J. OPERATOR THEORY
28(1992), 337–355

ON THE EXPONENTIAL RANK AND EXPONENTIAL LENGTH OF C^*-ALGEBRAS

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INTRODUCTION

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{K}$, $\mathcal{L}(\mathcal{H})$ be the algebras of all compact operators and all bounded operators on $\mathcal{H}$, respectively. If X is a compact Hausdorff space, then the C^*-algebra $C(X, \mathcal{L}(\mathcal{H}))$ consisting of all norm-continuous maps from X to $\mathcal{L}(\mathcal{H})$ is $*$-isomorphic to $\mathcal{L}(\mathcal{H}) \otimes C(X)$. Concerning the structure of the unitary group of $C(X, \mathcal{L}(\mathcal{H}))$, J. R. Ringrose recently proved the following estimates on the *C^*-exponential rank* and the *C^*-exponential length* in [22]:

$$\mathrm{cer}(C(X, \mathcal{L}(\mathcal{H}))) \leqslant 3 \ \text{ and } \ \mathrm{cel}(C(X, \mathcal{L}(\mathcal{H})) \leqslant \frac{5\pi}{2},$$

where "$\mathrm{cer}(C(X, \mathcal{L}(\mathcal{H}))) \leqslant 3$" means that every unitary $u = u(\cdot)$ in $C(X, \mathcal{L}(\mathcal{H}))$ can be written as a product of at most three exponentials; i.e.,

$$u = \exp(\mathrm{i}h_1)\exp(\mathrm{i}h_2)\exp(\mathrm{i}h_3)$$

for three norm-continuous maps $h_i = h_i(\cdot)\,(1 \leqslant i \leqslant 3)$ from X to bounded self-adjoint operators, while "$\mathrm{cel}(C(X, \mathcal{L}(\mathcal{H})) \leqslant \frac{5\pi}{2}$" means that $\frac{5\pi}{2}$ is the supremum of

$$\inf\left\{\sum_{i=1}^{n} \sup_{t\in X} \|h_i(t)\| : u = \exp(\mathrm{i}h_1)\ldots\exp(\mathrm{i}h_n)\right\}$$

as u runs over the unitary group of $C(X, \mathcal{L}(\mathcal{H}))$.

For an arbitrary unital C^*-algebra $\mathcal{A}$ (consider the unitalization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ instead in case $\mathcal{A}$ is non-unital) the C^*-exponential rank of $\mathcal{A}$, denoted by $\mathrm{cer}(\mathcal{A})$, is defined

[21] to be the smallest integer n (or $n+\varepsilon$) such that every unitary element in the identity path component can be written as (or, respectively, can be approximated in norm within any positive number by) a product of at most n exponentials. The C^*-exponential length of $\mathcal{A}$, denoted by cel($\mathcal{A}$), is defined to be the supremum

$$\sup\left\{\inf\left\{\sum_{i=1}^{n}\|h_i\| : u=\exp(\mathrm{i}h_1)\dots\exp(\mathrm{i}h_1)\right\}\right\},$$

where $h_i (1\leqslant i\leqslant n)$ are self-adjoint elements in $\mathcal{A}$ and sup is taken as u runs over the identity path component of the unitary group of $\mathcal{A}$. The reader is reffered to the survey article [21], also to [22], [19], [20] for more information.

Our first main result in this article is Theorem 1.1, which improves and generalizes Ringrose's results mentioned above. For any σ-unital C^*-algebra $\mathcal{A}$ and any unital C^*-algebra $\mathcal{B}$ we consider $\mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{B}$, where $\mathcal{L}(\mathcal{H}_\mathcal{A})$ is the C^*-algebra of bounded operators on the Hilbert C^*-module $\mathcal{H}_\mathcal{A}$ whose adjoints exist. We will prove that

$$\mathrm{cer}(\mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{B})\leqslant 3,\ \mathrm{cel}(\mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{B})\leqslant\frac{5\pi}{2}\ \text{ if } \mathcal{A} \text{ is unital; and}$$

$$\mathrm{cer}(\mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{B})\leqslant 3+\varepsilon,\ \mathrm{cel}(\mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{B})\leqslant 3\pi\ \text{ if } \mathcal{A} \text{ is non-unital.}$$

Theorem 1.1 covers almost all pairs $(\mathcal{A},\mathcal{B})$ of C^*-algebras. In particular, if $\mathcal{B}=C(X)$, we conclude

$$\mathrm{cer}(C(X,\mathcal{L}(\mathcal{H}_\mathcal{A})))\leqslant 3,\ \mathrm{cel}(C(X,\mathcal{L}(\mathcal{H}_\mathcal{A})))\leqslant\frac{5\pi}{2}\ \text{ if } \mathcal{A} \text{ is unital; and}$$

$$\mathrm{cer}(C(X,\mathcal{L}(\mathcal{H}_\mathcal{A})))\leqslant 3+\varepsilon,\ \mathrm{cel}(C(X,\mathcal{L}(\mathcal{H}_\mathcal{A})))\leqslant 3\pi\ \text{ if } \mathcal{A} \text{ is non-unital.}$$

N. C. Phillips recently proved [19] that if $\mathcal{A}$ is a purely infinite, simple C^*-algebra, then

$$\mathrm{cer}(\mathcal{A})\leqslant 1+\varepsilon;\ \text{ and } \mathrm{cel}(\mathcal{A})=\pi \text{ in case } \mathcal{A} \text{ has a unit.}$$

Dealing with the C^*-algebra of norm-continuous maps from X to $\mathcal{A}$, we assert, Theorem 1.2, that

$$\mathrm{cel}(C(X,\mathcal{A}))\leqslant\frac{5\pi}{2},\ \mathrm{cer}(C(X,\mathcal{A}))\leqslant 3\ \text{ if } \mathcal{A} \text{ is unital; and}$$

$$\mathrm{cel}(C(X,\mathcal{A}))\leqslant 2\pi,\ \mathrm{cer}(C(X,\mathcal{A}))\leqslant 2+\varepsilon\ \text{ if } \mathcal{A} \text{ is non-unital.}$$

Theorem 1.2 includes many interesting special cases; for example, all type III factors, the Cuntz algebras $\mathcal{O}_n (2\leqslant n\leqslant\infty)$, simple Cuntz-Krieger algebras $\mathcal{O}_A$, and many generalized Calkin algebras. The reader is reffered to §1 for detailed statements and corollaries.

