), (b) and the fact that $\sigma_e(S \otimes A) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \|A\|_e \text{ or } |\lambda| \in \sigma(A)\}$ (cf. [18, Corollary 2]). ∎

COROLLARY 3.11. *Suppose $A > 0$ is compact.*

(a) *If $S \otimes A \overset{d}{\prec} S \otimes B$, then B is also compact.*

(b) *If $S \otimes A \overset{d}{\sim} S \otimes B$, then $A \cong B$.*

Note that (a), (b) and (c) in Corollary 3.10 and (b) in Corollary 3.11 have been obtained before by Williams [18] for arbitrary Hilbert spaces. Although we only consider separable spaces, our proof also works for nonseparable ones.

For quasisimilarity of pure quasinormal operators, we start with

PROPOSITION 3.12. *If $\{c_j\}_{j=1}^n$, $1 \leqslant n \leqslant \infty$, is a sequence of complex numbers with $|c_j| > 1$, then*

(a) $\sum_{j=1}^{n} \oplus c_j S \prec S$

and

(b) $S \oplus \left(\sum_{j=1}^{n} \oplus c_j S\right) \prec S$.

COROLLARY 3.13. *If $0 < \beta < 1$ then $S_\infty \prec \beta S$.*

The above corollary has been obtained by Sz.-Nagy and Foias [16]. The idea of the proof of Proposition 3.12 comes from theirs. We start with some preparatory work.

Recall that a function φ in H^∞ is an *outer function* if M_φ has dense range, where M_φ is the operator on H^2 defined by $M_\varphi(f) = \varphi f$. Let X be a countable infinite subset of the unit circle $\partial\mathbb{D}$ such that the point 1 does not belong to X but is the only accumulation point of X. For any $\alpha > 0$, define

$$g(z) = \alpha \exp\left\{\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log h(t) \frac{dt}{2\pi}\right\}, \quad \text{where } h(t) = \frac{1}{2}\operatorname{dist}(e^{it}, X).$$

Then g is an outer function, continuous on $\overline{\mathbb{D}} \setminus \{1\}$, $\|g\|_\infty \leqslant \alpha$, and $g(z)$ vanishes exactly for $z \in X$ (cf. [9, Section II. 4]). For $n \geqslant 0$, let e_n denote the function $e_n(z) = z^n$ on $\mathbb{D}$.

Proof of Proposition 3.12. We first consider the case $n = \infty$. To begin with, we choose a countable infinite subset Y of the unit circle $\partial\mathbb{D}$ such that the point 1 does not belong to Y but is the only accumulation point of Y. We label the points of Y by two subscripts:

$$Y = \{a_{mn} : 0 \leqslant m,\ n < \infty\},$$

and define, for $0 \leqslant j < \infty$,

$$Y_j = \{a_{jn} : 0 \leqslant n < \infty\} \quad \text{and } X_j = Y \setminus Y_j.$$

Note that $Y_k \subset X_j$ if $j \neq k$.

Next choose a sequence of positive numbers α_j, $0 \leqslant j < \infty$, such that $\sum_{j=0}^{\infty} \alpha_j^2 (1 - -|c_j|^{-2})^{-1} \leqslant 1$. By the remarks preceding the proof, we have a sequence of functions g_j, $0 \leqslant j < \infty$, which are continuous on $\overline{\mathbb{D}} \setminus \{1\}$, analytic on $\mathbb{D}$ and satisfy the following conditions:

(3.12.1) $\|g_j\|_\infty \leqslant \alpha_j$.

(3.12.2) $g_j(z) = 0$ if and only if $z \in X_j$.

(3.12.3) For every $\lambda \in X_j$, $g_j(z) = (z - \lambda)\varphi(z)$ for some φ in H^∞.

(3.12.4) g_j is an outer function.

Observe that for any function u in H^2 we have

$$u\left(\frac{z}{c}\right) = \int_{-\pi}^{\pi} e^{it} u(e^{it}) \left(e^{it} - \frac{z}{c}\right)^{-1} \frac{dt}{2\pi} \quad \text{for } |z| \leqslant 1,\ |c| > 1.$$

Hence

$$\left|u\left(\frac{z}{c}\right)\right| \leqslant \left(\int_{-\pi}^{\pi} |u(e^{it})|^2 \frac{dt}{2\pi}\right)^{\frac{1}{2}} \left(\int_{-\pi}^{\pi} \left|e^{it} - \frac{z}{c}\right|^{-2} \frac{dt}{2\pi}\right)^{\frac{1}{2}} =$$

$$= \|u\|_2 \left(1 - \left|\frac{z}{c}\right|^2\right)^{-\frac{1}{2}} \leqslant \|u\|_2 (1 - |c|^{-2})^{-\frac{1}{2}}.$$

We infer that if $u = \sum_{j=1}^{\infty} \oplus u_j \in H^2_\infty$, then the series

$$\sum_{j=1}^{\infty} g_j(z) u_j^{(c_j)}(z)$$

converges uniformly on $\overline{\mathbb{D}} \setminus \{1\}$ and also in $\mathcal{L}^2$-norm, where $u_j^{(c_j)}(z) = u_j\left(\frac{z}{c_j}\right)$. Indeed, letting $d_j = (1 - |c_j|^{-2})^{-\frac{1}{2}}$, we have

$$|g_j(z) u_j^{(c_j)}(z)| \leqslant \alpha_j d_j \|u_j\|_2, \quad \|g_j u_j^{(c_j)}\|_2 \leqslant \alpha_j d_j \|u_j\|_2$$

and

$$\sum_{j=1}^{\infty} \alpha_j d_j \|u_j\|_2 \leqslant \left(\sum_{j=1}^{\infty} (\alpha_j d_j)^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \|u_j\|_2^2\right)^{\frac{1}{2}} \leqslant \|u\|_2.$$

Thus the operator T from H^2_∞ to H^2 defined by

$$Tu = \sum_{j=1}^{\infty} g_j u_j^{(c_j)}$$

is such that $\|T\| \leqslant 1$. Moreover,

$$T\left(\sum_{j=1}^{\infty} \oplus c_j S\right) u = T\left(\sum_{j=1}^{\infty} \oplus (c_j e_1 u_j)\right) = \sum_{j=1}^{\infty} e_1 g_j u_j^{(c_j)} = STu,$$

that is, $T\left(\sum_{j=1}^{\infty} \oplus c_j S\right) = ST$.

We assert that T is an injection, that is, $\sum_{j=1}^{\infty} g_j u_j^{(c_j)} = 0$ for $u \in H^2_\infty$ implies $u = 0$. To prove this, let $f(z) = \sum_{j=1}^{\infty} g_j(z) u_j^{(c_j)}(z)$ for $z \in \overline{\mathbb{D}} \setminus \{1\}$. Then f is a continuous function and our hypothesis says $f \equiv 0$. In particular, $f = 0$ on Y_j, and hence, by (3.12.2), $u_j\left(\frac{z}{c_j}\right) = 0$ for all $z \in Y_j$. Thus the analytic function u_j vanishes on an infinite set which has an accumulation point inside the unit circle. Therefore it vanishes identically: $u_j \equiv 0$. Since j is arbitrary, we have $u = 0$ as asserted.

