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FIELDS OF AF-ALGEBRAS

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1. INTRODUCTION

In a pioneering work J. Dixmier and A. Douady classified fields of elementary C^*-algebras [12]. Fields of AF-algebras were considered by several authors, [1, 2, 10, 12], but the classification problem has not been completely solved.

If $X = \mathbb{S}^n$ then the isomorphism classes of homogeneous locally trivial fields of C^*-algebras on X with fiber A are in one-to-one correspondence with

$$\pi_{n-1}(\mathrm{Aut}(A))/\pi_0(\mathrm{Aut}(A))$$

[15] and $\pi_k(\mathrm{Aut}(A))$ has been computed for a large class of AF-algebras [20, 26]. A similar device holds if (X, x_0) is a pointed compact connected CW-complex. If we denote by $[X, \mathrm{Aut}(A)]$ the set of homotopy classes of basepoint preserving mappings $X \to \mathrm{Aut}(A)$, $\mathrm{Aut}(A)$ being pointed by the identity automorphism, then isomorphism classes of homogeneous locally trivial fields of C^*-algebras with fiber A over SX are in one-to-one correspondence with $[X, \mathrm{Aut}(A)]\,/\pi_0(\mathrm{Aut}(A))$.

The main result of this paper is the determination of $[X, \mathrm{Aut}(A)]$ up to an extension of groups if A is an AF-algebra satisfying certain technical conditions as in 4.1. We also determine the operator kernel of this extension [18].

The technique of proof generalises the technique developed in [20] and the resulting exact sequence has close trends with the exact sequence of the Universal Coefficient Theorem for Kasparov's KK-groups [24].

An important device of the proof is that $\mathrm{Aut}^0(A) \to \mathrm{End}^0(A)$ is a weak homotopy equivalence [20].

Let us suppose that A is simple and (X, x_0) is a homotopy cogroup; then our results are complete and give isomorphisms $[X, \mathrm{Aut}(A)] \simeq \mathrm{KK}^0\left(A, C_0(X \setminus \{x_0\}, A)\right)$

if $1 \notin A$ and $[X, \mathrm{Aut}(A)] \simeq \mathrm{KK}^1(A', C_0(X \setminus \{x_0\}, A))$ if $1 \in A$ (here A' denotes the mapping cone of the inclusion $\mathbf{C} \to A$).

The next section contains results about filtered modules, morphisms and extensions of filtered modules. The definitions and the results of this section are in the spirit of [20]. Their purpose is to give a framework for the next sections. The objects we introduce and the theorem we prove reduce to well known ones if the filtrations are trivial — and this indeed happens if A is simple. The reader interested only in this case may very well skip this sections. Section three contains preliminary results concerning cancellation and comparability of projections. The result we obtain are crucial in turning K-theory data in homotopy information, they are in the spirit of the programs of [4] and [23]. Sections four and five contain the exact sequence in the general case and the determination of $[X, \mathrm{Aut}(A)]$ for A simple and X a homotopy cogroup. The latest section contains a brief discussion of the Samelson product. It is proved that in general there exists no natural group structure on the set of isomorphism classes of locally trivial fields of C^*-algebras with fiber A. This contrasts with the results of J. Dixmier and A. Douady [12].

2. FILTERED MODULES, MORPHISMS AND EXTENSIONS OF FILTERED MODULES

Let Ω be a complete lattice, R a commutative ring with unit.

DEFINITION 2.1. An *Ω-filtered module* E over a ring R is a left R-module endowed E with family of submodules $(E_\omega)_{\omega\in\Omega}$ such that $\omega \to E_\omega$ is a morphism of lattices. An Ω-filtered Z-module will be called simply an Ω-filtered abelian group.

2.2. Let E, F be Ω-filtered R-modules. A morphism $f : E \to F$ such that $f(E_\omega) \subset F_\omega$ will be called *compatible*. The set of compatible morphisms $f : E \to F$ will be denoted by $\mathrm{Hom}_{R,c}(E, F)$.

2.3. Let $0 \to E \to F \to G \to 0$ be an exact sequence of Ω-filtered R-modules with compatible morphisms.

DEFINITION. The above exact sequence will be called a *compatible extension of* G by E if $E_\omega = F_\omega \cap E$ and $E_\omega/F_\omega \to G_\omega$ is an isomorphism.

Two compatible extensions $0 \to E \to F_j \to G \to 0$, $j \in \{0, 1\}$ are called equivalent if there exists a commutative diagram of compatible morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F_0 & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & F_1 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

A compatible extension $0 \to E \to F \xrightarrow{f} G \to 0$ will be called *trivial* if there exists a compatible morphism $f_1 : G \to F$ such that $f \circ f_1 = \mathrm{id}_G$.

The pointed set of equivalence classes of compatible extensions of G by E will be denoted by $\mathrm{Ext}_{R,c}(G, E)$.

2.4. We will show that $\mathrm{Ext}_{R,c}(G, E)$ is a group with the Baer sum as operation and with the trivial extension as neutral element. The approach is standard [18].

Let $g \in G$. Denote by $\Omega_g = \{\omega \in \Omega, g \in E_\omega\}$. Then Ω_g is a complete sublattice of Ω. Denote by $\omega(g)$ the least element of Ω_g and call it the *support* of g.

For x denoting the compatible extension $0 \to E \to F \xrightarrow{\rho} G \to 0$ and $g \in G$ choose $f(g) \in F_{\omega(g)}$ such that $\rho(f(g)) = g$, $f(0) = 0$.

Let $f(0) = 0$, $\xi(g_1, g_2) = f(g_1) + f(g_2) - f(g_1 + g_2)$, $\zeta(r, g) = rf(g) - f(rg)$. Then $(\xi, \zeta) \in Z^1_{R,c}(G, E)$ where by $Z^1_{R,c}(G, F)$ we denote the group of pairs (ξ, ζ), $\xi : G \times G \to E$, $\zeta : R \times G \to E$ satisfying:

(1) $\xi(g_1, g_2) + \xi(g_1 + g_2, g_3) = \xi(g_1, g_2 + g_3) + \xi(g_2, g_3)$

(2) $\xi(g_1, g_2) = \xi(g_2, g_1)$

(3) $\xi(0, g) = \xi(g, 0) = 0$, $\zeta(0, g) = \zeta(1, g) = 0$

(4) $\zeta(r_1 r_2, g) = \zeta(r_1, r_2 g) + r_1 \zeta(r_2, g)$

(5) $r\xi(g_1, g_2) = \xi(rg_1, rg_2) + \zeta(r, g_1) + \zeta(r, g_2) - \zeta(r, g_1 + g_2)$

(6) $\xi(g_1, g_2) \in E_\omega$, $\zeta(r, g) \in E_\omega$ for any $g_1, g_2, g \in G_\omega$, $r \in R$

$(g, g_1, g_2 \in G$, $r, r_1, r_2 \in R)$.

A different choice of f will give a pair (ξ_1, ζ_1) such that $(\xi_1, \zeta_1) \in (\xi, \zeta) + B^1_{R,c}(G, E)$, where $B^1_{R,c}(G, E) \subset Z^1_{R,c}(G, E)$ is the group of those pairs $(\xi, \zeta) \in Z^1_{R,c}(G, E)$ such that

$$\begin{aligned} \xi(g_1, g_2) &= e(g_1) + e(g_2) - e(g_1 + g_2), \\ \zeta(r, g) &= -\, e(rg) + re(g) \end{aligned}$$

for some function $e : G \to E$, $e(0) = 0$, $e(G_\omega) \subset E_\omega$ $(\forall)\omega \in \Omega$.

Moreover the class of (ξ, ζ) in $Z^1_{R,c}(G, E)/B^1_{R,c}(G, E)$ depends only on the equivalence class of x in $\mathrm{Ext}_{R,c}(G, E)$. This shows that we obtain a well defined function $c : \mathrm{Ext}_{R,c}(G, E) \to Z^1_{R,c}(G, E)/B^1_{R,c}(G, E)$.

Conversely, given (ξ, ζ) satisfying (1) to (6) then let $G \times E = F$ with the operations

$$(g_1, e_1) + (g_2, e_2) = (g_1 + g_2, e_1 + e_2 + \xi(g_1, g_2))$$

$$r(g, e) = (rg, re + \zeta(r, g)),\ \ e, e_1, e_2 \in E,\ \ g, g_1, g_2 \in G$$

and filtration $F_\omega = G_\omega \times E_\omega$. Note that F is an Ω-filtered R-module due to (1)–(6), satisfies an exact sequence $0 \to E_\omega \to F_\omega \to G_\omega \to 0$ for any $\omega \in \Omega$, and if we let

$f : G \to F$ to be given by $f(g) = (g, 0)$, then $\xi(g_1, g_2) = f(g_1) + f(g_2) - f(g_1 + g_2)$ and $\zeta(r, g) = rf(g) - f(rg)$. Since to the Baer sum of extensions corresponds the sum of cycles we get that c is an additive bijection and hence $\mathrm{Ext}_{R,c}(G, E)$ has a group structure. It is obvious that $\mathrm{Ext}_{R,c}(G, E) \simeq 0$ if G is a free R-module.

2.5. Let $\varphi : G_1 \to G$, $\psi : E \to E_1$, be compatible morphisms, then we obtain morphisms $Z^1_{R,c}(G, E) \to Z^1_{R,c}(G_1, E)$, $B^1_{R,c}(G, E) \to B^1_{R,c}(G_1, E)$, $Z^1_{R,c}(G, E) \to$ $\to Z^1_{R,c}(G, E_1)$, $B^1_{R,c}(G, E) \to B^1_{R,c}(G, E_1)$ defined by $\xi \to \xi \circ (\varphi \times \varphi)$, $\zeta \to \zeta \circ (\mathrm{id}_R \times \varphi)$ and $\xi \to \psi \circ \xi$, $\zeta \to \psi \circ \zeta$.

We shall denote by $\varphi^* : \mathrm{Ext}_{R,c}(G, E) \to \mathrm{Ext}_{R,c}(G_1, E)$ and $\psi_* : \mathrm{Ext}_{R,c}(G, E) \to$ $\to \mathrm{Ext}_{R,c}(G, E_1)$ the morphisms defined by φ and ψ.