Acknowledgement. The author wishes to thank N. C. Phillips for several very helpful e-mails and conversations on the subject, and to thank M. Rørdam for sending him useful information via an e-mail.

0. PRELIMINARIES

Let $\mathcal{A}$ be any C^*-algebra. An $\mathcal{A}$-valued inner product is defined on the set of norm-bounded sequences in $\mathcal{A}$ by

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i^* b_i.$$

$\mathcal{H}_{\mathcal{A}}$ denotes the set of all sequences such that $\langle \{a_i\}, \{a_i\} \rangle$ exists as an element of $\mathcal{A}$. Naturally, $\langle .,. \rangle$ induces a norm

$$\|\{a_i\}\| = \|\langle \{a_i\}, \{a_i\} \rangle^{\frac{1}{2}}\| = \left\| \left(\sum_{i=1}^{\infty} a_i^* a_i \right)^{\frac{1}{2}} \right\| \quad \forall \{a_i\} \in \mathcal{H}_{\mathcal{A}}.$$

In this way, $\mathcal{H}_{\mathcal{A}}$ forms a Hilbert (right) $\mathcal{A}$-module. If $\mathcal{A}$ is the algebra of complex numbers, then $\mathcal{H}_{\mathcal{A}}$ reduces to a separable Hilbert space $\mathcal{H}$. However, for certain operators on $\mathcal{H}_{\mathcal{A}}$ the adjoint operator T^* on $\mathcal{H}_{\mathcal{A}}$ defined naturally by

$$\langle T^* \{a_i\}, \{b_i\} \rangle = \langle \{a_i\}, T\{b_i\} \rangle \quad \text{for all } \{a_i\}, \{b_i\} \in \mathcal{H}_{\mathcal{A}}$$

may not exist. Let $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ denote the set of all bounded operators on $\mathcal{H}_{\mathcal{A}}$ whose adjoint operators exist, equipped with the naturally defined operator norm, then $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a C^*-algebra which is $*$-isomorphic to the multiplier algebra $M(\mathcal{A} \otimes \mathcal{K})$ ([11]). Here we point out that each element in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ can be identified with a *bounded* infinite matrix whose entries are elements in $\mathcal{A}$ (if $\mathcal{A}$ has a unit). For each pair of elements x and y in $\mathcal{H}_{\mathcal{A}}$, a bounded operator of rank one is defined by

$$\Theta_{x,y}(z) = x\langle y, z \rangle \text{ for any } z \in \mathcal{H}_{\mathcal{A}}.$$

Let $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ stand for the closed linear span of all operators with rank one. Then $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ is a C^*-algebra which is $*$-isomorphic to $\mathcal{A} \otimes \mathcal{K}$. It is quite natural from the construction to call $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ the algebra of *compact operators* on $\mathcal{H}_{\mathcal{A}}$. Then correspondingly

$$\mathcal{L}(\mathcal{H}_{\mathcal{A}}) / \mathcal{K}(\mathcal{H}_{\mathcal{A}}) \ \ (\cong M(\mathcal{A} \otimes \mathcal{K}) / \mathcal{A} \otimes \mathcal{K})$$

is called the generalized Calkin algebra associated with $\mathcal{A}$. This general setup plays important roles in advanced mathematics; for example, the extension theory, K-theory and KK-theory of C^*-algebras. The reader is reffered to [1, 11] for more details.

It is clear that a $*$-isomorphic copy of $\mathcal{L}(\mathcal{H})$ is embedded in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ via

$$\mathcal{L}(\mathcal{H}) \to 1 \otimes \mathcal{L}(\mathcal{H}) \hookrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}}),$$

where '1' is the identity of $\tilde{\mathcal{A}}$ (or $M(A)$). Similarly, $\mathcal{L}(\mathcal{H})$ is embedded in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ by

$$\mathcal{L}(\mathcal{H}) \to 1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1 \hookrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B},$$

where the second '1' is the identity of $\mathcal{B}$. There are two important topologies on $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. One is the norm topology and the other is the strict topology, defined in analogy with $*$-strong operator topology on $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, by

$$x_\lambda \xrightarrow{\text{strictly}} x \quad \text{iff} \quad \max\{\|(x_\lambda - x)a\|, \|a(x_\lambda - x)\|\} \to 0 \quad \forall a \in \mathcal{K}(\mathcal{H}_{\mathcal{A}}).$$

If $\mathcal{B} = C(X)$, then

$$\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes C(X) \cong C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}})),$$

where $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}))$ is the C^*-algebra of all norm-continuous maps from X to $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$. In this particular situation, there is only one C^*-norm on $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes C(X)$. Actually, our results in this article hold for any C^*-norm on $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ in case there are more than one. We will often use the notation $u = u(\cdot)$ to denote an element in $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}))$. In particular, a unitary in $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}))$ is regarded as a norm-continuous map from X to the unitary group of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, while every element in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is regarded as a constant map on X.

1. MAIN RESULTS

In this section, we state our main result and corollaries. The proofs will be given by a sequence of lemmas in the next section. Throughout, we will denote the unitary group of a unital C^*-algebra $\mathcal{B}$ by $U(\mathcal{B})$ and the identity path component of $U(\mathcal{B})$ by $U_0(\mathcal{B})$. X denotes a compact Hausdorff space.

1.1. THEOREM. *Let $\mathcal{B}$ be any unital C^*-algebra.*

a) *If $\mathcal{A}$ is a σ-unital C^*-algebra, then every unitary element in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ can be approximated (in any C^*-norm) by products of at most six symmetries (two of which are in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$), $\operatorname{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant 3\pi$, and $\operatorname{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant 3 + \varepsilon$.*

b) *If $\mathcal{A}$ is a unital C^*-algebra, then every unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ can be approximated (in any C^*-norm) by products of at most six symmetries (three of which are in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$), $\operatorname{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant \frac{5\pi}{2}$, and $\operatorname{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant 3$.*

N. C. Phillips recently proved [19] that if $\mathcal{A}$ is a purely infinite simple C^*-algebra, then $\operatorname{cer}(\mathcal{A}) \leqslant 1 + \varepsilon$. If, in addition, $\mathcal{A}$ has a unit, then $\operatorname{cel}(\mathcal{A}) \leqslant \pi$. Considering the C^*-algebra of all norm-continuous maps from X to $\mathcal{A}$ instead, we have the following:

