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AN ISOMETRY FROM THE HAAGERUP TENSOR PRODUCT INTO COMPLETELY BOUNDED OPERATORS

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The space of completely bounded multilinear forms on a C^*-algebra is the dual space of a Haagerup tensor product of the algebra with itself [8], [14]. Another close relationship between this tensor product and completely bounded operators is investigated in this paper. R. R. Smith has shown that the operator θ from $B(H)\otimes_{\mathrm{h}} \otimes_{\mathrm{h}} B(H)$ into $CB(K(H))$, defined by

$$\theta(x \otimes y) = xby \qquad (x, y \in B(H),\ b \in K(H)),$$

is an isometry [16]. Definitions and notation are given later in the introduction. His proof is based on a technical decomposition lemma for operators in $B(H^{\infty}, H)$ and $B(H, H^{\infty})$. We give a proof that θ is an isometry based on a lemma which approximates a completely bounded multilinear form by a completely bounded normal multilinear form in a weak-$*$ topology. The lemma depends on the representation of a completely bounded multilinear form [14] and the weak-$*$ density of the normal states in the set of all states on a von Neumann algebra.

If N is a C^*-algebra, the same algebraically defined map θ induces a linear operator φ from $N \otimes_{\mathrm{h}} N$ into $CB(N)$ with $\|\varphi\| = 1$. This follows by a standard Cauchy-Schwarz argument [4, p. 421] of which the essential part is

$$|\langle\theta(u)(x)\xi, \eta\rangle| = \left|\sum\langle xb_j\xi, a_j^*\eta\rangle\right| \leqslant \left(\sum\|xb_j\xi\|^2\right)^{1/2}\left(\sum\|a_j^*\eta\|^2\right)^{1/2} \leqslant$$
$$\leqslant \|x\|\left(\sum\langle b_j^*b_j\xi, \xi\rangle\right)^{1/2}\left(\sum\langle a_ja_j^*\eta, \eta\rangle\right)^{1/2} \leqslant$$
$$\leqslant \|x\| \cdot \left\|\sum a_ja_j^*\right\|^{1/2} \cdot \left\|\sum b_j^*b_j\right\|^{1/2} \cdot \|\xi\| \cdot \|\eta\|$$

where $u = \sum a_j \otimes b_j$. It is well known that this operator $\varphi(= \theta)$ is an isometry if N is a matrix algebra $M_n(\mathbb{C})$ [4, p. 418]. Further φ is not an isometry if the centre Z

of N is non-trivial because $1 \otimes z - z \otimes 1$ is in the kernel of φ for all z in Z. If N is a factor, then φ is an isometry from $N \otimes_{\mathrm{h}} N$ into $CB(N)$ (Theorem 3). An application of Haagerup's characterization of injectivity in terms of the decomposability of a completely bounded linear operator shows that if all operators from all unital matrix subalgebras of a factor N into N are isometric restrictions of operators $\varphi(u)$, then N is injective. This is the main result of Section 3. Section 4 is devoted to a sketch of the multilinear cases.

The remainder of this introduction is devoted to the basic definitions, notation, and some discussion of the results. The C^*-algebra of $n \times n$ complex matrices will be denoted by M_n or $M_n(\mathbb{C})$. The algebra of all bounded linear operators on a Hilbert space H is denoted by $B(H)$, and the algebra of compact operators by $K(H)$. If A and B are C^*-algebras, let $CB(A, B)$ denote the algebra of completely bounded linear operators from A into B. Recall that a linear operator φ is completely bounded if

$$\|\varphi\|_{\mathrm{cb}} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$$

is finite, where φ_n is the operator from $M_n(A) = A \otimes M_n(\mathbb{C})$ into $M_n(B)$ given by

$$\varphi_n((a_{ij})) = (\varphi(a_{ij}))$$

for all (a_{ij}) in $M_n(A)$. The Haagerup tensor product $A \otimes_{\mathrm{h}} B$ of two C^*-algebras A and B is the completion of the algebraic tensor product $A \otimes B$ in the norm

$$\|u\|_{\mathrm{h}} = \inf\left\{ \left\|\sum_1^n x_j x_j^*\right\|^{1/2} \left\|\sum_1^n y_j^* y_j\right\|^{1/2} : u = \sum_1^n x_j \otimes y_j, x_j \in A, y_j \in B \right\}.$$

For the definitions of completely bounded multilinear operators, multiple Haagerup tensor products, and the representation theorems see [5], [14], [4], [8].

