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THE AUTOMORPHISM GROUP OF A FREE PRODUCT OF GROUPS AND SIMPLE C^*-ALGEBRAS

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1. INTRODUCTION

The aim of this paper is to introduce a new class of discrete groups G for which the reduced C^*-algebra $C_r^*(G)$ is simple with unique trace.

Powers [14] showed that the reduced C^*-algebra $C_r^*(F_2)$ of the free nonabelian group on two generators is simple with unique trace. His result was generalized by Choi [5], for the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, and by Paschke and Salinas [13] for a free product $G_1 * G_2$, where G_1 has at least two elements and G_2 at least three elements. De la Harpe [7] introduced the class of Powers groups, which contains all the previous examples. He showed that the Powers groups have the reduced C^*-algebra simple with unique trace. In [4], the results of de la Harpe are generalized for a larger class of groups, called weak Powers groups.

Another direction in looking for discrete groups for which the reduced C^*-algebras are simple with unique trace is given by Theorem 3 in [1].

THEOREM. (Akemann and Lee) *Let G be a discrete group which contains a normal free nonabelian subgroup with trivial centralizer. Then the reduced C^*-algebra of G is simple with unique trace.*

The natural question, asked by de la Harpe [7], is the following.

QUESTION. *Can one replace the free group in the previous theorem by a (weak) Powers group?*

Here we prove that one can replace the free group in the theorem of Akemann and Lee by any proper free product (i.e. of nontrivial groups), which is not the infinite dihedral group (i.e. $\mathbb{Z}_2 * \mathbb{Z}_2$). This can also be done in Corollary 5 of [1] which deals with a group that has normal subgroups a family of free nonabelian groups such that the intersection of their centralizers is trivial.

The paper is divided in four sections. In Section 2 we introduce some notations, we state the principal theorem and we show how the results described above can be obtained from it. In Section 3 we prove some combinatorial results related to free products, and in Section 4 there is the proof of the principal theorem.

REMARK. After our work has been completed, we received a paper of E. Bédos [3], where simplicity is proved in a more general case, but nothing is obtained about the uniqueness of the trace. More precisely, Bédos proved ([3], Corollary 3.6) that any discrete group posessing a normal C^*-simple subgroup with trivial centralizer is C^*-simple (a discrete group is called C^*-simple if its reduced C^*-algebra is simple). The techniques used in [3] and in our paper are quite different.

The technique used in this paper, which appears in almost all papers proving both simplicity and uniqueness of the trace ([1], [4], [5], [7], [13], [14] etc.) is based on the construction of a certain averaging process (see Theorem 1). This is somehow a necessary condition, due to the following result of Haagerup and Zsidó (see [6]): if A is a simple unital C^*-algebra with at most one tracial state, then for each $a \in A$, the closed convex hull of $\{uau^* : u \in A \text{ unitary}\}$ contains a scalar multiple of 1_A (namely $\mathrm{tr}(a) \cdot 1_A$, if tr is a trace on A).

Bédos uses reduced twisted crossed products of C^*-algebras by discrete groups (see [12]) and a result of Kishimoto ([8], Theorem 3.1), which says that the reduced crossed product of a simple C^*-algebra by a discrete group of outer automorphism is simple.

Unfortunately, there seems to be a lack of literature on conditions under which a reduced crossed product of a C^*-algebra with unique trace has a unique trace. Hence, in order to pursue further investigation in de la Harpe's question, one has either to find such conditions, or to use the averaging process method, where the difficulty is concentrated in group combinatorial tricks.

2. NOTATIONS AND THE PRINCIPAL RESULTS

We begin with some general notations. All groups are discrete.

