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MULTIPLIERS AND INVARIANT SUBSPACES IN THE DIRICHLET SPACE

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1. INTRODUCTION

The Dirichlet space D is the space of all analytic functions f on the open unit disc $\mathbb{D}$ that have a finite Dirichlet integral, i.e.

$$D(f) = \iint_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z) < \infty$$

where $\mathrm{d}A(re^{it}) = \frac{1}{\pi} r\,\mathrm{d}r\,\mathrm{d}t$ denotes the normalized area measure on $\mathbb{D}$. The operator of multiplication by z on D is called the Dirichlet shift, and we denote it by (M_z, D).

In this paper we study the invariant subspace structure of (M_z, D). It is one of our main results that every invariant subspace $\mathcal{M}$ is generated by a multiplier φ, i.e. an analytic function φ on $\mathbb{D}$ such that $\varphi f \in D$ whenever $f \in D$. Our proof builds on results from [9] and [11], where the somewhat more general situation of the Dirichlet-type spaces $D(\mu)$ (definition below) is considered. These spaces $D(\mu)$ naturally occur when one considers the totality of all invariant subspaces of (M_z, D). Hence it will be useful to know (and hardly any more difficult to prove) that almost all of our results are valid in this more general situation.

Let ζ be a point on the unit circle $\mathbb{T} = \{|z| = 1\}$. In [11], Section 2, it was shown that, if $f \in H^2$ and if

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{it}) - \alpha}{e^{it} - \zeta} \right|^2 \mathrm{d}t < \infty \tag{1.1}$$

for some complex number α, then the oricyclic limit (and therefore in particular the nontangential limit) of f at ζ exists and equals α. Thus, as in [11] we can define the *local Dirichlet integral* of the H^2-function f at ζ by

$$D_\zeta(f) = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta}\right|^2 dt. \tag{1.2}$$

Throughout this paper μ will always be a nonnegative finite Borel measure on the unit circle $\mathbf{T}$. The Dirichlet-type space $D(\mu)$ is the space of all H^2-functions such that $D_\zeta(f) \in L^1(\mu)$. A norm on $D(\mu)$ can be defined by

$$\|f\|_\mu^2 = \|f\|_{H^2}^2 + \int_{\mathbf{T}} D_\zeta(f)\,d\mu(\zeta).$$

Note that if $f \in D(\mu)$, then $D_\zeta(f) < \infty$ a.e. $[\mu]$, so that $f(\zeta)$ exists a.e. $[\mu]$ as the oricyclic limit of f at ζ.

Let m denote the normalized Lebesgue measure on $\mathbf{T}$, i.e. $dm(e^{it}) = \frac{1}{2\pi}dt$. By a formula of Douglas [5] one has

$$D(f) = \int_{\mathbf{T}} D_\zeta(f)\,dm(\zeta).$$

In fact, more generally it turns out that

$$D_\zeta(f) = \iint_{\mathbf{D}} |f'(z)|^2 \frac{1 - |z|^2}{|\zeta - z|^2}\,dA(z) \tag{1.3}$$

for arbitrary H^2-functions f (see Proposition 2.2 of [11]). These formulas explain the name "local Dirichlet integral", and we see that the Dirichlet space D equals the space $D(m)$.

In the following we shall use $(M_z, D(\mu))$ to denote the bounded linear transformation on $D(\mu)$ which takes a function f to the function zf. The lattice of all (closed) invariant subspaces of $(M_z, D(\mu))$ will be denoted by $\mathrm{Lat}(M_z, D(\mu))$. A multiplier of $D(\mu)$ is a function φ on $\mathbf{D}$ such that $\varphi f \in D(\mu)$ whenever $f \in D(\mu)$.

If $f, g \in D(\mu)$, then $[f]$ denotes the smallest invariant subspace of $(M_z, D(\mu))$ containing f, while $[f, g]$ is used for the span of $[f]$ and $[g]$. If $[f] = D(\mu)$ then f is called *cyclic* (or *cyclic in* $D(\mu)$).

Let $\mathcal{M} \in \mathrm{Lat}(M_z, D(\mu))$, $\mathcal{M} \neq (0)$. In Theorem 3.2 we shall show that $\mathcal{M} \ominus z\mathcal{M} (= \mathcal{M} \cap (z\mathcal{M})^\perp)$ is one dimensional. This generalizes Theorem 2 of [10] where this result was established for (M_z, D). If $\mathcal{M}$ contains a function that does not vanish at the origin, then a well known (and simple) argument shows that $f \in \mathcal{M} \ominus z\mathcal{M}$, $f \neq 0$, if and only if f is a solution to the extremal problem

$$\inf\left\{\frac{\|g\|_\mu}{|g(0)|} : g \in \mathcal{M}\right\}. \tag{1.4}$$

Thus it follows from Theorem 7.1 of [9] that $\mathcal{M} = \varphi D(\mu_\varphi)$, where φ is a unit vector solving the extremal problem (1.4), and μ_φ is the measure defined by $\mathrm{d}\mu_\varphi(\zeta) = |\varphi(\zeta)|^2 \mathrm{d}\mu(\zeta)$. This theorem is very reminiscent of Beurling's Theorem for the invariant subspaces of H^2. In fact the special case $\mu = 0$ corresponds to a part of Beurling's Theorem. What is lacking is information about the extremal functions satisfying (1.4). In the H^2 situation one knows that these are constant multiples of inner functions. Theorem 3.1 says that any solution to (1.4) has to be a multiplier of $D(\mu)$; and it follows that every invariant subspace of $(M_z, D(\mu))$ is generated by a multiplier.

In light of this "Beurling type theorem" for the invariant subspaces of $(M_z, D(\mu))$ one might wonder whether making use of the inner-outer factorization of H^2-functions obscures the fact that one should really be trying to characterize "Dirichlet-inner functions" (i.e. solutions to the extremal problem (1.4)). However, there seems to be a natural split between the cases of H^2-inner and outer functions: If B is a nontrivial inner function such that the invariant subspace $D(\mu) \cap BH^2$ is nonzero, then it properly contains the invariant subspace $D(\mu) \cap B^2 H^2$. On the other hand, if f and f^2 are outer functions in $D(\mu)$, then $[f] = [f^2]$ (see Theorem 4.3). Thus, if it is true that invariant subspaces are determined by certain zero sets in $\mathbb{D}^-$ (and all our results support such a conjecture), then one might say that inner functions "count multiplicity" of zero sets while outer functions don't.

Beurling showed in [1] that if f is in the Dirichlet space D, then the set of points in the unit circle for which f does not have a finite radial limit, has logarithmic capacity zero. Let $Z(f)$ denote the set of points in $\mathbb{T}$ where the radial limit of f is zero. Brown and Shields [4] prove that if $Z(f)$ has positive logarithmic capacity, then f is not cyclic in D. Furthermore, Brown and Shields conjecture that a function f is cyclic in D if and only if f is outer and $Z(f)$ has logarithmic capacity zero.

All of our results support this conjecture. In particular, we give an affirmative answer to Question 14 of [4] by showing that any nonvanishing univalent function in the Dirichlet space must be cyclic in D (see Corollary 4.4). This is interesting, because it was known before (see [1]) that for univalent functions f in D, $Z(f)$ has logarithmic capacity zero.