If N is a von Neumann algebra, let $CB(N \times N, \mathbb{C})$ denote the space of completely bounded bilinear forms on $N \times N$ and let $CB_\sigma(N \times N, \mathbb{C})$ denote the space of completely bounded normal bilinear forms on N with both of these spaces having the completely bounded norm. A multilinear form is said to be *normal* if it is separately ultraweakly continuous [4]. Note that $CB(N \times N, \mathbb{C})$ is the dual space of $N \otimes_{\mathrm{h}} N$ by [8], [14] and that this induces a natural weak-$*$ topology on $CB(N \times N, \mathbb{C})$. The unit ball of $CB_\sigma(N \times N, \mathbb{C})$ is weak-$*$ dense in the unit ball of $CB(N \times N, \mathbb{C})$ (Proposition 1). That the map θ from $B(H) \otimes_{\mathrm{h}} B(H)$ into $CB(K(H))$ is an isometry follows easily from this (Corollary 2). This result may also be interpreted in the notation of [8]. If N is a factor, the isometry of the operator φ, where $\varphi : N \otimes_{\mathrm{h}} N \to CB(N)$ is defined by $\varphi(a \otimes b)(x) = axb$, is proved using results of S. Popa [15] and R. Longo [11] on the existence of injective subfactors of a factor.

We wish to thank R. R. Smith for sending us the excellent preprint [16], and E. Christensen for suggesting the use of injective subfactors to avoid an assumption of injectivity. There is overlap between our Secton 1 and some of the results of D. P. Blecher and R. R. Smith [2]; their results are a detailed investigation of the Haagerup tensor product and its duality and representation properties.

1. COMPLETELY BOUNDED OPERATORS ON THE COMPACTS

In this section another proof is given of R. R. Smith's result [16] that $B(H)\otimes_{\mathrm{h}}\otimes_{\mathrm{h}}B(H)$ has a natural isometric embedding into $CB(K(H))$. The proof depends on Proposition 1, which is analogous to the result that the unit ball of a Banach space is weak-$*$ dense in the unit ball of the second dual.

PROPOSITION 1. *If N is a von Neumann algebra, the closed unit ball of $CB_\sigma(N\times \times N,\mathbb{C})$ is weak $*$-dense in the closed unit ball of $CB(N\times N,\mathbb{C})$.*

Proof. By the representation theorem for completely bounded bilinear forms a $\varphi\in CB(N\times N,\mathbb{C})$ with $\|\varphi\|_{\mathrm{cb}}=1$ has a representation

$$\varphi(x,y)=\langle\rho(x)a\rho(y)\xi,\eta\rangle$$

for all x and y in N, where ρ is a representation of N on a Hilbert space H_ρ, ξ and η are in H_ρ, and a is in $B(H_\rho)$ satisfying $\|a\|=\|\xi\|=\|\eta\|=1$ [8], [4], [14]. Now the algebraic tensor product $N\otimes N$ is norm dense in $N\otimes_{\mathrm{h}}N$. Hence it is sufficient to show that if $\varepsilon>0$ and F is a finite subset of N, there is a normal representation π of N on a Hilbert space H_π, vectors μ and ν in H_π, and b in $K(H_\pi)$ with $\|\mu\|=\|\nu\|=1$ and $\|b\|\leqslant 1+\varepsilon$ such that $\langle\rho(x)a\rho(y)\xi,\eta\rangle=\langle\pi(x)b\pi(y)\mu,\nu\rangle$ for all $x,y\in F$.

Let X be the linear span of F in N. The closed unit ball of X is compact, since X is finite dimensional, so there are a finite number of elements $x_j, 1\leqslant j\leqslant m$, in X such that for each x in the unit ball of X there exist integers j and k such that

$$\|xx^*-x_jx_j^*\|\quad\text{and}\quad\|x^*x-x_k^*x_k\|$$

are less than $\frac{\delta}{3}$, where $\delta>0$ depends on ε only and is chosen later.
Since the set of normal states on a von Neumann algebra is weak-$*$ dense in the set of all states, there is a normal representation π of N on a Hilbert space H_π and unit vectors μ and ν in H_π such that

$$|\langle\pi(x_jx_j^*)\nu,\nu\rangle-\langle\rho(x_jx_j^*)\eta,\eta\rangle|<\frac{\delta}{3}$$

and

$$|\langle \pi(x_j^* x_j)\mu, \mu\rangle - \langle \rho(x_j^* x_j)\xi, \xi\rangle| < \frac{\delta}{3}$$

for $1 \leqslant j \leqslant m$. Hence

$$|\langle \pi(xx^*)\nu, \nu\rangle - \langle \rho(xx^*)\eta, \eta\rangle| \leqslant \delta\|x\|^2$$