In the following, let H be an arbitrary group. Then:

– By e we always denote the identity element of H. $H^* := H \setminus \{e\}$ and $\mathrm{ord} h$ stands for the order of the element $h \in H$;

– $\mathbb{C}[H]$ stands for the group algebra of H. We identify the elements of H and their image in the group algebra;

- For any $X = \sum_{h \in H} c_h h \in \mathbb{C}[H]$ (the sum has finite support), denote $\mathrm{supp}X :=$ $:= \{h \in H | c_h \neq 0\}$;

- $C_r^*(H)$ denotes the reduced C^*-algebras of H, that is the closure of $\mathbb{C}[H]$ in the norm we obtain regarding its elements as left convolutors on the Hilbert space $\ell^2(H)$. This norm will be denoted by $\|.\|_H$. (Therefore there is an inclusion $\mathbb{C}[H] \hookrightarrow C_r^*(H)$ with dense range). Note that the elements $h \in H$ become unitary elements of $C_r^*(H)$;

- $\tau : C_r^*(H) \to \mathbb{C}$ is the canonical trace, i.e. the continuous extension of $\tau_0 : \mathbb{C}[H] \to \mathbb{C}$, $\tau_0 \left(\sum_{h \in H} c_h h \right) = c_e$;

- If $h, k \in H$, their commutator will be denoted by $[h,k] = hkh^{-1}k^{-1}$ and $\mathrm{Ad}h \in \mathrm{Int}(H) \subset \mathrm{Aut}(H)$ is given by $\mathrm{Ad}h(k) = hkh^{-1}$;

- Given a nonvoid set $M \subset H$, its centralizer is

$$Z_H(M) = \{h \in H \,|\, [h, x] = e, \text{ for all } x \in M\};$$

- In order to simplify the notations, we shall write $\overline{h}$ instead of $\mathrm{Ad}h$, for all $h \in H$. Therefore there is a surjective morphism $^- : H \to \mathrm{Int}\,(H)$. It is easy to check that its restriction to a subgroup $K \subset H$ is one-to-one if and only if $Z_H(K) = \{e\}$. The same holds for the map $^- : H \to \mathrm{Int}\,(H)|_K \subset \mathrm{Aut}\,(K)$, if K is a normal subgroup of H.

- Given a subgroup $K \subset H$, by an averaging process (of $C_r^*(H)$) with elements of K we shall mean a $\mathbb{C}$-linear map $\theta : C_r^*(H) \to C_r^*(H)$ given by

$$\theta(h) = \frac{1}{n} \sum_{i=1}^{n} k_i h k_i^{-1}, \ h \in H \tag{1}$$

where $\{k_i\}_{1 \leqslant i \leqslant n} \subset K$ is a fixed set. (As a matter of fact, one defines θ by (1), only on $\mathbb{C}[H]$, but since

$$\|\theta(X)\|_H \leqslant \|X\|_H, \text{ for all } X \in \mathbb{C}[H],$$

θ can be extended by continuity to $C_r^*(H)$).

It is easy to see that:

- if θ, θ' are averaging processes of $C_r^*(H)$ with elements of the subgroup $K \subset H$, then:

a) $\|\theta(X)\|_H \leqslant \|X\|_H$, for all $X \in C_r^*(H)$;

b) $\theta \circ \theta'$ is also an averaging process with elements of K;

- if H_1, H_2 are groups and $H_1 \subset H_2$, then there is an isometric embedding $C_r^*(H_1) \hookrightarrow C_r^*(H_2)$.

We can now fix the setting. Let A and B be nontrivial groups, $\Gamma = A * B$ their free product, $G = \mathrm{Aut}\,(\Gamma)$, the group of automorphisms of Γ, and $\pi : A * B \to A \times B$

the canonical morphisms, with components π_1 and π_2. Denote $\ker \pi = F$. It is known ([15], I, 1.3, Prop. 4) that F is a free group with basis

$$\{[a,b] \mid a \in A^*, b \in B^*\}.$$

Therefore, if $A * B \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then F is nonabelian.

For a nonvoid set $M \subset \Gamma$, we shall denote

$$\tilde{M} := \bigcup_{w \in \Gamma} wMw^{-1}.$$

Since $Z_\Gamma(\Gamma) = \{e\}$, the map $x \in \Gamma \mapsto \overline{x} \in \operatorname{Int}(\Gamma)$ is an isomorphism.

The principal result, the proof of which will be given in Section 4, is the following:

THEOREM 1. *Let A and B be nontrivial groups such that $A * B \neq \mathbf{Z}_2 * \mathbf{Z}_2$. Then for any finite nonvoid set $M \subset G^* = \operatorname{Aut}(A * B) \setminus \{\operatorname{id}_{A*B}\}$ and for any value $\varepsilon > 0$, there is an averaging process θ of $C_r^*(G)$ with elements of* $\operatorname{Int}(A * B)$ *such that*

$$\|\theta(g)\|_G \leqslant \varepsilon \text{ for all } g \in M.$$

In the sequel, we shall prove the results announced in Section 1.