Let $f, g \in D(\mu)$. We know from the above mentioned theorem that $[f] \cap [g]$ and $[f, g]$ are singly generated. If f and g are outer functions, then we define the outer functions $f \wedge g$ and $f \vee g$ by $|f \wedge g(\mathrm{e}^{\mathrm{i}t})| = \min\{|f(\mathrm{e}^{\mathrm{i}t})|, |g(\mathrm{e}^{\mathrm{i}t})|\}$ and $|f \vee g(\mathrm{e}^{\mathrm{i}t})| = \max\{|f(\mathrm{e}^{\mathrm{i}t})|, |g(\mathrm{e}^{\mathrm{i}t})|\}$. We shall show that $f \wedge g$ and $f \vee g$ are in $D(\mu)$ whenever both f and g are in $D(\mu)$ (Lemma 2.2), and that $[f] \cap [g] = [f \wedge g]$ and $[f, g] = [f \vee g]$ (Theorems 4.1 and 4.5). Finally, in Section 5 we prove that every invariant subspace of $(M_z, D(\mu))$ is of the form $[f] \cap BH^2$ where f is outer and B is inner (Theorem

5.3). In Proposition 5.4 we show that knowledge about the space $D(\mu)$, μ singular, is interesting when dealing with singular inner factors of functions in D.

In the following sections we shall use several elementary facts without further reference. One is that the product of two bounded $D(\mu)$-functions is again in $D(\mu)$, i.e. $D(\mu) \cap H^\infty$ is an algebra. Another fact that we shall frequently use is that a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subseteq D(\mu)$ converges weakly if and only if it converges pointwise in $\mathbb{D}$ and is norm-bounded (see Proposition 2, p. 272 of [4]). Furthermore if $\mu \neq 0$ and if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in \mathbb{D}$, then for the norm-boundedness in $D(\mu)$ one only needs to check that $\int_{\mathbb{T}} D_\zeta(f_n)\,\mathrm{d}\mu(\zeta) \leqslant C$ for some constant C and all $n \in \mathbb{N}$, because the H^2-norm of a function f is bounded by a constant times

$$\left(|f(0)|^2 + \int_{\mathbb{T}} D_\zeta(f)\,\mathrm{d}\mu(\zeta)\right)^{\frac{1}{2}}$$

(see Lemma 3.1 of [9]). In any case, the estimates for the H^2-norm would usually simply follow from pointwise estimates.

2. CUT-OFF FUNCTIONS

All of our results of the later sections will follow in part from some rather technical estimates for the local Dirichlet integral, which we shall obtain by use of a formula that was proved in [11]. This is done in Lemmas 2.1 and 4.2, and even though the details appear cumbersome the methods are of an elementary nature.

Let $f, g \in H^2$ be outer functions, then we define the outer functions $f \wedge g$ and $f \vee g$ by $|(f\wedge g)(\mathrm{e}^{\mathrm{i}t})| = \min\{|f(\mathrm{e}^{\mathrm{i}t})|, |g(\mathrm{e}^{\mathrm{i}t})|\}$ and $|(f\vee g)(\mathrm{e}^{\mathrm{i}t})| = \max\{|f(\mathrm{e}^{\mathrm{i}t})|, |g(\mathrm{e}^{\mathrm{i}t})|\}$. This means that for $z \in \mathbb{D}$

$$f \wedge g(z) = \exp\left\{\frac{1}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\log|(f\wedge g)(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t\right\},$$

$$f \vee g(z) = \exp\left\{\frac{1}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\log|(f\vee g)(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t\right\}.$$

Let $h = Bf$ be the inner-outer factorizarion of the H^2-function h and define the two functions $\varphi = B(f \wedge 1)$ and $\psi = 1/(f \vee 1)$. Then $h = \varphi/\psi$ is the quotient of two bounded functions, and in [4] it was shown that the Dirichlet integrals of φ and ψ are both bounded by the Dirichlet integral of h. In [11] it was proved that these estimates hold even for the local Dirichlet integrals of φ, ψ and $1/\psi$.

We note that because $|\psi(z)| \leqslant 1$ for all $z \in \mathbf{D}$, we have $|\psi'(z)| \leqslant |(1/\psi)'(z)|$. Hence it follows from (1.3) that $D_\zeta(\psi) \leqslant D_\zeta(1/\psi)$. Thus we see that at least in the case where h has no non-trivial inner factor the estimates mentioned above follow from the estimates for the local Dirichlet integrals of $f \wedge 1$ and $f \vee 1$.

In Lemma 2.2 we shall extend these results and show that the local Dirichlet integrals of $f \wedge g$ and $f \vee g$ are bounded by the sum of the local Dirichlet integrals of f and g. This will follow from the formula for the local Dirichlet integral that was proved in [11]. For the convenience of the reader we shall restate the part that we will use. In fact we shall state the results of Theorem 3.1, Lemma 3.4, and Proposition 3.5 of [11] in two formulas. To simplify subsequent notation we define

$$F(x,y) = \begin{cases} e^x - e^y - e^y(x-y) & \text{if } x, y \in \mathbf{R} \\ e^x & \text{if } x \in \mathbf{R}, y = -\infty \end{cases}.$$

We note that from the representation $F(x,y) = e^y(e^{x-y} - 1 - (x-y))$ it is easy to see that $F(x,y) \geqslant 0$ for all x, y. Let $h = Bf$ be the inner-outer factorization of $h \in H^2$ and write $u(e^{it}) = \log|f(e^{it})|$. Then for $\zeta \in \mathbf{T}$

$$D_\zeta(Bf) = D_\zeta(B)|f(\zeta)|^2 + D_\zeta(f), \tag{2.1}$$

$$D_\zeta(f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(2u(e^{it}), 2u(\zeta))}{|e^{it} - \zeta|^2}\,dt \tag{2.2}$$

where $D_\zeta(B)$ coincides with the angular derivative of the inner function B.

In order to obtain the estimates for $D_\zeta(f \wedge g)$ and $D_\zeta(f \vee g)$ we shall have to deal with the integrand of the term on the right hand side of (2.2).

LEMMA 2.1. *Let $x_1, x_2 \in \mathbf{R}$ and $y_1, y_2 \in \mathbf{R} \cup \{\infty\}$, then*

$$F(\max(x_1,x_2), \max(y_1,y_2)) \leqslant \max\{F(x_1,y_1), F(x_2,y_2)\}$$

$$F(\min(x_1,x_2), \min(y_1,y_2)) \leqslant \max\{F(x_1,y_1), F(x_2,y_2)\}.$$

Proof. We observe that both inequalities are trivially true, if either $\max(x_1,x_2) = x_1$ and $\max(y_1,y_2) = y_1$ or $\max(x_1,x_2) = x_2$ and $\max(y_1,y_2) = y_2$.

By symmetry we only need to consider one of the other two cases. Without loss of generality we assume $\max(x_1,x_2) = x_1$ and $\max(y_1,y_2) = y_2$. Furthermore, we note that for each fixed $x \in \mathbf{R}$, $F(x,\cdot)$ is continuous at $-\infty$. Thus, by a limit argument the result will follow in the stated generality, once it has been established for $y_1, y_2 > -\infty$.

It will be convenient to use the partial derivatives $F_x(x,y) = e^x - e^y$, $F_y(x,y) = e^y(y-x)$.

To prove the two inequalities we consider four (not necessarily distinct) cases.

(1) $x_1 = \max(x_1, x_2) \leqslant \max(y_1, y_2) = y_2$.

In this case $F_x(t, y_2) \leqslant 0$ for $x_2 \leqslant t \leqslant x_1 \leqslant y_2$, hence

$$F(\max(x_1,x_2), \max(y_1,y_2)) = F(x_1,y_2) \leqslant F(x_2,y_2).$$

(2) $x_1 = \max(x_1, x_2) \geqslant \max(y_1, y_2) = y_2$.

Now $F_y(x_1, t) \leqslant 0$ for $y_1 \leqslant t \leqslant y_2 \leqslant x_1$, thus,

$$F(\max(x_1,x_2), \max(y_1,y_2)) = F(x_1,y_2) \leqslant F(x_1,y_1).$$

(3) $x_2 = \min(x_1, x_2) \leqslant \min(y_1, y_2) = y_1$.

Note that $F_y(x_2, t) \geqslant 0$ for $x_2 \leqslant y_1 \leqslant t \leqslant y_2$. This implies

$$F(\min(x_1,x_2), \min(y_1,y_2)) = F(x_2,y_1) \leqslant F(x_2,y_2).$$

(4) $x_2 = \min(x_1, x_2) \geqslant \min(y_1, y_2) = y_1$.