LEMMA 2.6. *Let F, E_n be Ω-filtered R-modules, $\varphi_n \in \mathrm{Hom}_{R,c}(E_n, E_{n+1})$, $E =$ $= \varinjlim(E_n, \varphi_n)$. Then there exists an exact sequence*

$$0 \to \varprojlim{}^1 (\mathrm{Hom}_{R,c}(E_n, F), \varphi_n^*) \to \mathrm{Ext}_{R,c}(E, F) \to \varprojlim (\mathrm{Ext}_{R,c}(E_n, F), \varphi_n^*) \to 0.$$

Proof. Let $X_n \in \mathrm{Ext}_{R,c}(E_n, F)$ be such that $\varphi_n^*(X_{n+1}) = X_n$. There exists an infinite commutative diagram

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & G_n & \longrightarrow & E_n & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \longrightarrow & F & \longrightarrow & G_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

such that $0 \to F \to G_n \to E_n \to 0$ represents X_n in $\mathrm{Ext}_{R,c}(E_n, F)$. Let $G =$ $= \varinjlim(G_n, \psi_n)$.

Then the image of the exact sequence $0 \to F \to G_n \to E_n \to 0$ in $\mathrm{Ext}_{R,c}(E_n, F)$ is X_n.

Let us identify the kernel of $\eta : \mathrm{Ext}_{R,c}(E, F) \xrightarrow{\eta} \varprojlim (\mathrm{Ext}_{R,c}(E_n, F), \varphi_n^*)$. To this end, denote by $\chi_n : E_n \to E$ the obvious morphism. Suppose that $0 \to F \to G \to$ $\to E \to 0$ defines an element $X \in \mathrm{Ext}_{R,c}(E, F)$ such that $\chi_n^*(X) = 0$ for any n. Then there exists a compatible morphism $\tau_n : E_n \to G$ making the following diagram commutative

$$\begin{array}{ccccccccc}
 & & & & & & E_n & & \\
 & & & & & \swarrow{\scriptstyle \tau_n} & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

Let $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{R,c}(E_n, F)$, $\lambda_n = \tau_{n+1} \circ \varphi_n - \tau_n$. Denote by $d : \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{R,c}(E_n, F) \to \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{R,c}(E_n, F)$ the morphism $d((f_n)_{n \in \mathbb{N}}) =$

$= (f_{n+1} \circ \varphi_n - f_n)_{n\in\mathbf{N}}$. The two different choices of τ_n define sequences λ differing by an element in $\operatorname{Im} d$. This shows that there exists a well defined morphism $\ker\eta \to \varprojlim^1 (\mathrm{Hom}_{R,\mathrm{c}}(E_n, F), \varphi_n^*)$ which turns out to be an isomorphism.

2.7. We shall also need another group, *the group of extensions with order unit.* It is defined as in [20] (see below).

Let E, G be filtered Ω-filtered modules, $u \in G$ an element such that $u \in G_\omega$ if and only if $\omega = \sup\Omega$ (where $\sup\Omega$ is the largest element of Ω). Such an element will be called an *order unit* of G and will remain fixed in the following discussion.

By a *compatible extension with order unit of G by E* we shall mean a compatible extension $0 \to E \to F \xrightarrow{f} G \to 0$ for which an order unit of F is chosen such that $f(v) = u$. We shall write in this case $0 \to E \to (F, v) \to (G, u) \to 0$.

Two compatible extensions with order unit $0 \to E \to (F_j, v_j) \to (G, u) \to 0$, $j \in \{0, 1\}$ are said to be equivalent if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & (F_0, v_0) & \longrightarrow & (G, u) & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & (F_0, v_0) & \longrightarrow & (G, u) & \longrightarrow & 0 \end{array}$$

such that f is a compatible morphism and $f(v_0) = v_1$.

A compatible extension with order unit $0 \to E \to (F, v) \to (G, u) \to 0$ will be called *trivial* if there exists a compatible morphism $f_1 : G \to F$ such that $f_1(u) = v$ and $f \circ f_1 = \mathrm{id}_G$.

We shall denote by $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E)$ the set of equivalence classes of compatible extensions with order unit of G by E. The unit of G will be clear from the context. We shall omit R when $R = \mathbf{Z}$.

PROPOSITION 2.8. a) $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G,\ )$ *is a covariant functor from the category of Ω-filtered R-modules with compatible morphisms to abelian groups.*

b) $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(\ , E)$ *is a covariant functor from the category of Ω-filtered R-modules with unit preserving compatible morphism to abelian groups.*

c) *There exists an exact sequence:* $0 \to \mathrm{Hom}(G/Ru, E) \cap \mathrm{Hom}_{R,\mathrm{c}}(G, E) \to$ $\to \mathrm{Hom}_{R,\mathrm{c}}(G, E) \to \mathrm{Hom}_R(Ru, E) \to \mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E) \to \mathrm{Ext}_{R,\mathrm{c}}(G, E) \to 0$.

Proof. Let $f : E \to E_1$ be a compatible morphism between the Ω-filtered R-modules E and E_1. Let $0 \to E \to (F, v) \to (G, u) \to 0$ be a compatible extension with order unit. Denote by x its class in $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E)$. There exists by 2.5 a commutative diagram of compatible morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f' & & \| & & \\ 0 & \longrightarrow & E_1 & \longrightarrow & F_1 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Then $f_*(x)$ is defined to be the class of $0 \to E \to (F_1, f'(v)) \to (G, u) \to 0$ in $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E_1)$.

Let $\varphi : (G_1, u_1) \to (G, u)$ be a unit preserving compatible morphism, G, G_1 being Ω-filtered R-modules with order units. Let $x \in \mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E)$ be represented by $0 \to E \to (F, v) \xrightarrow{h} (G, u) \to 0$. Denote $F_1 \subset F \oplus G_1$ the submodule consisting of those pairs (f, g_1) such that $h(f) = \varphi(g_1)$. Let $v_1 = (v, u_1)$ be the order unit of F_1, then $\varphi^*(x)$ is represented by $0 \to E \to (F_1, v_1) \to (G_1, u_1) \to 0$.

Let $0 \to E \to (F_j, v_j) \to (G, u) \to 0$ represent $x_j \in \mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E)$, $j \in \{0, 1\}$. Denote by $d_2 : G \to G \times G$ the "diagonal" map: $d_2(g) = (g, g)$, and by $\sigma_2 : E \times E \to E$ the "addition" map: $\sigma_2(e_1, e_2) = e_1 + e_2$. Let $x \in \mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G \times G, E \times E)$ be represented by

$$0 \to E \times E \to (F_1 \times F_2, (v_1, v_2)) \to (G \times G, (u, u)) \to 0.$$

Then $x_1 + x_2$ is defined to be $d_2^*(\sigma_{2*}(x)) = \sigma_{2*}(d_2^*(x))$. (The last relation is proved as in Lemma III.1.6 of [18].) This proves a) and b).

The morphism $\mathcal{E}_G : \mathrm{Hom}_R(Ru, E) \to \mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E)$ is defined as follows. Let $e \in E$ and denote by f_e the morphism $f_e(r) = re$. Then $\mathcal{E}_G(f_e)$ is the class of $0 \to E \to (E \times G, (-e, u)) \to (G, u) \to 0$. The morphism $\mathrm{Ext}^{\mathrm{u}}_{R,\mathrm{c}}(G, E) \to \mathrm{Ext}_{R,\mathrm{c}}(G, E)$ is defined by "forgetting" the unit. The exactness is obvious.

3. PRELIMINARY RESULTS

We shall denote by $U(A)$ the group of those unitaries $u \in M(A)$ such that $u - 1 \in A$.

3.1. We shall fix from now on an AF-algebra A with the following properties:

a) For any ideals $I \subset J \subset A$, $I \neq J$, J/I is not type I.

b) Either $1 \in A$ or A is *completely nonunital* in the sense that for any projection $e \in A$, $(1 - e)A(1 - e)$ is a full corner in A.

DEFINITION 3.2. [20, Definition 2.2] Let (G, G_+) be an ordered group. We shall say that (G, G_+) has *large denominators* if for any $g \in G_+$ and $n \in \mathbb{N}$ there exists $g_1 \in G_+$ and $m \in \mathbb{N}$ such that $ng_1 \leqslant g \leqslant mg_1$.

PROPOSITION 3.3. *Let A be an AF-algebra. Then A satisfies* 3.1.a) *if and only if* $\mathrm{K}_0(A)$ *has large denominators.*

Proof. Suppose that $\pi : A \to B(H)$ is an irreducible representation such that $\mathcal{K}(H) \subset \pi(A)$ ($\mathcal{K}(H)$ denotes the algebra of compact operators on H).

Using L. Brown's lifting projection theorem for AF-algebras [7, 13] we find a projection $e \in A$ such that $\pi(e)$ is a rank one projection in $\mathcal{K}(H)$. Then $g = [e]$ does not satisfy the conditions of Definition 3.1.

Conversely, let e be a projection in $M_q(A)$ for some $q \in \mathbf{N}$. Denote by J' the ideal generated by e in $M_q(A)$. Fix $n \in \mathbf{N}$ and let $J = eJ'e = \overline{\cup J_k}$ with J_k finite dimensional. Denote by I_k the ideal of J_k consisting of those factors of J_k having dimension $\geqslant n$. It follows that $I_k \subset I_{k+1}$ and hence $I = \overline{\cup I_k}$ is an ideal of J. Denote as in [11] by $r(n)$ the least integer m with the property that $\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(m)} = 0$ for any $a_1, \ldots, a_m \in M_n(\mathbf{C})$ (S_m is the symmetric group of order m). It follows that for any $x_1, \ldots, x_m \in J/I$, $m \geqslant r(n)$, $\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)} = 0$, since this is true for x_j in the dense subalgebra J_k/I_k. The proof of [11, Proposition 3.6.3] shows that J/I has only finite dimensional representations (of dimension $\leqslant n$). The assumption on A shows that $I = J$ and hence that $e \in J_k$ for some large k. Choose a minimal projection from each factor of J_k and denote by p their sum. Then $n[p] \leqslant [e] \leqslant m[p]$ for some large $m \in \mathbf{N}$.

3.4. Let X be a locally compact space and $B = (A(x), x \in X, \Gamma)$ be a locally trivial field of C^*-algebras such that $A(x) \simeq A$ for any $x \in X$, [11, Ch. X]. B may be viewed as a fiber bundle with structure group $\operatorname{Aut}(A)$. Denote by $\operatorname{Aut}^0(A)$ the connected component of the identity in $\operatorname{Aut}(A)$. Recall that $\operatorname{Aut}^0(A) = \overline{\operatorname{Inn}(A)}$ [2].