1.2. THEOREM. *Suppose that $\mathcal{A}$ is a purely infinite, simple C^*-algebras.*

a) *If $\mathcal{A}$ is unital, then every unitary in $U_0(C(X,\mathcal{A}))$ can be approximated by products of six symmetries (three of which are elements of $U_0(\mathcal{A})$), $\mathrm{cel}(C(X,\mathcal{A})) \leqslant \frac{5\pi}{2}$, and $\mathrm{cer}(C(X,\mathcal{A})) \leqslant 3$.*

b) *If $\mathcal{A}$ is non-unital, then every unitary in $U_0(C(X,\mathcal{A}))$ can be approximated by products of four symmetries (two of which are elements of $U_0(\mathcal{A})$), $\mathrm{cel}(C(X,\mathcal{A})) \leqslant 2\pi$, and $\mathrm{cer}(C(X,\mathcal{A})) \leqslant 2+\varepsilon$.*

We recall that a simple C^*-algebra $\mathcal{A}$ is said to be purely infinite if there exists an infinite projection in $x\mathcal{A}x$ for each nonzero element x in $\mathcal{A}$ ([4, 25]). There are a dozen equivalent conditions in [16] characterizing purely infinite, simple C^*-algebras. The author proved [27] that a σ-unital, purely infinite, simple C^*-algebra is either unital or stable. The author also proved [27, 30] that a simple C^*-algebra is purely infinite if and only if $RR(\mathcal{A}) = 0$ and every nonzero projection in $\mathcal{A}$ is infinite. Particular examples include the Cuntz algebras $\mathcal{O}_n (2 \leqslant n \leqslant \infty)$, certain Cuntz-Krieger algebras, the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}$, all generalized Calkin algebras associated with σ-unital, purely infinite, simple C^*-algebras ([26, 27], [24], or [16] together with [13], or [14]).

Several corrolaries are in order. If we take $\mathcal{B} = C(X)$ in Theorem 1.1, then the following corollary generalizes a recent result of [22] for the special case $\mathcal{H}_{\mathcal{A}} = \mathcal{H}$.

COROLLARY 1.3. *If $\mathcal{A}$ is a σ-unital C^*-algebra and X is a compact Hausdorff space, then*

$$\mathrm{cel}(C(X,\mathcal{L}(\mathcal{H}_{\mathcal{A}}))) \leqslant 3\pi \quad \textit{and} \quad \mathrm{cer}(C(X,\mathcal{L}(\mathcal{H}_{\mathcal{A}}))) \leqslant 3+\varepsilon.$$

If $\mathcal{A}$ is a unital C^*-algebra, then

$$\mathrm{cel}(C(X,\mathcal{L}(\mathcal{H}_{\mathcal{A}}))) \leqslant \frac{5\pi}{2} \quad \textit{and} \quad \mathrm{cer}(C(X,\mathcal{L}(\mathcal{H}_{\mathcal{A}}))) \leqslant 3.$$

If $\mathcal{A}$ is the algebra of complex numbers and hence $\mathcal{H}_{\mathcal{A}} = \mathcal{H}$, then a recent result of J. R. Ringrose [22] is included; i.e.,

$$\mathrm{cel}(C(X,\mathcal{L}(\mathcal{H}))) \leqslant \frac{5\pi}{2} \quad \text{and} \quad \mathrm{cer}(C(X,\mathcal{L}(\mathcal{H}))) \leqslant 3.$$

If we take $\mathcal{B}$ in Theorem 1.1 to be the complex numbers, we have the following quite general result.

COROLLARY 1.4. *If $\mathcal{A}$ is a σ-unital C^*-algebra, then*

$$\mathrm{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})) \leqslant 3\pi \quad \textit{and} \quad \mathrm{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})) \leqslant 3+\varepsilon.$$

If $\mathcal{A}$ is unital, then

$$\mathrm{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})) \leqslant \frac{5\pi}{2} \text{ and } \mathrm{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})) \leqslant 3.$$

COROLLARY 1.5. *If $\mathcal{A}$ is σ-unital and $\mathcal{B}$ is unital, then the unitary group of $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ is contractible (in any C^*-norm).*

Proof. Since $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \cong \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B} \otimes C(X)$, it follows from Theorem A that every unitary element in $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B})$ can be written as a product of three exponentials $\exp(ih_1)\exp(ih_2)\exp(ih_3)$ for three self-adjoint elements h_1, h_2 and h_3. Then the standard path

$$\{\exp(ith_1)\exp(ith_2)\exp(ith_3) \;:\; 0 \leqslant t \leqslant 1\}$$

connects the unitary with the identity.

REMARKS 1.6. (i) It was proved in [17, 7] that if $\mathcal{A}$ is σ-unital, then the unitary group of $C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}))$ is connected for any comact Hausdorff space X. It follows that the unitary group of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is a contractible topological space. This result also follows from Corollary 1.5, for which our proof in the next section is rather elementary, not involving K-theory.

(ii) In [19] N. C. Phillips proved that $\mathrm{cer}(C(S^1, \mathcal{A})) \geqslant 2$ if $\mathcal{A}$ is a purely infinite, simple C^*-algebra with trivial K_1 (in particular if $\mathcal{A} = \mathcal{O}_2$). By a similar argument he pointed out to us via an e-mail that $\mathrm{cer}(C(S^1, \mathcal{L}(\mathcal{H}))) \geqslant 2$. At this stage, we do not know whether the upper bounds we give in Theorem 1.1 and Theorem 1.2 can be improved to $\mathrm{cer}(\cdot) = 2$ or $\mathrm{cer}(\cdot) \leqslant 2 + \varepsilon$ in general except for some special classes of C^*-algebras [32].

(iii) If $\mathcal{A}$ is a σ-unital, simple C^*-algebra such that $\mathcal{A} \otimes \mathcal{K}$ contains a nonzero projection (most known simple C^*-algebras satisfy this condition), then $\mathcal{A} \otimes \mathcal{K} \cong$ $\cong \mathcal{A}_1 \otimes \mathcal{K}$ for a unital, hereditary C^*-subalgebra $\mathcal{A}_1$ of $\mathcal{A} \otimes \mathcal{K}$. Hence

$$\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B} \cong \mathcal{L}(\mathcal{H}_{\mathcal{A}_1}) \otimes \mathcal{B}.$$

Therefore, Theorem 1.1(b) covers all the cases. Particular examples include simple AF algebras, C^*-algebras of real rank zero (unital or not), all von Neumann algebras.

For some special cases of Theorem 1.1, we can reach better estimates for the cer(,) and cel($\cdot$) than the ones in Theorem 1.1 with more techniques involved [32]. Now we turn to some corollaries of Theorem 1.2.