Note that the range of T contains the function $e_n g_1$ $(0 \leqslant n < \infty)$: $e_n g_1 = T(c_1^n e_n \oplus 0 \oplus \ldots)$. Since g_1 is an outer function by (3.12.4), these functions span H^2. Thus T is a quasiaffinity completing the proof of (a).

To prove (b), let T and g_0 be as above, and define, for $v \in H^2$ and $u \in H^2_\infty$,

$$X(v \oplus u) = M_{g_0} v - Tu.$$

Clearly, X is an operator in $\mathcal{B}(H^2 \oplus H^2_\infty, H^2)$ with dense range. A simple calculation shows that $X(S \oplus \sum_{j=1}^{\infty} \oplus c_j S) = SX$.

We assert that X is injective, that is,

$$g_0 v - \sum_{j=1}^{\infty} g_j u_j^{(c_j)} = 0 \quad \text{for } v \in H^2 \text{ and } u \in H^2_\infty \text{ implies } v = 0 \text{ and } u = 0.$$

Suppose $v \neq 0$. Then $u_k^{(c_k)} \neq 0$ for some $k \geqslant 1$. Let λ in Y_k be such that $u_k^{(c_k)}(\lambda) \neq 0$. Our hypothesis says that $g_0(z)v(z) = \sum_{j=1}^{\infty} g_j(z) u_j^{(c_j)}(z) \equiv f(z)$ for z in $\mathbf{D}$. Since, by (3.12.3), $g_0(z) = (z-\lambda)\varphi(z)$ for some φ in H^∞, we deduce that $\varphi(z)v(z) = f(z)(z-\lambda)^{-1}$ for all z in $\mathbf{D}$. Thus $f(z)(z-\lambda)^{-1}$ is a function in H^2, which contradicts the fact that

$$\lim_{z \to \lambda} f(z) = f(\lambda) = g_k(\lambda) u_k\left(\frac{\lambda}{c_k}\right) \neq 0.$$

Hence $v = 0$, and the above proof of (a) shows that $u = 0$. This completes the proof for $n = \infty$.

For finite n, it is obvious that if we restrict T and X to the subspaces H^2_n of H^2_∞ and $H^2 \oplus H^2_n$ of $H^2 \oplus H^2_\infty$, respectively, then the resulting operators T_n and X_n will be quasiaffinities satisfying $T_n\left(\sum_{j=1}^{n} \oplus c_j S\right) = ST_n$ and $X_n\left(S \oplus \sum_{j=1}^{n} \oplus c_j S\right) = SX_n$. ∎

LEMMA 3.14. *Suppose N is a normal operator on an infinite-dimensional space H and T is an operator on K.*

(a) *If T has dense range and $m(N) \geqslant \|T\|$, then $S \otimes N \prec (S \otimes N) \oplus (S \otimes T)$.*

(b) *If T and N are both invertible and $m(T) \geqslant \|N\|$, then $(S \otimes N) \oplus (S \otimes T) \prec \prec S \otimes N$.*

Proof. (a) Note that $m(N) \geqslant \|T\| > 0$ implies that N is invertible. Since N is normal and $\dim H = \infty$, there exist, by [10, Problem 142], infinite-dimensional reducing subspaces H_j, $j = 1, 2 \ldots$, for N such that $H = \sum_{j=1}^{\infty} \oplus H_j$. Let U_j be a coisometry from H_j onto K. For each $n \geqslant 1$ and $f = \sum_{j=1}^{\infty} \oplus f_j \in H$, define

$$X_0\left(\sum_{j=1}^{\infty} \oplus f_j\right) = \sum_{j=1}^{\infty} \oplus \left(\frac{1}{3}\right)^{j-1} f_j, \tag{3.14.1}$$

(3.14.2) $$Y_1\left(\sum_{j=1}^{\infty}\oplus f_j\right)=\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j-1}U_jf_j \text{ and } Y_{n+1}=TY_nN^{-1}.$$

Clearly, $X=X_0^{(\infty)}$ is a positive definite operator on $H^{(\infty)}$ and $X(S\otimes N)=(S\otimes N)X$. On the other hand, observe that

$$\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j-1}\|U_jf_j\|\leqslant\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j-1}\|f_j\|\leqslant$$

$$\leqslant\left(\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{2j-2}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\|f_j\|^2\right)^{\frac{1}{2}}=\left(\frac{4}{3}\right)^{\frac{1}{2}}\|f\|,$$

whence Y_1 is bounded. Also,

$$\|Y_{n+1}\|\leqslant\|T\|\,\|Y_n\|\,\|N^{-1}\|=\|T\|\,\|Y_n\|m(N)^{-1}\leqslant\|Y_n\|.$$

Thus $Y=\sum_{n=1}^{\infty}\oplus Y_n$ is an operator from $H^{(\infty)}$ to $K^{(\infty)}$ satisfying $Y(S\otimes N)=(S\otimes T)Y$.

Define the operator Z from $H^{(\infty)}$ to $H^{(\infty)}\oplus K^{(\infty)}$ by $Zf=Xf\oplus Yf$ for $f\in$ $\in H^{(\infty)}$. A simple calculation shows that $(S\otimes N)Z=Z((S\otimes N)\oplus(S\otimes T))$. Since X is injective, so is Z. We assert that Z has dense range, that is,

(3.14.3)

$$Z^*(f\oplus g)=X^*f+Y^*g=0 \text{ for } f=\sum_{n=1}^{\infty}\oplus f_n\in H^{(\infty)} \text{ and } g=\sum_{n=1}^{\infty}\oplus g_n\in K^{(\infty)}$$

$$\text{implies } f=0 \text{ and } g=0.$$

From (3.14.1) and (3.14.2), (3.14.3) is equivalent to

(3.14.4) $$X_0^*f_n+Y_n^*g_n=0 \text{ implies } f_n=0 \text{ and } g_n=0 \text{ for any } n.$$

To prove (3.14.4), observe that N is invertible and T^* is injective. Thus it is sufficient to show that

(3.14.5) $X_0^*h+Y_1^*k=0$ for $h=\sum_{j=1}^{\infty}\oplus h_j\in H$ and $k\in K$ implies $h=0$ and $k=0$.

Since $Y_1^*k=\sum_{j=1}^{\infty}\oplus\left(\frac{1}{2}\right)^{j-1}U_j^*k$, our hypothesis in (3.14.5) leads to

$$0=\left(\frac{1}{3}\right)^{j-1}h_j+\left(\frac{1}{2}\right)^{j-1}U_j^*k \quad \text{for all } j.$$

This implies $\left(\frac{3}{2}\right)^{j-1}\|k\| = \|h_j\|$. Hence $h_j = 0$ for all j and $k = 0$. This completes the proof of (a).