Let Ω be the lattice of ideals of A. Ω can be identified with the lattice of ideals of $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators.

Let $\varphi \in \operatorname{Aut}^0(A)$, then $\varphi(\omega) = \omega$ for any $\omega \in \Omega$. Suppose that our bundle admits a restriction of the structure group to $\operatorname{Aut}^0(A)$ (this always happens is X is simply connected). Denote by ξ the associated $\operatorname{Aut}^0(A)$ principal bundle.

The $\operatorname{Aut}^0(A)$-equivariant inclusion $\omega \subset A$ gives rise to an inclusion $\xi[\omega] \subset \xi[A]$ of fiber bundles. (Our notation and terminology are taken from [15].)

Denote by $B(B_\omega)$ the C^*-algebra of continuous sections of $\xi[A] = B$ ($\xi[\omega]$) see [11, Ch. X]. We obtain an Ω-filtration of $\mathrm{K}_0(B)$ by $\mathrm{K}_0(B)_\omega = \operatorname{Ran}(\mathrm{K}_0(B_\omega) \to \mathrm{K}_0(B))$.

The following proposition is the key for translating homotopy information into K-theory language.

Proposition 3.5. *Let (X, x_0) be a pointed compact connected CW-complex, A, B, B_ω as above. Denote by $\eta : \mathrm{K}_0(B) \to \mathrm{K}_0(A)$ the "evaluation at x_0" morphism.*

a) *If $a, a' \in \mathrm{K}_0(B)$, $\eta(a') \geqslant \eta(a)$ and $m(\eta(a') - \eta(a)) \geqslant \eta(a)$, for some $m \in \mathbf{N}$, then $a' \geqslant a$.*

b) *Let $a, a' \in \mathrm{K}_0(B)$, $a = [e]$ for e a projection in a matrix algebra of B, $\eta(a) = \eta(a')$. Denote by ω the ideal generated by $e(x_0)$ in $A \otimes \mathcal{K}$. Then $a' \geqslant 0$ if and only if $a' - a \in \mathrm{K}_0(B_\omega)$.*

c) *Suppose ξ is trivial, then B has the cancellation property for projections and $\pi_j(U(B)) \to \mathrm{K}_{j+1}(B)$ is an isomorphism for any $j \geqslant 0$.*

Proof. The idea of the proof is to identify elements $b \in B$ satisfying certain properties with sections in an appropiately defined fiber bundle.

We may assume that B is stable (i.e. $B \simeq B \otimes \mathcal{K}$).

a) Let e, e' be projections in B representing a, a' in $\mathrm{K}_0(B)$. We may assume that $e(x_0) \leqslant e'(x_0)$. Denote by $V(x) = \{v(x) \in A(x),\ v^*(x)v(x) = e(x),\ v(x)v(x)^* \leqslant \leqslant e'(x)\}$. $V(x)$ is a sort of "generalised Stieffel manifold". Let $V = \bigcup_{x \in X} V(x) \subset \xi[A]$ with the induced topology, then V becomes a locally trivial fiber bundle on X. It is easy to prove that $U(e(x_0)Ae(x_0)) \ni u \to ue \in V(x_0)$ is locally trivial fiber bundle with fiber $U(e'(x_0)Ae'(x_0))/U((e'(x_0) - e(x_0))A(e'(x_0) - e(x_0)))$ (the proof is similar to Lemma 1.2 of [20]). The hypothesis shows that the ideals generated by $e'(x_0)$ and $e'(x_0) - e(x_0)$ in $A \otimes \mathcal{K}$ coincide. The exact sequence of homotopy groups [27] and Proposition 2.4.b) of [20] show

$$U((e'(x_0) - e(x_0))A(e'(x_0) - e(x_0))) \to U(e'(x_0)Ae'(x_0))$$

is weak homotopy equivalence and hence $\pi_k(V(x)) \to \pi_k(V(x_0)) \simeq \{0\}$ for any $k \geqslant 0$ and $x \in X$. A standard argument [15, Theorem 7.1 p. 21] shows that V has a cross-section. This cross-section defines a partial isometry from e to a subprojection of e'.

b) Let $e, e' \in B$ be projections such that $e(x_0) = e'(x_0)$. Then $e, e' \in B_\omega$ and hence $[e] - [e'] \in \mathrm{K}_0(B)_\omega$. Conversely, suppose that $a = [e]$, $a' - a \in \mathrm{K}_0(B)_\omega$, ω being the ideal generated by $e(x_0)$ in A. $\mathrm{Aut}^0(A)$ acts on $\omega^+ = \omega + \mathbb{C}1$. Let $\tilde{B}_\omega$ be the C^*-algebra of continuous sections in $\xi[\omega^+]$. The split exact sequence $0 \to B_\omega \to \tilde{B}_\omega \to C(X) \to 0$ shows that the element $a' = (a' - a) + a$ of $\mathrm{K}_0(B_\omega)$ may be represented as $[e_1'] - [e_1]$, e_1 and e_1' being projections in $\tilde{B}_\omega \otimes \mathcal{K}$ such that $\chi(e_1') = \chi(e_1)$ ($\chi = \chi_0 \otimes \mathrm{id}_{\mathcal{K}}$). Denote by $\chi_x : \omega(x)^+ \otimes \mathcal{K} \to \mathcal{K}$ the quotient morphism ($\omega(x)^+$ is the fiber of $\xi[\omega^+]$ at x). Define $W(x) = \{w(x) \in \omega(x)^+ \otimes \mathcal{K},\ \chi_x(w(x)) = = \chi_x(e_1'(x))\ (= \chi_x(e_1(x))),\ w(x)w(x) = e_1(x),\ w(x)w(x) \leqslant e_1'(x)\}$. It follows as in b) that W_x is homeomorphic to $W_{x_0} \neq \varnothing$ and $\pi_k(W_x) \simeq \{0\}$. This shows that $W = \bigcup_{x \in X} W_x$ has a cross-section and that e_1 is equivalent to a subprojection of e_1. We obtain $a' \geqslant 0$.

c) The cancellation property follows from standard results in topology. Indeed, suppose that e_1, e_2 are projections in $M_q(B)$ such that $[e_1] = [e_2]$ in $\mathrm{K}_0(B)$. We write $A = \cup A_n$, A_n being finite dimensional C^*-algebras. Let d denote the dimension of X. We may suppose that $e_1, e_2 \in C(X, A_n)$ for some large n. Also, since $\mathrm{K}_0(A)$ has large denominators we may suppose that the dimensions of the projections e_1

and e_2 in $K_0(C(X, A_n))$ are large enough (i.e. greater than $\frac{d}{2}$ and that $[e_1] = [e_2]$ in $K_0(C(X, A_n)))$. This can be done by increasing n if necessary. But stable isomorphic vector bundles of large dimension are isomorphic (see [15, Theorem 8.1.7, page 100]).

Let us observe that $s = \mathrm{tsr}(C(X)) < \infty$ (see [22] for definition and notations). It follows that $\pi_0(U(M_q(C(X))))$ is isomorphic to $K_1(C(X))$ for $q \geqslant s+2$, [22, Theorem 10.12] and use also the fact that the topological stable rank and the Bass stable rank coincide for C^*-algebras [14]. This shows that $\pi_j(U(B)) \simeq \varinjlim \pi_j(U(C(X, A_n))) \simeq \simeq \varinjlim \pi_0(U(S^j C(X, A_n))) \simeq \varinjlim K_{j+1}(C(X, A_n)) \simeq K_{j+1}(B)$ since the A_n may be chosen such that the dimensions of the blocks of A_n increase to ∞ (this is due to the assumption that $K_0(A)$ has large denominators).

Let $A = \overline{\cup A_n}$ with A_n finite dimensional. Denote by $i_{m,n}$ the inclusion $A_n \to A_m$ and by i_n the inclusion $A_n \to A$. Let $\mathrm{Hom}^0(A_n, A)$ denote the connected component of i_n in $\mathrm{Hom}(A_n, A)$. Let $B = C(X, A)$, $J = C_0(X \setminus \{x_0\}, A)$. Denote by $[X, \mathrm{Hom}^0(A_n, A)]$ the homotopy classes of base-point preserving continuous functions $\varphi : X \to \mathrm{Hom}^0(A_n, A)$ the base-point of $\mathrm{Hom}^0(A_n, A)$ being i_n. Such a continuous function defines a morphism $\Phi_n(\varphi) : A_n \to B$. We shall denote by $j_n : A_n \to B$ the morphism $\Phi_n(\varphi)$ for $\phi(x) = i_n$ $(\forall) x \in X$.

The following lemma shows the power of the previous proposition.

LEMMA 3.6. *The map*

$$[X, \mathrm{Hom}^0(A_n, A)] \ni [\varphi] \to K_0(\Phi_n(\varphi)) - K_0(j_n) \in \mathrm{Hom}_c(K_0(A_n), K_0(J))$$

is well defined and bijective if the filtration of $K_0(J)$ *is* $K_0(J)_\omega = K_0(C_0(X \setminus \{x_0\}, \omega))$, *the filtration of* $K_0(A_n)$ *is* $K_0(A_n)_\omega = K_0(i_n)^{-1}(K_0(\omega))$, *and* A *is completely nonunital.*