COROLLARY 1.7. *If $\mathcal{M}$ is a type III factor (von Neumann algebra), then*

$$\mathrm{cel}(C(X, \mathcal{M})) \leqslant \frac{5\pi}{2} \text{ and } \mathrm{cer}(C(X, \mathcal{M})) \leqslant 3.$$

COROLLARY 1.8. *If $\mathcal{O}_n (2 \leqslant n \leqslant \infty)$ are the Cuntz algebras, then*

$$\mathrm{cel}(C(X, \mathcal{O}_n)) \leqslant \frac{5\pi}{2} \text{ and } \mathrm{cer}(C(X, \mathcal{O}_n)) \leqslant 3.$$

COROLLARY 1.9. *If $\mathcal{A}$ is a σ-unital, purely infinite, simple C^*-algebra, then*

$$\mathrm{cel}(C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}) / \mathcal{K}(\mathcal{H}_{\mathcal{A}}))) \leqslant \frac{5\pi}{2} \text{ and } \mathrm{cer}(C(X, \mathcal{L}(\mathcal{H}_{\mathcal{A}}) / \mathcal{K}(\mathcal{H}_{\mathcal{A}}))) \leqslant 3.$$

2. THE PROOFS

In this section, we will prove Theorem 1.1 and Theorem 1.2 by a sequence of lemmas, which have a interest in their own.

First of all, we point out that if the identity of a unital C^*-algebra $\mathcal{B}$ is a (finite or infinite) direct sum $\sum p_i$ of mutually orthogonal projections, called a decomposition of the identity, then element x can be written as $\sum_{i,j} p_i x p_j$. It is easy to show that the map defined by

$$\varphi(x) = \begin{pmatrix} p_1xp_1 & p_1xp_2 & p_1xp_3 & \cdots \\ p_2xp_1 & p_2xp_2 & p_2xp_3 & \cdots \\ p_3xp_1 & p_3xp_2 & p_3xp_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \forall x \in \mathcal{B}$$

is a $*$-isomorphism, where we use the same rules of matrix operations as the ones in the matrix C^*-algebra over $\mathcal{B}$. $\varphi(x)$ is said to be the matrix form of x with respect to the decomposition $1 = \sum p_i$. The following Lemma 2.1 is useful for investigating the unitary group and the space of projections in an infinite C^*-algebra.

LEMMA 2.1. *Let $\mathcal{D}$ be a unital C^*-algebra and p be a nontrivial projection in $\mathcal{D}$. If a unitary element in $\mathcal{D}$ has the matrix form*

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$$

with respect to $p \oplus (1-p) = 1$, then we have the following:

(i) *a, b and c are partial isometries in $\mathcal{D}$ such that $aa^* = p$, $b^*b = p$, $bb^* + cc^* = = 1 - p$, $a^*a + c^*c = 1 - p$, $ac^* = 0$ and $b^*c = 0$.*

(ii)*$(ba+c)(ba+c)^* = (ba+c)^*(ba+c) = 1-p$. In other words, $ba+c$ is a (local) unitary in $(1-p)\mathcal{D}(1-p)$.*

(iii)

$$s_1 = \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix} \text{ and } s_2 = \begin{pmatrix} 0 & a \\ a^* & c^*c \end{pmatrix} \text{ are symmetries of } \mathcal{D}.$$

(iv)

$$\begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix} \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \begin{pmatrix} 0 & a \\ a^* & c^*c \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & ba+c \end{pmatrix}.$$

Proof. (i) follows from the equalities $uu^* = u^*u = 1$ and matrix multiplication. Using the identities in (i), one can easily show that

$$(ba+c)(ba+c)^* = (ba+c)^*(ba+c) = 1-p,\ s_i = s_i^* \text{ and } s_i^2 = 1 \quad (i = 1,2).$$

Hence, (ii) and (iii) are clear. (iv) follows from (i) and the matrix multiplication.

From now on, the notation '$p \sim q$' is reserved for the Murray-von Neumann equivalence between two projections in C^*-algebra.

LEMMA 2.2. *Let $\mathcal{A}$ be a C^*-algebra. If p is a projection in $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ equivalent to the identity and u is a unitary in $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ with a matrix form*

$$\begin{pmatrix} p & 0 \\ 0 & u \end{pmatrix} \text{ (with respect to } p + (1-p) = 1),$$

then there exists a unitary w and four symmetries in $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ such that

$$w^*uw = s_1s_2s_3s_4,$$

where s_2 and s_3 are symmetries in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$.

Proof. Let $\{e_k\}_{k=1}^\infty$ be a sequence of infinite dimensional, mutually orthogonal projections in $\mathcal{L}(\mathcal{H})$. Obviously, we can write the identity of $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ as the following sums

$$\sum_{k=1}^\infty 1 \otimes e_k \otimes 1 = 1 \otimes \left(\sum_{k=1}^\infty e_k \right) \otimes 1,$$

where $1 \otimes e_k \otimes 1$ is of course equivalent to the identity. Since $p \sim 1$, we can choose projections p_k $(k \geqslant 1)$ in $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ such that

$$1 - p \leqslant p_1 \text{ and } 1 - p_1 = \sum_{k=2}^\infty p_k,$$

where $p_k \sim 1$ for each $k \geqslant 1$. Of course $u(1-p_1) = (1-p_1)u = 1-p_1$. For each $k \geqslant 1$, let v_k be a partial isometry in $\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}$ such that

$$v_kv_k^* = p_k \text{ and } v_k^*v_k = 1 \otimes e_k \otimes 1.$$

Set $w = \sum_{k=1}^{\infty} v_k$. Then w is a unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ such that

$$w^* u w = u_1 \oplus \sum_{k=2}^{\infty} 1 \otimes e_k \otimes 1,$$

where u_1 is a unitary of $(1 \otimes e_1 \otimes 1)\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}(1 \otimes e_1 \otimes 1)$. With respect to the decomposition $1 = \sum_{k=1}^{\infty}(1 \otimes e_k \otimes 1)$ we write

$$w^* u w = s_1 s_2 s_3 s_4,$$

where s_1, s_2, s_3 and s_4 are four symmetries whose matrices have the following forms:

$$s_1 = \begin{pmatrix} 0 & u_1 & & & & \\ u_1^* & 0 & & & & \\ & & 0 & u_1 & & \\ & & u_1^* & 0 & & \\ & & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

$$s_3 = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & & & & & \\ & 0 & u_1^* & & & \\ & u_1 & 0 & & & \\ & & & 0 & u_1^* & \\ & & & u_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}.$$

It is a routine to show that $s_i \in \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$, $s_i = s_i^*$, $s_i^2 = 1\,(i = 1, 2, 3, 4)$. Clearly, s_2 and s_3 are elements in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$.