and

$$|\langle \pi(x^* x)\mu, \mu\rangle - \langle \rho(x^* x)\xi, \xi\rangle| \leqslant \delta\|x\|^2$$

for all $x \in X$, where

$$\delta = 1 - (1 + \varepsilon)^{-1}.$$

Hence

$$|\langle a\rho(y)\xi, \rho(x)^*\eta| \leqslant \|\rho(y)\xi\| \cdot \|\rho(x)^*\eta\| \leqslant (1 - \delta)^{-1}\|\pi(y)\mu\| \cdot \|\pi(x)^*\nu\|$$

for all $x, y \in X$ by the inequalities above. This inequality shows that the sesquilinear form F from $\pi(X)\mu \times \pi(X)^*\nu$ into $\mathbb{C}$ given by

$$F(\pi(y)\mu, \pi(x)^*\nu) = \langle a\rho(y)\xi, \rho(x)^*\eta\rangle$$

is bounded by $(1-\delta)^{-1} = 1+\varepsilon$, so there is a continuous linear operator d from $\pi(X)\mu$ into $\pi(X)^*\nu$ such that

$$\langle a\rho(y)\xi, \rho(x)^*\eta\rangle = \langle d\pi(y)\mu, \pi(x)^*\nu\rangle$$

for all $x, y \in X$ and $\|d\| \leqslant 1 + \varepsilon$. Let p be the projection from H_π onto $\pi(X)\mu$ and let $b = dp$ be in $K(H_\pi)$. This completes the proof. ■

Note that we have proved more than was stated: given a completely bounded form on $N \times N$ there is a normal completely bounded form equal to the given form on a finite number of elements and with only slightly greater norm.
Further the normal form arises from a finite rank operator b. If this operator is written as a linear combination of rank one operators, then the given completely bounded form is obtained as a sum of products of normal states.

COROLLARY 2. *The operator θ from $B(H) \otimes_{\mathrm{h}} B(H)$ into $CB(K(H))$ defined by*

$$\theta(x \otimes y)b = xby \qquad (x, y \in B(H), b \in K(H))$$

is an isometry.

Proof. The standard Cauchy-Schwarz argument in the survey paper [5, p. 421] shows that $\|\theta\| \leqslant 1$. It is thus sufficent to prove that if $u = \sum x_j \otimes y_j$ is in $B(H)\otimes$

$\otimes B(H)$ with $\|u\|_h = 1$ and $\varepsilon > 0$, then $\|\theta(u)\|_{cb} \geqslant (1-\varepsilon)(1+\varepsilon)^{-1}$. By the Hahn-Banach Theorem and the duality between $B(H) \otimes_h B(H)$ and $CB(B(H) \times B(H), \mathbf{C})$ [8], [14], there is a completely bounded bilinear form φ on $B(H) \times B(H)$ such that $\|\varphi\|_{cb} = \varphi(u) = 1$. Now by the proof of Proposition 1, there is a normal representation π of $B(H)$ on H_π, unit vectors μ and ν in H_π, and b in $B(H_\pi)$ of finite rank with $\|b\| \leqslant 1 + \varepsilon$ such that

$$\sum \langle \pi(x_j) b \pi(y_j) \mu, \nu \rangle = 1.$$

Since π is a normal representation, it is the amplification of the identity representation of $B(H)$ on H, so H_π is a direct sum of copies of H. The unit vectors μ and ν may be decomposed into components, which lie in the direct summands H. Only a countable number of these components are non-zero, and it is only this part of the amplification that plays a role. Hence μ and ν can be written $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$ with μ_j and ν_j in H. Let $b = (b_{ij})$ be the corresponding decomposition of the operator b into a matrix with entries b_{ij} from $B(H)$ each of finite rank, since b is of finite rank. If e_n is the orthogonal projection onto the first H^n direct summand in the amplification H_π, then $\|e_n b e_n - b\|$ tends to zero as n tends to infinity because b is a compact operator. Hence on writing out the inner product,

$$\left| \sum_k \sum_{i,j=1}^{n} \langle x_k b_{ij} y_k \mu_j, \nu_i \rangle - 1 \right| < \varepsilon$$

for a large enough choice of n.