In this final case $F_x(t, y_1) \geqslant 0$ for $y_1 \leqslant x_2 \leqslant t \leqslant x_1$, so

$$F(\min(x_1,x_2), \min(y_1,y_2)) = F(x_2,y_1) \leqslant F(x_1,y_1). \qquad \blacksquare$$

The lemma about the local Dirichlet integrals of $D_\zeta(f \vee g)$ and $D_\zeta(f \wedge g)$ follows immediately.

LEMMA 2.2. *Let f, g be outer functions in H^2. Then for every $\zeta \in \mathbf{T}$, $D_\zeta(f \vee g) \leqslant D_\zeta(f) + D_\zeta(g)$ and $D_\zeta(f \wedge g) \leqslant D_\zeta(f) + D_\zeta(g)$.*

Furthermore, if f, g are outer functions in $D(\mu)$, then $f \vee g$ and $f \wedge g$ are in $D(\mu)$ and $\|f \vee g\|_\mu^2 \leqslant \|f\|_\mu^2 + \|g\|_\mu^2$ and $\|f \wedge g\|_\mu^2 \leqslant \|f\|_\mu^2 + \|g\|_\mu^2$.

Proof. Fix $\zeta \in \mathbf{T}$. The inequalities are trivially true if either $D_\zeta(f)$ or $D_\zeta(g)$ is infinite, thus we shall assume that both quantities are finite. We use formula (2.2) and Lemma 2.1. $f \vee g$ is an outer function, hence if $u(e^{it}) = \log|f(e^{it})|$ and $v(e^{it}) = \log|g(e^{it})|$, then

$$\begin{aligned} D_\zeta(f \vee g) &= \frac{1}{2\pi}\int \frac{F(\max\{2u(e^{it}), 2v(e^{it})\}, \max\{2u(\zeta), 2v(\zeta)\})}{|e^{it} - \zeta|^2}\,dt \leqslant \\ &\leqslant \frac{1}{2\pi}\int \frac{\max\{F(2u(e^{it}), 2u(\zeta)), F(2v(e^{it}), 2v(\zeta))\}}{|e^{it} - \zeta|^2}\,dt \leqslant \\ &\leqslant D_\zeta(f) + D_\zeta(g). \end{aligned}$$

The inequality for $D_\zeta(f \wedge g)$ follows in exactly the same way. The second part of the lemma follows by integrating the above inequalities with respect to μ and by noting that a similar estimate for the H^2-norm follows trivially. ■

If $h = Bf$ is the inner-outer factorization of $h \in H^2$ and $N \in \mathbf{R}$ is a constant, then we consider the cut-off function $h_N = B(f \wedge e^N)$.

COROLLARY 2.3. *If $h \in H^2$, $\zeta \in \mathbf{T}$, and $N \in \mathbf{R}$, then $D_\zeta(h_N) \leqslant D_\zeta(h)$. Consequently $\|h_N\|_\mu \leqslant \|h\|_\mu$ whenever $h \in D(\mu)$.*

Proof. Of course $D_\zeta(e^N) = 0$. Thus with the notation from above, Lemma 2.2 and formula (2.1) imply that

$$D_\zeta(h_N) = D_\zeta(B)|(f \wedge e^N)(\zeta)|^2 + D_\zeta(f \wedge e^N) \leqslant \\ \leqslant D_\zeta(B)|f(\zeta)|^2 + D_\zeta(f) = D_\zeta(h).$$

The $D(\mu)$ inequality follows as before. ■

Corollary 2.3 and the following results will enable us to reduce many questions about Dirichlet functions to bounded Dirichlet functions. First recall a result from [11].

LEMMA 2.4. (=Corollary 5.5, [11]) *Let $f, g \in D(\mu)$. If $|g(z)| \leqslant c|f(z)|$ for some constant $c > 0$ and all $z \in \mathbf{D}$, then $[g] \subseteq [f]$.*

LEMMA 2.5. *If $\varphi \in H^\infty$ and $\psi \in H^2$, then for each $N \in \mathbf{R}$ and $\zeta \in \mathbf{T}$*

$$D_\zeta(\psi_N \varphi) \leqslant 4D_\zeta(\psi\varphi) + 6\|\varphi\|_\infty^2 D_\zeta(\psi)$$

Proof. Let $N \in \mathbf{R}$ and $\zeta \in \mathbf{T}$ be fixed. The assertion is trivially true if either $D_\zeta(\psi)$ or $D_\zeta(\psi\varphi)$ is infinite, so we may assume that both quantities are finite. Now note that

$$\frac{\psi_N\varphi - \psi_N\varphi(\zeta)}{z-\zeta} = \psi_N \frac{\varphi - \varphi(\zeta)}{z-\zeta} + \varphi(\zeta)\frac{\psi_N - \psi_N(\zeta)}{z-\zeta}$$

and that

$$\psi\frac{\varphi - \varphi(\zeta)}{z-\zeta} = \frac{\psi\varphi - \psi\varphi(\zeta)}{z-\zeta} - \varphi(\zeta)\frac{\psi - \psi(\zeta)}{z-\zeta}.$$

Hence we see that

$$D_\zeta(\psi_N\varphi) \leqslant 2\left(\left\|\psi_N\frac{\varphi-\varphi(\zeta)}{z-\zeta}\right\|_{H^2}^2 + \|\varphi\|_\infty^2 D_\zeta(\psi_N)\right) \leqslant \\ \leqslant 2\left\|\psi\frac{\varphi-\varphi(\zeta)}{z-\zeta}\right\|_{H^2}^2 + 2\|\varphi\|_\infty^2 D_\zeta(\psi) \leqslant$$

$$\leqslant 4\left(D_\zeta(\psi\varphi) + \|\varphi\|_\infty^2 D_\zeta(\psi)\right) + 2\|\varphi\|_\infty^2 D_\zeta(\psi). \qquad \blacksquare$$

In [11] we stated the following result without proof. We include the proof here as it is the basis of much of the rest of the paper. We remark that in [8] it was proved that every function in the Dirichlet space D can be factored as a product of a bounded and a cyclic function in D. In that case the new (and more difficult part) of the result is that the bounded factor generates the same invariant subspace as the original function (and the observation that one over the cyclic factor is also a cyclic vector).

LEMMA 2.6. *Let $f \in D(\mu)$. If $f = \varphi/\psi$, where $\varphi, \psi \in H^\infty \cap D(\mu)$ and $1/\psi \in$ $\in D(\mu)$, then $[f] = [\varphi]$ and ψ and $1/\psi$ are cyclic vectors in $D(\mu)$.*

Note that for any $N \in \mathbf{R}$ the cut-off functions $\varphi = f_N$ and $\psi = e^N/(h \vee e^N)$ satisfy the hypothesis of the lemma; here h is the outer factor of f.

Proof. By a scaling argument we may assume $\|\varphi\|_\infty, \|\psi\|_\infty \leqslant 1$.

For all $z \in \mathbf{D}$ we have $|\varphi(z)| \leqslant |f(z)|$, thus the inclusion $[\varphi] \subseteq [f]$ follows from Lemma 2.4.