Since the proof of (b) is similar to that of (a), we only sketch it and leave the details to the reader. Let X be as in the proof of (a). Define $W = \sum_{n=1}^{\infty} \oplus W_n$ from $K^{(\infty)}$ to $H^{(\infty)}$ by

$$W_1 f = \sum_{n=1}^{\infty} \oplus \left(\frac{1}{2}\right)^{n-1} U_n^* f \quad \text{and} \quad W_{n+1} = N W_n T^{-1} \text{ for } n \geqslant 1.$$

Let $V(f \oplus g) = Xf + Wg$ for f in $H^{(\infty)}$ and g in $K^{(\infty)}$. It is easy to check that V is a quasiaffinity and $V((S \otimes N) \oplus (S \otimes T)) = (S \otimes N)V$. ■

PROPOSITION 3.15. *If $A, B > 0$ satisfy any of the following conditions*

(a) $\|A\|_e > \|B\|$,

(b) $\|A\|_e = \|B\|$ *and* $\|B\| \notin \sigma_p(B)$,

(c) $\|A\|_e = \|B\|$ *and* dim ker$(A - \|A\|_e) = \infty$,

then $S \otimes A \prec (S \otimes A) \oplus (S \otimes B)$.

Proof. Suppose A and B act on the spaces H and K, respectively. We shall find a reducing subspace M for A such that

$$S \otimes (A|M) \prec (S \otimes (A|M)) \oplus (S \otimes B). \tag{3.15.1}$$

(a) In this case, let $M = E_A([\|B\|, \infty))H$. Then dim $M = \infty$ and $m(A|M) \geqslant \|B\|$. (3.15.1) follows from Lemma 3.14.

(b) Let $[m(B), \|B\|) = \bigcup_{n=1}^{\infty} [b_n, b_{n+1})$ where $\{b_n\}$ is a sequence of strictly increasing positive numbers satisfying $K_n = E_B([b_n, b_{n+1}))K \neq \{0\}$. Let $B_n = B|K_n$. Choose mutually orthogonal infinite-dimensional reducing subspaces H_n, $n = 1, 2, \ldots$, for A such that $m(A|H_n) \geqslant \|B|K_n\|$. Again, from Lemma 3.14, we have

$$S \otimes (A|H_n) \prec (S \otimes (A|H_n)) \oplus (S \otimes (B|K_n)) \quad \text{for each } n \geqslant 1.$$

Let $M = \sum_{n=1}^{\infty} \oplus H_n$. Clearly, (3.15.1) holds.

(c) It is easy to see that $M \equiv \ker(A - \|A\|_e)$ satisfies (3.15.1). ■

The next proposition generalizes Proposition 3.12.

PROPOSITION 3.16. *If $A, B > 0$ satisfy either*

(a) $m(A) > m(B)$ *or*

(b) $m(B) = m(A)$ *and* $m(A) \notin \sigma_p(A)$, *then* $(S \otimes A) \oplus (S \otimes B) \prec S \otimes B$.

To prove the above proposition, we need the following lemma.

LEMMA 3.17. *If* $A > 0$ *acts on the space* H *and* $m(A) > 0$, *then* $\|A\|(S \otimes I_H) \prec \prec S \otimes A \prec m(A)(S \otimes I_H)$.

Proof. Let $X = \sum_{n=1}^{\infty} \oplus (m(A)A^{-1})^{n-1}$ and $Y = \sum_{n=1}^{\infty} \oplus \left(\frac{A}{\|A\|}\right)^{n-1}$ on $H^{(\infty)}$. It is easy to see that X and Y are quasiaffinities. A simple calculation shows that $X(S \otimes A) = (S \otimes m(A)I_H)X$ and $Y(S \otimes \|A\|I_H) = (S \otimes A)Y$. ■

Proof of Proposition 3.16. Assume that A and B act on spaces H and K, respectively. We shall find a reducing subspace M for B such that

$$(S \otimes A) \oplus (S \otimes (B|M)) \prec S \otimes (B|M). \tag{3.16.1}$$

(a) Assume $m(A) > m(B)$. If $\ker(B - \alpha) \neq \{0\}$ for some α in $[m(B), m(A)]$, let $M = \ker(B - \alpha)$. Then, from Lemma 3.17 and Proposition 3.12, we have

$$(S \otimes A) \oplus (S \otimes (B|M)) \prec (S \otimes m(A)) \oplus (S \otimes (B|M)) \prec S \otimes (B|M).$$

Otherwise, choose $\beta > 0$ such that $M \equiv E_B([\beta, m(A)])K$ has infinite dimension. Since $\|B|M\| \leqslant m(A)$, Lemma 3.14 leads to (3.16.1).

(b) Assume $m(B) = m(A) \notin \sigma_p(A)$. Choose $\{a_n\}$ such that $\|A\| = a_1 > a_2 > > \cdots > a_n > \cdots$ and $(m(A), \|A\|] = \bigcup_{n=1}^{\infty} (a_{n+1}, a_n]$. Let $H_n = E_A((a_{n+1}, a_n])H$ and $A_n = A|H_n$. If $\ker(B - m(B)) \neq \{0\}$, let $M = \ker(B - m(B))$. Then, from Lemma 3.17 and Proposition 3.12, we have

$$(S \otimes A) \oplus (S \otimes (B|M)) \cong \left(\sum_{n=1}^{\infty} \oplus (S \otimes A_n)\right) \oplus (S \otimes (B|M)) \prec$$

$$\prec \left(\sum_{n=1}^{\infty} \oplus (S \otimes m(A_n)I_{H_n})\right) \oplus (S \otimes (B|M)) \prec S \otimes (B|M).$$

Otherwise, there exist mutually orthogonal reducing subspaces K_n for B such that if $B_n = B|K_n$ then $m(B_n) < m(A_n)$. Thus (a) implies that $(S \otimes A_n) \oplus (S \otimes B_n) \prec \prec S \otimes B_n$, and hence $M = \sum_{n=1}^{\infty} \oplus K_n$ also satisfies (3.16.1). ■

Proof of Theorem 3.1. The necessity follows from Corollaries 3.7 and 3.10.