By the definition of the amplification $\theta(u)_n$ of $\theta(u)$,

$$\sum_k \sum_{i,j=1}^{n} \langle x_k b_{ij} y_k \mu_j, \nu_i \rangle = \langle \theta(u)_n (b_{ij})_n (\mu_1, \ldots, \mu_n), (\nu_1, \ldots, \nu_n) \rangle$$

where $(b_{ij})_n$ is the element in $M_n(K(H))$ with (i,j) entry b_{ij}, so

$$|\langle \theta(u)_n (b_{ij})_n (\mu_1, \ldots, \mu_n), (\nu_1, \ldots, \nu_n) \rangle| \geqslant 1 - \varepsilon.$$

Note that $(\mu_1, \ldots, \mu_n)$ and $(\nu_1, \ldots, \nu_n)$ are really columns and that

$$\|(b_{ij})_n\| = \|b\| \leqslant 1 + \varepsilon,$$
$$\|(\mu_1, \ldots, \mu_n)\| \leqslant \|\mu\| = 1 \quad \text{and} \quad \|(\nu_1, \ldots, \nu_n)\| \leqslant \|\nu\| = 1.$$

Thus $\|\theta(u)_n\| \geqslant (1-\varepsilon)(1+\varepsilon)^{-1}$ as required. ∎

2. THE HAAGERUP TENSOR PRODUCT OF A FACTOR

For notational convenience and to avoid restriction of domain signs for an operator which will already have subscripts to denote n-fold amplification, let φ from $N \otimes_h N$ into $CB(N)$ be defined by $\varphi(a \otimes b)(x) = axb$ for all a, b, x in N. Thus $\varphi(u)$ is $\theta(u)$ restricted to N for $u \in N$.

THEOREM 3. *If N is a factor with separable predual, then φ from $N \otimes_h N$ into $CB(N)$ is an isometry.*

Proof. The inequality $\|\varphi(u)\|_{cb} \leqslant \|u\|_h$ was proved in the introduction so it is sufficient to consider the reverse inequality. Let N act standardly on the Hilbert space H, let u be in the algebraic tensor product $N \otimes N$ with $\|u\|_h = 1$, which is dense in $N \otimes_h N$, and let $\varepsilon > 0$. Since θ is an isometry ([16] and Corollary 2) there is an n in $\mathbb{N}$, x in $M_n(K(H))$ and ξ, η in H^n with

$$\|x\| = \|\xi\| = \|\eta\| = 1 \text{ such that } |\langle \theta(u)_n(x)\xi, \eta\rangle| \geqslant 1 - \varepsilon.$$

Now there is an injective subfactor R of N' such that $(R \cup N)'' = B(H)$. If N' (and N) are injective, then of course R is N'. If N' is of type II then Popa's Theorem [15, Corollary 4.1] gives the existence of a hyperfinite subfactor R of N' with trivial relative commutant, i.e. $R' \cap N' = \mathbb{C}1$. If N is infinite, then Longo's Theorem [12, Corollary 4.5] implies there is an injective subfactor R of N' with trivial relative commutant; he actually proves considerably more than this. In each case $(R \cup N)'' = (R' \cap N')' = B(H)$ as required. By the Kaplansky density theorem [17] there is a $y \in M_n(\text{Algebra}(N, R))$ with $\|y\| = 1$ such that $|\langle \theta(u)_n(y)\xi, \eta\rangle| \geqslant 1 - 2\varepsilon$. Note that y involves only a finite number of elements of R. Connes' results on injective von Neumann algebras [6] and another use of the Kaplansky density Theorem [17, p. 25] give a finite dimensional subalgebra F of R and a z in $M_n(\text{Algebra}(N, F))$ with $\|z\| \leqslant 1$ such that

$$|\langle \theta(u)_n(z)\xi, \eta\rangle| \geqslant 1 - 3\varepsilon.$$

Since N is a factor, the algebra Algebra(N, N') generated by N and N' is isomorphic to the tensor product $N \otimes N'$ via the natural map [17, p. 228]. Hence Algebra(N, F), which is isomorphic to $N \otimes F$, is isomorphic to a subalgebra of $M_m(N)$ for a suitable integer m with x in N mapped onto the diagonal matrix diag(x) in $M_m(N)$. Let α be this isomorphism from Algebra(N, F) onto a unital subalgebra of $M_m(N)$ with $\alpha(x) = \text{diag}(x)$ for all x in N. Recall the choice of z in $M_n(\text{Algebra}(N, F))$. If x and y are in N, then

$$\alpha_n(\theta(x \otimes y)_n(z)) = \alpha_n(\text{diag}(x) z \text{diag}(y)) =$$
$$= \text{diag}(x)\alpha_n(z)\text{diag}(y) = \theta(x \otimes y)_{mn}(\alpha_n(z)).$$

