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ANALYTIC AUTOMORPHISMS OF A FIELD OF COMPACTS

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1. INTRODUCTION

A basic notion in the theory of operator algebras is the construction of crossed product algebras from covariant systems, be it with groups, groupoids, groups acting on C^*-algebras, groups acting on von Neumann algebras, or foliations of spaces, to name just a few examples. (cf. [4], [15], and [17].) The unfortunate fact is that the construction of $*$-algebras by this process tends to lose information; the construction may produce isomorphic $*$-crossed products from non-isomorphic covariant systems. (See for instance, Wang's example in [20] of two smooth, free actions of the real line on the plane with non-isomorphic foliations yielding isomorphic C^*-crossed products.) Recent work indicates that this deficiency results from restricting one's attention to $*$-algebras, as certain non-self-adjoint algebras arising naturally in the crossed product construction often retain more information about the underlying covariant system. When the group of integers acts on an abelian C^*-algebra, the problem goes back to Arveson in [1] and Averson and Josephson in [3], where two non-self-adjoint crossed products are shown to be unitarily equivalent only if the ergodic transformations of the covariant systems are measureably conjugate. Similarly, DeAlba and Peters consider integers acting on a finite dimensional algebra in [6], while in [10], the author examines topological ordered groups acting freely on the maximal ideal space of abelian C^*-algebras. In all these instances the non-self-adjoint algebras retain much more structural information about the covariant systems.

In each of these instances, a locally compact group G acts on a C^*-algebra A, with a regularly closed subsemigroup Σ of G determining the so-called analytic subalgebra $A\times_\alpha\Sigma$ of the crossed product $A\times_\alpha G$, as described in the work by McAsey and Muhly in [14]. This non-self-adjoint algebra is defined as the completion of the convolution algebra $L^1(\Sigma, A)$, which sits as a subalgebra in $*$-algebra $L^1(G, A)$ used to define the crossed product, where the group action α introduces a "twist" in the convolution product. The present work is motivated by an attempt to understand the simplest case where the group is neither discrete nor freely acting; namely, where the real line acts on a one-torus by rotation. Here, $A \times_\alpha G$ is the transformation group C^*-algebra $C^*(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ which by Green [8] is canonically isomorphic to $C(\mathbf{T}) \otimes \mathcal{K}$, with $\mathbf{T}$ the circle of complex numbers of modulus one and $\mathcal{K}$ the algebra of compact operators on the Hilbert space $L^2([0,1])$. It will be shown that this isomorphism maps $A \times_\alpha \mathbf{R}^+$ to the subalgebra of $C(\mathbf{T}) \otimes \mathcal{K}$ consisting of those $\mathcal{K}$-valued continuous functions that extend to analytic functions on the open disk, and taking values at the origin in the Volterra algebra of "lower triangular operators" on $L^2([0,1])$. The analytic automorphisms of $C^*(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ will be characterized, that is, those C^*-automorphisms which preserve the analytic subalgebra $A \times_\alpha \mathbf{R}^+$. It turns out that each such automorphism induces naturally an orientation-preserving homeomorphism of $\mathbf{R}/\mathbf{Z}$, there by recovering the essential structural information of the underlying space.

The analysis does not depend in any essential way on the crossed product structure of $A \times_\alpha \mathbf{R}^+$, so Section 2 begins by replacing $A \times_\alpha \mathbf{R}^+$ with a more general analytic subalgebra $A_\mathcal{N}$ in $C(\mathbf{T}) \otimes \mathcal{K}$ of $\mathcal{K}$-valued analytic functions on the disk with value at the origin in the nest algebra determined by the nest $\mathcal{N}$. Theorem 1 describes an obvious class of analytic isomorphisms for this algebra, which may be described as a rotation of the disk followed by a unitary transformation of "shifted" nests. Section 3 is devoted to showing that these are the only analytic isomorphisms, while Section 4 restricts attention to the group action case. Proposition 7 describes the canonical mapping of $A \times_\alpha \mathbf{R}^+$ mentioned above, while Theorem 8 characterizes the analytic automorphisms for the real action. Theorem 9 characterizes the analytic automorphisms for the case of a cyclic action by the integers on a finite space, wherein the algebra $A_\mathcal{N}$ is the algebra of analytic functions from the disk into n by n matrices, taking lower triangular matrix values at the origin. This finite cyclic action and matrix algebra were examined earlier by McAsey in [13], in a study of invariant subspaces.

2. NESTS AND ANALYTIC SUBALGEBRAS

Let $\mathcal{H}$ denote a separable Hilbert space, with $\mathcal{K}$ the algebra of compact operators on $\mathcal{H}$. A *nest* $\mathcal{N}$ in $\mathcal{H}$ is a linearly ordered, strongly closed lattice of projections containing 0 and I. Let $\mathcal{K}_\mathcal{N}$ denote the subalgebra of compacts contained in the nest algebra determined by $\mathcal{N}$. That is, $\mathcal{K}_\mathcal{N} = \{T \in \mathcal{K} : TP = PTP,\ P \in \mathcal{N}\}$. By the Erdos Density Theorem, $\mathcal{K}_\mathcal{N}$ determines $\mathcal{N}$ as its lattice of invariant projections. (cf.