LEMMA 2.3. *If $\mathcal{A}$ is a σ-unital C^*-algebra and p is a proper projection in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes \otimes\mathcal{B}$ (i.e., $p \sim 1 - p \sim 1$), then every unitary u in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ with the matrix form*

$$u = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \text{ (with respect to } p + (1 - p) = 1)$$

can be written as a product of five symmetries in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$, say $u = s_1 s_2 s_3 s_4 s_5$. Furthermore, $s_2 = ws'w^*$ and $s_3 = ws''w^*$ where w is a unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ and s', s'' are symmetries in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$.

Proof. It follows immediately from Lemma 2.1 that

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & ba + c \end{pmatrix}.$$

Set $s_1 = \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix}$. By Lemma 2.2 there exist a unitary w and four symmetries s_i' $(i = 1, 2, 3, 4, 5)$ in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ such that

$$\begin{pmatrix} p & 0 \\ 0 & ba + c \end{pmatrix} = ws_2's_3's_4's_5'w^*,$$

where s_3' and s_4' are symmetries in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$. Set $s_2 = ws_3'w^*$, $s_3 = ws_4'w^*$, $s_4 = = ws_4's_3's_2's_3's_4'w^*$, and $s_5 = ws_5'w^*$. Then $u = s_1s_2s_3s_4s_5$, as desired.

The following Lemma 2.4 was first observed by M. Rørdam [23].

LEMMA 2.4. *If $\mathcal{D}$ is a unital C^*-algebra and s_1, s_2 are two symmetries in $\mathcal{D}$, then* $cer(s_1s_2) \leqslant 1 + \varepsilon$.

LEMMA 2.5. *Let $\mathcal{B}$ be any unital C^*-algebra.*

(i) *If $\mathcal{A}$ is a σ-unital C^*-algebra and if p and q are two projections in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$, then for any positive number ε there exist proper projections p' and q' in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ such that $p' \leqslant p$, $q' \leqslant q$ and $\|p'q'\| < \varepsilon$.*

(ii) *If $\mathcal{A}$ is a unital C^*-algebra and if u is a unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$, then there exist two sequences of mutually orthogonal, rank one projections $\{e_{n_i,n_i}\}$ and $\{e_{m_j,m_j}\}$ in $\mathcal{K}$ such that $\left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right)\left(\sum_{j=1}^{\infty} e_{m_j,m_j}\right) = 0$ and*

$$\left\| \left[1 \otimes \left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right) \otimes 1\right] u \left[1 \otimes \left(\sum_{j=1}^{\infty} e_{m_j,m_j}\right) \otimes 1\right] \right\| < \varepsilon.$$

Here $\{e_{ij}\}$ is the set of matrix units in $\mathcal{L}(\mathcal{H})$ with respect to an orthonormal basis of $\mathcal{H}$.

Proof. The proof for (i) uses the proof of [7, Lemma 4] with minor modifications. We leave it to the reader to go through the details. We give more details for (ii), since we will use the explicit construction later.

Let $e = \sum_{i=1}^{\infty} e_{2i-1,2i-1}$ and $f = \sum_{i=1}^{\infty} e_{2i,2i}$. Then $e \oplus f$ is the identity of $\mathcal{L}(\mathcal{H})$. Since $1 \otimes e_{nn}$ converges to zero in the strict topology of $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$, for any fixed i we have

$$\lim_{n\to\infty} \|(1 \otimes e_{nn} \otimes 1)u(1 \otimes e_{2i,2i} \otimes 1)\| = 0; \text{ and}$$
$$\lim_{n\to\infty} \|(1 \otimes e_{2i-1,2i-1} \otimes 1)u(1 \otimes e_{nn} \otimes 1)\| = 0.$$

Thus, we can find a subsequence $\{e_{n_i,n_i}\}$ of $\{e_{2i-1,2i-1}\}$ and a subsequence $\{e_{m_j,m_j}\}$ of $\{e_{2i,2i}\}$ such that

$$\left\|(1 \otimes e_{n_i,n_i} \otimes 1)\, u\, (1 \otimes e_{m_j,m_j} \otimes 1)\right\| < \frac{\varepsilon}{2^{i+j}}.$$

It follows that

$$\left\| \left[1 \otimes \left(\sum_{i=1}^{\infty} e_{n_i,n_i} \right) \otimes 1 \right] u \left[1 \otimes \left(\sum_{j=1}^{\infty} e_{m_j,m_j} \right) \otimes 1 \right] \right\| =$$

$$= \left\| \sum_{i,j} (1 \otimes e_{n_i,n_i} \otimes 1)\, u \,(1 \otimes e_{m_j,m_j} \otimes 1) \right\| < \varepsilon.$$

2.6. THE PROOF FOR THEOREM 1.1.

The proof for (a). Assume that $\mathcal{A}$ is a σ-unital (not necessarily unital). Let p be any proper projection and u any unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$. Set $q = upu^*$. Using Lemma 2.5 (i), for any positive number δ one gets two proper projections p' and q' such that

$$p' \leqslant 1 - p, \quad q' \leqslant q \quad \text{and} \quad \|p'q'\| < \sqrt{\delta}.$$

Set $u^*q'u = p_0$. Then

$$p_0 \leqslant p, \quad up_0u^* = q' \quad \text{and} \quad \|p'up_0u^*\| < \sqrt{\delta}.$$

It is obvious that $p' \sim q' \sim p_0$ and $p'p_0 = 0$. Let v' be a partial isometry in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ with initial projection p' and final projections p_0. Set

$$s_0 = v' + v'^* + 1 - p' - p_0.$$

Then s_0 is a symmetry in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ such that $s_0p_0s_0 = p'$. It follows that

$$\|p_0s_0up_0u^*s_0p_0\| < \delta.$$

If δ is small enough, we can find, by [9, 2.1], a unitary u_0 such that

$$\|u_0 - 1\| < \varepsilon \quad \text{and} \quad p_1 := u_0^*s_0up_0u^*s_0u_0 \leqslant 1 - p_0.$$

Here we point out that there is a symmetry s_0' in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ such that $s_0's_0up_0u^*s_0s_0' = p_1 \leqslant 1 - p_0$ as long as $\delta < \frac{1}{2}$ (see [33, Proposition 3], for example), since

$$\|p_1 - s_0up_0u^*s_0\| \leqslant 2\|u_0 - 1\| < 2\delta < 1.$$

This observation is irrelevant to the proof here but is useful in [33].