To prove the sufficiency, assume that A and B act on spaces H and K, respectively. Let

$$H_0 = \ker(A - m(A)),\ H_1 = E_A((m(A), \|A\|_e])H,\ H_2 = E_A((\|A\|_e, \|A\|])H,$$

$$K_0 = \ker(B - m(B)),\ K_1 = E_B((m(B), \|B\|_e])K,\ K_2 = E_B((\|B\|_e, \|B\|])K,$$

and

$$A_j = A|H_j,\ B_j = B|K_j,\ j = 0, 1, 2.$$

Observe, from Proposition 3.16, that

$$(S \otimes A_0) \oplus (S \otimes A_1) \prec S \otimes A_0 \cong S \otimes B_0 \quad \text{if } H_0 \neq \{0\}, \tag{3.1.1}$$

and

$$(S \otimes A_1) \oplus (S \otimes B_1) \prec S \otimes B_1 \quad \text{if } H_0 = \{0\}. \tag{3.1.2}$$

Assume $\Gamma(A)$ is infinite. From Lemma 3.14, we have

$$S \otimes A_2 \cong S \otimes B_2 \prec (S \otimes B_1) \oplus (S \otimes B_2). \tag{3.1.3}$$

Then (3.1.1) and (3.1.3) lead to

$$S \otimes A \cong (S \otimes A_0) \oplus (S \otimes A_1) \oplus (S \otimes A_2) \prec$$

$$\prec (S \otimes B_0) \oplus (S \otimes B_1) \oplus (S \otimes B_2) \cong S \otimes B \quad \text{if } H_0 \neq \{0\},$$

and (3.1.2) and (3.1.3) lead to

$$S \otimes A \cong (S \otimes A_1) \oplus (S \otimes A_2) \prec (S \otimes A_1) \oplus (S \otimes B_1) \oplus (S \otimes B_2) \prec$$

$$\prec (S \otimes B_1) \oplus (S \otimes B_2) \cong S \otimes B \quad \text{if } H_0 = \{0\}.$$

On the other hand, if $\Gamma(A)$ is finite, then we can easily reduce our analysis to the case when $H_2 = K_2 = \{0\}$ and either $\dim \ker(A - \|A\|_e) = \dim \ker(B - \|B\|_e) = 0$ or $\dim \ker(A - \|A\|_e) = \dim \ker(B - \|B\|_e) = \infty$. From Proposition 3.15, we have

$$S \otimes A_1 \prec (S \otimes A_1) \oplus (S \otimes B_1). \tag{3.1.4}$$

Then (3.1.1) and (3.1.4) lead to

$$S \otimes A \cong (S \otimes A_0) \oplus (S \otimes A_1) \prec (S \otimes A_0) \oplus (S \otimes A_1) \oplus (S \otimes B_1) \prec$$

$$\prec (S \otimes B_0) \oplus (S \otimes B_1) \cong S \otimes B \quad \text{if } H_0 \neq \{0\},$$

and (3.1.2) and (3.1.4) lead to

$$S \otimes A \cong S \otimes A_1 \prec (S \otimes A_1) \oplus (S \otimes B_1) \prec S \otimes B_1 \cong S \otimes B \quad \text{if } H_0 = \{0\}.$$

Thus we conclude that $S \otimes A \prec S \otimes B$. By symmetry, the proof is complete. ■

Using similar methods, we can prove the following result.

THEOREM 3.18. *Suppose $T_1 = S \otimes A$ and $T_2 = S \otimes B$, where $A, B > 0$. Then*

(a) $T_1 \overset{i}{\prec} T_2$ *if and only if* $m(A) \geqslant m(B)$ *and* $\dim\ker(A - m(B)) \leqslant \dim\ker(B - m(B))$.

(b) $T_1 \overset{i}{\sim} T_2$ *if and only if* $m(A) = m(B)$ *and* $\dim\ker(A - m(A)) = \dim\ker(B - m(B))$.

(c) $T_1 \overset{d}{\prec} T_2$ *if and only if the following hold:*

(c1) $\|A\|_e \geqslant \|B\|_e$.

(c2) *If* $\{\lambda_j(B)\}_{j=1}^m$ *are the eigenvalues of* B *in* $(\|A\|_e, \infty)$ *arranged in descending order, then* $m \leqslant n(A)$ *and* $\lambda_j(B) \leqslant \lambda_j(A)$ *for any finite* $j \leqslant m$.

(c3) *If* $n(A) < \infty$, *then* $\dim\ker(B - \|A\|_e) - n(A) + m \leqslant \dim\ker(A - \|A\|_e)$.

(d) $T_1 \overset{d}{\sim} T_2$ *if and only if the following hold:*

(d1) $\|A\|_e = \|B\|_e$.

(d2) $\dim\ker(A - \lambda) = \dim\ker(B - \lambda)$ *for* $\lambda > \|A\|_e$.

(d3) *If* $\Gamma(A)$ *is finite, then* $\dim\ker(A - \|A\|_e) = \dim\ker(B - \|B\|_e)$.

(e) $T_1 \prec T_2$ *if and only if* $T_1 \overset{i}{\prec} T_2$ *and* $T_1 \overset{d}{\prec} T_2$.

In particular, the preceding theorem generalizes the corresponding result for unilateral shifts (cf. [20, Lemma 2] and [5, Proposition 1.4]).

To end this section, we give the following corollaries.

COROLLARY 3.19. *For every pure quasinormal operator T, there exists a diagonal positive definite operator D such that T is quasisimilar to $S \otimes D$.*

COROLLARY 3.20. *Suppose $T_1 = S \otimes A$ and $T_2 = S \otimes B$, where $A, B > 0$. Then the following hold:*

(a) *Either* $T_1 \overset{i}{\prec} T_2$ *or* $T_2 \overset{i}{\prec} T_1$.

(b) $\{T_1 \oplus T_2\}' \neq \{T_1\}' \oplus \{T_2\}'$.

(c) $XT_1 = T_2X$ *has no nontrivial solution if and only if*

(i) $m(B) > \|A\|$,

(ii) $m(B) = \|A\| \notin \sigma_p(A)$, *or*

(iii) $m(B) = \|A\| \notin \sigma_p(B)$.

Proof. That (a) follows from Theorem 3.18 (a) and that (b) from (a) are immediate. As for (c), one direction is proved in Lemmas 3.2, 3.3 and 3.4. To prove the other, assume that $m(B) < \|A\|$ or $m(B) = \|A\| \in \sigma_p(A) \cap \sigma_p(B)$. In either case, we shall find nontrivial reducing subspaces L and M for A and B, respectively, such that

$$S \otimes (A|L) \overset{i}{\prec} S \otimes (B|M). \tag{3.20.1}$$

Suppose that A and B act on H and K, respectively. If $m(B) < \|A\|$, let $L = E_A((c, \|A\|])H$ and $M = K$, where $m(B) < c < \|A\|$. Then $m(A|L) > m(B)$ and (3.20.1) follows from Proposition 3.16. For the case $m(B) = \|A\| \in \sigma_p(A) \cap \sigma_p(B)$, let L and M be any one-dimensional subspace of $\ker(A - \|A\|)$ and $\ker(B - m(B))$, respectively. Clearly, (3.20.1) also holds. ∎

4. QUASISIMILARITY IN GENERAL

We start with a well-known result on normal operators (cf. [8, Lemma 4.1]).