Thus

$$\alpha_n(\theta(u)_n(z)) = \theta(u)_{mn}(\alpha_n(z)) = \varphi(u)_{mn}(\alpha_n(z))$$

so that

$$\|\varphi(u)\|_{\mathrm{cb}} \geqslant \|\theta(u)_{mn}(\alpha_n(z))\| = \|\theta(u)_n(z)\| \geqslant \left| \langle \theta(u)_n(z)\xi, \eta\rangle \right| \geqslant 1 - 3\varepsilon$$

because $\|\alpha_n(z)\| = \|z\| \leqslant 1$ and $\|\xi\| = \|\eta\| = 1$. This completes the proof. ∎

REMARKS. Our first version of Theorem 3 applied to the hyperfinite II_1 factor and depended on the matrix case [5, p. 418].

3. A CHARACTERISATION OF INJECTIVITY IN TERMS OF COMPLETELY BOUNDED MAPS ON MATRIX SUBFACTORS

In this section completely bounded maps from matrix subfactors to von Neumann factors are considered, and a characterisation of injectivity in terms of the Haagerup tensor norm is given.

LEMMA 4. *Let N be an injective von Neumann algebra, let A be a C^*-algebra with unit, and let $T : A \to N$ be a completely bounded linear map. There exist completely positive maps*

$$S_i : A \to N,$$

$i = 1$ or 2, with $\|S_i\| = \|T\|_{\mathrm{cb}}$ such that

$$\Phi : M_2(A) \to M_2(N)$$

given by

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{pmatrix}$$

is a completely positive linear map.

Proof. Let N be contained in $B(H)$, where H is a Hilbert space, and let $E : B(H) \to N$ be a conditional expectation. Let $\|T\|_{\mathrm{cb}} = 1$. Let $CP(A, B(H))$ denote the space of completely positive linear maps from A to $B(H)$. There exist φ_1 and $\varphi_2 \in CP(A, B(H))$ such that

$$\|\varphi_i\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}} = 1 = \varphi_i(1)$$

$i = 1$ or 2, and $\Phi_1 : M_2(A) \to M_2(B(H))$ given by

$$\Phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi_1(a) & T(b) \\ T^*(c) & \varphi_2(d) \end{pmatrix}$$

is completely positive. (See [13], Theorem 7.3.)

Let $S_i = E\varphi_i$, $i = 1$ or 2. Then S_i, being a composition of completely positive maps, is completely positive. Hence

$$\|S_i\|_{\mathrm{cb}} = \|E\varphi_i(1)\| = 1 = \|T\|_{\mathrm{cb}}.$$

Furthermore $\begin{pmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{pmatrix} = E_2\Phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So $\Phi = E_2\Phi_1$ is completely positive. ■

Relations between the Haagerup tensor product and completely bounded maps are used in the following theorem.

THEOREM 5. *Let N be an injective factor. Let $M_n(\mathbb{C})$ be a unital subalgebra of N and let $\varphi : M_n(\mathbb{C}) \to N$ be completely bounded. Then there exist $u_1, \ldots, u_d, w_1, \ldots, w_d \in N$ such that $d \leqslant 4n^2$,*

$$\varphi(x) = \sum_j u_j x w_j$$

for $x \in M_n(\mathbb{C})$, and

$$\|\varphi\|_{\mathrm{cb}}^2 = \left\|\sum_j u_j \otimes w_j\right\|_{\mathrm{h}}^2 = \left\|\sum_j u_j u_j^*\right\| \left\|\sum_j w_j^* w_j\right\|.$$

Proof. Assume that $\|\varphi\|_{\mathrm{cb}} = 1$. By Lemma 1, there exist completely positive linear maps $S_i : M_n(\mathbb{C}) \to N$ $(i = 1, 2)$ such that $\|S_i\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$ and Φ given by

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} S_1(a) & \varphi(b) \\ \varphi^*(c) & S_2(d) \end{pmatrix}$$

is a completely positive map from $M_2(M_n(\mathbb{C}))$. There exist $g_1, \ldots, g_d \in M_2(N)$ with $d \leqslant 4n^2$ such that

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_j g_j^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_j$$

(See [10], Proposition 2.1, and [13], Proposition 4.7.)

Let $g_j = \begin{pmatrix} u_j^* & x_j \\ y_j & w_j \end{pmatrix}$ where $j = 1, \ldots, d$; $u_j, x_j, y_j, w_j \in N$.