Now let us continue the proof for (a). With respect to the decomposition $p_0 \oplus (1 - p_0) = 1$ we write the following matrix forms

$$u_0^*s_0u = \begin{pmatrix} x & a \\ b & c \end{pmatrix} \quad \text{and} \quad p_0 = \begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the relation

$$\begin{pmatrix} x & a \\ b & c \end{pmatrix} \begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^* & b^* \\ a^* & c^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p_1 \end{pmatrix},$$

we immediately conclude that $xx^* = 0$, and hence $x = 0$. It follows that

$$u = s_0 u_0 \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}.$$

Appllying Lemma 2.3, we can write

$$u = s_0 u_0 s_1 s_2 s_3 s_4 s_5,$$

where s_k $(0 \leqslant k \leqslant 5)$ are symmetries in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$. Furthermore, $s_2 s_3 = ws's''w^*$ for some unitary w in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$ and two symmetries s' and s'' in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$. Rewrite

$$(*) \qquad u = (\mathrm{i}s_0)(u_0)(\mathrm{i}s_1)(s_2 s_3)(\mathrm{i}s_4)(\mathrm{i}s_5).$$

Then

$$\mathrm{i}s_0 = \exp(\mathrm{i}h_0),\ u_0 = \exp(\mathrm{i}h_0'),\ \mathrm{i}s_1 = \exp(\mathrm{i}h_1),$$

$$s_2 s_3 = \exp(\mathrm{i}h_2),\ \mathrm{i}s_4 = \exp(\mathrm{i}h_3),\ \text{and } \mathrm{i}s_5 = \exp(\mathrm{i}h_4),$$

where h_0', h_0, h_1, h_2, h_3 and h_4 are self-adjoint elements in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$. Since $\|u_0 - 1\|$ can be arbitrarily small, we can choose h_0' such that $\|h_0'\|$ be as small as desired. We can choose h_2 in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$ such that $\|h_2\| \leqslant \pi$ (see [22]) and choose h_0, h_1, h_3 and h_4 with $\|h_k\| \leqslant \pi/2$ for $k = 0, 1, 3, 4$. (This is clear, since the spectrum $\sigma(\exp(\mathrm{i}h_k)) = \{-\mathrm{i}, \mathrm{i}\}$.) Thus we have

$$\mathrm{cel}(u) \leqslant [\|h_0'\| + \|h_0\| + \|h_1\| + \|h_2\| + \|h_3\| + \|h_4\|] \leqslant \|h_0'\| + 3\pi.$$

It follows from the definition [22] that $\mathrm{cel}(u) \leqslant 3\pi$. Since u is an arbitrary unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$, we conclude that

$$\mathrm{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant 3\pi.$$

Since u can be approximated in norm by products of six symmetries as in $(*)$, it follows from Lemma 2.4 that

$$\mathrm{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}) \leqslant 3 + \varepsilon.$$

The proof for (b). Assume that $\mathcal{A}$ is unital. Let $\delta > 0$ again be any number and u any unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}$. Using Lemma 2.5 (ii), there are two disjoint sequences $\{n_i\}$ and $\{m_j\}$ such that $\left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right)\left(\sum_{j=1}^{\infty} e_{m_j,m_j}\right) = 0$ and

$$\left\|\left[1 \otimes \left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right) \otimes 1\right] u \left[1 \otimes \left(\sum_{j=1}^{\infty} e_{m_j,m_j}\right) \otimes 1\right]\right\| < \sqrt{\delta}.$$

Let $p = 1 \otimes \left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right) \otimes 1$ and $p_0 = 1 \otimes \left(\sum_{j=1}^{\infty} e_{m_j,m_j}\right) \otimes 1$. Then

$$\|pup_0u^*p\| < \delta.$$

Since $pp_0 = 0$ and $p \sim p_0$, we can define, in a standard way as in the proof for (a), a symmetry v in $\mathcal{L}(\mathcal{H})$ such that

$$v\left(\sum_{i=1}^{\infty} e_{n_i,n_i}\right) v = \sum_{j=1}^{\infty} e_{m_j,m_j}.$$

Set $s_0 = 1 \otimes v \otimes 1$. Then s_0 is a symmetry in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$ such that $s_0p_0s_0 = p$. It follows that

$$\|p_0s_0up_0u^*s_0p_0\| < \delta,$$

and hence there exists a unitary u_0 arbitrarily close to the identity norm (as long as δ is small enough) such that $u_0^*s_0up_0u^*s_0u_0 \leqslant 1 - p_0$ ([9, 2.1]). Here again there is a symmetry s_0' such that $s_0's_0up_0u^*s_0u_0 \leqslant 1 - p_0$ (see [33, 3]). It follows that

$$u^*s_0u = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$$

with respect to the decomposition $p_0 + (1 - p_0) = 1$. Now by Lemma 2.3 we can write

$$u = s_0u_0s_1s_2s_3s_4s_5.$$

Here, first, we can assume that s_1 and s_2 are symmetries in $1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$ (by a permutation of s_1, s_2, s_3); and, secondly, we can take the unitary w in Lemma 2.3 to be the identity, since $p_0 \in 1 \otimes \mathcal{L}(\mathcal{H}) \otimes 1$. Rewrite

$$u = (\mathrm{i}s_0u_0s_0)(s_0s_1s_2)(\mathrm{i}s_3)(\mathrm{i}s_4)(\mathrm{i}s_5) =$$

$$= \exp(\mathrm{i}h_0)\exp(\mathrm{i}h_1)\exp(\mathrm{i}h_2)\exp(\mathrm{i}h_3)\exp(\mathrm{i}h_4),$$

where $s_0s_1s_2$ is a unitary in $1\otimes\mathcal{L}(\mathcal{H})\otimes 1$ and h_k $(0\leqslant k\leqslant 4)$ are self-adjoint elements with

$$\|h_1\|\leqslant\pi,\ \|h_k\|\leqslant\frac{\pi}{2}\ \text{for}\ k=2,3,4,$$

while $\|h_0\|$ can be arbitrarily small (see [22] for the existence of h_1). Therefore, we conclude that

$$\mathrm{cel}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes\mathcal{B})\leqslant\frac{5\pi}{2}.$$

It follows from [22, 1.8, 1.9] that

$$\mathrm{cer}(\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes\mathcal{B})\leqslant 3.$$

Before giving the proof for Theorem 1.2, we point out that u in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes\mathcal{B}$ can be written as a product of at most seven symmetries $s_0s_0's_1s_2s_3s_4s_5$ if one replaces the unitary u_0 above in the proof for Theorem 1.1 by a symmetry s_0'. Since this observation will be useful in [33], we single it out as Corollary 2.9 later.