LEMMA 4.1. *The following are equivalent for two normal operators* N_1 *and* N_2*:*
(a) $N_1 \overset{\mathrm{i}}{\sim} N_2$;
(b) $N_1 \overset{\mathrm{d}}{\sim} N_2$;
(c) $N_1 \sim N_2$;
(d) $N_1 \approx N_2$;
(e) $N_1 \cong N_2$.

The next lemma slightly generalizes [18, Theorem 2] with a simple proof. Note that a more general result is true with $S \otimes A$ replaced by any pure hyponormal operator (cf. [17]).

LEMMA 4.2. *Suppose* T *is cohyponormal on* $H (TT^* \geqslant T^*T)$ *and* A *is an injection on* K. *If* $X \in \mathcal{B}(H, K^{(\infty)})$ *and* $XT = (S \otimes A)X$, *then* $X = 0$.

Proof. Let P_n be the projection from $K^{(\infty)}$ onto $\{0\} \oplus \ldots \oplus \{0\} \oplus \underset{nth}{K} \oplus \{0\} \oplus \ldots$ and $X_n = P_n X$. A simple calculation shows that

$$X_1 T = 0 \quad \text{and} \quad X_{n+1} T = A X_n \ \text{ for } n \geqslant 1.$$

Using induction, we have $X_{n+1} T^n = A^n X_1$. Thus $X_{n+1} T^{n+1} = 0$. Since T is cohyponormal, $\operatorname{ran} T$ and $\operatorname{ran} T^{n+1}$ have the same closure. It follows that $0 = X_{n+1} T = AX_n$ for $n \geqslant 1$. The injectivity of A implies that $X_n = 0$ for all n. Therefore, $X = 0$. ∎

An immediate consequence of Lemmas 4.1 and 4.2 is

COROLLARY 4.3. *Suppose* N_1 *and* N_2 *are normal and* $A, B > 0$. *If* $T_1 = N_1 \oplus (S \otimes A)$ *and* $T_2 = N_2 \oplus (S \otimes B)$ *are quasisimilar, then* $N_1 \cong N_2$ *and* $S \otimes A \overset{\mathrm{d}}{\sim} S \otimes B$.

It follows from the preceding corollary and Corollary 3.10 that quasisimilar quasinormal operators have equal essential spectra, the main result in [18]. It also suggests that to study the problem when two quasinormal operators T_1 and T_2 are quasisimilar, it suffices to consider that

$$(*) \qquad T_1 = N \oplus (S \otimes A) \text{ and } T_2 = N \oplus (S \otimes B),$$

where N is normal, $A, B > 0$, and $S \otimes A \overset{d}{\sim} S \otimes B$.

Before stating the principal results, a few terminology and notations are needed.

Any contraction can be decomposed as $U_s \oplus U_a \oplus T$, where U_s and U_a are singular and absolutely continuous unitary operators and T is completely nonunitary (c.n.u.), that is, T has no nontrivial unitary direct summand. For a c.n.u. contraction T and $f \in H^\infty$, the functional calculus $f(T)$ is well-defined. We say that c.n.u. contraction T is *of class* C_0 if there is a nonzero f in H^∞ such that $f(T) = 0$. For these and other properties of contractions, readers are referred to [15].

Let N be a normal operator acting on the space H and $E(\cdot)$ be its spectral measure. Then N can be decomposed as a direct sum

$$(**) \qquad N = N_0 \oplus N_a \oplus N_s \oplus N_1$$

on $H = H_0 \oplus H_a \oplus H_s \oplus H_1$, where $N_0 = N|E(\mathbf{D})H$, $N_1 = N|E(\mathbf{C} \setminus \overline{\mathbf{D}})H$, $N_a \oplus N_s = N|E(\partial\mathbf{D})H$, and N_a and N_s are absolutely continuous and singular unitary operators. Note that N_0 is a c.n.u. contraction and some of these summands may not appear.

Our main result is the following.

THEOREM 4.4. *Let $T_1 = N \oplus (S \otimes A)$ and $T_2 = N \oplus (S \otimes B)$ be as in $(*)$. Let $\alpha = \min(m(A), m(B))$ and $d = \max(\dim\ker(A - \alpha), \dim\ker(B - \alpha))$. Then T_1 and T_2 are quasisimilar if and only if one of the following conditions holds:*

(a) $S \otimes A \overset{i}{\sim} S \otimes B$;

(b) $d = 0$ *and* $\sigma(N)$ *has a limit point in* $\alpha\overline{\mathbf{D}}$;

(c) $d > 0$ *and the absolutely continuous unitary part of* $(1/\alpha)N$ *does not vanish*;

(d) $d > 0$ *and the c.n.u. part of* $(1/\alpha)N$ *is not of class* C_0.

Note that Theorem 4.4 implies that a quasinormal operator quasisimilar to an isometry is actually unitarily equivalent to it (cf. [21, Proposition 4.2]).

We split the proof of Theorem 4.4 into lemmas and propositions. The next two propositions take care of c.n.u. normal N. Recall that an operator T is *algebraic* if $p(T) = 0$ for some nonzero polynomial p.

PROPOSITION 4.5. *If N is normal, nonalgebraic and $\|N\| < 1$, then $S_n \oplus N \prec N$ for any n, $1 \leqslant n \leqslant \infty$.*

Proof. By the spectral theorem for normal operators, it is sufficient to assume that $N = N_\mu$, where μ is a regular Borel measure on the complex plane $\mathbf{C}$ with compact support K and $N_\mu(g)(z) = zg(z)$ for $g \in \mathcal{L}^2(\mu)$.

Since N is nonalgebraic, $K = \sigma(N)$ has a limit point, say, z_0 in $\mathbb{D}$. Choose positive numbers c_m, $m = 0, 1, \ldots$, decreasing to 0 such that the sets $D_0 \equiv \{z : z = z_0 \text{ or } |z - z_0| > c_0\}$ and $D_m \equiv \{z : c_m < |z - z_0| \leqslant c_{m-1}\}$ all have positive measure. Let $f = \sum_{m=0}^{\infty} (\mu(D_m))^{\frac{1}{2}} \chi_{D_m} \exp\left(\frac{-1}{c_m}\right)$. Then f is a positive bounded measurable function. Observe that if $\varphi \in H^2$ and $z \in K$ then

$$\varphi(z) = \int_{-\pi}^{\pi} e^{it}\varphi(e^{it})(e^{it} - z)^{-1} \frac{dt}{2\pi},$$

and hence

$$|\varphi(z)| \leqslant \|\varphi\|_2 (1 - |z|^2)^{-\frac{1}{2}}.$$

We infer that $\int |\varphi|^2 d\mu \leqslant a\mu(K)\|\varphi\|_2^2$, where $a = \max\{(1 - |z|^2)^{-1} : z \in K\}$. Thus

$$X(\varphi \oplus g) = \varphi + fg \tag{4.5.1}$$

is an operator from $H^2 \oplus \mathcal{L}^2(\mu)$ to $\mathcal{L}^2(\mu)$. Obviously, $X(S \oplus N) = NX$.