To show that $[f] \subseteq [\varphi]$ we shall prove that $\left\{\left(\frac{1}{\psi}\right)_N \varphi\right\}_{N\in\mathbf{N}}$ is contained in $[\varphi]$ and converges weakly to f. First, we note that it follows from Lemma 2.5 applied with $\varphi \in H^\infty$ and $1/\psi \in H^2$ that $D_\zeta\left(\left(\frac{1}{\psi}\right)_N \varphi\right) \leqslant 4D_\zeta(f) + 6D_\zeta\left(\frac{1}{\psi}\right)$ for all $N \in \mathbf{R}$ and $\zeta \in \mathbf{T}$. We integrate with respect to μ and obtain that $\left(\frac{1}{\psi}\right)_N \varphi \in D(\mu)$, in fact we have $\left\|\left(\frac{1}{\psi}\right)_N \varphi\right\|_\mu^2 \leqslant c\left(\|f\|_\mu^2 + \left\|\frac{1}{\psi}\right\|_\mu^2\right)$ for some constant c. Again we use Lemma 2.4 to see that for each $N \in \mathbf{R}$, $\left(\frac{1}{\psi}\right)_N \varphi \in [\varphi]$. It is easy to see that for each $z \in \mathbf{D}$, $\left(\frac{1}{\psi}\right)_N (z) \to \left(\frac{1}{\psi}\right)(z)$ as $N \to \infty$. Thus, the sequence $\left\{\left(\frac{1}{\psi}\right)_N \varphi\right\}_{N\in\mathbf{N}}$ is bounded in $D(\mu)$ and converges pointwise to f, hence it converges weakly to f. This shows that $[f] \subseteq [\varphi]$.

Finally, we have to show that ψ and $1/\psi$ are cyclic in $D(\mu)$. Lemma 2.4 together with the fact that $|(1/\psi)(z)|$ is bounded below by 1 for $z \in \mathbf{D}$ implies that $1/\psi$ is cyclic. Furthermore, the same argument as in the previous paragraph shows that $\left\{\left(\frac{1}{\psi}\right)_N \psi\right\}_{N\in\mathbf{N}} \subseteq [\psi]$ converges weakly to 1. Thus ψ is cyclic as well. $\blacksquare$

For the special case of the Dirichlet space, i.e. $\mu =$ Lebesgue measure on $\mathbf{T}$ the following corollary was shown in [3].

COROLLARY 2.7. *If f and $1/f$ are contained in $D(\mu)$, then both are cyclic.*

Proof. We use Lemma 2.6. It follows from the assumption that f has no inner factor, hence $f = \varphi/\psi$ and $1/f = \psi/\varphi$, where φ and ψ are cyclic. Consequently $[f] = [\varphi] = D(\mu) = [\psi] = [1/f]$. ■

For later reference we shall need one more lemma about cut-off functions.

LEMMA 2.8. *Let $\{f_n\}_{n\in\mathbf{N}} \subseteq H^2$ and let f be an outer function in H^∞, $\|f\|_\infty \leqslant 1$. If $\{f_n\}_{n\in\mathbf{N}} \to f$ in H^2, then the cut-off functions $\{(f_n)_0\}_{n\in\mathbf{N}}$ satisfy $\{(f_n)_0(z)\} \to$ $\to f(z)$ for all $z \in \mathbf{D}$. Consequently if $\{f_n\}_{n\in\mathbf{N}} \to f$ in $D(\mu)$, then $\{(f_n)_0\} \to f$ (weakly) in $D(\mu)$.*

Proof. Clearly $\{f_n(z)\} \to f(z)$ for every $z \in \mathbf{D}$. Hence if we let $h_n = (f_n)_0/f_n$, then we need to show that $\{h_n(z)\} \to 1$ for every $z \in \mathbf{D}$. Furthermore, $\|h_n\|_\infty \leqslant 1$ for all n, thus it suffices to show that $\{|h_n(0)|\} \to 1$. This can be done in exactly the same way as in the proof of Lemma 6, p. 283 of [4].

The second part of the Lemma follows, because convergence in $D(\mu)$ implies convergence in H^2. ■

3. INVARIANT SUBSPACES ARE GENERATED BY MULTIPLIERS

Theorem 7.1 of [9] states that any invariant subspace $\mathcal{M}$ of $(M_z, D(\mu))$ with $\dim \mathcal{M} \ominus z\mathcal{M} = 1$ is of the form $\mathcal{M} = [\varphi] = \varphi D(\mu_\varphi)$, where $\varphi \in \mathcal{M} \ominus z\mathcal{M}$ and $\mathrm{d}\mu_\varphi = |\varphi|^2 \mathrm{d}\mu$. Furthermore if φ has norm one, then we have for every $g \in D(\mu_\varphi)$

$$\|\varphi g\|_\mu = \|g\|_{\mu_\varphi}. \tag{3.1}$$

We note that this implies that φ defines a bounded multiplication operator on $D(\mu)$, whenever $\varphi \in H^\infty$. In fact, if φ is bounded, then $D(\mu) \subseteq D(\mu_\varphi)$, hence

$$\varphi D(\mu) \subseteq \varphi D(\mu_\varphi) = \mathcal{M} \subseteq D(\mu).$$

Using (3.1) one can check that the multiplier norm of φ is bounded by the maximum of $\|\varphi\|_\mu$ and $\|\varphi\|_\infty$.

We shall show that such functions φ are indeed multipliers of $D(\mu)$. Furthermore, later we shall show that every non-zero invariant subspace of $(M_z, D(\mu))$ satisfies the hypothesis of the above quoted theorem, i.e. $\dim \mathcal{M} \ominus z\mathcal{M} = 1$.

THEOREM 3.1. *If $f \in D(\mu)$ is orthogonal in $D(\mu)$ to $z^n f$ for every $n > 0$, then f is a multiplier of $D(\mu)$.*

Proof. We assume $f \neq 0$. The assumptions in the Theorem imply that $f \in [f] \ominus z[f]$. Furthermore, it is true for any operator T that a cyclic invariant subspace

$\mathcal{M}$ satisfies $\dim \mathcal{M} \ominus T\mathcal{M} \leqslant 1$. Thus, $\dim[f] \ominus z[f] = 1$ and by what has been said above we only need to show that f is a bounded function.

We denote the power series coefficients of a function $h \in D(\mu)$ by $\widehat{h}(n)$, $n = 0, 1, \ldots$.

Let k be the smallest nonnegative integer such that $\widehat{f}(k) \neq 0$, i.e. there exists a function $g \in D(\mu)$ such that $f = z^k g$ and $g(0) \neq 0$. Then all functions in $[f]$ are of the form $z^k h$ for some $h \in D(\mu)$, and it is well known (and easy to check) that $f = z^k g$ is a solution to the extremal problem

$$\inf\left\{\frac{\|z^k h\|_\mu}{|h(0)|} : z^k h \in [f]\right\}. \tag{3.2}$$

In the following we shall show that for any unbounded function $z^k h \in [f]$ with $h(0) \neq 0$ there exists $N > 0$ such that the (bounded) cut-off function $z^k h_N = (z^k h)_N \in [f]$ satisfies

$$\frac{\|z^k h_N\|_\mu}{|h_N(0)|} < \frac{\|z^k h\|_\mu}{|h(0)|}. \tag{3.3}$$

This will of course establish that the infimum in (3.2) can only be attained by a bounded function, i.e. $f \in H^\infty$.

To verify (3.3) we square, multiply through with $|h(0)|^2$, and subtract $\|z^k h_N\|_\mu^2$ from both sides of the inequality to see that the assertion is equivalent to

$$\|z^k h_N\|_\mu^2 \left(\frac{|h(0)|^2}{|h_N(0)|^2} - 1\right) < \|z^k h\|_\mu^2 - \|z^k h_N\|_\mu^2.$$

Thus, we have to show that for large N

$$\|z^k h_N\|_\mu^2 < \frac{\|z^k h\|_\mu^2 - \|z^k h_N\|_\mu^2}{\dfrac{|h(0)|^2}{|h_N(0)|^2} - 1}. \tag{3.4}$$

It follows from Corollary 2.3 that the left hand side is bounded by $\|z^k h\|_\mu^2$. We shall finish the proof of (3.3) by establishing that the right hand side of (3.4) is larger than ce^{2N} for large N.