By matrix multiplication,

$$S_1(a) = \sum_j (u_j a u_j^* + y_j^* c u_j^* + u_j b y_j + y_j^* d y_j)$$

for all $a, b, c, d \in M_n(\mathbb{C})$. Put $a = b = c = 0$ and $d = 1$ to see $y_j = 0$ for all j. Similarly, by considering S_2 and putting $d = b = c = 0$ and $a = 1$, we see $x_j = 0$ for all j.

Therefore

$$\|S_1\|_{\mathrm{cb}} = \|S_1(1)\| = \Big\|\sum_j u_j u_j^*\Big\| \quad \text{and} \quad \|S_2\|_{\mathrm{cb}} = \|S_2(1)\| = \Big\|\sum_j w_j^* w_j\Big\|.$$

Hence

$$\|\varphi\|_{\mathrm{cb}}^2 = \|S_1\|_{\mathrm{cb}}\|S_2\|_{\mathrm{cb}} = \Big\|\sum_j u_j u_j^*\Big\| \, \Big\|\sum_j w_j^* w_j\Big\|$$

with $\varphi(b) = \sum_j u_j b w_j$ for $b \in M_n(\mathbb{C})$.

Thus by the Cauchy-Schwarz inequality

$$\Big\|\sum_j u_j \otimes w_j\Big\|_{\mathrm{h}}^2 \geqslant \|\varphi\|_{\mathrm{cb}}^2 = \Big\|\sum_j u_j u_j^*\Big\| \, \Big\|\sum_j w_j^* w_j\Big\| \geqslant \Big\|\sum_j u_j \otimes w_j\Big\|_{\mathrm{h}}^2. \qquad \blacksquare$$

The converse is proven using results due to Haagerup. Recall the definition of the decomposable norm on completely bounded operators from l_n^∞ to a von Neumann algebra N : $\|T\|_{\mathrm{dec}}$ is the infimum of $\lambda \geqslant 0$ such that there exist completely positive operators S_i with $\|S_i\| \leqslant \lambda$ and

$$x \to \begin{pmatrix} S_1(x) & T(x) \\ T^*(x) & S(x) \end{pmatrix} : l_n^\infty \to M_2(N)$$

is completely positive.

THEOREM 6. *Let N be a von Neumann factor. If for each unital matrix subalgebra $M_n(\mathbb{C})$ of N and each completely bounded map $\varphi : M_n(\mathbb{C}) \to N$ there exist $u_1, \ldots, u_k, w_1, \ldots, w_k$ in N such that*

$$\varphi(x) = \sum_j u_j x w_j,$$

for $x \in N$, and

$$\|\varphi\|_{\mathrm{cb}}^2 = \Big\|\sum_j u_j u_j^*\Big\| \, \Big\|\sum_j w_j^* w_j\Big\|,$$

then N is injective.

Proof. Let n be an integer and let $T : l_n^\infty \to N$ be completely bounded. Identify l_n^∞ with the main diagonal of $M_n(\mathbb{C})$, let E be the conditional expectation from

$M_n(\mathbb{C})$ onto l_n^∞, and let $\varphi = TE$. Clearly $\|\varphi\|_{cb} = \|T\|_{cb}$. Hence there exist $u_1, \ldots, u_k, w_1, \ldots, w_k \in N$ such that for $b \in l_n^\infty$

$$T(b) = \sum_j u_j b w_j$$

and

$$\|T\|_{cb}^2 = \left\|\sum_j u_j u_j^*\right\| \left\|\sum_j w_j^* w_j\right\|.$$

In addition replacing u_j by $\frac{u_j}{\alpha}$ and w_j by αw_j where $\left\|\sum_j w_j^* w_j\right\| = \alpha^4 \left\|\sum_j u_j u_j^*\right\|$ ensures $\left\|\sum_j u_j u_j^*\right\| = \left\|\sum_j w_j^* w_j\right\|$. Define a completely positive map $\Phi : M_2(l_n^\infty) \to$ $\to M_2(N)$ by

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_j \begin{pmatrix} u_j & 0 \\ 0 & w_j^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_j^* & 0 \\ 0 & w_j \end{pmatrix}.$$

Then

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{pmatrix}$$

where $S_1(a) = \sum_j u_j a u_j^*$ and $S_2(b) = \sum_j w_j^* b w_j$. Since S_1 and S_2 are completely positive $\|S_1\|_{cb} = \left\|\sum_j u_j u_j^*\right\|$ and $\|S_2\|_{cb} = \left\|\sum_j w_j^* w_j\right\|$, and so

$$\|T\|_{cb}^2 = \|S_1\|_{cb}\|S_2\|_{cb}.$$

Now consider the map $P : l_n^\infty \to M_2(l_n^\infty)$ given by $P(x) = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$; then P is completely positive. Let $R = \Phi P : l_n^\infty \to M_2(N)$. Being a composition of completely positive maps, R is completely positive. Since $\|S_i\|_{cb} = \|T\|_{cb}$, $\|T\|_{\text{dec}} \leqslant \|T\|_{cb}$, so N is injective ([11], Theorem 2.1).