We need the following lemma, the proof of which is standard ([7], Proposition 3):

LEMMA 2. *Let H be a group with the property that for any finite nonvoid set $M \subset H^*$ and any value $\varepsilon > 0$, there is an averaging process θ with elements of H such that:*

$$\|\theta(h)\|_H < \varepsilon, \text{ for all } h \in M.$$

Then $C_r^*(H)$ is simple with unique trace.

COROLLARY 3. *Let A and B be nontrivial groups such that $A * B \neq \mathbf{Z}_2 * \mathbf{Z}_2$. If a (discrete) group H contains $A * B$ as a normal subgroup with trivial centralizer, then $C_r^*(H)$ is simple with unique trace.*

Proof. Since $\Gamma = A * B$ is normal in H, there is an inclusion:

$$\operatorname{Int}(\Gamma) \subset H' \subset \operatorname{Aut}(\Gamma) \tag{2}$$

where $H' = \{\operatorname{Ad} h_{|\Gamma} \mid h \in H\}$. Moreover H and H' are isomorphic because $Z_H(\Gamma) = \{e\}$. Therefore, the problem reduces to prove that $C_r^*(H')$ is simple with unique trace for a group H' satisfying (2). But this follows easily by Theorem 1, due to Lemma 2. ∎

REMARK. *It is not hard to see that the groups H that satisfy the hypothesis of Corollary 3 are in one-to-one correspondence with the subgroups of* $\mathrm{Out}(A * B) :=$ $:= \mathrm{Aut}\,(A * B)/\,\mathrm{Int}\,(A * B)$.

COROLLARY 4. *Denote by $\mathcal{F}$ the family of groups that are proper free products and are not equal to $\mathbb{Z}_2 * \mathbb{Z}_2$. Let H be a group having as normal subgroups a family $\{\Gamma_i\}_{i\in I}$ of groups (I any nonvoid set), such that $\Gamma_i \in \mathcal{F}$ for all $i \in I$ and*

$$\bigcap_{i\in I} Z_H(\Gamma_i) = \{e\}.$$

Then $C_r^*(H)$ is simple with unique trace.

Proof. We shall prove that H satisfies the hypothesis of Lemma 2.

Note first the following

$$(3)\qquad \begin{cases} \text{for any } i \in I, \text{ given a nonvoid finite set } M \subset H \setminus Z_H(\Gamma_i) \\ \text{and a value } \varepsilon > 0, \text{ there is an averaging process } \theta \text{ with} \\ \text{elements of } \Gamma_i \text{ such that } \|\theta(h)\|_H < \varepsilon \text{ for all } h \in M. \end{cases}$$

To see this, denote $M' = \{\,\mathrm{Ad}\, h_{|\Gamma_i} | h \in M\}$. Then $M' \subset \mathrm{Aut}\,(\Gamma_i)$ because Γ_i is normal in H, and $\mathrm{id}_{\Gamma_i} \notin M'$ because $M \cap Z_H(\Gamma_i) = \emptyset$. By Theorem 1 we obtain an averaging process θ'

$$\theta'(\alpha) = \frac{1}{n}\sum_{k=1}^{n} \overline{g}_k \alpha \overline{g}_k^{-1},\ \alpha \in \mathrm{Aut}\,(\Gamma_i),$$

with $\{g_k\}_{k=1,2,\ldots,n} \subset \Gamma_i$ such that $\|\theta'(h')\|_{G_i} < \varepsilon$, for all $h' \in M'$, where $G_i =$ $= \mathrm{Aut}\,(\Gamma_i)$.