We assert that X is an injection, that is,

$$\varphi + fg = 0 \text{ for } \varphi \in H^2 \text{ and } g \in \mathcal{L}^2(\mu) \text{ implies } \varphi = 0 \text{ and } g = 0. \tag{4.5.2}$$

If $\varphi \neq 0$, write $\varphi(z) = (z - z_0)^n Q(z)$, where $n \geqslant 0$ and Q is an analytic function on $\mathbb{D}$ with $Q(z_0) \neq 0$. Choose $\delta > 0$ such that $|Q(z)| \geqslant \delta$ on some neighborhood of z_0. Our hypothesis in (4.5.2) implies that

$$\|g\|^2 = \int \left|\frac{\varphi}{f}\right|^2 d\mu \geqslant \int_{D_m} \left|\frac{\varphi}{f}\right|^2 d\mu \geqslant \int_{D_m} \delta^2 |z - z_0|^{2n} |f(z)|^{-2} d\mu(z) \geqslant$$

$$\geqslant \delta^2 c_m^{2n} \mu(D_m) \frac{\exp\left(\frac{2}{c_m}\right)}{\mu(D_m)} = \delta^2 c_m^{2n} \exp\left(\frac{2}{c_m}\right),$$

which is a contradiction for sufficiently large m. Thus $\varphi = 0$ and hence $g = 0$.

From (4.5.1) we see that the range of X contains fg for any $g \in \mathcal{L}^2(\mu)$. Since f is positive and bounded, these functions are dense in $\mathcal{L}^2(\mu)$. Thus X is a quasiaffinity. This proves our assertion for $n = 1$. Using induction, we have $S_n \oplus N \prec N$ for any finite n. If $n = \infty$, choose β such that $1 > \beta > \|N\|$, then $S \oplus \left(\frac{N}{\beta}\right) \prec \frac{N}{\beta}$ and $S_\infty \prec \beta S$ (Corollary 3.13) lead us to $S_\infty \oplus N \prec \beta S \oplus N \prec N$. ∎

PROPOSITION 4.6. *If N is a normal operator such that $\sigma(N) \cap \mathbb{D} = \{a_k\}$ satisfies $\sum_k (1 - |a_k|) = \infty$, then $S_n \oplus N \prec N$ for any n, $1 \leqslant n \leqslant \infty$.*

In order to prove the above proposition, we need the following two lemmas.

LEMMA 4.7. *If $\Delta \subset \mathbf{D}$ is a compact neighborhood of $a \in \mathbf{D}$ and $k > 0$, then there exists a universal constant $\delta(a, \Delta, k) > 0$ such that*

$$|B(a)| \geqslant \delta(a, \Delta, k)$$

for any Blaschke product B with zeros $\{z_n\}$ satisfying

$$\sum_n (1 - |z_n|) \leqslant k \quad \text{and} \quad \{z_n\} \cap \Delta = \emptyset.$$

Proof. Choose $\varepsilon > 0$ such that Δ includes a closed disc centered at a of radius ε. Let τ be the greatest integer les[illegible] $2k/(1-|a|)$. Our assumption $\sum_n (1-|z_n|) \leqslant k$ implies that $(1+|a|)/2 > |z_n|$ [illegible]ost τ indices n. Because

$$\frac{|z_n - a|}{|1 - \bar{z}_n a|} \geqslant \frac{|z_n| - |a|}{1 - |z_n||a|} \;\text{[illegible]}\; + 4\frac{1 - |z_n|}{1 - |a|}\Bigg)^{-1} \geqslant \exp\left(-4\frac{1 - |z_n|}{1 - |a|}\right)$$

for $|z_n| \geqslant (1 + |a|)/2$, a straightforward computation shows that

$$|B(a)| = \prod_n \frac{|z_n - a|}{|1 - \bar{z}_n a|} \geqslant \left(\frac{\varepsilon}{2}\right)^\tau \exp\left[-\frac{4}{1 - |a|}\sum_n (1 - |z_n|)\right] \geqslant$$

$$\geqslant \left(\frac{\varepsilon}{2}\right)^\tau \exp\left(-\frac{4k}{1 - |a|}\right).$$

Thus $\delta(a, \Delta, k) = (\varepsilon/2)^\tau \exp[-4k/(1 - |a|)]$ satisfies

$$|B(a)| \geqslant \delta(a, \Delta, k)$$

completing the proof. ∎

LEMMA 4.8. *Let $\{a_k\}$ be a sequence of distinct points in $\mathbf{D}$ such that $\sum_k (1 - |a_k|) = \infty$ and $\{a_k\}$ only clusters on $\partial\mathbf{D}$. Then there exists a bounded sequence $\{b_k\}$ of positive numbers such that*

$$\limsup_{k \to \infty} |B(a_k)|/kb_k \geqslant 1$$

for any Blaschke product B.

Proof. Let Δ_k be the closed disc centered at a_k of radius $\frac{1}{3}\operatorname{dist}(a_k, \partial\mathbf{D} \cup \{a_h\}_{h \neq k})$. It is immediate that

$$\Delta_h \cap \Delta_k = \emptyset \quad \text{if } h \neq k, \tag{4.8.1}$$

and

$$(4.8.2)\qquad \sum_n (1-|z_n|)=\infty \quad \text{if } \{z_n\}\subset \mathbb{D} \text{ and } \{z_n\}\cap \Delta_k\neq\emptyset \text{ for all } k\geqslant k_0.$$

Define

$$b_k=\delta(a_k,\Delta_k,k)/k,$$

where $\delta(a,\Delta,k)$ is the function given by Lemma 4.7.

Let B be a Blaschke product with zeros (counted with multiplicity) $\{z_n\}$. (4.8.1) and (4.8.2) imply that $\{z_n\}\cap\Delta_k=\emptyset$ for infinitely many k. On the other hand, if $\{z_n\}\cap\Delta_k=\emptyset$ and $k\geqslant\sum_n(1-|z_n|)$, then (by Lemma 4.7)

$$|B(a_k)|/b_k\geqslant k.$$

We conclude that

$$\limsup_{k\to\infty}|B(a_k)|/kb_k\geqslant 1. \qquad \blacksquare$$

Proof of Proposition 4.6. Clearly, we can restrict our attention to the case when N is the diagonal normal operator with distinct eigenvalues $\{a_k\}$. We can always decompose $\{a_k\}$ as a disjoint union $\bigcup_{j=1}^{\infty}\{a_{k,j}\}$ in such a way that $\sum_k(1-|a_{k,j}|)=\infty$ for all $j\geqslant 1$, and write $N=\sum_j\oplus N_j$, where N_j has eigenvalues $\{a_{k,j}\}$ $(j=1,2,\ldots)$. Observe that $S\oplus N_j\prec N_j$ for all j imply that $S_\infty\oplus N\prec N$ (the case when $1<n<\infty$ can be similarly analyzed). This reduces the proof to the case $n=1$. The operator N can be identified with the multiplication by the variable on the space $\mathcal{L}^2(\mu)$, where μ is the discrete positive Borel measure defined by $\mu(\{a_k\})=(1-|a_k|^2)/k^2$ $(k\geqslant 1)$ and $\mu(\mathbb{C}\setminus\{a_k\})=0$. Let

$$f(a_k)=b_k\exp[-(1-|a_k|)^{-2}],$$

where $\{b_k\}$ is the sequence given by Lemma 4.8. Trivially, $f\in\mathcal{L}^\infty(\mu)$.