Let g be the outer factor of h and write $u(e^{it}) = \log|g(e^{it})|$, $u_N(e^{it}) = \log|g_N(e^{it})|$, and $F_N = \{t \in [0, 2\pi] : u(e^{it}) > N\}$. The definition of the cut-off function g_N implies that u and u_N agree on the complement of F_N, hence we obtain

$$\log\frac{|h(0)|^2}{|h_N(0)|^2} = 2(\log|g(0)| - \log|g_N(0)|) = 2\frac{1}{2\pi}\int_{F_N}(u(e^{it}) - N)\,dt.$$

We note that this easily implies that $\frac{|h(0)|^2}{|h_N(0)|^2} \searrow 1$ as $N \to \infty$.

Consequently, there exist $c > 0$ and $N_0 \in \mathbf{R}$ such that for $N \geqslant N_0$

$$2\frac{1}{2\pi}\int\limits_{F_N}(u(\mathrm{e}^{\mathrm{i}t}) - N)\,\mathrm{d}t > c\left(\frac{|h(0)|^2}{|h_N(0)|^2} - 1\right).$$

Recall from Corollary 2.3 that for each $N \in \mathbf{R}$ and $\zeta \in \mathbf{T}$ one has $D_\zeta(h_N) \leqslant D_\zeta(h)$. This implies

$$\|h\|_\mu^2 - \|h_N\|_\mu^2 = \|h\|_{H^2}^2 - \|h_N\|_{H^2}^2 + \int (D_\zeta(h) - D_\zeta(h_N))\,\mathrm{d}\mu(\zeta) \geqslant$$
$$\geqslant \frac{1}{2\pi}\int\limits_0^{2\pi}(|g(\mathrm{e}^{\mathrm{i}t})|^2 - |g_N(\mathrm{e}^{\mathrm{i}t})|^2)\,\mathrm{d}t = \frac{1}{2\pi}\int\limits_{F_N}\left(\mathrm{e}^{2u(\mathrm{e}^{\mathrm{i}t})} - \mathrm{e}^{2N}\right)\mathrm{d}t \geqslant$$
$$\geqslant \mathrm{e}^{2N}\frac{1}{2\pi}\int\limits_{F_N} 2(u(\mathrm{e}^{\mathrm{i}t}) - N)\,\mathrm{d}t,$$

because $\mathrm{e}^x - 1 \geqslant x$ for all $x \in \mathbf{R}$. We have thus seen that for all $N \geqslant N_0$

$$\frac{\|h\|_\mu^2 - \|h_N\|_\mu^2}{\frac{|h(0)|^2}{|h_N(0)|^2} - 1} > c\mathrm{e}^{2N}.$$

As mentioned above this implies that our function f which solves the extremal problem (3.2) must be bounded, and this concludes the proof of Theorem 3.1. ∎

Recall from [7] that an operator T on a Hilbert space is called *cellularly indecomposable*, if any two non-zero invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of T have a non-zero intersection. In [10] it was shown that the Dirichlet shift is cellularly indecomposable. Using Theorem 3.1 it is now easy to see that all the operators $(M_z, D(\mu))$ are cellularly indecomposable. Indeed, it is clear that every non-zero invariant subspace contains non-zero functions which satisfy the hypothesis of Theorem 3.1. Thus, every non-trivial invariant subspace of $(M_z, D(\mu))$ contains non-zero multipliers. This implies that any two non-zero invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of $(M_z, D(\mu))$ have a non-zero intersection, because by Lemma 2.4 the intersection will contain all products of the form $\varphi\psi$, where $\varphi \in \mathcal{M}$ and $\psi \in \mathcal{N}$ are multipliers.

THEOREM 3.2. *The operator* $(M_z, D(\mu))$ *is cellularly indecomposable. Every non-zero invariant subspace* $\mathcal{M}$ *satisfies* $\dim \mathcal{M} \ominus z\mathcal{M} = 1$, *and consequently is of the form* $\mathcal{M} = [\varphi] = \varphi D(\mu_\varphi)$, *where* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$ *is a multiplier of* $D(\mu)$.

Furthermore if $\varphi = B\psi$ *is the inner-outer factorization of* φ, *then* $|\varphi(\mathrm{e}^{\mathrm{i}t})| = |\psi(\mathrm{e}^{\mathrm{i}t})|$ *a.e.* $[\mu]$ *so that* $D(\mu_\varphi) = D(\mu_\psi)$ *with equality of norms.*

We note that it is a consequence of this Theorem that $D(\mu) \subseteq D(\mu_\varphi)$.

Proof. We have already seen that $(M_z, D(\mu))$ is cellularly indecomposable. Let $\mathcal{M} \in \mathrm{Lat}(M_z, D(\mu))$, $\mathcal{M} \neq (0)$, we have to prove that $\dim \mathcal{M} \ominus z\mathcal{M} = 1$. But this follows from a theorem of Bourdon [2], which states that under some extra assumptions which the spaces $D(\mu)$ clearly meet, all non-zero invariant subspaces of a cellularly indecomposable operator satisfy $\dim \mathcal{M} \ominus z\mathcal{M} = 1$.

Now let $\varphi = B\psi$ be the inner-outer factorization of φ. Since $B\psi \in D(\mu)$ we have that $D_\zeta(B\psi) < \infty$ a.e. $[\mu]$. From formula (2.1) we know that $D_\zeta(B\psi) = D_\zeta(B)|\psi(\zeta)|^2 + D_\zeta(\psi)$, where the local Dirichlet integral $D_\zeta(B)$ of the inner function B equals its angular derivative. Hence, if $D_\zeta(B)$ is finite, then $|B(\zeta)| = 1$. We see that if $D_\zeta(B\psi)$ is finite, then either $|\psi(\zeta)| = 0$ or $|B(\zeta)| = 1$, in either case $|\varphi(\zeta)| = |B\psi(\zeta)| = |\psi(\zeta)|$. ■

4. INVARIANT SUBSPACES GENERATED BY OUTER FUNCTIONS

In this section we shall prove results about spans and intersections of invariant subspaces generated by outer functions. We shall start out with the result concerning the spans.

THEOREM 4.1. *Let f and g be outer functions in $D(\mu)$. Then*

$$[f, g] = [f \vee g].$$

Proof. We have $|f(z)|, |g(z)| \leqslant |(f \vee g)(z)|$ for every $z \in \mathbb{D}$. By Lemma 2.4 this means that $f, g \in [f \vee g]$.

The outer inclusion will follow from Theorem 3.2 and Lemma 2.2. The span $[f, g]$ contains outer functions; thus by Theorem 3.2 it is of the form $[f, g] = hD(\mu_h)$ for some outer function $h \in D(\mu)$. We have to show that $f \vee g \in hD(\mu_h)$, i.e. $h_1 = (f \vee g)/h \in D(\mu_h)$.

We know $f_1 = f/h$, $g_1 = g/h \in D(\mu_h)$. From the definition of $f \vee g$ we have $h_1 = f_1 \vee g_1$, thus by Lemma 2.2

$$\int D_\zeta(h_1)\,\mathrm{d}\mu_h \leqslant \int D_\zeta(f_1)\,\mathrm{d}\mu_h + \int D_\zeta(g_1)\,\mathrm{d}\mu_h.$$

This implies $h_1 \in D(\mu_h)$ and $f \vee g = h_1 h \in hD(\mu_h) = [f, g]$. ■

It follows from Corollary 5.6 of [11] that a bounded function f in $D(\mu)$ is cyclic if and only if f^2 is cyclic. Theorem 4.3 is a generalization of this. It and Theorem 4.5 also are generalizations of results of Brown and Shields (see [4], p. 290, Proposition 14 and its corollaries).

First we shall prove the crucial lemma.