[5]). With $\mathbf{T}$ the group of complex numbers of modulus one, let $C(\mathbf{T},\mathcal{K})$ denote the C^*-algebra of continuous functions from $\mathbf{T}$ into $\mathcal{K}$. Note that polynomials in z and z^{-1} with coefficients T_k in $\mathcal{K}$, of the form $\sum_{k=-n}^{n} z^k T_k$, are norm dense in $C(\mathbf{T},\mathcal{K})$. Define the analytic subalgebra $A_{\mathcal{N}}$ in $C(\mathbf{T},\mathcal{K})$ as the norm closure of polynomials in z alone, of the form $\sum_{k=0}^{n} z^k T_k$, with T_0 in $\mathcal{K}_{\mathcal{N}}$ and the remaining T_k in $\mathcal{K}$, $k > 0$. Of course, $A_{\mathcal{N}}$ is a non-self-adjoint algebra. By extending the Fourier transform on $\mathbf{T}$ to operator-valued functions, it is easy to see that $A_{\mathcal{N}}$ is precisely the algebra of functions in $C(\mathbf{T},\mathcal{K})$ whose negative Fourier coefficients are zero and whose zero-th coefficient is an element of $\mathcal{K}_{\mathcal{N}}$. Equivalently, $A_{\mathcal{N}}$ is the algebra of functions in $C(\mathbf{T},\mathcal{K})$ with operator-valued analytic extensions to the interior of the unit disk in the complex plane, taking values in $\mathcal{K}_{\mathcal{N}}$ at the origin.

If $\mathcal{N}$ and $\mathcal{M}$ are two nests in $\mathcal{H}$ with U a unitary such that $U\mathcal{N}U^* = \mathcal{M}$, then the $*$-automorphism ρ defined by

$$\rho(F)(z) = UF(z)U^*, \quad F \in C(\mathbf{T},\mathcal{K}),\ z \in \mathbf{T},$$

maps $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$. Any $*$-automorphism mapping $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$ will be called an analytic automorphism relative to analytic subalgebras $A_{\mathcal{N}}$ and $A_{\mathcal{M}}$. A slightly more general analytic automorphism is obtained by also allowing a rotation of the circle $\mathbf{T}$ by an element λ in $\mathbf{T}$, so that z maps to $z\lambda$. The interesting analytic automorphisms occur when there is a unitary which shifts the "bottom half" of the first nest $\mathcal{N}$ onto the "top half" of $\mathcal{M}$, and vice versa, as in the following theorem.

THEOREM 1. *Fix λ in $\mathbf{T}$, P and Q projections in nests $\mathcal{N}$ and $\mathcal{M}$, respectively, and U a unitary such that $UP\mathcal{N}U^* = Q^\perp\mathcal{M}$ and $UP^\perp\mathcal{N}U^* = Q\mathcal{M}$. Then the C^*-automorphism ρ of $C(\mathbf{T},\mathcal{K})$ given by*

$$\rho(F)(z) = (Q + z\lambda Q^\perp)UF(z\lambda)U^*(Q + z\lambda Q^\perp)^*, \quad F \in C(\mathbf{T},\mathcal{K}),\ z \in \mathbf{T},$$

is an analytic automorphism mapping $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$.

Proof. It sufficies to consider how ρ acts on the dense set of polynomials in $A_{\mathcal{N}}$ of the form $F(z) = \sum_{k=0}^{n} z^k T_k$, with T_k in $\mathcal{K}$ and T_0 in $\mathcal{K}_{\mathcal{N}}$. The image under ρ is the polynomial function

$$z^{-1}\lambda^{-1}QUT_0U^*Q^\perp + z^0(QUT_0U^*Q + \lambda QUT_1U^*Q^\perp) + \text{order } z \text{ and higher.}$$

Since T_0 is in $\mathcal{K}_{\mathcal{N}}$, $0 = UP^\perp T_0PU^* = QUT_0U^*Q^\perp$, so the z^{-1} coefficient is zero. Since Q is in $\mathcal{M}$, the term $QUT_1U^*Q^\perp$ is in $\mathcal{K}_{\mathcal{M}}$, as is $QUT_0U^*Q = UP^\perp T_0P^\perp U^*$, so the z^0 coefficient lies in $\mathcal{K}_{\mathcal{M}}$. Thus polynomials in $A_{\mathcal{N}}$ map to polynomials in $A_{\mathcal{M}}$. To see this map is onto all polynomials in $A_{\mathcal{M}}$, note the inverse of ρ takes a function $G(z)$ to the function

$$\rho^{-1}(G)(z) = (P + z\lambda^{-1}P^{\perp})U^*G(z\lambda^{-1})U(P + z\lambda^{-1}P^{\perp})^*, \quad z \in \mathbf{T},$$

which is of the same form as ρ, with the roles of $\mathcal{N}$ and $\mathcal{M}$ reversed. ∎