Define the averaging process θ of $C_r^*(H)$ with elements of Γ_i by

$$\theta(h) = \frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1},\ h \in H.$$

Since, for all $h \in H$,

$$\|\theta'(\,\mathrm{Ad}\, h_{|\Gamma_i})\|_{G_i} = \left\|\frac{1}{n}\sum_{k=1}^{n}\overline{g}_k(\,\mathrm{Ad}\, h_{|\Gamma_i})\overline{g}_k^{-1}\right\|_{G_i} =$$

$$= \left\|\left[\frac{1}{n}\sum_{k=1}^{n}\overline{g_k\,\mathrm{Ad}\, h(g_k^{-1})}\right](\,\mathrm{Ad}\, h_{|\Gamma_i})\right\|_{G_i} = \left\|\frac{1}{n}\sum_{k=1}^{n}\overline{g_k\,\mathrm{Ad}\, h(g_k^{-1})}\right\|_{G_i} =$$

$$= \left\|\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1} h^{-1}\right\|_{\Gamma_i} = \left\|\left(\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1}\right) h^{-1}\right\|_{H} =$$

$$= \left\|\frac{1}{n}\sum_{k=1}^{n} g_k h g_k^{-1}\right\|_{H} = \|\theta(h)\|_H,$$

we see that (3) holds.

Consider now a finite nonvoid set $M \subset H^*$ and a value $\varepsilon > 0$. There is a finite family $\{i_1, \ldots, i_N\} \subset I$ such that

$$M \cap \left(\bigcap_{k=1}^{N} Z_H(\Gamma_{i_k}) \right) = \emptyset.$$

Denote $M_n = M \setminus \bigcap_{k=1}^{n} Z_H(\Gamma_{i_k})$ for $n = 1, 2, \ldots, N$ and $M_0 = \emptyset$ (hence $M_N = M$). Deleting some indices i_k, we may assume that

$$M_{n+1} \setminus M_n \neq \emptyset, \text{ for all } 0 \leqslant n < N.$$

Note that

$$M_1 \subset H \setminus Z_H(\Gamma_{i_1}),$$

and

$$M_n \setminus M_{n-1} \subset \bigcap_{k=1}^{n-1} Z_H(\Gamma_{i_k}) \setminus Z_H(\Gamma_{i_n})$$

for $2 \leqslant n \leqslant N$. By (3), there are averaging processes θ_n with elements of Γ_{i_n}, $i \leqslant n \leqslant N$ such that

$$\|\theta_n(h)\|_H < \varepsilon \text{ for all } h \in M_n \setminus M_{n-1}.$$

Then $\theta = \theta_N \circ \theta_{N-1} \circ \cdots \circ \theta_1$ will be the needed averaging process for M. ■

3. COMBINATORIAL LEMMAS

Let $\Gamma = A * B$ be as above a free product of nontrivial groups. Each g can be uniquely written in the reduced form as $g = g_1 \ldots g_m$, where for $1 \leqslant j \leqslant m$, g_j is an element of A^* or B^* and two adjacent g_j's are not both in A^* or both in B^*. In this case, define the lenght of g to be $|g| = m$. (We set $|e| = 0$). We also define the beginning of g to be $l(g) := g_1$, and the end of g to be $r(g) := g_m$.

For $v_1, v_2, \ldots, v_k \in \Gamma, k \geqslant 2$, we say that the product $v_1 v_2 \cdots v_k$ is reduced if none of the factors is e and, for all $i = 1, \ldots, k-1$, the end of v_i and the beginning of v_{i+1} are in different groups. We say that the product $v_1 v_2 \cdots v_k$ is reduced $\mod \{v_j | j \in J\}$, where $J \subset \{1, \ldots, k\}$, if some v_j's, $j \in J$, may equal e and after deleting those equal to e, we obtain a reduced product. By a statement like "the product $(v_1 v_2) v_3$ is reduced" we mean that "the product $w v_3$ is reduced, where $w = v_1 v_2$".

Let w_1, w_2 be two words in Γ^*. We say that in the product w_1w_2 there is a consolidation if $r(w_1)$ and $l(w_2)$ are both in the same group (A or B) and $r(w_1)l(w_2) \neq$ $\neq e$. We say that there is a cancellation if $r(w_1)l(w_2) = e$.

Let X be a subset of a group H. Then X is a free family if and only if $X \cap X^{-1} = \emptyset$ and no product $w = x_1x_2\cdots x_n$ is equal to e, where $n \geqslant 1$, $\{x_1, \ldots, x_n\} \subset X \cup X^{-1}$ and $x_ix_{i+1} \neq e$, for all $1 \leqslant i \leqslant n-1$. ($X^{-1} := \{x^{-1} | x \in X\}$).