Note that in this case,

$$\left\|\sum_j u_j \otimes w_j\right\|_h^2 \geqslant \|\varphi\|_{cb}^2 = \left\|\sum_j u_j u_j^{\star *}\right\| \left\|\sum_j w_j^{\star *} w_j\right\| \geqslant \left\|\sum_j u_j \otimes w_j\right\|_h^2.$$

4. THE MULTILINEAR CASE

There are multilinear versions of Proposition 1 and Corollary 2 above, and rather than inflicting the full proofs of these on the reader, we shall briefly explain here how these results may be obtained.

Let $CB(N^n, \mathbb{C})$ and $CB_\sigma(N^n, \mathbb{C})$ denote the spaces of completely bounded n-linear forms from N^n into $\mathbb{C}$ and of completely bounded normal n-linear forms from N^n into $\mathbb{C}$, and let $\otimes_{\mathrm{h}}^n N$ denote the n-fold Haagerup tensor product of N with itself. Then $CB(N^n, \mathbb{C})$ is the dual of $\otimes_{\mathrm{h}}^n N$ [14].

THEOREM 7. *The closed unit ball of* $CB_\sigma(N^n, \mathbb{C})$ *is weak-$*$ dense in the closed unit ball of* $CB(N^n, \mathbb{C})$.

Proof. To prove this it is sufficent to show that if φ is in the closed unit ball of $CB(N^n, \mathbb{C})$, F is a finite subset of N, and $\varepsilon > 0$, then there is a ψ in $CB_\sigma(N^n, \mathbb{C})$ such that $\varphi = \psi$ on F^n and $\|\psi\|_{\mathrm{cb}} \leqslant 1+\varepsilon$. Let X be the linear subspace of N spanned by F. The proof is by induction on m, $1 \leqslant m \leqslant [n/2]$, in the following statement.

For each finite subset F of N, for each $\varepsilon > 0$, and each completely bounded n linear form φ on N with $\|\varphi\|_{\mathrm{cb}} = 1$, there is a completely bounded n-linear form ψ on N such that $\psi = \varphi$ on F^n, $\|\psi\|_{\mathrm{cb}} \leqslant 1+\varepsilon$, and ψ is separately ultraweakly continuous in the variable x_j for $1 \leqslant j \leqslant m$ and $n-m \leqslant j \leqslant n$.

The representation theorem for a completely bounded n-linear operator [14] gives a representation ρ of N on a Hilbert space H_ρ, unit vectors ξ and η in H_ρ, and continuous linear operators a_j on H_ρ with $\|a_j\| \leqslant 1$, $1 \leqslant j \leqslant n-1$, such that

$$\varphi(x_1, \dots, x_n) = \langle \rho(x_1)a_1\rho(x_2)\cdots a_{n-1}\rho(x_n)\xi, \eta\rangle$$

for all x_j in N. Now choose ν and μ as in Proposition 1. Taking k-fold amplifications with $x_1, \dots, x_n$ in $M_k(N)$, $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ unit vectors in $\mathbb{C}^k$ yields

$$\begin{aligned}\langle \varphi_k(x_1, \dots, x_n)\alpha, \beta\rangle &= \sum \alpha_j\overline{\beta}_i(\varphi_k(x_1, \dots, x_n))_{ij} = \\ &= \langle \rho_k(x_1)(a_1 \otimes I)\rho_k(x_2)\cdots(a_{n-1}\otimes I)\rho_k(x_n)(\alpha_j\xi), (\beta_i\eta)\rangle\end{aligned}$$

by the matrix multiplication definition of the k-fold amplification of φ_k, where $(\alpha_j\xi)$ and $(\beta_i\eta)$ are in the k-fold direct sum of H_ρ (see [4], [14] for typical similar calculations).