3. ANALYTIC AUTOMORPHISMS

This section establishs that every analytic automorphism of $C(\mathbf{T}, \mathcal{K})$ is of the form described in Theorem 1. It is convenient to recall some elementary facts about the double centralizers of the algebras under consideration; that is, their multiplier algebras. First, note that $\operatorname{alg}\mathcal{N}$, the algebra of bounded operators in $\mathcal{B}(\mathcal{H})$ leaving invariant the range subspaces of projections in $\mathcal{N}$, is exactly the multiplier algebra of $\mathcal{K}_{\mathcal{N}} = \mathcal{K} \cap \operatorname{alg}\mathcal{N}$, since $K_{\mathcal{N}}$ contains a bounded approximate identity and is strong* dense in $\operatorname{alg}\mathcal{N}$. (cf. [5]). The diagonal algebra $D_{\mathcal{N}}$, defined as $\operatorname{alg}\mathcal{N} \cap \operatorname{alg}\mathcal{N}^*$, is the commutant of the nest of projections. As for $C(\mathbf{T}, \mathcal{K})$, it is easy to compute that its multiplier algebra is the C^*-algebra of strong* continuous functions from the circle into the bounded operators on $\mathcal{H}$, herein denoted by $C_*(\mathbf{T}, \mathcal{B}(\mathcal{H}))$, which acts pointwise on $C(\mathbf{T}, \mathcal{K})$. As the subalgebra $A_{\mathcal{N}}$ has a bounded approximate identity for $C(\mathbf{T}, \mathcal{K})$, there is a natural inclusion of its multiplier algebra into $C_*(\mathbf{T}, \mathcal{B}(\mathcal{H}))$, by Proposition 2.5 of [14]. By the extended Fourier transform, $M(A_{\mathcal{N}})$ is identified as the subalgebra of functions in $C_*(\mathbf{T}, \mathcal{B}(\mathcal{H}))$ whose negative Fourier coefficients vanish, and whose zero-th coefficient lies in $\operatorname{alg}\mathcal{N}$. That is,

$$M(A_{\mathcal{N}}) = \{F \in C_*(\mathbf{T}, \mathcal{B}(\mathcal{H})) : \hat{F}(n) = 0,\ n < 0,\ \hat{F}(0) \in \operatorname{alg}\mathcal{N}\}.$$

Every automorphism of the C^*-algebra $C(\mathbf{T}, \mathcal{K})$ extends to an automorphism of its multiplier algebra, (cf. [15]), and an analytic automorphism ρ mapping $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$ extends to an automorphism of $C_*(\mathbf{T}, \mathcal{B}(\mathcal{H}))$ mapping $M(A_{\mathcal{N}})$ onto $M(A_{\mathcal{M}})$. Since ρ is a *-map, its extension maps the diagonal $M(A_{\mathcal{N}}) \cap M(A_{\mathcal{N}})^*$ onto $M(A_{\mathcal{M}}) \cap M(A_{\mathcal{M}})^*$. Again by the Fourier transform, note that $M(A_{\mathcal{N}}) \cap M(A_{\mathcal{N}})^*$ is just the subalgebra in $C_*(\mathbf{T}, \mathcal{B}(\mathcal{H}))$ of constant functions $F(z) = T_0$, with T_0 any element in the diagonal $D_{\mathcal{N}}$ of the nest $\mathcal{N}$.

Given a homeomorphism θ of the circle and a strongly continuous path $z \mapsto U_z$ of unitaires, it is easy to write down an automorphism of $C(\mathbf{T}, \mathcal{K})$; forcing it to be analytic puts many restrictions on the homeomorphism and the unitaires.

PROPOSITION 2. *Let $\rho(F)(z) = U_z F(\theta(z)) U_z^*$ define a C^*-automorphism of $C(\mathbf{T}, \mathcal{K})$, where θ is a homeomorphism of $\mathbf{T}$ and $z \mapsto U_z$ is a strongly continuous path of unitaires. If ρ is analytic, mapping $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$, then θ is analytic, the unitaires U_z satisfy $U_z D_{\mathcal{N}} U_z^* = D_{\mathcal{M}}$, and the unitaires $W_z = U_z U_1^*$ are in the von Neumann algebra generated by the nest $\mathcal{M}$. Moreover, there exist linear maps a_{-1}, a_0 and a_1 on $\mathcal{K}$, with*

$$W_z T W_z^* = \theta(z)^{-1} a_{-1}(T) + a_0(T) + \theta(z) a_1(T), \quad T \in \mathcal{K}, \ z \in \mathbf{T}.$$

Proof. The polynomial $F(z) = zI$ is an element of $M(A_{\mathcal{N}})$ so its image $G(z) = \theta(z)I$ under the extension of ρ must be in $M(A_{\mathcal{M}})$. Thus the negative Fourier coefficients of the function $\theta(z)$ vanish, so θ is analytic.

To see that W_z is in the von Neumann algebra generated by the nest $\mathcal{M}$, which is just the commutant of the diagonal $D_{\mathcal{M}}$, fix an operator S in $D_{\mathcal{M}}$, and let the constant function $G(z) = S$ denote an element of $M(A_{\mathcal{M}}) \cap M(A_{\mathcal{M}})^*$. Since the multiplier extension of ρ maps $M(A_{\mathcal{N}}) \cap M(A_{\mathcal{N}})^*$ onto $M(A_{\mathcal{M}}) \cap M(A_{\mathcal{M}})^*$, there exists an operator T in $D_{\mathcal{N}}$ and a constant function $F(z) = T$ in $M(A_{\mathcal{N}}) \cap M(A_{\mathcal{N}})^*$ with $\rho(F) = G$. Thus

$$U_z T U_z^* = S = U_1 T U_1^*, \quad z \in \mathbf{T},$$

so $W_z = U_z U_1^*$ commutes with S, for all S in $D_{\mathcal{M}}$. Thus W_z is in the commutant. Moreover, since the extension of ρ is onto, every S in $D_{\mathcal{M}}$ is achieved as above by some T in $D_{\mathcal{N}}$, so $U_z D_{\mathcal{N}} U_z^* = D_{\mathcal{M}}$.