The following remark will be useful in Section 4. It is exercise 12 in [10], Section 1.4.

REMARK. *If $\{a, b\}$ is a free subset in a group H, then $\{a^n b^{-n}\}_{n \geqslant 1}$ is also free.*

We define a function $q_A : \Gamma \backslash A \to A$ in the following way: for any $g \in \Gamma \backslash A$, there are unique elements $w \in \Gamma^*, a_1, a_2 \in A$ such that $g = a_1wa_2$ is reduced $\bmod \{a_1, a_2\}$, and $l(w), r(w) \in B^*$; then $q_A(g) := a_1a_2$.

LEMMA 5. *Let A and B be nontrivial groups such that $A * B \neq \mathbf{Z}_2 * \mathbf{Z}_2$, and let $\alpha \in \operatorname{Aut}(A * B)$. Then $\alpha_{|F} = \mathrm{id}_F$ implies $\alpha = \mathrm{id}_{A*B}$.*

Proof. See Proposition 3 in [11]. ■

LEMMA 6. *Let $a \in A^*$ and $w \in \Gamma \setminus A$ be elements of infinite order. Assume $s \in \mathbf{N}^*$ is such that*

$$q_A(w) \notin \{a^{ns} | n \in \mathbf{Z}\}.$$

Then $\{a^s, w^s\}$ is a free family in Γ.

Proof. If $w \in B^*$, the conclusion is clear. Therefore we assume from now on that $|w| \geqslant 2$. Then:

$$l(w^n) = l(w), r(w^n) = r(w), \text{ for all } n \in \mathbf{N}^*, \tag{4}$$

as will be checked at the end of the proof.

Write $w = a_1w_1a_2$ reduced $\bmod \{a_1, a_2\}$ with $a_1, a_2 \in A$. By (4), one obtains that

$$w^{ns} = \begin{cases} a_1w_na_2, \text{ with } l(w_n), r(w_n) \in B^* & \text{if } n \geqslant 1 \\ a_2^{-1}w_na_1^{-1}, \text{ with } l(w_n), r(w_n) \in B^* & \text{if } n \leqslant -1 \end{cases}. \tag{5}$$

Let z be any word in a^s and w^s. Conjugating it by a high power of w^s, we can put it in the form:

$$z = w^{sn_1}a^{sm_1}w^{sn_2}a^{sm_2}\cdots a^{sm_{p-1}}w^{sn_p}$$

where $p \geqslant 2$, $n_i, m_j \in \mathbf{Z}^*$ for $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant p-1$. Using (5), we obtain that

$$z = c_1w_{n_1}c_2w_{n_2}\cdots c_{p-1}w_{n_p}c_p \tag{6}$$

where $c_1 \in \{a_1, a_2^{-1}\}, c_p \in \{a_2, a_1^{-1}\}$ and c_i, $i = 2, \dots, p-1$ are elements of A of the form $a_2 a^{sm} a_1, a_2 a^{sm} a_2^{-1}, a_1^{-1} a^{sm} a_1$ or $a_1^{-1} a^{sm} a_2^{-1}$. By the hypothesis, $c_i \neq e$ for $i = 2, \dots, p-1$, therefore in (6) z is reduced $\mod \{c_1, c_p\}$, hence $z \neq e$.

It remains to prove (4).

If $|w|$ is even, then w begins and ends with letters from different groups, hence in the product $ww \cdots w = w^n$ there appears neither consolidation nor cancellation. Therefore (4) holds.

If $|w|$ is odd, then w begins and ends with letters from the same group. If in ww there is only a consolidation, since $|w| \geqslant 2$, one sees easily that (4) holds. Otherwise, one can show by induction that w has a reduced form $w = v\tilde{w}v^{-1}$ with $\tilde{w}, v \in \Gamma^*$, such that $\operatorname{ord} \tilde{w}$ is infinite and in $\tilde{w}\tilde{w}$ there is only a consolidation. If $|\tilde{w}| \geqslant 2$, we have seen above that $l(\tilde{w}^n) = l(\tilde{w})$, $r(\tilde{w}^n) = r(\tilde{w})$. Therefore $w^n = v(\tilde{w}^n)v^{-1}$ is reduced for $n \in \mathbf{N}^*$, and this holds even if $|\tilde{w}| = 1$. Hence $l(w^n) = l(v), r(w^n) = r(v^{-1})$ and then (4) holds too. ■