As in the case of Proposition 4.5, we define

$$X(\varphi\oplus g)=\varphi+fg$$

from $H^2\oplus\mathcal{L}^2(\mu)$ to $\mathcal{L}^2(\mu)$. Observe that

$$\int|\varphi|^2\mathrm{d}\mu=\sum_k|\varphi(a_k)|^2\mu(\{a_k\})\leqslant\|\varphi\|_2^2\sum_k(1-|a_k|^2)^{-1}(1-|a_k|^2)/k^2\leqslant$$

$$\leqslant\|\varphi\|_2^2\sum_k 1/k^2,$$

so that X is bounded. Clearly, $\operatorname{ran} X$ contains $\{fg : g \in \mathcal{L}^2(\mu)\}$, which is dense in $\mathcal{L}^2(\mu)$, and $X(S \oplus N) = NX$.

It only remains to show that X is injective. Assume that $\varphi + fg = 0$ for some nonzero $\varphi \in H^2$ and some $g \in \mathcal{L}^2(\mu)$, and let $\varphi = B\Psi$ be the canonical factorization of φ, where B is a Blaschke product and Ψ is in H^2 which never vanishes on $\mathbb{D}$. Then Ψ^{-1} is in the Nevanlinna class, and therefore

$$|\Psi(z)^{-1}| = O(\exp\{c/[1-|z|]\})$$

for some $c > 0$ and $z \in \mathbb{D}$ (cf. [9, Section II.5]). Thus,

$$\lim_{k\to\infty} |\Psi(a_k)|(1-|a_k|^2)^{\frac{1}{2}}/\exp[-(1-|a_k|)^{-2}] = \infty.$$

Since $g \in \mathcal{L}^2(\mu)$, the sequence $\{g(a_k)(1-|a_k|^2)^{\frac{1}{2}}/k\}$ is square-summable, and therefore bounded. But the above observation and Lemma 4.8 indicate that

$$\begin{aligned}
&\limsup_{k\to\infty} |g(a_k)|(1-|a_k|^2)^{\frac{1}{2}}/k = \\
&= \limsup_{k\to\infty}\{|\varphi(a_k)|/f(a_k)\}\cdot(1-|a_k|^2)^{\frac{1}{2}}/k = \\
&= \limsup_{k\to\infty}\{|B(a_k)|/kb_k\}\cdot\{|\Psi(a_k)|(1-|a_k|^2)^{\frac{1}{2}}/\exp[-(1-|a_k|)^{-2}]\} = \infty,
\end{aligned}$$

a contradiction. Thus, $\varphi = 0$ and hence $g = 0$, whence we conclude that X is a quasiaffinity. ∎

LEMMA 4.9. *Suppose N is a normal operator on H with $m(N) = 1$. If $N = N_1$ or $N_s \oplus N_1$ in the decomposition (**), then the equation $XS = NX$ has no nontrivial solution in $\mathcal{B}(H^2, H)$.*

Proof. Let $E(\cdot)$ be the spectral measure of N. For each $n = 1, 2, \ldots$, let $P_n = E\left(\left\{z : 1 \leqslant |z| \leqslant 1 + \frac{1}{n}\right\}\right)$. From Lemma 3.2 and the fact that $m(N|P_n^{\perp}H) \geqslant 1 + \frac{1}{n} > 1$, we have

$$X = P_n X \quad\text{and}\quad P_n XS = NP_n X.$$

Since P_n converges to $P \equiv E(\partial\mathbb{D})$ in the strong operator topology, it follows that

$$X = PX \quad\text{and}\quad PXS = NPX.$$

(a) If $N = N_1$, then $P = 0$ whence $X = PX = 0$.

(b) If $N = N_s \oplus N_1$, then $PXS = N_s PX$. It follows that $X = PX = 0$ (Cf. [8, Theorems 2 and 3]). ∎

The next proposition will be used to settle the cases (c) and (d) in Theorem 4.4.

PROPOSITION 4.10. *Let $N = N_0 \oplus N_a \oplus N_s \oplus N_1$ be as in* (**). *Then the following conditions on N are equivalent:*

(a) *N_a does not vanish or N_0 is not of class C_0;*

(b) *$S_n \oplus N \prec N$ for all n, $1 \leqslant n \leqslant \infty$;*

(c) *$S_n \oplus N \prec N$ for some n, $1 \leqslant n \leqslant \infty$.*

Proof. That (a) implies (b) follows from [12, Corollary 3] and Propositions 4.5 and 4.6. Indeed, in the latter case, if $\sigma(N_0)$ has a limit point in $\mathbf{D}$, then, choosing α, $0 < \alpha < 1$, such that $\alpha\mathbf{D}$ contains this limit point and letting $N_\alpha = N_0 | E(\alpha\mathbf{D})H$, we have, by Proposition 4.5, $S_n \oplus N_\alpha \prec N_\alpha$ whence $S_n \oplus N \prec N$. On the other hand, if $\sigma(N_0)$ has no limit point in $\mathbf{D}$ and N_0 is not of class C_0, then $S_n \oplus N \prec N$ follows from Proposition 4.6. That (b) implies (c) is trivial. To prove (c) implies (a), note that if N_a vanishes, then Lemma 4.9 implies that $S_n \overset{i}{\prec} N_0$. Thus N_0 is a c.n.u. contraction not of class C_0. ∎

PROPOSITION 4.11. *Suppose $\{a_n : n = 1, 2, \ldots\}$ is a sequence of complex numbers such that $|a_m| > \alpha \equiv \inf\{|a_n| : n = 1, 2, \ldots\}$ for all m, and N is a normal operator. Then the following statements are equivalent:*

(a) $\sum_{n=1}^{\infty} \oplus a_n S \overset{i}{\prec} N$;

(b) *$\sigma(N)$ has a limit point in $\alpha\overline{\mathbf{D}}$;*

(c) $N \oplus \left(\sum_{n=1}^{\infty} \oplus a_n S\right) \prec N$.

Proof. Suppose N acts on H and has spectral measure $E(\cdot)$.

(a) implies (b). Assume that $\sigma(N)$ has no limit point in $\alpha\overline{\mathbf{D}}$. Choose β and m such that $\beta > |a_m| > \alpha \geqslant 0$ and $\sigma(N)$ has no limit point in $\beta\overline{\mathbf{D}}$. Let $N_\beta = N | E(\beta\mathbf{D})H$. Note that N_β is algebraic and $\|N_\beta\| \leqslant \beta$. From Lemma 3.2, we have that $a_m S \overset{i}{\prec} N_\beta$ which contradicts the fact that S is nonalgebraic.