Finally, let F denote a function in $A_{\mathcal{N}}$ of the form $F(z) = zU_1^* T U_1$, with T in $\mathcal{K}$. Its image under ρ is a function $\rho(F)(z) = \theta(z) W_z T W_z^*$ which must be analytic, so expanding as a power series in $\theta(z)$ yields

$$W_z T W_z^* = \sum_{k=-1}^{\infty} \theta(z)^k a_k(T).$$

Considering T^* instead, observe that $W_z T^* W_z^*$ must also have a series expansion beginning with a term of order $\theta(z)^{-1}$; but starring the last equation also gives an expansion for $W_z T^* W_z^*$ with negative terms $\theta(z)^{-2} a_2(T)^*$, $\theta(z)^{-3} a_3(T)^*$, and so on, which must vanish. ∎

These essentially polynomial actions on the compacts are interesting in themselves, in that they arise only in a very simple form, as the following lemma shows.

LEMMA 3. *Let $z \mapsto W_z$ be a path of unitaires from the circle $\mathbf{T}$ into an abelian von Neumann algebra, with $W_1 = I$, such that there are linear maps a_{-1}, a_0, a_1 of $\mathcal{K}$ to $\mathcal{K}$ with*

$$W_z T W_z^* = z^{-1} a_{-1}(T) + a_0(T) + z a_1(T), \quad T \in \mathcal{K}, \ z \in \mathbf{T}.$$

Then there exists projection Q in the von Neumann algebra such that

$$W_zTW_z^* = (Q + zQ^\perp)T(Q + zQ^\perp)^*, \quad T \in \mathcal{K}, \ z \in \mathbf{T}.$$

Proof. When T is a Hilbert-Schmidt operator, so is $W_zTW_z^*$, hence the linear maps a_i take Hilbert-Schmidts to Hilbert-Schmidts; indeed, by the polynomial form above, the a_i act as bounded operators on the space of Hilbert-Schmidt operators. Thus, one may consider Ad_{W_z} and a_i as elements of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\overline{\mathcal{H}})$, the algebra of bounded operators on the Hilbert tensor product $\mathcal{H} \otimes \overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ is the conjugate space of $\mathcal{H}$.

Rewriting the expansion for Ad_{W_z} in terms of operators on $\mathcal{H} \otimes \overline{\mathcal{H}}$ yields

$$W_z \otimes W_z = z^{-1}a_{-1} + a_0 + za_1,$$

where $W_z \otimes W_z$ is an element of $\mathcal{W} \otimes \mathcal{W}$, with $\mathcal{W}$ denoting the abelian von Neumann algebra containing the W_z. Uniqueness of the series expansion shows the a_i also lie in $\mathcal{W} \otimes \mathcal{W}$. Taking adjoints gives

$$W_z^* \otimes W_z^* = z^{-1}a_1^* + a_0^* + za_{-1}^*,$$

and their product is

$$\begin{aligned} I = (W_z \otimes W_z)(W_z^* \otimes W_z^*) = \\ = z^{-2}a_{-1}a_1^* + z^{-1}(a_{-1}a_0^* + a_0a_1^*) + (a_{-1}a_{-1}^* + a_0a_0^* + a_1a_1^*) + z^1(a_1a_0^* + \\ + a_0a_{-1}^*) + z^2a_1a_{-1}^*. \end{aligned}$$

Noting that the terms with non-zero powers of z must vanish, one obtains $a_ia_j = 0$ for $i \neq j$. Moreover, expanding the identity $I = W_1 \otimes W_1$ in terms of the a_i yields that a_{-1}, a_0 and a_1 are disjoint projections in $\mathcal{W} \otimes \mathcal{W}$ summing to I.

Since $\mathcal{W}$ is an abelian von Neumann algebra, it is isomorphic to an algebra $L^\infty(\Omega)$ for some measure space Ω. Representing the unitary W_z in $\mathcal{W}$ as a function f_z in $L^\infty(\Omega)$, with $f_1 = 1$, and the projections a_i in $\mathcal{W} \otimes \mathcal{W}$ as characteristic functions χ_i in $L^\infty(\Omega \times \Omega)$ give the a.e. equality

$$f_z(s)\overline{f_z(t)} = z^{-1}\chi_{-1}(s,t) + \chi_0(s,t) + z\chi_1(s,t), \quad z \in \mathbf{T}, s,t \in \Omega.$$

Thus by Fubini's theorem, for almost all t in Ω, the functions q_{-1}^t, q_0^t, q_1^t on Ω, defined by $q_i^t(s) = \chi_i(s,t)$, are elements of $L^\infty(\Omega)$, representing disjoint projections Q_{-1}^t, Q_0^t and Q_1^t in $\mathcal{W}$ which sum to the identity. Moreover, for almost all r, s, and t in Ω and all z in $\mathbf{T}$,

$$\begin{aligned} f_z(r)\overline{f_z(s)} &= f_z(r)\overline{f_z(t)}\,\overline{f_z(s)\overline{f_z(t)}} = \\ &= (z^{-1}q_{-1}^t(r) + q_0^t(r) + zq_1^t(r))(z^{-1}q_1^t(s) + q_0^t(s) + zq_{-1}^t(s)), \end{aligned}$$

so for almost all t in Ω and all z in $\mathbf{T}$,

$$W_z \otimes W_z = (z^{-1}Q^t_{-1} + Q^t_0 + zQ^t_1) \otimes (z^{-1}Q^t_{-1} + Q^t_0 + zQ^t_1).$$