Proof. Let $p_1, \ldots, p_k$ be the minimal projections of A_n. Denote by ω_1 the ideal generated by p_1 in A. It follows from Proposition 3.5 b) that $K_0(\Phi(\varphi))([p_1]) - -K_0(j_n)([p_1]) \in K_0(J)_{\omega_1}$. Conversely, suppose that $f \in \mathrm{Hom}_c(K_0(A_n), K_0(J))$. It follows from Proposition 3.5 b) that there exists a projection $e_1' \in B$ such that $[e_1'] = [j_n(p_1)] + f([p_1])$. Suppose that $A_n = A_n^{(1)} \oplus \cdots \oplus A_n^{(k)}$ and $A_n^{(q)}$ is a factor of type I_{m_q}. Using Proposition 3.5 b) one obtaines by induction $m_1 + m_2 + \cdots + m_k$ orthogonal projections $\tilde{e}_{11}^{(1)}, \ldots, \tilde{e}_{m_1 m_1}^{(1)}, \tilde{e}_{11}^{(2)}, \ldots, \tilde{e}_{m_k m_k}^{(k)} \in B$ such that $[\tilde{e}_{rr}^{(1)}] = e_1'$. It follows from Proposition 3.5 c) that $\tilde{e}_{11}^{(1)}$ is equivalent to $\tilde{e}_{rr}^{(1)}$. Choose $\tilde{e}_{r1}^{(1)}$ such that $\tilde{e}_{r1}^{(1)*} \tilde{e}_{r1}^{(1)} = \tilde{e}_{11}^{(1)}$ and $\tilde{e}_{r1}^{(1)} \tilde{e}_{r1}^{(1)*} = \tilde{e}_{rr}^{(1)}$, $r \in \{2, \ldots, m_1\}$. Denote by $\tilde{e}_{rq}^{(1)} = \tilde{e}_{r1}^{(1)} \left(\tilde{e}_{q1}^{(1)}\right)^*$, $q \in \{2, \ldots, m_1\}$, $\tilde{e}_{1r}^{(1)} = \tilde{e}_{r1}^{(1)*}$. Let $\tilde{e}_{rq}^{(1)}$, $1 \in \{1, \ldots, k\}$, $r, q \in \{1, \ldots, m\}$ denote a matrix unit of A_n. There exists $\varphi : X \to \mathrm{Hom}^0(A_n, A)$, $\varphi_x\left(e_{rq}^{(1)}\right) = \tilde{e}_{rq}^{(1)}(x)$. Moreover, any map $X \to \mathrm{Hom}^0(A_n, A)$ is homotopic to a base point preserving map. This shows the surjectivity of $[X, \mathrm{Hom}^0(A_n, A)] \to \mathrm{Hom}_c(K_0(A_n), K_0(J))$.

In order to prove that it is injective let us observe that if $\varphi, \psi \in \mathrm{Hom}(A_n, B)$ have the property that $\mathrm{K}_0(\varphi) = \mathrm{K}_0(\psi)$ then it follows from Proposition 3.5 c) using a standard trick of O. Bratteli that φ and ψ are unitary conjugated: $\varphi = \mathrm{ad}_u \circ \psi$ with $u \in U(J)$. Let $e \in \psi(1)$. Then there exists a unitary v in $U((1-e)J(1-e))$ such that uv is in the connected component of the identity (use Proposition 3.5 c) and 3.1 b)) and $\varphi = \mathrm{ad}_{uv} \circ \psi$. Hence φ is homotopic to ψ.

Recall the pointed space (X, x_0) is a homotopy-cogroup ([27] where the term of H'-space is used) if there exists a homotopy associative comultiplication $\theta : X \to X \vee X$ such that if $\mathrm{ct} : X \to X$ is the constant map $\mathrm{ct}(x) = x_0$ and $q_1 = \mathrm{id} \vee \mathrm{ct} : X \vee X \to X$ ($q_2 = \mathrm{ct} \vee \mathrm{id} : X \vee X \to X$) then $\mathrm{id}, q_1 \circ \theta$ and $q_2 \circ \theta$ are homotopic. Moreover it is supposed that exists $\beta : X \to X$, "the inverse", such that $(\beta \vee \mathrm{id}) \circ \theta$ and $(\mathrm{id} \vee \beta) \circ \theta$ are homotopic to the constant map (all maps and homotopies are understood to be base-point preserving).

REMARK. If (X, x_0) is a homotopy cogroup then $[X, \mathrm{Hom}^0(A_n, A)]$ is a group [27] and $[X, \mathrm{Hom}^0(A_n, A)] \to \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(J))$ is actually a group-morphism.

4. THE EXACT SEQUENCE

Denote by $\mathrm{Map}(X, \mathrm{Aut}^0(A))$ the space of *base-point* preserving continuous mappings $X \to \mathrm{Aut}^0(A)$. Let $B = C(X, A)$ be identified with $C(X) \otimes A$ and let $J \subset B$, $J = C(X \setminus \{x_0\}) \otimes B$.

4.1. There exists a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Map}(X, \mathrm{Aut}^0(A)) & & \\ & \swarrow^{\Phi} & & \searrow & \\ \mathrm{Hom}(A, B) & & \xrightarrow{\;T\;} & & \mathrm{End}(B) = \mathrm{Hom}(B, B). \end{array}$$

The first vertical arrow associates to a continuous function $x \ni x \to \varphi_x \in \mathrm{Aut}(A)$ the morphism $\Phi(\varphi) : A \to B$ given by $\Phi(\varphi)(a)(x) = \varphi_x(a)$. The horizontal arrow associates to a morphism $\psi : A \to B$ the morphism $T(\psi) : B = C(X) \otimes A \to B$ defined by $T(\psi)(f \otimes a) = f\psi(a)$.

Passing to K-groups one obtaines the following commutative diagram

$$\begin{array}{ccccc} & & [X, \mathrm{Aut}^0(A)] & & \\ & \swarrow^{\alpha_i'} & & \searrow^{\alpha_i} & \\ \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)) & & \xrightarrow{\;\mu_X\;} & & \mathrm{Hom}_{\mathrm{K}_0(X)}(\mathrm{K}_0(B), \mathrm{K}_0(B)) \end{array}$$

($i = 0$ if $1 \notin A$, $i = 1$ if $1 \in A$).

Since $\mathrm{K}_0(B) \simeq \mathrm{K}^0(X) \otimes \mathrm{K}_0(A)$ it follows that $\mathrm{K}_0(B)$ is a $\mathrm{K}^0(X)$-module. This module structure can be described directly as follows: let $[e] \in \mathrm{K}_0(B)$, $[p] \in \mathrm{K}^0(X)$

with $e \in M_q(B)$, $p \in M_r(C(X))$, then $[p][e]$ is the class of $(p \otimes I_q)(I_r \otimes e) \in M_{rq}(B)$ in $\mathrm{K}_0(B)$. This shows that $\alpha_i([\varphi])$ is indeed $\mathrm{K}^0(X)$-linear.

$[X, \mathrm{Aut}(A)]$ is a group with the law $[\varphi][\psi] = [\varphi \circ \psi]$. It is clear from definition that $\alpha_i([\varphi][\psi]) = \alpha_i([\varphi])\alpha_i([\psi])$. μ_X can be described by $\mu_X(f)(x \otimes z) = xf(z)$ for $f \in \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$, $x \in \mathrm{K}^0(X)$, $z \in \mathrm{K}_0(A)$.

Let us denote by G^i (respectively $G^{i'}$) the range of α_i (respectively α_i'). Since $\mu_X : \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \to \mathrm{Hom}_{\mathrm{K}^0(X)}(\mathrm{K}_0(B), \mathrm{K}_0(B))$ is bijective it follows that in order to find G' it is enough to determine $G^{i'}$.

Denote as before by Ω the lattice of ideals of A and observe that $\mathrm{K}_0(A)$, $\mathrm{K}_j(B)$, $\mathrm{K}_j(J)$ have natural Ω-filtrations ($j \in \{0,1\}$):

$$\mathrm{K}_0(A)_\omega = \mathrm{K}_0(\omega),\ \mathrm{K}_j(B)_\omega = \mathrm{K}_j(C(X,\omega)) \simeq \mathrm{K}^j(X) \otimes \mathrm{K}_0(\omega),$$
$$\mathrm{K}_j(J)_\omega = \mathrm{K}_j(C_0(X \setminus \{x_0\}, \omega)) \simeq \tilde{\mathrm{K}}^j(X) \otimes \mathrm{K}_0(\omega).$$

If $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ is constant denote by $\iota = \iota_X$ the embedding $\mathrm{K}_0(A) \to \to \mathrm{K}_0(B)$ defined by $\mathrm{K}_0(\Phi(\varphi))$. It is equal to the composition of $\mathrm{K}_0(A) \ni [e] \to \to [1] \otimes [e] \in \mathrm{K}^0(X) \otimes \mathrm{K}_0(A)$ with the isomorphism $\mathrm{K}^0(X) \otimes \mathrm{K}_0(A) \simeq \mathrm{K}_0(B)$.

4.2. The following constructions are needed in order to determine the kernel of α_i.

Let $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$. Denote by $E_\varphi \subset M_\varphi \subset C([0,1] \times X, A)$ the C^*-algebras defined by $E_\varphi = \{f,\ (\exists) a \in A$ such that $f(0,x) = a,\ f(t, x_0) = a,\ f(1,x) = = \varphi_x(a)$ for any $x \in X,\ t \in [0,1]\}$, $M_\varphi = \{f,\ f(1,x) = \varphi_x(f(0,x))$ for any $x \in X\}$. M_φ is the mapping torus of $\tilde{\varphi} = T \circ \Phi(\varphi) \in \mathrm{End}(B)$ [5].

Let $\rho : E_\varphi \to A$, $\rho(f) = f(0, x_0)$, $\chi : M_\varphi \to B$, $\chi(f) = f|\{0\} \times X$. Then there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SJ & \longrightarrow & E_\varphi & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & SB & \longrightarrow & M_\varphi & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Let us recall that the connecting morphisms of the K-theory exact sequence of the bottom row are the composition of $\mathrm{id} - \mathrm{K}_j(\tilde{\varphi}) : \mathrm{K}_j(B) \to \mathrm{K}_j(B)$ and of $\mathrm{K}_j(B) \to \mathrm{K}_{j-1}(SB)$ [5, Proposition 10.4.1].

Observe that if we denote by $\mathrm{K}_*(\tilde{\varphi}) = \mathrm{K}_0(\tilde{\varphi}) \oplus \mathrm{K}_1(\tilde{\varphi}) : \mathrm{K}_*(B) = \mathrm{K}_0(B) \oplus \oplus \mathrm{K}_1(B) \to \mathrm{K}_*(B)$ then $\mathrm{K}_*(\tilde{\varphi})$ is the unique $\mathrm{K}^*(X) = \mathrm{K}^0(X) \oplus \mathrm{K}^1(X)$-linear extension of $\mathrm{K}_0(\tilde{\varphi})$. Thus, if $\mathrm{K}_0(\tilde{\varphi}) = \mathrm{id}_{\mathrm{K}_0(B)}$, then also $\mathrm{K}_1(\tilde{\varphi}) = \mathrm{id}_{\mathrm{K}_1(B)}$.