LEMMA 4.2. *If $f \in H^2$ is an outer function, $N \in \mathbf{R}$, and $\zeta \in \mathbf{T}$, then*

$$D_\zeta\left(\left(\frac{1}{f}\right)_N f^2\right) \leqslant 4D_\zeta(f).$$

Proof. Fix $\zeta \in \mathbf{T}$. Without loss of generality we may assume that $D_\zeta(f) < \infty$. Furthermore, assume for a moment that the lemma has been verified for $N = 0$. Then for $N \in \mathbf{R}$

$$\begin{aligned} D_\zeta\left(\left(\frac{1}{f}\right)_N f^2\right) &= D_\zeta\left(\mathrm{e}^N\left(\frac{1}{\mathrm{e}^N f}\right)_0 f^2\right) = \\ &= D_\zeta\left(\mathrm{e}^{-N}\left(\frac{1}{\mathrm{e}^N f}\right)_0 (\mathrm{e}^N f)^2\right) = \\ &= \mathrm{e}^{-2N} D_\zeta\left(\left(\frac{1}{\mathrm{e}^N f}\right)_0 (\mathrm{e}^N f)^2\right) \leqslant \\ &\leqslant 4\mathrm{e}^{-2N} D_\zeta(\mathrm{e}^N f) = \quad \text{(by the assumption)} \\ &= 4D_\zeta(f). \end{aligned}$$

Hence it suffices to show the lemma for $N = 0$.

First consider the case $f(\zeta) = 0$. Since we clearly have $\left|\left(\frac{1}{f}\right)_0 f^2(z)\right| \leqslant$ $\leqslant |f(z)|$, $\forall z \in \mathbf{T}$ the definition of the local Dirichlet integral at ζ implies in this case $D_\zeta\left(\left(\frac{1}{f}\right)_0 f^2\right) \leqslant D_\zeta(f) \leqslant 4D_\zeta(f)$.

Now assume $f(\zeta) \neq 0$. In this case we shall use formula (2.2). For $z \in \mathbf{T}$ write $u(z) = \log|f(z)|$. Then $\left|\left(\frac{1}{f}\right)_0 f^2\right| = \mathrm{e}^v$, where

$$v(z) = \begin{cases} u(z) & u(z) \geqslant 0 \\ 2u(z) & u(z) < 0 \end{cases}.$$

$u(z) \neq -\infty$ a.e., thus, it follows from formula (2.2) that we are done once we have shown that

$$F(2v(z), 2v(\zeta)) \leqslant 4F(2u(z), 2u(\zeta)) \quad \forall z \in \mathbf{T},\ u(z) \neq -\infty.$$

If we define the real valued function g on $\mathbf{R}$ by $g(x) = x$ if $x \geqslant 0$ and $g(x) = 2x$ if $x < 0$, then we have to show that

$$G(x, y) = 4F(x, y) - F(g(x), g(y)) \geqslant 0 \quad \forall x, y \in \mathbf{R}.$$

Here we do not need to consider the case $y = -\infty$, because by our assumption $u(\zeta) \neq -\infty$. Recall that $F(x, y) = \mathrm{e}^x - \mathrm{e}^y - \mathrm{e}^y(x - y)$.

We distinguish four cases:

(1) $x, y \geqslant 0$.

In this case $G(x,y) = 3F(x,y) \geqslant 0$.

(2) $x, y < 0$.

A short computation shows that $G_y(x,y) = 4e^y(1-e^y)(y-x)$. This has the same sign as $y-x$. Since $G(x,x)=0$, we conclude that $G(x,y) \geqslant 0$.

(3) $x < 0$, $y \geqslant 0$.

Note that G is continuous on $\mathbb{R}^2$, hence case (2) implies that $G(x,0) \geqslant 0$. We compute the partial derivative G_y, $G_y(x,y) = e^y(3y-2x) \geqslant 0$, i.e. $G(x,y)$ is a non-decreasing function of y.

(4) $x \geqslant 0$, $y < 0$.

Again we compute G_y, $G_y(x,y) = 4e^y(y-x) - 2e^{2y}(2y-x) \leqslant 4e^y(1-e^y)(y-x) \leqslant \leqslant 0$. Thus in this case $G(x,y)$ is a decreasing function of y. By case (1) $G(x,0) \geqslant \geqslant 0$, hence the continuity of G implies that $G(x,y) \geqslant 0$. ■

THEOREM 4.3. *If $f \in D(\mu)$ is an outer function, and if $\alpha > 0$ such that $f^\alpha \in \in D(\mu)$, then $[f] = [f^\alpha]$.*

Proof. Write $f = \varphi/\psi$, where $\varphi, \psi \in D(\mu) \cap H^\infty$ are cut-off functions such that $1/\psi \in D(\mu)$, $\|\varphi\|_\infty, \|\psi\|_\infty \leqslant 1$. Then φ^α and ψ^α are the corresponding cut-off functions for f^α. It follows from Lemma 2.6 that $[f] = [\varphi]$ and $[f^\alpha] = [\varphi^\alpha]$. Thus, we may assume that f is bounded.

We shall first prove the theorem in the case where $\alpha = 2$. We have to show that $[f] = [f^2]$. By Lemma 2.4 it is clear that $[f^2] \subseteq [f]$. To show the reverse inclusion we shall use cut-off functions and verify that $\left\{\left(\frac{1}{f}\right)_N f^2\right\}_{N\in\mathbb{N}}$ converges weakly to f. As in the proof of Lemma 2.6 it follows from Lemmas 2.5 and 2.4 that $\left\{\left(\frac{1}{f}\right)_N f^2\right\}_{N\in\mathbb{N}} \subseteq [f^2]$, and $\left\{\left(\frac{1}{f}\right)_N f^2(z)\right\}_{N\in\mathbb{N}} \to f(z)$ for each $z \in \mathbb{D}$. Furthermore, by Lemma 4.2 we know that the sequence $\left\{\left(\frac{1}{f}\right)_N f^2\right\}_{N\in\mathbb{N}}$ is norm bounded in $D(\mu)$, hence it must converge weakly to f.

Next assume that $\alpha \geqslant 1$. A repeated application of the result of the previous paragraph shows that $[f] = [f^{2^n}]$ for any $n \geqslant 0$. But we assumed that f is bounded, thus, by Lemma 2.4 $[f] = [f^{2^n}] \subseteq [f^\alpha] \subseteq [f]$ whenever $2^n \geqslant \alpha \geqslant 1$. This shows that $[f] = [f^\alpha]$ for any $\alpha \geqslant 1$. Now if $\alpha \leqslant 1$ then $1/\alpha \geqslant 1$, and we can apply the above argument to $g = f^\alpha$ and $g^{1/\alpha} = f$. ■

The following answers a question of Brown and Shields, see [4], Question 14, p. 297.

COROLLARY 4.4. *If $f \in D$ is univalent, then f is cyclic in D if and only if $f(z) \neq 0$ for all $z \in \mathbb{D}$.*

Proof. Clearly if f vanishes somewhere in the unit disk then f cannot be cyclic. If $f(z) \neq 0$ for all $z \in \mathbb{D}$, then since f is univalent, f must be an outer function (see [6], Theorem 3.17). It follows from Theorem 4.3 that the result will follow once we show that $f^\alpha \to 1$ in D as $\alpha \to 0$. It is clear that $f^\alpha(z) \to 1$ for every $z \in \mathbb{D}$. Furthermore for $0 < \alpha < 1$

$$\begin{aligned} D(f^\alpha - 1) = D(f^\alpha) = \iint_{\mathbb{D}} |(f^\alpha(z))'|^2 \,\mathrm{d}A(z) = \\ = \alpha^2 \iint_{\mathbb{D}} |f(z)|^{2\alpha-2}|f'(z)|^2 \,\mathrm{d}A(z) = \\ = \alpha^2 \iint_{f(\mathbb{D})} |w|^{2\alpha-2}\,\mathrm{d}A(w) \leqslant \quad \text{(by a change of variables)} \\ \leqslant \alpha^2 \iint_{\mathbb{D}} |w|^{2\alpha-2}\mathrm{d}A(w) + \alpha^2 \iint_{f(\mathbb{D})\cap\mathbb{C}-\mathbb{D}} |w|^{2\alpha-2}\mathrm{d}A(w) \leqslant \\ \leqslant \alpha + \alpha^2 \iint_{f(\mathbb{D})\cap\mathbb{C}-\mathbb{D}} \mathrm{d}A(w) \leqslant \\ \leqslant \alpha + \alpha^2 D(f) \to 0 \end{aligned}$$

as $\alpha \to 0$. ∎

It is clear that a somewhat more general result holds, because one can use a more general change of variables formula than the one being used. We leave the details to the reader.