By fixing such a t where the equalities hold, and expanding the tensor product, while noting conjugate linearity in the second term, one obtains

$$\begin{aligned} W_z \otimes W_z =& z^{-2}Q^t_{-1} \otimes Q^t_1 + z^{-1}(Q^t_{-1} \otimes Q^t_0 + Q^t_0 \otimes Q^t_1) + \dots \\ =& z^{-1}a_{-1} + a_0 + za_1, \end{aligned}$$

so the z^{-2} coefficient $Q^t_{-1} \otimes Q^t_1$ must vanish. Thus either Q^t_{-1} or Q^t_1 is zero: in the first case, take $Q = Q^t_0$, with $Q^\perp = Q^t_1$; in the second case, take $Q = Q^t_{-1}$ with $Q^\perp = Q^t_0$. Either way,

$$W_z \otimes W_z = (Q + zQ^\perp) \otimes (Q + zQ^\perp)$$

and so

$$W_zTW_z^* = (Q + zQ^\perp)T(Q + zQ^\perp)^*, \quad T \in \mathcal{K},\ z \in \mathbf{T}. \qquad \blacksquare$$

The following lemma will be convenient in the proof of the subsequent theorem, though the result is no doubt well known.

LEMMA 4. *Let $\mathcal{M}$ be a nest and Q a projection in the von Neumann algebra generated by $\mathcal{M}$. Suppose that for every compact operator T, the operator $QTQ^\perp$ is a member of* alg$\mathcal{M}$*. Then Q is in the nest $\mathcal{M}$.*

Proof. Let ξ be a separating vector for the von Neumann algebra generated by the nest $\mathcal{M}$. The rank one operator $(Q\xi) \otimes (Q^\perp\xi)$ is in alg$\mathcal{M}$ so by Lemma 3.7 of [5], there is a projection P in $\mathcal{M}$, with immediate predecessor P_-, such that $PQ\xi = Q\xi$ and $(P_-)^\perp Q^\perp \xi = Q^\perp \xi$. Since ξ is separating, $PQ = Q$ and $(P_-)^\perp Q^\perp = = Q^\perp$. Combining the two yields $Q = P - (P - P_-)Q^\perp$ and since $P - P_-$ is an atom, either $Q = P$ or $Q = P_-$. Either way, Q is in the nest. $\blacksquare$

The above propositions combine to give the following main theorem on the structure of the analytic automorphisms.

THEOREM 5. *Let ρ be an analytic automorphism of the C^*-algebra $C(\mathbf{T}, \mathcal{K})$ mapping the analytic subalgebra $A_\mathcal{N}$ onto $A_\mathcal{M}$, where $\mathcal{N}$ and $\mathcal{M}$ are fixed nests. If $\mathcal{M}$ (equivalently $\mathcal{N}$) is a non-trivial nest, then there exists a constant λ in $\mathbf{T}$, projections P and Q in $\mathcal{N}$ and $\mathcal{M}$ respectively, and a unitary U with $UP\mathcal{N}U^* = Q^\perp\mathcal{M}$ and $UP^\perp\mathcal{N}U^* = Q\mathcal{M}$ such that*

$$\rho(F)(z) = (Q + z\lambda Q^\perp)UF(z\lambda)U^*(Q + z\lambda Q^\perp)^*, \quad F \in C(\mathbf{T}, \mathcal{K}),\ z \in \mathbf{T}.$$

Proof. By Theorem 4.1 of [19], since the space $\mathbf{T}$ has trivial second cohomology, the C^*-automorphism ρ can be written as

$$\rho(F)(z) = U_z F(\theta(z)) U_z^*, \quad F \in C(\mathbf{T}, \mathcal{K}),\ z \in \mathbf{T},$$

where U_z is a continuous path of unitaries on $\mathbf{T}$ and θ is a homeomorphism of $\mathbf{T}$. By Proposition 2, θ is an analytic map, and U_z can be factored as $U_z = W_z U$, where $U = U_1$, $U D_{\mathcal{N}} U^* = D_{\mathcal{M}}$ and W_z is in the von Neumann algebra generated by the nest $\mathcal{M}$, with

$$W_z T W_z^* = \theta(z)^{-1} a_{-1}(T) + a_0(T) + \theta(z) a_1(T), \quad T \in \mathcal{K},\ z \in \mathbf{T}.$$

By Lemma 3, there exists a projection Q in the von Neumann algebra generated by $\mathcal{M}$ with

$$W_z T W_z^* = (Q + \theta(z) Q^\perp) T (Q + \theta(z) Q^\perp)^*, \quad T \in \mathcal{K},\ z \in \mathbf{T}.$$

Thus,

$$\rho(F)(z) = (Q + \theta(z) Q^\perp) U F(\theta(z)) U^* (Q + (\theta(z) Q^\perp)^*, \quad F \in C(\mathbf{T}, \mathcal{K}),\ z \in \mathbf{T}.$$