To prove Theorem 1.2, we borrow an idea of N. C. Philips in [19] to prove the following lemma.

LEMMA 2.7. *Suppose that $\mathcal{B}$ is a unital C^*-algebra and p is a projection in $\mathcal{B}$ such that $n[p]\leqslant[1-p]$ for any integer $n\geqslant 1$ (in other words, $1-p$ contains n mutually orthogonal subprojections equivalent to p for any integer $n\geqslant 1$). If $u=u(\cdot)$ is a unitary in $C(X,\mathcal{B})$ such that $u(1-p)=(1-p)u=1-p$ and $pup\in U_0(C(X,p\mathcal{B}p))$, then for any number $\varepsilon>0$ there are four symmetries s_1,s_2,s_3, and s_4 in $C(X,\mathcal{B})$ such that $\|u-s_1s_2s_3s_4\|<\varepsilon$, where we can choose s_1 and s_2 from $U_0(\mathcal{B})$.*

Proof. Let $\{w_t\colon 0\leqslant t\leqslant 1\}$ be a continuous path of unitaries in $U_0(C(X,p\mathcal{B}p))$ such that $w_1=p$ and $w_0=pup$. Let $0=t_0<t_1<t_2<\ldots<t_n=1$ be a subdivision of $[0,1]$ such that

$$\max_{1\leqslant i\leqslant n}\|u_i-u_{i-1}\|<\varepsilon,$$

where $w_{t_i}:=u_i$ for $0\leqslant i\leqslant n$. Set $p_1=p$. Choose subprojections $p_2,p_3,\ldots,p_{2n+1}$ of $1-p$ and partial isometries $v_2,v_3,\ldots,v_{2n+1}$ in $\mathcal{B}$ such that

$$p_ip_j=0\ (1\leqslant i<j\leqslant 2n+1),\ \ v_iv_i^*=p_i\ \textit{and}\ v_i^*v_i=p\ (1\leqslant i\leqslant 2n+1),$$

where $v_1=p_1$. Set

$$x_1=\sum_{i=1}^{n}\left(v_{2i-1}u_iv_{2i}^*+v_{2i}u_i^*v_{2i-1}^*\right)+\left(1-\sum_{i=1}^{2n}p_i\right),$$

$$x_2=\sum_{i=1}^{n}\left(v_{2i-1}v_{2i}^*+v_{2i}v_{2i-1}^*\right)+\left(1-\sum_{i=1}^{2n}p_i\right),$$

$$x_3 = p_1 + \sum_{i=1}^{n} \left(v_{2i}u_i^* v_{2i+1}^* + v_{2i+1}u_i v_{2i}^*\right) + \left(1 - \sum_{i=1}^{2n+1} p_i\right),$$

$$x_4 = p_1 + \sum_{i=1}^{n} \left(v_{2i}v_{2i+1}^* + v_{2i+1}v_{2i}^*\right) + \left(1 - \sum_{i=1}^{2n+1} p_i\right).$$

It is easily checked that $x_i = x_i^*$, $x_i^2 = 1\,(i = 1,2,3,4)$. Moreover, x_2 and x_3 are actually in $U_0(\mathcal{B})$. Furthermore, one can check by computation that

$$\|u - x_1x_2x_3x_4\| = \left\|\sum_{i=1}^{n} v_{2i-1}\left(u_i - u_{i-1}\right) v_{2i-1}^*\right\| = \max_{1\leqslant i\leqslant n} \|u_i - u_{i-1}\| < \varepsilon.$$

To verify this estimate, the reader may consider the matrix forms of u and x_i $(1 \leqslant i \leqslant 4)$ with respect to the decomposition $p_1 + p_2 + \ldots + p_{2n-1} + \left(1 - \sum_{i=0}^{2n+1} p_i\right) = 1$, or by direct computation. Set $s_1 = x_2$, $s_2 = x_3$, $s_3 = x_3x_2x_1x_2x_3$, $s_4 = x_4$, as desired.

2.8. THE PROOF FOR THEOREM 1.2.

The proof for (a). Assume that $\mathcal{A}$ is unital. Let $u = u(\cdot)\colon X \to U(\mathcal{A})$ be any unitary element in $U_0(C(X, \mathcal{A}))$ and let ε be a given positive number. Let p be any nontrivial projection (i.e., $p \neq 0, 1$) in $\mathcal{A}$. It follows from the proof for [31, Lemma 2.4] that for any number $\delta > 0$ there exist projections $0 \neq p_0 < p$ and $0 \neq q_0 < 1 - p$ such that

$$\|q_0 u p_0 u^* q_0\| < \delta, \text{ (i.e., } \sup_{t\in X} \|q_0 u(t) p_0 u(t)^* q_0\| < \delta).$$

Since $\mathcal{A}$ is purely infinite and simple and $p_0q_0 = 0$, there is a partial isometry $v \in \mathcal{A}$ such that $vp_0v^* < q_0$. Let $s_1 = v + v^* + 1 - p_0 - vp_0v^*$. Then s_1 is a symmetry of $U_0(\mathcal{A})$ such that $s_1p_0s_1 = vp_0v^* < q_0$. It follows that

$$\|p_0 s_1 u p_0 u^* s_1 p_0\| < \delta.$$

By [9, 2.1], if δ is small enough, there is a unitary $u_0 = u_0(\cdot) \in U_0(C(X, \mathcal{A}))$ such that

$$\|u_0 - 1\| < \varepsilon \text{ and } u_0^* s_1 u p_0 u^* s_1 u_0 \leqslant 1 - p_0.$$

Here we notice that the role of u_0 can be replaced by a symmetry s_0'. It is easy to check as in (2.6) that with respect to the decomposition $p_0 + (1 - p_0) = 1$ we can write $u_0^* s_1 u$ into a matrix with the form