THEOREM 4.5. *If f and g are outer functions in $D(\mu)$, then*

$$[f \wedge g] = [f] \cap [g].$$

If in addition $fg \in D(\mu)$, then also

$$[fg] = [f] \cap [g].$$

Proof. We shall first show that $[f] \cap [g] \subseteq [fg]$ whenever $fg \in D(\mu)$.

$[f] \cap [g]$ is an invariant subspace of $(M_z, D(\mu))$; thus, by Theorem 3.2 it is generated by a bounded function h, i.e. $[h] = [f] \cap [g]$. Since $(f \wedge 1)(g \wedge 1) \in [h]$ it is clear that h is an outer function, and we may assume that $\|h\|_\infty \leqslant 1$. We shall show that $h^2 \in [fg]$. This will conclude the proof of $[f] \cap [g] = [h] \subseteq [fg]$ because of Theorem 4.3.

By assumption we can find sequences of polynomials $\{p_n\}_{n\in\mathbf{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ such that $\{p_n f\} \to h$ and $\{q_n g\} \to h$ in $D(\mu)$. We assumed that $|h(z)| \leqslant 1$ for all $z \in \mathbf{D}$, thus, by Lemma 2.8 the chopped-off sequences $\{(p_n f)_0\}_{n\in\mathbf{N}}$ and $\{(q_n g)_0\}_{n\in\mathbf{N}}$ converge weakly to h. For each $n \in \mathbf{N}$ we have

$$(p_n f)_0 (q_n g)_0 \in [p_n q_n f g] \subseteq [fg].$$

By use of (1.3) we see that the sequence $\{(p_n f)_0 (q_n g)_0\}_{n\in\mathbf{N}}$ is norm bounded in $D(\mu)$, hence it converges weakly to h^2. Thus, $h^2 \in [fg]$ as promised.

Now we can use the above to prove that $[f \wedge g] = [f] \cap [g]$ for general outer functions $f, g \in D(\mu)$. We observe that the following inclusions hold:

$$\begin{aligned}
[f \wedge g] &\subseteq [f] \cap [g] \quad \text{(by Lemma 2.4, since } |f \wedge g| \leqslant |f|, |g|) \\
&= [f \wedge 1] \cap [g \wedge 1] \quad \text{(by Lemma 2.6)} \\
&\subseteq [(f \wedge 1)(g \wedge 1)] \quad \text{(by the first part of this proof)} \\
&\subseteq [(f \wedge 1) \wedge (g \wedge 1)] \quad \text{(Lemma 2.4)} \\
&\subseteq [f \wedge g] \quad \text{(Lemma 2.4);}
\end{aligned}$$

thus we must have equality throughout.

To finish the proof of Corollary 4.5 we have to show that $fg \in D(\mu)$ implies that $[fg] \subseteq [f] \cap [g]$.

The assumption $fg \in D(\mu)$ implies by Lemma 2.2 that $f(g\wedge 1) = (fg)\wedge f \in D(\mu)$. Now write $f = \varphi/\psi$, where $\varphi, \psi \in H^\infty \cap D(\mu)$ and $1/\psi \in D(\mu)$. Then we can apply Lemma 2.6 to $f(g \wedge 1) = (\varphi(g \wedge 1))/\psi$, and we see that $[f(g \wedge 1)] = [\varphi(g \wedge 1)] \subseteq$ $\subseteq [g]$. Next note that $|((fg) \wedge 1)(e^{it})| > |(f(g \wedge 1))(e^{it})|$ only if $|f(e^{it})| \leqslant 1$. Hence $|(fg) \wedge 1| \leqslant |(f(g \wedge 1)) \vee g|$. Thus by Lemmas 2.6, 2.4, and Theorem 4.1 we have

$$[fg] = [(fg) \wedge 1] \subseteq [(f(g \wedge 1) \vee g] = [f(g \wedge 1), g] \subseteq [g, g] = [g].$$

Of course, by symmetry we must then also have $[fg] \subseteq [f]$. ■

5. INNER FUNCTIONS AND INVARIANT SUBSPACES

If B is an arbitrary inner function then B may or may not be in $D(\mu)$. For example, an inner function is in D if and only if B is a finite Blaschke product. However, B may still occur as a factor of some functions in $D(\mu)$. The set of all such functions is $D(\mu) \cap BH^2$ and it is easy to check that this is an invariant subspace of $(M_z, D(\mu))$ (possibly the zero space). Similarly, if $\mathcal{N}$ is any invariant subspace of $(M_z, D(\mu))$, then $\mathcal{N} \cap BH^2 \in \mathrm{Lat}(M_z, D(\mu))$. Thus it is natural to ask:

QUESTION 5.1. *Is every invariant subspace of $(M_z, D(\mu))$ of the form $\mathcal{N} \cap BH^2$, where B is an inner function and $\mathcal{N} \in \mathrm{Lat}(M_z, D(\mu))$ contains an outer function?*

We shall answer this question affirmatively. Without going into any further detail we remark that this implies that Question 12 and 14 of [12], p. 345 are equivalent to one another.

We start out with a lemma.

LEMMA 5.2. *If B is an inner function, and if $\mathcal{N} \in \mathrm{Lat}(M_z, D(\mu))$ contains an outer function, then*

$$B\mathcal{N} \cap D(\mu) = \mathcal{N} \cap BH^2.$$

Furthermore, $\mathcal{N} \cap BH^2 = (0)$ if and only if either $\mathcal{N} = (0)$ or $D(\mu) \cap BH^2 = (0)$.

Proof. If $g \in \mathcal{N}$ such that $Bg \in D(\mu)$, then by Lemma 2.4 $Bg \in [g] \subseteq \mathcal{N}$. This together with the trivial observation that $B\mathcal{N} \subseteq BH^2$ implies that $B\mathcal{N} \cap D(\mu) \subseteq \mathcal{N} \cap BH^2$.

In order to show the other inclusion we recall from Theorem 3.2 that $\mathcal{N}$ is of the form $\mathcal{N} = fD(\mu_f)$ for some $f \in D(\mu)$. Since $\mathcal{N}$ contains an outer function, it is clear that f must be an outer function.

Let $h \in \mathcal{N} \cap BH^2$. We must show that $h \in B\mathcal{N}$. The equality $\mathcal{N} = fD(\mu_f)$ implies that h is of the form $h = fg$, where $g \in D(\mu_f)$. The function f is outer, thus $h = BH^2$ implies that g has B as an inner factor. Inner factors increase the local Dirichlet integral (see formula (2.1)), hence $g/B \in D(\mu_f)$. This implies that $fg/B \in \mathcal{N}$, thus $h = B(fg/B) \in B\mathcal{N}$.