The map θ is analytic. If $\theta(0) = 0$, then there exists λ in $\mathbf{T}$ with $\theta(z) = z\lambda$. This will be shown to be the only possibility for such an analytic θ, by contradiction. If $\theta(0) \neq 0$, fix S a compact operator, let

$$T = U^* Q S Q^\perp U + \theta(0)^{-1} U^* (QSQ + Q^\perp S Q^\perp) U + \theta(0)^{-2} U^* Q^\perp S Q U$$

and let $F(z) = zT$ define an element of $A_{\mathcal{N}}$. The image of F under ρ is an analytic function taking value S at the origin. Thus the image of $A_{\mathcal{N}}$, namely $A_{\mathcal{M}}$, contains analytic functions with values at the origin spanning K. But by definition, $A_{\mathcal{M}}$ contains only analytic functions taking values at the origin in $\mathcal{K}_{\mathcal{M}}$, contradicting that $\mathcal{M}$ is non-trivial.

To show that Q is in the nest $\mathcal{M}$, let S be any compact operator and let $F(z) = zU^*SU$ define an element of $A_{\mathcal{N}}$; its image under ρ is an analytic function taking the value $QSQ^\perp$ at the origin. By definiton of $A_{\mathcal{M}}$, $QSQ^\perp$ must lie in $\mathcal{K}_{\mathcal{M}}$ for all compact operators S, so by Lemma 4, Q is in the nest $\mathcal{M}$.

Finally, define the projection $P = U^* Q^\perp U$. If $F(z) = T$ is any constant function in $A_{\mathcal{N}}$, with T a compact operator in $K_{\mathcal{N}}$, then its image, given by

$$\rho(F)(z) = z^{-1}\lambda^{-1} QUTU^* Q^\perp + (QUTU^*Q + Q^\perp UTU^* Q^\perp) + z\lambda Q^\perp UTU^* Q,$$

is an element of $A_{\mathcal{M}}$. Thus the z^{-1} coefficient must vanish, so $P^\perp T P = 0$ for all T in $\mathcal{K}_{\mathcal{N}}$, which shows P is in the nest $\mathcal{N}$ of invariant projections for $\mathcal{K}_{\mathcal{N}}$. Moreover, the z^0 coefficient must lie in $\mathcal{K}_{\mathcal{M}}$, so $QUP^{\perp}TP^{\perp}U^*Q$ and $Q^{\perp}UPTPU^*Q^{\perp}$ lie in $\mathcal{K}_{\mathcal{M}}$, for all T in $\mathcal{K}_{\mathcal{N}}$. By restricting to subnests, one obtains UT_1U^* in $\mathcal{K}_{Q\mathcal{M}}$ for all T_1 in $\mathcal{K}_{P^{\perp}\mathcal{N}}$ and UT_2U^* in $\mathcal{K}_{Q^{\perp}\mathcal{M}}$ for all T_2 in $\mathcal{K}_{P\mathcal{N}}$. Thus $UP^{\perp}\mathcal{N}U^* = Q\mathcal{M}$ and $UP\mathcal{N}U^* = Q^{\perp}\mathcal{M}$. ∎

The diagonal determined by an analytic subalgebra is often of particular interest, so it is useful to describe how the analytic automorphism acts on it.

Corollary 6. *Let ρ be an analytic automorphism of the C^*-algebra $C(\mathbf{T},\mathcal{K})$ mapping the analytic subalgebra $A_{\mathcal{N}}$ onto $A_{\mathcal{M}}$, where $\mathcal{N}$ and $\mathcal{M}$ are fixed nests. Then the extension of ρ to the multiplier maps the diagonal $D_{\mathcal{N}}$ onto $D_{\mathcal{M}}$ via adjunction by a unitary shift mod 1 of the nests. That is, there exist projections P and Q in $\mathcal{N}$ and $\mathcal{M}$ respectively, and a unitary U with $UP\mathcal{N}U^* = Q^{\perp}\mathcal{M}$ and $UP^{\perp}\mathcal{N}U^* = Q\mathcal{M}$ such that*

$$\rho(T) = UTU^*, \quad T \in D_{\mathcal{N}}.$$

Proof. $D_{\mathcal{N}}$ is identified with the constant maps in $C_*(\mathbf{T},\mathcal{B}(\mathcal{H}))$ taking values in $D_{\mathcal{N}}$. If T is a operator in $D_{\mathcal{N}}$, the constant map $F(z) = T$ is taken under ρ to the function

$$\rho(F)(z) = (Q + z\lambda Q^{\perp})UTU^*(Q + z\lambda Q^{\perp})^*.$$

But UTU^* is an element of $D_{\mathcal{M}}$ and commutes with Q and $Q^{\perp}$, so $\rho(F)$ is the constant function taking value UTU^*. ∎

Note that the unitary U in the above theorem induces a homeomorphism of the topological spaces $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{M}}$ obtained by endowing the nests $\mathcal{N}$ and $\mathcal{M}$ with the strong operator topology and identifying the endpoints $0 \sim I$. A point x in $\tilde{\mathcal{N}}$ corresponds to one (or at most two) projection P_x in $\mathcal{N}$; its image under the homeomorphism is the point in $\tilde{\mathcal{M}}$ corresponding to the projection Q_x, which equals $Q + UP_xU^*$ if $P_x \leqslant P$, or $Q + UP_xU^* - I$ if $P_x \geqslant P$. Of course, $0 \sim I$ in $\tilde{\mathcal{N}}$ maps to Q and P maps to $0 \sim I$ in $\tilde{\mathcal{M}}$. Thus the interval $[0, P]$ maps increasingly to $[Q, I]$, while the interval $[P, I]$ maps increasingly to $[0, Q]$. This is what is meant by saying U is a shift mod 1 on the nests. This will be particularly transparent in the following section, where group actions are considered.