We obtain for any $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ such that $\alpha_i([\varphi]) = \mathrm{id}_{\mathrm{K}_0(B)}$ a commutative diagram with exact rows:

$$\begin{array}{cccccccccc} (1) & 0 & \longrightarrow & \mathrm{K}_1(J) & \longrightarrow & \mathrm{K}_0(E_\varphi) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & 0 \\ & & & \downarrow \simeq & & \downarrow & & \downarrow \iota & & \\ (2) & 0 & \longrightarrow & \mathrm{K}_1(B) & \longrightarrow & \mathrm{K}_0(M_\varphi) & \longrightarrow & \mathrm{K}_0(B) & \longrightarrow & 0. \end{array}$$

If we denote $E_{\varphi,\omega} = E_\varphi \cap C([0,1] \times X, \omega)$, $M_{\varphi,\omega} = M_\varphi \cap C([0,1] \times X, \omega)$ then $\mathrm{K}_0(E_\varphi)$ and $\mathrm{K}_0(M_\varphi)$ are natural equipped with Ω-filtration. Moreover there exists an obvious morphism of $C(X)$ into the center of $M(M_\varphi)$ giving a $\mathrm{K}^0(X)$-module structure on $\mathrm{K}_0(M_\varphi)$.

Let us denote by $\gamma_0'(\varphi)$ the class of (1) in $\mathrm{Ext}_{\mathrm{c}}(\mathrm{K}_0(A), \mathrm{K}_1(J))$ and by $\gamma_0(\varphi)$ the class of (2) in $\mathrm{Ext}_{\mathrm{K}^0(X),\mathrm{c}}(\mathrm{K}_0(B), \mathrm{K}_0(B))$ for $[\varphi] \in \ker\alpha_0$. If A has a unit then E_φ and M_φ are also unital and the quotient morphisms ρ and χ are unit preserving. Note also that $\mathrm{K}_0(E_\varphi)$ and $\mathrm{K}_0(M_\varphi)$ have order units given by the classes of the units. This shows that if $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$, $\alpha_1([\varphi]) = \mathrm{id}_{\mathrm{K}_0(A)}$ then we can define $\gamma_1'(\varphi) \in \mathrm{Ext}^{\mathrm{u}}_{\mathrm{K}^0(X),\mathrm{c}}(\mathrm{K}_0(B), \mathrm{K}_0(B))$ and $\gamma_1'(\varphi) \in \mathrm{Ext}^{\mathrm{u}}_{\mathrm{c}}(\mathrm{K}_0(A), \mathrm{K}_0(J))$ by regarding (1) and (2) as compatible extensions with order unit.

Let $\varphi, \psi \in \mathrm{Map}(X, \mathrm{Aut}(A))$, $\tilde{\varphi} = T \circ \Phi(\varphi)$, $\tilde{\psi} = T \circ \Phi(\varphi)$. Denote by σ_2 the composition of $SB \oplus SB \simeq C_0\left(\left(0, \frac{1}{2}\right), B\right) \oplus C_0\left(\left(\frac{1}{2}, 1\right), B\right) \to SB$. Then, if we denote by $D = \{f \in C([0,2], B),\ f(1) = \tilde{\psi}(f(0)),\ f(2) = \tilde{\varphi}(f(1))\}$, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SB \oplus SB & \longrightarrow & M_\psi \oplus M_\varphi & \longrightarrow & B \oplus B & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}\oplus\mathrm{id}} & & \\
0 & \longrightarrow & SB \oplus SB & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & SB & \longrightarrow & M_{\varphi\circ\psi} & \longrightarrow & B & \longrightarrow & 0.
\end{array}$$

If $\mathrm{K}_0(\tilde{\varphi}) = \mathrm{K}_0(\tilde{\psi}) = \mathrm{id}_B$ then the corresponding diagram of K_0-groups shows that $\gamma_i(\varphi \circ \psi) = \gamma_i(\varphi) + \gamma_i(\psi)$, $i \in \{0, 1\}$. It is obvious that if φ is homotopic to the constant map $x \to \mathrm{id}_A$ then $\gamma_i(\varphi) = 0$. Also observe that there exists obvious morphisms

$$r_0 : \mathrm{Ext}_{\mathrm{K}^0(X),\mathrm{c}}(\mathrm{K}_0(B), \mathrm{K}_1(B)) \to \mathrm{Ext}_{\mathrm{c}}(\mathrm{K}_0(A), \mathrm{K}_1(J))$$

and

$$r_0 : \mathrm{Ext}^{\mathrm{u}}_{\mathrm{K}^0(X),\mathrm{c}}(\mathrm{K}_0(B), \mathrm{K}_1(B)) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{c}}(\mathrm{K}_0(A), \mathrm{K}_1(J))$$

obtained by composing the "forgetfull" morphism $\mathrm{Ext}_{\mathrm{K}^0(X),\mathrm{c}}(\cdot,\cdot) \to \mathrm{Ext}_{\mathrm{c}}(\cdot,\cdot)$ $(\mathrm{Ext}^{\mathrm{u}}_{\mathrm{K}^0(X),\mathrm{c}}(\cdot,\cdot) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{c}}(\cdot,\cdot))$ with ι^* and using the isomorphism $\mathrm{K}_1(J) \simeq \mathrm{K}_1(B)$. It follows that $\gamma_i' = r_i \circ \gamma_i$ is also a morphism. It also follows that $\gamma_0, \gamma_0', \gamma_1$ and γ_1' depend only on the class of φ in $[X, \mathrm{Aut}(A)]$.

The preceding discussion is partially included in the following lemma:

LEMMA 4.3. a) *There exist commutative diagram of morphisms:*

$$\begin{array}{ccc} & \ker\alpha_0 & \\ \swarrow^{\gamma_0} & & \searrow^{\gamma_0'} \\ \mathrm{Ext}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B),\mathrm{K}_1(B)) & \xrightarrow{r_0} & \mathrm{Ext}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) \end{array}$$

for A non unital, and

$$\begin{array}{ccc} & \ker\alpha_1 & \\ \swarrow^{\gamma_1} & & \searrow^{\gamma_1'} \\ \mathrm{Ext}^{u}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B),\mathrm{K}_1(B)) & \xrightarrow{r_1} & \mathrm{Ext}^{u}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) \end{array}$$

for A unital. r_i is an isomorphism $i \in \{0,1\}$.

b) $\gamma_i([\psi]\xi[\psi]^{-1}) = \alpha_i([\psi])^{*-1}\mathrm{K}_1(\tilde{\psi})_*(\gamma_i(\xi))$ *for* $\psi \in \mathrm{Map}(X,\mathrm{Aut}(A))$, $\xi \in \ker\alpha_i$.

Proof. Let $A = \overline{\cup A_n}$ with A_n finite dimensional.

Let us observe that there exists by Lemma 2.6 a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B),\mathrm{K}_1(B)) & \xrightarrow{r_0} & \mathrm{Ext}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) \\ \downarrow\simeq & & \downarrow\simeq \\ \varprojlim{}^1\mathrm{Hom}_{\mathrm{K}^0(X),c}(\mathrm{K}^0(X)\otimes\mathrm{K}_0(A_n),\mathrm{K}_1(B)) & \xrightarrow{\sim} & \varprojlim{}^1\mathrm{Hom}_c(\mathrm{K}_0(A_n),\mathrm{K}_1(J)) \end{array}$$

from which it follows that r_0 is also an isomorphism.

Using Lemma 2.8 we get a commutative diagram with exact rows:

$$\begin{array}{cccccc} \longrightarrow & \mathrm{Hom}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) & \longrightarrow & \mathrm{Hom}(\mathbf{Z},\mathrm{K}_1(J)) & \longrightarrow & \\ & \uparrow & & \uparrow & & \\ \longrightarrow & \mathrm{Hom}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B),\mathrm{K}_1(B)) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^0(X)}(\mathrm{K}^0(X),\mathrm{K}_1(B)) & \longrightarrow & \\ \longrightarrow & \mathrm{Ext}^{u}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) & \longrightarrow & \mathrm{Ext}_c(\mathrm{K}_0(A),\mathrm{K}_1(J)) & \longrightarrow & 0 \\ & \uparrow r_1 & & \uparrow r_0 & & \\ \longrightarrow & \mathrm{Ext}^{u}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B),\mathrm{K}_1(B)) & \longrightarrow & \mathrm{Ext}_{\mathrm{K}^0(X)}(\mathrm{K}^0(B),\mathrm{K}_1(B)) & \longrightarrow & 0. \end{array}$$

From the Five Lemma we obtain that r_1 is also an isomorphism. The equality of b) follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SB & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow S\tilde{\psi} & & \uparrow \varphi & & \uparrow \tilde{\psi} & & \\ 0 & \longrightarrow & SB & \longrightarrow & M & \underset{\psi\varphi\psi^{-1}}{\longrightarrow} & B & \longrightarrow & 0 \end{array}$$

if $[\varphi] = \xi$.

PROPOSITION 4.4. *$G'^0 = \iota + \mathrm{Hom}_c(\mathrm{K}_0(A),\mathrm{K}_0(J))$ and γ_0' is an isomorphism.*

Proof. Let $A = \overline{\cup A_n}$ with A_n finite dimensional. Denote as before by $i_{m,n}$: : $A_n \to A_m$ the inclusion of A_n in A_m and by i_n the inclusion of A_n in A. It is proved in [20], Lemma 1.2 that the restriction $i_{m,n} : \mathrm{Hom}^0(A_m,A) \to \mathrm{Hom}^0(A_n,A)$

is a fibration, and hence $\mathrm{Map}(X, \mathrm{Hom}^0(A_m, A)) \to \mathrm{Map}(X, \mathrm{Hom}^0(A_n, A))$ is also a fibration [27, Theorem 7.10, p. 31].

. Let $\mathrm{End}(A)$ denote the space $\mathrm{Hom}(A, A)$ pointed by id_A and $\mathrm{End}^0(A)$ denote the connected component of id_A in $\mathrm{End}(A)$. Then [20, Lemma 1.4] $\mathrm{Map}(X, \mathrm{Aut}^0(A)) \to$ $\to$ $\mathrm{Map}(X, \mathrm{End}^0(A))$ is a weak homotopy equivalence. It is obvious that $\mathrm{Map}(X, \mathrm{End}^0(A))$ is homeomorphic to the inverse limit $\varprojlim \mathrm{Map}(X, \mathrm{Hom}^0(A_n, A))$.