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix},$$

where

$$a = a(\cdot)\colon X \to p_0 U(\mathcal{A})(1-p_0),$$
$$b = b(\cdot)\colon X \to (1-p_0) U(\mathcal{A}) p_0,$$
$$c = c(\cdot)\colon X \to (1-p_0) U(\mathcal{A})(1-p_0)$$

are norm-continuous mappings. Set

$$s_2 = \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix}.$$

Then it follows from Lemma 2.1 that

$$u = s_1 u_0 s_2 \begin{pmatrix} p_0 & 0 \\ 0 & ba+c \end{pmatrix}.$$

Since $\mathcal{A}$ is purely infinite and simple, it is clear that $n[1] \leqslant [p_0]$ for any $n \geqslant 1$ ([5, 1.5]). Hence, Lemma 2.7 applies. We can approximate $\begin{pmatrix} p_0 & 0 \\ 0 & ba+c \end{pmatrix}$ within ε in norm by a product $s_3 s_4 s_5 s_6$ of four symmetries from $U_0(\mathcal{A})$. Then u is approximated by a product of six symmetries within ε, since $\|u_0 - 1\| < \varepsilon$. Thus, there exists a unitary u_1 in $C(X, \mathcal{A})$ such that $\|u_1 - 1\| < \varepsilon$ and

$$u = u_1 s_1 s_2 s_3 s_4 s_5 s_6 = (u_1)(\mathrm{i} s_1 s_3 s_4)(\mathrm{i} s_4 s_3 s_2 s_3 s_4)(\mathrm{i} s_5)(\mathrm{i} s_6).$$

If ε is small enough, for an arbitrary small positive number δ_0 we can write

$$u_1 = \exp(\mathrm{i} h_0) \ \text{ with } \ \|h_0\| < \delta_0,$$

$$\mathrm{i} s_1 s_3 s_4 = \exp(\mathrm{i} h_1) \ \text{ with } \ \|h_1\| < \pi,$$

$$\mathrm{i} s_4 s_3 s_2 s_3 s_4 = \exp(\mathrm{i} h_2) \ \text{ with } \ \|h_2\| < \frac{\pi}{2},$$

$$\mathrm{i} s_5 = \exp(\mathrm{i} h_3) \ \text{ with } \ \|h_3\| < \frac{\pi}{2},$$

$$\mathrm{i} s_6 = \exp(\mathrm{i} h_4) \ \text{ with } \ \|h_4\| < \frac{\pi}{2},$$

where $h_i \, (0 \leqslant i \leqslant 4)$ are self-adjoint elements of $C(X, \mathcal{A})$, and δ_0 is an arbitrarily small number. In fact, since $\mathrm{i} s_1 s_3 s_4$ is a unitary of $U_0(\mathcal{A})$, we use the recent result of [19]; viz., every unitary in $U_0(\mathcal{A})$ can be approximated by unitaries of finite spectrum. It is then clear that such an h_1 exists. It is obvious that $h_i \, (i = 2, 3, 4)$ can be chosen with $\|h_i\| \leqslant \frac{\pi}{2}$. Therefore, we conclude that

$$\mathrm{cel}(C(X, \mathcal{A})) \leqslant \frac{5\pi}{2}.$$

It follows from [22, 1.8, 1.9] that

$$\mathrm{cer}(C(X,\mathcal{A}))\leqslant 3.$$

The proof for (b). Assume that $\mathcal{A}$ is not unital. It was proved [27, 30] that $RR(\mathcal{A}) = 0$ and hence $\mathcal{A}$ has an approximate identity consisting of projections. If $u = u(\cdot)$ is any unitary in $U(C(X,\tilde{\mathcal{A}}))$ where $\tilde{\mathcal{A}}$ is the C^*-algebra obtained by adjoining an identity to $\mathcal{A}$, then by the proof for [31, 2.6] there exists a projection $p_0 \in \mathcal{A}$ such that u is close within any given number $\varepsilon > 0$ in norm to another unitary u_0 in $C(X,\tilde{\mathcal{A}})$ such that $u_0(1-p_0) = (1-p_0)u_0 = \lambda(1-p_0)$ for some complex number λ with $|\lambda| = 1$.

Assume that u is any unitary in $U_0(C(X\,\tilde{\mathcal{A}}))$. Using the same argument as in the proof for [31, 2.9], we can properly choose p_0 so that $p_0up_0 \in U_0(C(X,p_0\mathcal{A}p_0))$. Of course $n[p_0]\leqslant[1-p_0]$ for all $n\geqslant 1$. It then follows from Lemma 2.7 that $\overline{\lambda}u$ can be approximated with ε in norm by a product of four symmetries in $C(X,\tilde{\mathcal{A}})$, say $s_1s_2s_3s_4$, where s_1 and s_2 can be chosen from $U_0(\tilde{\mathcal{A}})$. Then it again follows from [19] that $\lambda s_1s_2 = \exp(\mathrm{i}h_1)$ with $h_1 = h_1^*$ and $\|h_1\|\leqslant\pi$. It is clear that $\mathrm{i}s_3 =$ $= \exp(\mathrm{i}h_2)$, $\mathrm{i}s_4 = \exp(\mathrm{i}h_3)$ with $h_2 = h_2^*$, $h_3 = h_3^*$ and $\|h_2\|$, $\|h_3\|\leqslant\frac{\pi}{2}$. Therefore,

$$\mathrm{cel}(C(X,\mathcal{A}))\leqslant 2\pi \quad\text{and}\quad \mathrm{cer}(C(X,\mathcal{A}))\leqslant 2+\varepsilon.$$

Now we have finished the proof for Theorem 1.2.

We conclude the article with the following by-products, which is significant and will be useful in [33].

COROLLARY 2.9.

(i) *Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 1.1. Then every unitary in $\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes\mathcal{B}$ can be approximated in norm by products of six symmetries and can be precisely written as a product of at most seven symmetries.*

(ii) *Let X and $\mathcal{A}$ be as in Theorem 1.2. Them every unitary in the identity path component of $U(C(X,\tilde{\mathcal{A}}))$ can be approximated by products of six symmetries.*

(iii) *$\mathcal{L}(\mathcal{H}_{\mathcal{A}})\otimes\mathcal{B}$ is generated algebraically by projections; and $C(X,\mathcal{A})$ is generated by projections.*

Proof. The conclusion of (i) follows from the proof of Theorem 1.1, if the unitary u_0 is replaced by a symmetry s_0' wherever u_0 appears. The conclusion (ii) is involved in the proof for Theorem 1.2. The conclusion (iii) follows from the fact that every element in a unital C^*-algebra is a linear combination of unitaries in the identity path component [10, 4.1.7].

Partially supported by NSF.

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Received January 31, 1991; revised November 27, 1991.