4. GROUP ACTIONS

We now consider an application of the above results to automorphisms of C^*-dynamical systems. The real line $\mathbf{R}$ acts on the circle $\mathbf{R}/\mathbf{Z}$ by translation modulo the integers, with the action given by $(t, x) \mapsto t + x \bmod \mathbf{Z}$, for all t in $\mathbf{R}$ and x in

$\mathbf{R}/\mathbf{Z}$. The C^*-crossed product, denoted $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$, is defined as the enveloping C^*-algebra of the Banach $*$-algebra $L^1(R, C(\mathbf{R}/\mathbf{Z}))$, with a certain twisted convolution product and twisted $*$-operation. (cf. [15]). Following [14], the analytic subalgebra for the crossed product is the non-self-adjoint subalgebra $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}^+$ obtained as the norm closure in $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$ of the image of the functions in $L^1(\mathbf{R}, C(\mathbf{R}/\mathbf{Z}))$ with support in $\mathbf{R}^+$. By the construction in Theorem 2.5 in [8], there is a C^*-isomorphism φ of $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$ onto $C(\mathbf{T}, \mathcal{K})$, with $\mathcal{K}$ the algebra of compact operators on the Hilbert space $L^2([0,1])$, where the measure space $[0,1]$ is a cross section of the space $\mathbf{R}/\mathbf{Z}$. For each f in $L^1(\mathbf{R}, C(\mathbf{R}/\mathbf{Z}))$, $\varphi(f)$ is defined as power series expansion in z,

$$\varphi(f)(z) = \sum_{k=-\infty}^{\infty} z^k T_k(f),$$

where $T_k(f)$ is a Hilbert-Schmidt operator on $L^2([0,1])$, with kernel $K_k(f)$ defined on $[0,1] \times [0,1]$ by $K_k(f)(s,t) = f_{s-t+k}(s)$. It happens that φ maps the analytic subalgebra of the C^*-crossed product onto an analytic subalgebra of the form discussed in the earlier section.

PROPOSITION 7. *The C^*-isomorphism $\varphi : C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R} \to C(\mathbf{T}, \mathcal{K})$ defined above maps $C(\mathbf{R}/\mathbf{Z}) \times \mathbf{R}^+$ onto $A_{\mathcal{N}}$, the analytic subalgebra of $C(\mathbf{T}, \mathcal{K})$ determined by the continuous nest of projections $\mathcal{N} = \{P_t : t \in [0,1]\}$, with P_t the projection onto the subspace $L^2([t,1])$ in $L^2([0,1])$.*

Proof. For any f in $L^1(\mathbf{R}, C(\mathbf{R}/\mathbf{Z}))$ with support in $\mathbf{R}^+$, the kernel function $K_k(f)$ defined above is zero for all k less than zero, and $K_0(f)(s,t) = 0$ for all $s < t$. Thus $\varphi(f)$ is an analytic function in z, with zero-th order coefficient $T_0(f)$ a lower-triangular Hilbert-Schmidt operator in $\mathcal{K}_{\mathcal{N}}$, so $\varphi(f)$ is in $A_{\mathcal{N}}$. Since $L^1(\mathbf{R}^+, C(\mathbf{R}/\mathbf{Z}))$ is dense in the analytic crossed product, φ maps $C(\mathbf{R}/\mathbf{Z}) \times \mathbf{R}^+$ into $A_{\mathcal{N}}$.

To show that φ is onto $A_{\mathcal{N}}$, it sufficies to demonstrate a dense family of analytic polynomials in the image. For finitely many non-negative integers k, let K_k be a continuous function on the square $[0,1] \times [0,1]$ which vanishes on the boundary, and let K_0 have support in the lower triangle; that is, $K_0(s,t) = 0$ for all $s < t$. Thus each K_k represents a Hilbert-Schmidt operator T_k on $L^2([0,1])$, with T_0 in $\mathcal{K}_{\mathcal{N}}$, so the polynomial $\sum z^k T_k$ is an element of $A_{\mathcal{N}}$. The set of all polynomials constructed in this way is dense in $A_{\mathcal{N}}$, since the set of such of T_k is dense in $\mathcal{K}$, for each $k > 0$, while the set of such T_0 is dense in $\mathcal{K}_{\mathcal{N}}$.