The proof of [27, Theorem 4.8, p. 433] shows that

$$H = \varprojlim{}^1 \pi_1(\mathrm{Map}(X, \mathrm{Hom}^0(A_n, A)))$$

acts free on the pointed set $\pi_0(\mathrm{Map}(X, \mathrm{End}^0(A)))$ and that

$$\pi_0(\mathrm{Map}(X, \mathrm{End}^0(A)) \to \varprojlim \pi_0(\mathrm{Map}(X, \mathrm{Hom}^0(A_n, A)))$$

gives a bijection

$$\pi_0(\mathrm{Map}(X, \mathrm{End}^0(A))/H \to \varprojlim \pi_0(\mathrm{Map}(X, \mathrm{Hom}^0(A_n, A))).$$

Let us observe that $\pi_1(\mathrm{Map}(X, \mathrm{Hom}^0(A_n, A)))$ is naturally isomorphic to

$$[SX, \mathrm{Hom}^0(A_n, A)] \simeq \mathrm{Hom_c}(\mathrm{K}_0(A_n), \mathrm{K}_1(J))$$

by Lemma 3.6 (use also Remark 3.7). It follows also from Lemma 3.6 that there exists a commutative diagram

$$\begin{array}{ccc} [X, \mathrm{End}^0(A)] & \longrightarrow & \varprojlim \pi_0(\mathrm{Map}(X, \mathrm{Hom}^0(A_n, A))) \\ \downarrow{\scriptstyle \alpha_0 - \iota} & & \downarrow \\ \mathrm{Hom_c}(\mathrm{K}_0(A), \mathrm{K}_0(J)) & \longrightarrow & \varprojlim \mathrm{Hom_c}(\mathrm{K}_0(A_n), \mathrm{K}_0(J)) \end{array}$$

in which the bottom arrow is an isomorphism and the right vertical arrow is a bijection.

We obtain the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim{}^1 \mathrm{Hom_c}(\mathrm{K}_0(A_n), \mathrm{K}_1(J)) & \to & [X, \mathrm{End}^0(A)] & \to & \iota + \mathrm{Hom_c}(\mathrm{K}_0(A), \mathrm{K}_0(J)) & \to & 0 \\ & & \downarrow{\scriptstyle \simeq} & & \uparrow & & & & \\ & & \mathrm{Ext_c}(\mathrm{K}_0(A), \mathrm{K}_1(J)) & \xleftarrow{\gamma_0'} & \ker \alpha_0 & & & & \end{array}$$

in which the first vertical arrow is an isomorphism by Lemma 2.6 and the top horizontal line is an exact sequence of pointed sets.

The proof of this proposition will be concluded if we show that this diagram is commutative.

Let $\varphi \in \mathrm{Map}(X, \mathrm{Aut}^0(A))$, $[\varphi] \in \ker\alpha_0$. Our assumption shows that $\varphi|A_n \in$ $\in \mathrm{Map}(X, \mathrm{Hom}^0(A_n, A))$ is homotopic to the function $x \to i_n$ via a homotopy $\psi_n \in$ $\in \mathrm{Map}([0,1] \times X|[0,1] \times \{x_0\}, \mathrm{Hom}^0(A_n, A))$ i.e. $\psi_n|\{1\} \times X = \varphi|A_n$, $\psi_n(0,x) =$ $= i_n \cdot \psi_n$ defines a morphism $f_n : A_n \to E_\varphi$ such that $\rho \circ f_n = i_n$ (ρ is the quotient map $E_\varphi \to A$). It follows from Lemma 2.6 that $\gamma_0'([\varphi])$ is represented in $\varprojlim^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_1(J))$ by the sequence $(\lambda_n)_{n\in\mathbf{N}} = (k_0(f_{n+1}) \circ \mathrm{K}_0(i_{n+1,n}) -$ $-\mathrm{K}_0(f_n))_{n\in\mathbf{N}}$ (we identify $\mathrm{K}_0(SJ)$ with $\mathrm{K}_1(J)$ by Bott periodicity).

Let us define $\eta_n' : [0,1] \times X \to \mathrm{Hom}^0(A_n, A)$ by $\eta_n'(t,x) = \psi_{n+1}(2t,x)|A_n$ for $t \in \left[0, \frac{1}{2}\right]$ and by $\eta_n'(t,x) = \psi_n(2-2t,x)$ for $t \in \left[\frac{1}{2}, 1\right]$ then $\eta_n'(0,x) = \eta_n'(t,x_0) =$ $= \eta_n'(1,x) = i_n$ and thus η_n' factors to a mapping $\eta_n : SX \to \mathrm{Hom}^0(A_n, A)$ of pointed spaces. Then [27, Theorem 4.8, p. 433] φ is represented in $\varprojlim^1[SX, \mathrm{Hom}^0(A_n, A)]$ by the sequence $([\eta_n])_{n\in\mathbf{N}}$. Since $[\eta_n]$ is sent to $\lambda_n \in \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_1(J))$ under the isomorphism of Lemma 3.6 (see also 3.7) the commutativity of the diagram follows.

We now turn to the case A is unital.

Let us first note that $A \otimes \mathcal{K}$ is completely non unital and that there exists a morphism $\mathrm{Aut}(A) \to \mathrm{Aut}(A \otimes \mathcal{K})$ given by $\eta \to \eta \otimes \mathrm{id}_{\mathcal{K}}$. We have denoted as usual by $\mathcal{K}$, the C^*-algebra of compact operators on a separable Hilbert space.

Denote by $\sigma : [X, \mathrm{Aut}(A)] \to [X, \mathrm{Aut}(A \otimes \mathcal{K})]$ the corresponding morphism. The following lemma is folklore and identifies the range of this morphism. We sketch its proof for the convenience of the reader.

LEMMA 4.5. *Let* $\psi \in \mathrm{Map}(X, \mathrm{Aut}(A \otimes \mathcal{K}))$ *then* $[\psi]$ *is in the range of* σ *if and only if* $\mathrm{K}_0([\varphi])([1]) = [1]$.

Proof. One implication is obvious.

Let $(e_{n,m})_{n,m\in\mathbf{N}}$ denote a matrix unit of $\mathcal{K}$. Denote as before $B = C(X, A)$ and let $\tilde{\psi} : B \otimes \mathcal{K} \to B \otimes \mathcal{K}$ be the morphism defined by ψ, $f_{m,n} = 1 \otimes e_{n,m} \in B \otimes \mathcal{K}$. If $\tilde{\psi}(f_{m,n}) = f_{m,n}$ for any n, m then it follows that there exists $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ such that $\tilde{\psi} = \tilde{\varphi} \otimes \mathrm{id}_{\mathcal{K}}$ ($\tilde{\varphi}$ is the morphism $B \to B$ defined by φ). We obtain that ψ is in the range of $\mathrm{Map}(X, \mathrm{Aut}(A)) \to \mathrm{Map}(X, \mathrm{Aut}(A \otimes \mathcal{K}))$. In general, the assumption that $\mathrm{K}_0(\psi)([1]) = \alpha_0([\psi])([1]) = [1]$ shows that $\tilde{\psi}(f_{00})$ is equivalent to f_{00} (use Proposition 3.5). (If we identify $\mathrm{K}_0(A)$ with $\mathrm{K}_0(A \otimes \mathcal{K})$ by stability, then $[1] = [f_{00}]$.) Let $v \in B \otimes \mathrm{K}$ be such that $v^*v = f_{00}$, $vv^* = \tilde{\psi}(f_{00})$. Let $u = \sum_{n\in\mathbf{N}} \tilde{\psi}(f_{n0})vf_{n0}$, the convergence being in the strict topology of $M(B \otimes \mathcal{K})$, then u is a unitary in $M(B \otimes \mathcal{K})$ and $\mathrm{ad}_u(f_{n,m}) = \tilde{\psi}(f_{n,m})$. Since the unitary group of $M(D \otimes \mathcal{K})$ is contractible for any C^*-algebra D [9, 19] it follows that there exists a path of unitaries $u_t \in M(B \otimes \mathcal{K})$ connecting u to $1 \in M(B \otimes \mathcal{K})$ and such that the value of u_t

in x_0 is a multiple of the identity of $M(A\otimes\mathcal{K})$ (use the fact that $U(M(A\otimes\mathcal{K}))$ is a direct summand of $U(M(B\otimes\mathcal{K}))$. Then $t \to \mathrm{ad}_{u_t} \circ \psi$ is a homotopy of ψ in $\mathrm{Map}(X, \mathrm{Aut}(A\otimes \otimes\mathcal{K}))$ to a mapping in the range of $\mathrm{Map}(X, \mathrm{Aut}(A)) \to \mathrm{Map}(X, \mathrm{Aut}(A\otimes\mathcal{K}))$.

We get the following corollary:

COROLLARY 4.6. a) $G^{1'} = G^{0'} \cap \{\xi \in \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)),\ \alpha_1(\xi)([1]) = [1]\}$.
b) *The restriction of* σ *to* $\ker\alpha_1$ *maps* $\ker\alpha_1$ *onto* $\ker\alpha_0$.

Proof. Use Proposition 4.1 and Lemma 4.5.

LEMMA 4.7. γ_1' *is an isomorphism.*

Proof. We first prove that γ_1' is injective.

Suppose that there exists $\tau \in \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(E_\varphi))$ such that $\mathrm{K}_0(\rho)\circ\tau = = \mathrm{id}_{\mathrm{K}_0(A)}$ and $\tau([1]) = [1]$, this means precisely that τ is a unital splitting of $\gamma_1'([\varphi]) = =$ the class of $0 \to \mathrm{K}_0(SJ) \to \mathrm{K}_0(E_\varphi) \overset{\mathrm{K}_0(\rho)}{\to} \mathrm{K}_0(A) \to 0$.