4. PROOF OF THE MAIN RESULT

We define, for $n \in \mathbf{N}^*$ and $v \in \Gamma$ the following averaging process:

$$\theta_{n,v}(\alpha) = \frac{1}{n} \sum_{k=1}^{n} \overline{v}^{nk} \alpha \overline{v}^{-nk}, \quad \alpha \in \operatorname{Aut}(\Gamma).$$

Theorem 1 will be proved using Lemma 7 and the Theorem of Akemann and Lee. All these results rely on the following consequence of the theorem of Akemann and Ostrand [2]: if $\{h_i\}_{1 \leqslant i \leqslant n}$ is a free family in a (discrete) group H, then

$$\left\| \sum_{i=1}^{n} h_i \right\|_H = 2\sqrt{n-1}.$$

LEMMA 7. *Assume $a \in A^*$ and $\alpha \in \operatorname{Aut}(A * B)$ are such that a is an element of infinite order and $\alpha(a) \notin A$ or $\alpha^{-1}(a) \notin A$. If $n \in \mathbf{N}^*$ is such that*

$$q_A(\{\alpha(a), \alpha^{-1}(a)\} \setminus A) \cap \{a^{nk} | k \in \mathbf{Z}^*\} = \emptyset$$

then $\|\theta_{n,a}(\alpha)\|_G = 2\dfrac{\sqrt{n-1}}{n}$.

Proof. Denote $\alpha(a) = w_1, \alpha^{-1}(a) = w_2$. Since

$$\beta\overline{v} = \overline{\beta(v)}\beta, \text{ for } \beta \in \operatorname{Aut}(\Gamma), v \in \Gamma,$$

we obtain:

$$\|\theta_{n,a}(\alpha)\|_G = \left\| \left[\frac{1}{n} \sum_{k=1}^{n} \overline{a^{nk}\alpha(a^{nk})^{-1}} \right] \alpha \right\|_G = \\ = \left\| \frac{1}{n} \sum_{k=1}^{n} \overline{a^{nk} w_1^{-nk}} \right\|_G = \left\| \frac{1}{n} \sum_{k=1}^{n} a^{nk} w_1^{-nk} \right\|_\Gamma$$

respectively:

$$\|\theta_{n,a}(\alpha)\|_G = \left\| \alpha \left[\frac{1}{n} \sum_{k=1}^{n} \overline{\alpha^{-1}(a)^{nk} a^{-nk}} \right] \right\|_G = \\ = \left\| \frac{1}{n} \sum_{k=1}^{n} w_2^{nk} a^{-nk} \right\|_\Gamma .$$

By Lemma 6 applied to w_1 or w_2 and by the Remark from Section 3, $\{a^{nk} w_1^{-nk}\}_{k \geqslant 1}$ or $\{w_2^{nk} a^{-nk}\}_{k \geqslant 1}$ is a free family in Γ (note that $\operatorname{ord} w_1 = \operatorname{ord} w_2 = = \operatorname{ord} a$), therefore the conclusion follows from the result of Akemann and Ostrand. ∎

LEMMA 8. *Assume $\alpha \in \operatorname{Aut}(A * B)$ is such that one of the situations below holds:*

(i) $\alpha(A) \subset \tilde{A}$ *and* $\alpha(B) \subset \tilde{B}$;

(ii) $\alpha(A) \subset \tilde{B}$ *and* $\alpha(B) \subset \tilde{A}$.

Then $\alpha(F) \subset F$.

Proof. We shall verify only the case (i), the other being similar.