The statement of the last sentence of the lemma follows, because trivially $\mathcal{N} \cap BH^2 = \mathcal{N} \cap (D(\mu) \cap BH^2)$ and because $(M_z, D(\mu))$ is cellularly indecomposable (see Theorem 3.2). ∎

If $\mathcal{M}$ is an invariant subspace of $(M_z, D(\mu))$, then by Theorem 3.2 $\mathcal{M} = [g] = gD(\mu_g)$ for some $g \in D(\mu)$. The greatest common inner divisor of $\mathcal{M}$, i.e. the maximal factor which is an inner factor of every function in $\mathcal{M}$, is the same as the inner factor of g. Let B denote this inner factor, so that $g = Bf$, where f is an outer function. Inner factors increase the local Dirichlet integral, hence the set $\mathcal{M}/B$ is contained in $D(\mu)$ and contains the outer function f. Trivially, $\mathcal{M} = B(\mathcal{M}/B) \cap D(\mu)$. Let $\mathcal{N}$ be the closure of $\mathcal{M}/B$ in $D(\mu)$. With this choice for $\mathcal{N}$ one always has $\mathcal{M} \subseteq B\mathcal{N} \cap D(\mu) = \mathcal{N} \cap BH^2$. The more difficult part of Question 5.1 is to show that the inclusion cannot be proper.

THEOREM 5.3. (a) *If B is an inner function and f is an outer function such that $Bf \in D(\mu)$, then*

$$[Bf] = B[f] \cap D(\mu) = [f] \cap BH^2.$$

(b) *Let $\mathcal{M} \in \mathrm{Lat}(M_z, D(\mu))$, and let B be the greatest common inner divisor of $\mathcal{M}$. Then, there is an outer function $f \in D(\mu)$ such that*

$$\mathcal{M} = [Bf] = B[f] \cap D = [f] \cap BH^2.$$

In fact, f can be chosen so that f and Bf are multipliers of $D(\mu)$.

Proof. (a) As mentioned above the inner factor increases the local Dirichlet integral, so by Lemma 5.2 we have $[Bf] \subseteq B[f] \cap D(\mu) = [f] \cap BH^2$.

To finish the proof of (a) let $g \in [f]$ such that $Bg \in D(\mu)$, we must show that $Bg \in [Bf]$. We use cut-off functions to write $g = \varphi/\psi$, where $\varphi, \psi, B\varphi \in D(\mu) \cap H^\infty$ and $1/\psi \in D(\mu)$. By Lemma 2.6 $[\varphi] = [g] \subseteq [f]$ and $[Bg] = [B\varphi]$. Thus it suffices to show that $B\varphi \in [Bf]$, and we may as well assume that $\|\varphi\|_\infty \leqslant 1$.

First we shall consider the case where φ is an outer function. By assumption there exists a sequence of polynomials $\{p_n\}$ such that $p_n f \to \varphi$. By Lemma 2.8 the sequence of cut-off functions $\{f_n\} = \{(p_n f)_0\} \subseteq [f]$ converges to φ weakly and satisfies $\|f_n\|_\infty \leqslant 1$. Furthermore $Bf_n = (p_n Bf)_0 \in [Bf]$, and so by Lemma 2.4 $\varphi B f_n \in [Bf]$. We have

$$|(\varphi B f_n)'| \leqslant |(B\varphi)'| + |f_n'|,$$

hence by (1.3) the sequence $\{\varphi B f_n\}$ is norm bounded in $D(\mu)$ and converges pointwise to $B\varphi^2$. We have thus seen that $B\varphi^2 \in [Bf]$. But this implies that $B\varphi \in [Bf]$, because $\left\{\left(\frac{1}{\varphi}\right)_N \varphi^2 B\right\}_{N\in\mathbb{N}} \subseteq [B\varphi^2] \subseteq [Bf]$ converges weakly to $B\varphi$. Indeed, as the pointwise convergence is clear we only need to show that the sequence is norm bounded. From the formulas for the local Dirichlet integral ((2.1) and (2.2)) we see

$$\begin{aligned} D_\zeta\left(\left(\frac{1}{\varphi}\right)_N \varphi^2 B\right) &= D_\zeta(B)\left|\left(\frac{1}{\varphi}\right)_N \varphi^2(\zeta)\right|^2 + D_\zeta\left(\left(\frac{1}{\varphi}\right)_N \varphi^2\right) \leqslant \\ &\leqslant D_\zeta(B)|\varphi(\zeta)|^2 + 4D_\zeta(\varphi) \leqslant \quad \text{(by Lemma 4.2)} \\ &\leqslant 4D_\zeta(B\varphi). \end{aligned}$$

Hence $\left\|\left(\frac{1}{\varphi}\right)_N \varphi^2 B\right\|_\mu^2 \leqslant 4\|B\varphi\|_\mu^2$.

Finally suppose φ has an inner factor, say $\varphi = S\psi$, where ψ is the outer factor. Since $\varphi = S\psi \in [f]$ and f is outer, Lemma 5.2 implies that $\psi \in [f]$. Also $B\varphi = BS\psi \in D(\mu)$ implies that $B\psi \in D(\mu)$. By what was shown above it follows that $B\psi \in [Bf]$. But then Lemma 2.4 implies that $B\varphi = BS\psi \in [Bf]$.

(b) By Theorem 3.2 the generator Bf of $\mathcal{M}$ is a multiplier of D. Inner factors increase the Dirichlet integral, hence f must be a multiplier as well. This proves (b).

■

We close this section by describing a situation where information about $D(\mu)$ for a singular measure μ might be important for solving a question about D. It is an open problem to determine the zero sets of Dirichlet space functions, i.e. given a sequence $\{\alpha_n\}_{n\in\mathbf{N}} \subseteq \mathbf{D}$, when does there exist a nonzero function $f \in D$ such that $f(\alpha_n) = 0$ for each $n \in \mathbf{N}$? This is equivalent to asking, when is there a function $g \in D$ such that $Bg \in D$, where B is the Blaschke product with zeros $\{\alpha_n\}$. More generally one might ask, which inner functions appear as factors of Dirichlet space functions. Regarding the singular inner factor we can relate this question to the space $D(\mu)$ as follows.

PROPOSITION 5.4. *Let μ be a Borel measure on $\mathbf{T}$ that is nonnegative, finite, and singular, and let S_μ be the singular inner function associated with μ, i.e.*

$$S_\mu(z) = \exp\left\{-\int_{\mathbf{T}} \frac{e^{it}+z}{e^{it}-z}\, d\mu(e^{it})\right\}, \quad z \in \mathbf{D}.$$

If $f \in H^2$, then $S_\mu f \in D$ if and only if $f \in D(\mu) \cap D$ and $f = 0$ a.e. $[\mu]$.

Proof. We recall from [11], Theorem 3.1 that

$$D_\zeta(S_\mu) = \int_{\mathbf{T}} \frac{2}{|e^{it}-\zeta|^2}\, d\mu(e^{it}).$$

Thus formula (2.1) implies that

$$D_\zeta(S_\mu f) = \int_{\mathbf{T}} \frac{2}{|e^{it}-\zeta|^2}\, d\mu(e^{it})|f(\zeta)|^2 + D_\zeta(f).$$

We integrate this identity with respect to $dm(\zeta)$. An application of Fubini's Theorem shows that the Dirichlet integral of $S_\mu f$ equals

$$D(S_\mu f) = 2\int_{\mathbf{T}} \left(\int_{\mathbf{T}} \frac{|f(\zeta)|^2}{|e^{it}-\zeta|^2}\, dm(\zeta)\right) d\mu(e^{it}) + D(f). \tag{5.1}$$

Now suppose $D(S_\mu f) < \infty$, then $D(f) < \infty$, i.e. $f \in D$. But we also must have

$$\int_{\mathbf{T}} \frac{|f(\zeta)|^2}{|e^{it}-\zeta|^2}\, dm(\zeta) < \infty \text{ a.e. } [\mu].$$

Thus from the definition of the local Dirichlet integral (see (1.1), (1.2), and the remark made between these formulas) it follows that

$$D_{e^{it}}(f) = \int_{\mathbf{T}} \frac{|f(\zeta)|^2}{|e^{it} - \zeta|^2} \, dm(\zeta)$$

and that $f = 0$ a.e. $[\mu]$. Hence since the first term on the right hand side of (5.1) has to be finite we see that $f \in D(\mu)$.

The converse follows in a similar manner from formula (5.1). ■

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