Let $A = \overline{\cup A_n}$ with A_n finite dimensional and let $i_{m,n}$, i_n have the same meaning as in the discussion preceding Lemma 3.6. It is an immediate consequence of Proposition 3.5 that there exists a morphism $\eta_n : A_n \to E_\varphi$ such that $\rho\circ\eta_n = i_n$ and $\mathrm{K}_0(\eta_n) = \tau\circ\mathrm{K}_0(i_n)$, moreover any two such morphisms are unitary conjugated. Using induction on n one can define morphisms $\eta_n : A_n \to E_\varphi$ as above such that $\eta_m|A_n = \eta_n$ for any $m \geqslant n$. This can be done at follows. Suppose that we have defined $\eta_1, \dots, \eta_n$ as above. Choose $\eta_{n+1}' : A_{n+1} \to E_\varphi$ arbitrarily such that $\rho\circ\eta_{n+1}' = i_{n+1}$ and $\mathrm{K}_0(\eta_{n+1}') = \tau\circ\mathrm{K}_0(i_n)$. Then there exists $u \in U(A)$ such that $\eta_{n+1}'|A_n = \mathrm{ad}_u\circ\eta_n$. Let $\eta_{n+1} = \mathrm{ad}_u^{-1}\circ\eta_{n+1}'$. The sequence $(\eta_n)_{n\in\mathbb{N}}$ defines a lifting $\eta : A \to E_\varphi$ for $\rho : \rho\circ\eta = \mathrm{id}_A$. The map η defines by evaluating at $(t,x) \in [0,1]\times X$ a mapping $\psi : [0,1]\times X \to \mathrm{End}^0(A)$. Then ψ defines a path connecting φ to the constant mapping in $\mathrm{Map}(X, \mathrm{End}^0(A))$. Using the isomorphism $[X, \mathrm{Aut}^0(A)] \simeq [X, \mathrm{End}^0(A)]$ [20, Lemma 1.4] we obtain that $\ker\gamma_1' \simeq \{0\}$.

To prove the surjectivity of γ_1' consider the diagram

$$\begin{array}{ccccccc}
\mathrm{K}_1(J) & \overset{\mathrm{Ad}}{\longrightarrow} & \ker\alpha_1 & \longrightarrow & \ker\alpha_0 & \longrightarrow & 0 \\
\Vert & & \downarrow{\scriptstyle\gamma_1'} & & \downarrow{\scriptstyle\gamma_0'} & & \\
\mathrm{Hom}(\mathbb{Z}, \mathrm{K}_1(J)) & \longrightarrow & \mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_1(J)) & \longrightarrow & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_1(J)) & \longrightarrow & 0.
\end{array}$$

The bottom line is exact by 2.8 c). Ad is defined as follows, let $u \in U(J)$, u being represented by a function $u : X \to U(A)$ such that $u(x_0) = 1$. Let $\mathrm{Ad}([u])$ be the class of $x \to \mathrm{ad}_{u(x)} \in \mathrm{Aut}(A)$. It follows that the diagram is commutative and hence γ_1' is onto (a simple diagram chase).

We put together the results of this section in the following theorem.

4.8. Let (X, x_0) be a pointed compact connected CW-complex and A an AF--algebra which satisfies 3.1 a) and b). Let $i = 0$ if $1 \notin A$, $i = 1$ if $1 \in A$.

Denote by Ω the lattice of ideals of A, let $B = C(X, A)$, $J = C_0(X \setminus \{x_0\}, A)$. The groups $\mathrm{K}_j(B)$ and $\mathrm{K}_j(J)$ are Ω-filtered $\mathrm{K}^0(X)$-modules ($j \in \{0, 1\}$).

THEOREM. *The range of* α_0 *is* $G^0 = \mathrm{id}_{\mathrm{K}_0(B)} + \mathrm{Hom}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B), \mathrm{K}_0(J))$ *and the range of* α_1 *is* $G^1 = \{\eta \in G^0,\ \eta([1]) = [1]\}$.

The product in G^i is the composition of morphisms. Moreover

$$\gamma_0 : \ker\alpha_0 \to \mathrm{Ext}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B), \mathrm{K}_1(B))$$

and

$$\gamma_1 : \ker\alpha_1 \to \mathrm{Ext}^{u}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B), \mathrm{K}_1(B))$$

are isomorphisms. We obtain exact sequences

$$0 \to \mathrm{Ext}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B), \mathrm{K}_1(B)) \to [X, \mathrm{Aut}(A)] \to G^0 \to 0$$

if $1 \notin A$, and

$$0 \to \mathrm{Ext}^{u}_{\mathrm{K}^0(X),c}(\mathrm{K}_0(B), \mathrm{K}_1(B)) \to [X, \mathrm{Aut}(A)] \to G^1 \to 0$$

if $1 \in A$. These exact sequences are natural in (X, x_0); the kernels are determined by Lemma 4.3 b).

Here are some consequences of the naturality in X of the exact sequence.

COROLLARY 4.9. *Let* (X, x_0), (Y, y_0) *be pointed compact connected CW-complexes. Suppose* $f : (X, x_0) \to (Y, y_0)$ *induces an isomorphism of the K-groups then* $f^* : [Y, \mathrm{Aut}(A)] \to [X, \mathrm{Aut}(A)]$ *is an isomorphism. If* $\mathrm{K}^0(f)$ *is an isomorphism and* $\mathrm{K}^1(Y) = 0$ *then the exact sequences of the preceding theorem split.*

Proof. Denote by $G^i(X)$ the range of $\alpha_i : [X, \mathrm{Aut}^0(A)] \to \mathrm{End}(\mathrm{K}_0(C(X, A)))$.

By the naturality of the exact sequences there exist commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}^1(Y)) & \longrightarrow & [Y, \mathrm{Aut}(A)] & \longrightarrow & G^0(Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{K}^1(f)_*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle G^0(f)} & & \\
0 & \longrightarrow & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}^1(X)) & \longrightarrow & [X, \mathrm{Aut}(A)] & \longrightarrow & G^0(X) & \longrightarrow & 0
\end{array}$$

if $1 \notin A$ and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^u_c(\mathrm{K}_0(A), \mathrm{K}^1(Y)) & \longrightarrow & [Y, \mathrm{Aut}(A)] & \longrightarrow & G^1(Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{K}^1(f)} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle G^1(f)} & & \\
0 & \longrightarrow & \mathrm{Ext}^u_c(\mathrm{K}_0(A), \mathrm{K}^1(X)) & \longrightarrow & [X, \mathrm{Aut}(A)] & \longrightarrow & G^1(X) & \longrightarrow & 0
\end{array}$$

if $1 \in A$.

$G^0(f)$ is obtained from the commutative diagram

$$\begin{array}{ccc} \iota_Y + \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(C_0(Y \setminus \{y_0\}, A))) & \xrightarrow{\mu_Y} & G_0(Y) \\ \downarrow \mathrm{K}_0(f)_* & & \downarrow G^0(f) \\ \iota_X + \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(C_0(X \setminus \{x_0\}, A))) & \xrightarrow{\mu_X} & G_0(X) \end{array}$$

$\iota_X, \iota_Y, \mu_X, \mu_Y$ have the same meaning as in 5.1 and $f^* : C_0(Y \setminus \{y_0\}, A) \to \to C_0(X \setminus \{x_0\}, A)$ is given by $b \to b \circ f$.

The first part of the corollary is a consequence of the Five Lemma. The second part follows from the fact that $\alpha_i : [Y, \mathrm{Aut}(A)] \to G^i(Y)$ and $G^i(f) : G^i(Y) \to G^i(X)$ are isomorphisms and hence $f^* \circ \alpha_i^{-1} \circ G^i(f)^{-1}$ is well defined and gives the described splitting.

5. THE CASE A SIMPLE AND X A HOMOTOPY COGROUP

Let A be a simple AF-C^*-algebra not stably isomorphic to $\mathcal{K}$, (X, x_0) a pointed compact connected CW complex which is also a homotopy cogroup (see 3.7 for definition and notation).

Let B and J be as in the preceding sections and denote by A' the mapping cone of the inclusion $\mathbf{C} \to A$ if $1 \in A$. We shall prove that $[X, \mathrm{Aut}(A)]$ is naturally isomorphic to $\mathrm{KK}^0(A, J)$ if $1 \notin A$ or to $\mathrm{KK}^0(A', SJ)$ if $1 \in A$.

For the definition and the basic properties of the KK-bifunctor the reader is referred to the original papers of G. G. Kasparov [16, 17] or to the book of B. Blackadar [5]. Our approach uses Cuntz's "quasihomomorphism picture" of KK^0-groups [5, 8].

We shall first define natural transformations $c_0 : [X, \mathrm{Aut}(A)] \to \mathrm{KK}^0(A, J)$ if $1 \notin A$ and $c_1 : [X, \mathrm{Aut}(A)] \to \mathrm{KK}^0(A', SJ)$ if $1 \in A$.

Let $\varphi_0 \in \mathrm{Map}(X, \mathrm{Aut}(A))$ denote the constant function. For $\varphi \in \mathrm{Map}(X, \mathrm{Aut}(A))$ we shall denote by $\Phi(\varphi) \in \mathrm{End}(B)$ the morphism defined by φ, i.e. $\Phi(\varphi)(a)(x) = = \varphi_x(a)$ for any $a \in A$, $x \in X$. It follows that $\Phi(\varphi)(a) - \Phi(\varphi_0)(a) \in J$ for any $a \in A$ and hence the pair $(\Phi(\varphi), \Phi(\varphi_0))$ is a quasihomomorphism from A to J. We shall denote by $c_0([\varphi])$ the corresponding element in $\mathrm{KK}^0(A, J)$ [5, 8]. If A is unital denote by $\psi, \psi_0 : A' \to C([0,1] \times X, A) \subset M(SJ)$ the morphism defined as follows. Recall first that $A' = \{f : [0,1] \to A,\ f(0) = 0,\ f(1) \in C\}$. Then $\psi(f)(t,x) = \varphi_x(f(t))$, $\psi_0(f)(t,x) = f(t)$ for any $f \in A'$. It follows that $\psi(f) - \psi_0(f) \in SJ$ for any $f \in A'$. We shall define $c_1([\varphi])$ to be the class of the quasihomomorphism (ψ, ψ_0) in $\mathrm{KK}^0(A', SJ)$ [5, 8].

LEMMA 5.1. *c_i is a morphism* $(i \in \{0, 1\})$.

Proof. We shall prove the lemma for $i = 0$, for $i = 1$ the proof is similar.

It is a well known fact that the multiplication in $[X, \mathrm{Aut}(A)]$ may be defined also by $[\varphi][\psi] = [(\varphi \vee \psi) \circ \theta]$ (θ is the comultiplicator of X). There exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \oplus J & \longrightarrow & C(X \vee X, A) & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \nu & & \downarrow \theta^* & & \| & & \\ 0 & \longrightarrow & J \oplus J & \longrightarrow & C(X \vee X, A) & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

The quotient maps are obtained by evaluation at the base point.