(b) implies (c). We shall find a reducing subspace M for N such that

$$(N|M) \oplus \left(\sum_{n=1}^{\infty} \oplus a_n S\right) \prec N|M. \tag{4.11.1}$$

Let z_0 be a limit point of $\sigma(N)$ in $\alpha\overline{\mathbf{D}}$. Let $\{F_n : n = 1, 2, \ldots\}$ be a sequence of mutually disjoint Borel subsets of $\sigma(N)$ such that $T_n = N | E(F_n)H$ is nonalgebraic and

$$\sup\{|z - z_0| : z \in F_n\} < |a_n| - |z_0|.$$

We have

$$\|T_n\| \leqslant \|T_n - z_0\| + |z_0| < (|a_n| - |z_0|) + |z_0| = |a_n|.$$

From Proposition 4.5, we have that $a_n S \oplus T_n \prec T_n$ for every n. Let $M = E\left(\bigcup_{n=1}^{\infty} F_n\right)H$. Clearly, M satisfies (4.11.1).

(c) implies (a). Trivial. ∎

Now we are ready to take care of the case (b) in Theorem 4.4.

PROPOSITION 4.12. *Let $T_1 = N \oplus (S \otimes A)$ and $T_2 = N \oplus (S \otimes B)$ be as in* (*). *Suppose $m(A) > m(B)$ and $m(B) \notin \sigma_p(B)$. Then $T_1 \sim T_2$ if and only if $\sigma(N)$ has a limit point in $m(B)\overline{\mathbf{D}}$.*

Proof. From Theorem 3.18 and our assumption in (*), we have

$$S \otimes A \prec S \otimes B. \tag{4.12.1}$$

Moreover, by Corollary 3.19 there exists a diagonal positive definite operator T such that

$$S \otimes B \sim (S \otimes A) \oplus (S \otimes T) \tag{4.12.2}$$

and

$$m(B) = m(T) < T < m(A) \tag{4.12.3}$$

Assume $T_1 \sim T_2$. Then $N \oplus (S \otimes A) \oplus (S \otimes T) \prec N \oplus (S \otimes A)$. Thus $S \otimes T \overset{\mathrm{i}}{\prec} N \oplus \oplus (S \otimes A)$. From (4.12.3) and Corollary 3.5, we have $S \otimes T \overset{\mathrm{i}}{\prec} N$. Therefore, $\sigma(N)$ has a limit point in $m(B)\overline{\mathbf{D}}$ by Proposition 4.11.

Conversely, assume that $\sigma(N)$ has a limit point in $m(B)\overline{\mathbf{D}}$. Proposition 4.11 implies that

$$N \oplus (S \otimes T) \prec N. \tag{4.12.4}$$

It is obvious from (4.12.2) and (4.12.4) that

$$T_2 \sim N \oplus (S \otimes A) \oplus (S \otimes T) \prec N \oplus (S \otimes A) = T_1.$$

This, together with (4.12.1), implies $T_1 \sim T_2$. ∎

LEMMA 4.13. *Suppose T is a c.n.u. contraction and $0 \leqslant n < m \leqslant \infty$. If $S_m \overset{\mathrm{i}}{\prec} T \oplus S_n$, then T is not of class C_0.*

Proof. The case $n = 0$ (i.e., $S_m \overset{\mathrm{i}}{\prec} T$) is trivial. Assume from now on that $n > 0$. Suppose T acts on the space H and X is an injection such that $XS_m = (T \oplus S_n)X$.

Let P be the projection from $H \oplus H_n^2$ onto $H \oplus \{0\}$, $X_1 = PX$ and $X_2 = (1-P)X$. An easy calculation shows that

$$X_1 S_m = T X_1 \tag{4.13.1}$$

and

$$X_2 S_m = S_n X_2. \tag{4.13.2}$$

From (4.13.2) and the assumption that $n < m$, we infer that X_2 cannot be injective (cf. [20, Lemma 2]). Hence there exists Ψ in H_m^2 such that

$$X_2(\Psi) = 0 \quad \text{and} \quad \Psi \neq 0. \tag{4.13.3}$$

Suppose $f \in H^\infty$ such that $f(T) = 0$. From (4.13.1) and (4.13.3), we have $X_1(f\Psi) = 0$ and $X_2(f\Psi) = 0$. Thus $X(f\Psi) = 0$. Since $\Psi \neq 0$ and X is injective, we conclude that $f = 0$. Thus T cannot be of class C_0. ■

Proof of Theorem 4.4. That (a) and (∗) imply $T_1 \sim T_2$ follows from Theorem 3.18 (e). If (a) is false and $d = 0$, then (b) is equivalent to $T_1 \sim T_2$ follows from Proposition 4.12. For the rest of the proof, we assume that (a) is false and $d > 0$. In this case, $\alpha > 0$. For convenience, we may assume that $\alpha = 1$. Let $n = \dim\ker(A-1)$ and $m = \dim\ker(B-1)$. Since $S \otimes A$ and $S \otimes B$ are not injectively similar, Theorem 3.18 (b) implies that $n \neq m$. Assume that $n < m = d$. Then from Theorem 3.18 and (∗), we have

$$S \otimes A \prec S \otimes B. \tag{4.4.1}$$

Note that there exists an operator $T > 1$ such that

$$S \otimes A \sim S_n \oplus (S \otimes T) \quad \text{and} \quad S \otimes B \sim S_m \oplus (S \otimes T). \tag{4.4.2}$$

Let $N = N_0 \oplus N_a \oplus N_s \oplus N_1$ be as in (∗∗). If (c) or (d) holds, then Proposition 4.10 implies that

$$N \oplus S_{m-n} \prec N. \tag{4.4.3}$$

It is obvious from (4.4.2) and (4.4.3) that $T_2 \sim N \oplus S_m \oplus (S \otimes T) \prec N \oplus S_n \oplus (S \otimes T) \sim T_1$. This, together with (4.4.1), yields $T_1 \sim T_2$.

Conversely, assume that $T_1 \sim T_2$. If N_a does not appear, then Lemma 4.9 implies that $S_m \overset{i}{\prec} N_0 \oplus S_n$ and hence N_0 is not of class C_0 by Lemma 4.13. This shows that $T_1 \sim T_2$ implies (c) or (d) (under our assumption that (a) is false and $d > 0$). ■

During this research, the first and the third authors were partially supported by the National Science Council of the Republic of China.
The second author was partially supported by a grant of the National Science Foundation of the U.S.A.
Regretably, the second author passed away in April, 1991.
The other two authors would like to dedicate this paper to his memory.

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Received July, 23, 1990; revised January 18, 1991.