Let $a \in A^*, b \in B^*$. According to (i), there are $a' \in A^*, b' \in B^*$ and $v, w \in \Gamma$ such that

$$\alpha(a) = va'v^{-1}, \quad \alpha(b) = wb'w^{-1}.$$

Then:

$$\pi(\alpha(a)) = \pi(v)(a', e)\pi(v)^{-1} = (\pi_1(v)a'\pi_1(v)^{-1}, e) \\ \pi(\alpha(b)) = (e, \pi_2(w)b'\pi_2(w)^{-1})$$

hence

$$\pi(\alpha([a,b])) = [\pi(\alpha(a)), \pi(\alpha(b))] = e,$$

that is

$$\alpha([a,b]) \in \ker \pi = F.$$

But $\{[a,b] | a \in A^*, b \in B^*\}$ generate F, therefore we get that $\alpha(F) \subset F$. ∎

Proof of Theorem 1. Denote

$$G_1 = \{\alpha \in \mathrm{Aut}\,(A * B) \,|\, \alpha(F) = F\},$$

and, for $x \in \Gamma$,

$$G_2(x) = \{\alpha \in \mathrm{Aut}\,(A * B) \,|\, \alpha(x) \notin \tilde{A} \cup \tilde{B} \text{ or } \alpha^{-1}(x) \notin \tilde{A} \cup \tilde{B}\}.$$

We begin by infering some consequences of the preceeding lemmas.

G_1 is a subgroup of G, having $\overline{F} = \{\overline{f} = \mathrm{Ad}\, f | f \in F\}$ as normal subgroup (because $\alpha \overline{f} \alpha^{-1} = \overline{\alpha(f)}$). Since

$$\overline{} : A * B \to \mathrm{Aut}\,(A * B)$$

is one-to-one, $\overline{F}$ is isomorphic to F. Moreover, since

$$\begin{aligned} \alpha \in Z_{G_1}(\overline{F}) &\Leftrightarrow \alpha \overline{f} \alpha^{-1} = \overline{f} \text{ for all } f \in F \\ &\Leftrightarrow \overline{\alpha(f)} = \overline{f} \text{ for all } f \in F \\ &\Leftrightarrow \alpha(f) = f \text{ for all } f \in F, \end{aligned}$$

we get by Lemma 5 that $Z_{G_1}(\overline{F}) = \{e\}$. Consequently, G_1 has as normal subgroup with trivial centralizer the free nonabelian group $\overline{F}$, hence the proof of the theorem of Akemann and Lee shows that:

$$\text{(7)} \qquad \begin{cases} \text{for any finite nonvoid set } M \subset G_1^* \text{ and any } \varepsilon > 0, \text{ there is an} \\ \text{averaging process } \theta \text{ with elements of } \overline{F}, \text{ such that} \\ \|\theta(\alpha)\|_{G_1} < \varepsilon, \text{ for all } \alpha \in M. \end{cases}$$

Since for all $v \in \Gamma$ and $\alpha \in \mathrm{Aut}\,(\Gamma)$ one has:

$$\overline{v}\alpha\overline{v}^{-1}(x) = (v\alpha(v)^{-1})\alpha(x)(v\alpha(v)^{-1})^{-1}, \text{ for all } x \in \Gamma,$$

we see that for any $x \in \Gamma$,

$$\text{(8)} \qquad \alpha \in G_2(x) \Rightarrow \overline{v}\alpha\overline{v}^{-1} \in G_2(x), \text{ for all } v \in \Gamma.$$

Note that

$$\text{(9)} \qquad G = G_1 \cup \left(\bigcup_{x \in A^* \cup B^*} G_2(x) \right).$$

Indeed, assume by the contrary, that $\alpha \in \mathrm{Aut}\,(A*B)$ is an element not belonging to the set on the right hand side above. Due to Lemma 8, there are elements $a \in A^*$ and $b \in B^*$ such that one of the conditions below is fulfilled:

(i) $\alpha(a), \alpha(b) \in \tilde{A}$
(ii) $\alpha(a), \alpha(b) \in \tilde{B}$
(iii) $\alpha^{-1}(a), \alpha^{-1}(b) \in \tilde{A}$
(iv) $\alpha^{-1}(a), \alpha^{-1}(b) \in \tilde{B}$.