Hence

$$\begin{aligned}&|\langle \varphi_k(x_1, \dots, x_n)\alpha, \beta\rangle| \leqslant \\ (*)\qquad &\leqslant \|\rho_k(x_n)(\alpha_j\xi)\| \cdot \|\rho_k(x_1)^*(\beta_j\eta)\| \cdot \|\rho_k(x_2)\cdots\rho_k(x_{n-1})\| \leqslant \\ &\leqslant \|x_2\|\cdots\|x_{n-1}\| \cdot (1-\delta)^{-1} \cdot \|\pi_k(x_n)(\alpha_j\mu)\| \cdot \|\pi_k(x_1)^*(\beta_j\nu)\|\end{aligned}$$

for all x_j in $M_k(X)$, since

$$\|\rho_k(x_n)(\alpha_j\xi)\|^2 = \sum_i \left\| \rho \left[\sum_j \alpha_j x_{nij} \right] \xi \right\|^2 \leqslant$$
$$\leqslant (1-\delta)^{-1} \sum_i \left\| \pi \left[\sum_j \alpha_j x_{nij} \right] \mu \right\|^2 = (1-\delta)^{-1} \|\pi_k(x_n)(\alpha_j \mu)\|^2$$

by the choice of π, μ and ν. Using a linear space of sesquilinear forms running over $\otimes^{n-2}X$ in place of the single sesquilinear form of Proposition 1, yields an $(n-2)$-linear operator χ_0 from X^{n-2} into $B(\pi(X)\xi, \pi(X)^*\eta)$ such that

$$\varphi(x_1, \ldots, x_n) = \langle \pi(x_1)\chi_0(x_2, \ldots, x_{n-1})\pi(x_n)\mu, \nu \rangle$$

for all x_j in X. Further inequality $(*)$ above shows that χ_0 is completely bounded with

$$\|\chi_0\|_{\text{cb}} \leqslant (1-\delta)^{-1} = 1 + \varepsilon.$$

Let p and q be the projections from H_π onto $\pi(X)^*\nu$ and $\pi(X)\mu$, respectively, and let χ be a completely bounded $(n-2)$-linear extension of χ_0 to $B(H)^{n-2}$ into $B(\pi(X)\xi, \pi(X)^*\eta)$ with $\|\chi\|_{\text{cb}} = \|\chi_0\|_{\text{cb}} \leqslant 1 + \varepsilon$ [14]. Let

$$\psi(x_1, \ldots, x_n) = \langle \pi(x_1) p \chi(x_2, \ldots, x_{n-1}) q \pi(x_n)\mu, \nu \rangle$$

for all x_j in N. Then ψ is the n-linear form required by the first stage of the induction since $\|\psi\|_{\text{cb}} \leqslant 1 + \varepsilon$.

The general stage of the induction is similar *except* that in place of the state $x \mapsto \langle \rho(x)\xi, \xi \rangle$ on the factor, which is approximated by a normal state on a finite dimensional space X, there is a completely positive operator $x \mapsto q\rho(x)q$, with q a finite rank projection, that is weakly approximated for all x in X just on a *finite dimensional* subspace of H_ρ by a normal completely positive finite rank operator. That a completely positive normal operator from a von Neumann algebra into a matrix algebra can be approximated weakly on a finite dimensional subspace by a normal completely positive operator with the same norm follows from the results that a state may be weakly approximated by a normal state and [13, Theorem 5.1], which characterizes the complete positivity of operators into $M_n(\mathbf{C})$ in terms of an associated linear functional. It is easy to check that the normality of the operator and functional is preserved under this association [13, p. 63 and 64].

Similar techniques to those used in the proof of Corollary 2, and Theorem 3 yield the following results from Theorem 7.

THEOREM 8. *If θ from $\otimes_{\mathrm{h}}^{n+1} B(H)$ into $CB(K(H)^n, K(H))$ is defined by*

$$\theta(a_1, \ldots, a_{n+1})(x_1, \ldots, x_n) = a_1 x_1 a_2 x_2 \cdots x_n a_{n+1},$$

then θ is an isometry.

THEOREM 9. *If N is a factor with separable predual, and if φ from $\otimes_{\mathrm{h}}^{n+1} N$ into $CB(N^n, N)$ is defined by*

$$\varphi(a_1, \ldots, a_{n+1})(x_1, \ldots, x_n) = a_1 x_1 a_2 x_2 \cdots x_n a_{n+1},$$

then φ is an isometry.

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Added in proof:

1. After this paper was accepted Martin Mathieu informed us that Ulfe Haagerup had obtained Proposition 1 and Corollary 2 in an unpublished manuscript *The α--tensor product for C^*-algebras*, 1980. His paper contains a definition of what is now called the Haagerup tensor product.

2. There has been further progress since this paper was written and the following references should be consulted:

CHATTERJEE, A.; SMITH, R. R., The central Haagerup tensor product and maps between von Neumann algebras, *J. Functional Analisys*, **112**(1993), 97–120.
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