To show these polynomials are in the image, fix the K_k. Define a function f as follows: for each s in $[0,1]$ and t in $\mathbf{R}$, pick the unique k in $\mathbf{Z}$ so that $s - t + k$ lies in the half-open interval $[0,1)$, and set $f_t(s) = K_k(s, s-t+k)$. Since K_k vanishes on the boundaries of the square, $K_k = 0$ if $k < 0$, and only finitely many K_k's are non-zero, it is easy to check that f is in $L^1(\mathbf{R}^+, C(\mathbf{R}/\mathbf{Z}))$. Moreover, the choice of f was precisely such that $T_k(f) = T_k$, so $\varphi(f) = \sum z^k T_k$. Thus the image of $L^1(\mathbf{R}^+, C(\mathbf{R}/\mathbf{Z}))$ is a dense subset of $A_{\mathcal{N}}$. ■

Since the algebra $C(\mathbf{R}/\mathbf{Z})$ is a subset of multiplier algebra of $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$, the isomorphism φ described above can be extended to a map defined on $C(\mathbf{R}/\mathbf{Z})$. It is easy to check that for any f in $C(\mathbf{R}/\mathbf{Z})$, $\varphi(f)$ is the constant function in $C(\mathbf{T}, \mathcal{B}(\mathcal{H}))$, taking value $M_{f\circ\psi}$, the operator on $L^2([0,1])$ of multiplication by the function $f \circ \psi$, where ψ is the map of $[0,1]$ onto $\mathbf{R}/\mathbf{Z}$ obtained by identifying the endpoints. Moreover, using the isomorphism φ, the diagonal algebra $M(C(\mathbf{R}/\mathbf{Z})\times_\tau \mathbf{R}^+) \cap M(C(\mathbf{R}/\mathbf{Z})\times_\tau \mathbf{R}^+)^*$ can be identified in a natural way with $L^\infty(\mathbf{R}/\mathbf{Z})$, which acts by multiplication on the $C(\mathbf{R}/\mathbf{Z})$ elements of the multiplier.

Now a "shift mod 1" of the nest $\mathcal{N}$ corresponds to an orientation preserving transformation of the circle $\mathbf{R}/\mathbf{Z}$. In fact, the following is true.

THEOREM 8. *Let ρ be an analytic automorphism of $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$ mapping $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}^+$ to $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}^+$. Then there exists an orientation and measure class preserving homeomorphism ϕ of $\mathbf{R}/\mathbf{Z}$ such that the multiplier extension of ρ satisfies $\rho(f) = f \circ \phi$, for all f in $C(\mathbf{R}/\mathbf{Z})$. Moreover, every orientation and measure class preserving homeomorphism of $\mathbf{R}/\mathbf{Z}$ is obtained by some analytic automorphism.*

Proof. By Proposition 7, the analytic automorphism ρ can be pulled back to an analytic automorphism of $C(\mathbf{T}, \mathcal{K})$ mapping the analytic algebra $A_{\mathcal{N}}$ onto $A_{\mathcal{N}}$, where $\mathcal{N}$ is the continuous nest on $L^2([0,1])$. By Corollary 7, this pullback maps the diagonal $L^\infty([0,1])$ onto itself via a unitary shift mod 1. Thus a function f in $L^\infty([0,1])$ maps to $f \circ \tilde{\phi}$, where $\tilde{\phi}$ is the measure class preserving transformation of $[0,1]$ induced by U. Since U is a shift mod 1, there exist points p and q in $[0,1]$ such that $\tilde{\phi}$ maps the subinterval $[0,p]$ monotonically increasing onto the subinterval $[q,1]$, and maps the subinterval $[p,1]$ monotonically increasing onto the subinterval $[0,q]$. Pushing forward to $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$ shows that ρ maps a function f in the diagonal $L^\infty(\mathbf{R}/\mathbf{Z})$ to $f \circ \phi$, where ϕ is the measure class and orientation preserving homeomorphism of $\mathbf{R}/\mathbf{Z}$ obtained from $\tilde{\phi}$ by identifying endpoints.

Moreover, every such ϕ maybe obtained from a mod 1 shift $\tilde{\phi}$ of the interval $[0,1]$, which itself arises from a unitary shift mod 1 of the continuous nest, so every such ϕ arises from analytic automorphism. ■

Using Theorem 5 and Proposition 7, all analytic automorphisms of the C^*-crossed product $C(\mathbf{R}/\mathbf{Z}) \times_\tau \mathbf{R}$ can be described; it suffices to say that every such automorphism is a unique product of one each of the following three types of automorphisms. First, there is the "inner" automorphism coming by conjugation by a unitary in the diagonal

$L^\infty(\mathbf{R}/\mathbf{Z})$; second, there is an "outer" automorphism given by an orientation and measure-class preserving homeomorphism of $\mathbf{R}/\mathbf{Z}$ lifted to the crossed product; third, there is a cocycle automorphism coming from a rotation by λ of $\mathbf{T}$, the dual group of the stabilizer $\mathbf{Z}$ in $\mathbf{R}$.

The same analysis holds for the case of the group $\mathbf{Z}$ acting by addition mod n on the finite group $\mathbf{Z}_n$. In particular, the analytic map preserves an orientation on the underlying discrete space. The analogous theorem is as follows.

THEOREM 9. *Let ρ be an analytic automorphism of $C(\mathbf{Z}_n) \times_\tau \mathbf{Z}$ mapping $C(\mathbf{Z}_n) \times_\tau \mathbf{Z}^+$ to $C(\mathbf{Z}_n) \times_\tau \mathbf{Z}^+$. Then there exists a shift ϕ on $\mathbf{Z}_n$, $\phi(k) = k + k_0$* mod n, *such that ρ satisfies $\rho(f) = f \circ \phi$, for all f in $C(\mathbf{Z}_n)$. Moreover, every shift* mod n *on $\mathbf{Z}_n$ is obtained by some analytic automorphism.*

Proof. Follows as in the proof of Theorem 8, replacing the continuous nest on $[0,1]$ with a maximal nest on the finite dimensional Hilbert space $l^2(1,2,\ldots,n)$. ■

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