If we prove that each of these cases implies that α belongs to some $G_2(x)$ with $x \in A^* \cup B^*$, we get the contradiction. We shall do this only for the case (i). Hence there are $a_1, a_2 \in A^*$, $v, w \in \Gamma$ such that $\alpha(a) = va_1v^{-1}$, $\alpha(b) = wa_2w^{-1}$. Therefore $\alpha^{-1}(a_1) \in \tilde{A^*}$, $\alpha^{-1}(a_2) \in \tilde{B^*} \Rightarrow \alpha^{-1}(a_1a_2) \in \tilde{A^*} \cdot \tilde{B^*} \Rightarrow \alpha \in G_2(a_1a_2)$, because $\tilde{A^*} \cdot \tilde{B^*} \cap (\tilde{A} \cup \tilde{B}) = \emptyset$. Indeed,

$$\pi(\tilde{A^*} \cdot \tilde{B^*}) = \pi(\tilde{A^*}) \cdot \pi(\tilde{B^*}) = (A^* \times \{e\}) \cdot (\{e\} \times B^*) = A^* \times B^*,$$

while

$$\pi(\tilde{A} \cup \tilde{B}) = (A \times \{e\}) \cup (\{e\} \times B).$$

Denote by $\mathcal{G}$ the family of all nonvoid sets $G_2(x)$ with $x \in A^* \cup B^*$.

Note that $G_2(x) \neq \emptyset$ implies that $\operatorname{ord} x$ is infinite, because a consequence of Kurosh's Subgroup Theorem [9] is that any finite subgroup of $A * B$ is conjugated to a subgroup of A or B (hence if $\operatorname{ord} \alpha^{\pm 1}(x) = \operatorname{ord} x$ is finite, then $\alpha(x) \in \tilde{A} \cup \tilde{B}$ and $\alpha^{-1}(x) \in \tilde{A} \cup \tilde{B}$).

By Lemma 7 we infer that

$$(10) \qquad \begin{cases} \text{for any } \varepsilon > 0 \text{ and } G_0 \in \mathcal{G}, \text{ given a finite nonvoid set } M \subset G_0, \\ \text{there is an averaging process } \theta \text{ with elements of } \overline{\Gamma} \text{ such that} \\ \|\theta(\alpha)\|_G < \varepsilon, \text{ for all } \alpha \in M. \end{cases}$$

After this preparation, we are ready to prove the theorem. Let $M \subset G^*$ be a finite nonvoid set and $\varepsilon > 0$. By (9), M can be writen as a disjoint union

$$M = \bigcup_{i=1}^{N} M_i; \quad (N \geqslant 1)$$

where $M_1 = M \cap G_1 \subset G_1^*$, and for $i = 2, \dots, N$ there are groups $G_i \in \mathcal{G}$ such that $\emptyset \neq M_i \subset G_i$.

We shall prove by induction on n, for $n = 1, 2, \dots, N$, the statement:

$$(11)_n \qquad \begin{cases} \text{there is an averaging process } \theta_n \text{ with elements of } \overline{\Gamma} \text{ such that} \\ \|\theta_n(\alpha)\|_G < \varepsilon, \text{ for all } \alpha \in \bigcup_{i=1}^{n} M_i. \end{cases}$$

For $n = 1$, this holds, due to (7) (if $M_1 = \emptyset$, we take $\theta_1 = \mathrm{id}$). Assume $(11)_n$ holds for some n, $1 \leqslant n < N$. Denote

$$M_{(n+1)} := \bigcup \{\operatorname{supp} \theta_n(\alpha) \mid \alpha \in M_{n+1}\}.$$

Then $M_{(n+1)}$ is a finite nonvoid set and $M_{(n+1)} \subset G_{n+1}$ due to (8), therefore (10) gives an averaging process $\theta_{(n+1)}$ with elements of $\overline{\Gamma}$ such that

$$\|\theta_{(n+1)}(\alpha)\|_G < \varepsilon, \text{ for all } \alpha \in M_{(n+1)}. \tag{12}$$

We define $\theta_{n+1} = \theta_{(n+1)} \circ \theta_n$. Then for $\alpha \in \bigcup_{i=1}^{n} M_i$, one has

$$\|\theta_{n+1}(\alpha)\|_G = \|\theta_{(n+1)}(\theta_n(\alpha))\|_G \leqslant \|\theta_n(\alpha)\|_G < \varepsilon$$

and for $\alpha \in M_{n+1}$, one has

$$\|\theta_{n+1}(\alpha)\|_G = \|\theta_{(n+1)}(\theta_n(\alpha))\|_G < \varepsilon$$

by (12), because $\theta_n(\alpha)$ is a convex combination of elements of $M_{(n+1)}$.

Therefore, $(11)_n$ implies $(11)_{n+1}$, hence $(11)_N$ holds, and the theorem is proved. ∎

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