It follows from the assumptions on (X, x_0) that ν is a homotopy equivalence on each direct summand. This shows that if $(\xi, \zeta) \in \mathrm{KK}^0(A, J) \oplus \mathrm{KK}^0(A, J) \simeq \simeq \mathrm{KK}^0(A, J \oplus J)$ then $\nu_*(\xi, \zeta) = \xi + \zeta$. It follows from the definitions that

$$\nu_*(c_0([\varphi]), c_0([\psi])) = c_0([(\varphi \vee \psi) \circ \theta])$$

and hence $c_0([\varphi][\psi]) = c_0([\varphi]) + c_0([\psi])$.

THEOREM 5.2. *Suppose $A \neq \mathcal{K}$ is simple. The maps $c_0 : [X, \mathrm{Aut}(A)] \to \to \mathrm{KK}^0(A, J)$ if $1 \notin A$ and $c_1 : [X, \mathrm{Aut}(A)] \to \mathrm{KK}^1(A', J)$ if $1 \in A$ are isomorphisms for X a homotopy cogroup.*

Proof. Let ι be the composition $\mathrm{K}_0(A) \ni \xi \to [1] \otimes \xi \in \mathrm{K}^0(X) \otimes \mathrm{K}_0(A) \simeq \mathrm{K}_0(B)$.

There exists a commutative diagram for $1 \notin A$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_1(J)) & \longrightarrow & [X, \mathrm{Aut}(A)] & \longrightarrow & \iota + \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(J)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow c_0 & & \downarrow \mathrm{id} - \iota & & \\ 0 & \longrightarrow & \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_1(J)) & \longrightarrow & \mathrm{KK}(A, J) & \longrightarrow & \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(J)) & \longrightarrow & 0 \end{array}$$

in which the top line is exact by Proposition 4.4 and the bottom line is exact by the Universal Coefficient Theorem [24]. The first vertical arrow is a morphism since it is the restriction of c_0. The third vertical arrow is also a morphism. This can be viewed as follows. Let $F_1, F_2 \in \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(J))$,

$$\mu_X : \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \to \mathrm{Hom}_{\mathrm{K}^0(X)}(\mathrm{K}_0(B), \mathrm{K}_0(B))$$

be as in 5.1, then

$$\mu_X(\iota + F_j)(\iota(a) + b_0) = \iota(a) + F_j(a) + b_0$$

for any $a \in \mathrm{K}_0(A)$, $b_0 \in \mathrm{K}_0(J)$ since $\tilde{\mathrm{K}}^0(X)^2 = 0$. This shows that

$$\mu_X(\iota + F_1)\mu_X(\iota + F_2) = \mu_X(\iota + F_1 + F_2).$$

The filtrations are trivial if A is simple and hence

$$\mathrm{Ext_c}(\mathrm{K_0}(A), \mathrm{K_1}(J)) \simeq \mathrm{Ext}(\mathrm{K_0}(A), \mathrm{K_1}(J))$$

and

$$\mathrm{Hom_c}(\mathrm{K_0}(A), \mathrm{K_1}(J)) \simeq \mathrm{Hom}(\mathrm{K_1}(A), \mathrm{K_0}(J))$$

are isomorphisms. This shows that c_0 is an isomorphism.

Let us prove now that c_1 is an isomorphism. Since $\mathrm{K_1}(A') \simeq \mathrm{K_0}(A)/\mathbf{Z}[1]$, $\mathrm{K_0}(A') \simeq 0$ we obtain using Corollary 4.6 a), Lemma 4.7 and the Universal Coefficient Theorem [24] that there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext_c^u}(\mathrm{K_0}(A), \mathrm{K_1}(J)) & \to & [X, \mathrm{Aut}(A)] & \to & G_1 & \to & 0 \\ & & \downarrow h & & \downarrow c_1 & & \downarrow & & \\ 0 & \to & \mathrm{Ext}(\mathrm{K_0}(A)/\mathbf{Z}[1], \mathrm{K_1}(J)) & \to & \mathrm{KK}^0(A, SJ) & \to & \mathrm{Hom}(\mathrm{K_0}(A)/\mathbf{Z}[1], \mathrm{K_0}(J)) & \to & 0 \end{array}$$

Let us determine the morphism h.

Suppose that $\varphi \in \mathrm{Map}(X, \mathrm{Aut}^0(A))$, then $[\varphi] \in \ker \alpha_1$, if and only if $\mathrm{K_0}(\Phi(\varphi)) = \mathrm{K_0}(\Phi(\varphi_0)) = \iota$.

Then there exists a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^2 J & \longrightarrow & SE_\varphi & \longrightarrow & SA & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^2 J & \longrightarrow & SE_\varphi & \longrightarrow & SA & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbf{C} & & \mathbf{C} & & \end{array}$$

(We have denoted by E'_φ the mapping cone of the inclusion $\mathbf{C} \to E_\varphi$). The corresponding diagram of $\mathrm{K_1}$-groups shows that h associates to the class of the compatible extension with order unit

$$0 \to \mathrm{K_1}(J) \to (\mathrm{K_0}(E), [1]) \to (\mathrm{K_0}(A), [1]) \to 0$$

the class of

$$0 \to \mathrm{K_1}(J) \to \mathrm{K_0}(E_\varphi)/\mathbf{Z}[1] \to \mathrm{K_1}(A)/\mathbf{Z}[1] \to 0$$

in $\mathrm{Ext}(\mathrm{K_0}(A)/\mathbf{Z}[1], \mathrm{K_1}(J))$ [5]. The morphism h is obviously an isomorphism if A is simple. The Five Lemma shows that c_1 is also an isomorphism.

6. THE SAMELSON PRODUCT

In this section we shall briefly study the Samelson product. It turns out that it does not vanish in general and hence the classifying space of $\mathrm{Aut}^0(A)$ is not a H--space [27]. This shows that the set of isomorphism classes of locally trivial fields of C^*-algebras on X with fiber A cannot be endowed with a natural group structure for any compact CW-complex X [27, p. 475, 7.8].

Let us recall the definition of the Samelson product [27, p. 467] it is a pairing

$$\langle\cdot,\cdot\rangle : [X, \mathrm{Aut}(A)] \times [Y, \mathrm{Aut}(A)] \to [X \wedge Y, \mathrm{Aut}(A)]$$

defined by

$$\langle[\varphi], [\psi]\rangle = [\eta], \ \eta(x \wedge y) = \varphi(x)\psi(y)\varphi(x)^{-1}\psi(y)^{-1}.$$

If $X = \mathbb{S}^n$, $Y = \mathbb{S}^m$ this gives a pairing

$$\pi_n(\mathrm{Aut}(A)) \times \pi_m(\mathrm{Aut}(A)) \to \pi_{n+m}(\mathrm{Aut}(A)).$$

Let us observe that $\alpha(\langle a, b\rangle)$ depends only on $\alpha(a)$ and $\alpha(b)$ (we omit various subscripts or superscripts of α) and it is defined by

$$j \circ \mu_{X\wedge Y}(\alpha(\langle a, b\rangle)) = \left(\mu'_X(a)\mu'_Y(b)\mu'_X(a)^{-1}\mu'_Y(b)^{-1}\right) \circ j. \tag{6.1}$$

Here $j : \mathrm{K}_0(C(X \wedge Y, A)) \to \mathrm{K}_0(C(X \times Y, A))$ is the obvious inclusion, $\mu'_X(a)$ is obtained out of $\mu_X(\alpha(a)) : \mathrm{K}_0(C(X, A)) \to \mathrm{K}_0(C(X, A))$ as a $\mathrm{K}^0(X \times Y)$-linear morphism $\mu'_X(a) : \mathrm{K}_0(C(X \times Y, A)) \to \mathrm{K}_0(C(X \times Y, A))$ by extending the ring using $\mathrm{K}^0(X) \to \mathrm{K}^0(X \times Y)$. μ'_Y is defined similarly.

Moreover, since $\ker\alpha$ is represented by approximately inner loops we obtain the following result:

PROPOSITION 6.2. a) *$\mu(\langle a, b\rangle)$ depends only on $\alpha(a)$ and $\alpha(b)$ and its formula is given by* (6.1).

b) *$\langle\ker\alpha, [Y, \mathrm{Aut}(A)]\rangle$ and $\langle[Y, \mathrm{Aut}(A)], \ker\alpha\rangle$ are contained in* $\ker\alpha$.

c) $\langle\ker\alpha, \ker\alpha\rangle = 0$.

d) *$\bigoplus_{n\geqslant 0} \pi_n(\mathrm{Aut}(A))$ with the Samelson product is gradedly isomorphic to*

$$\mathrm{Aut}(\mathrm{K}_0(A), \Sigma(A)) \oplus \left(\bigoplus_{k\geqslant 1} \mathrm{Ext}^{p(k)}(\mathrm{K}_0(A), \mathrm{K}_0(A))\right)$$

with the product $\langle a, b\rangle' = aba^{-1}b^{-1}$ if a and b are of degree 0, $\langle a, b\rangle' = aba^{-1} - b$ if a is of degree 0 and b of degree $\geqslant 1$, and $\langle a, b\rangle' = ab - ba$ if a, b are both of degree $\geqslant 1$.

Here $\mathrm{Ext}^{p(k)}(\mathrm{K}_0(A), \mathrm{K}_0(A))$ denotes

$\mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(A))$ if k is even and $1 \notin A$,

$\mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(A)) \cap \mathrm{Hom}(\mathrm{K}_0(A)/\mathbf{Z}[1], \mathrm{K}_0(A))$ if k is even and $1 \in A$,

$\mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A))$ if k is odd, $1 \notin A$,

$\mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_0(A))$ if k is odd and $1 \in A$.

$\Sigma(A)$ is the scale of the ordered group $\mathrm{K}_0(A)$.

For the last part see also [20].

REMARK 6.3. The preceding theorem gives a necessary condition on A in order to exist a natural group structure on the set of isomorphism classes of locally trivial fields of AF-algebras with fiber A. Indeed, if such a natural group structure would exist then every field on $\mathbb{S}^n \vee \mathbb{S}^m$ would have an extension on $\mathbb{S}^n \times \mathbb{S}^m$ thus forcing the vanishing of the Samelson product on $\pi_{n-1}(\mathrm{Aut}(A)) \times \pi_{m-1}(\mathrm{Aut}(A))$ [27, p. 476, 7.10]. This cannot happen if A is simple and $\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(A))$ is not commutative. d) also identifies the action of $\pi_0(\mathrm{Aut}(A))$ on $[X, \mathrm{Aut}(A)]$ for $X = \mathbb{S}^n$.

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