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TAYLOR JOINT SPECTRUM FOR FAMILIES OF OPERATORS GENERATING NILPOTENT LIE ALGEBRAS

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1. INTRODUCTION

The analytic functional calculus for commutative operator families was constructed in the papers of J. L. Taylor [7], [8].

Let $a = (a_1, \dots a_n)$ be a commutative family of linear bounded operators in a complex Banach space X. According to [7], the exactness of corresponding chain Koszul complex $\mathrm{Kos}\,(a, X)$ is the generalization of the concept of invertibility from one operator to commutative operator families. The Taylor joint spectrum of a is the set $\sigma(a)$ of n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for which $\mathrm{Kos}\,(a - \lambda, X)$ is not exact.

The analytic functional calculus is the continuous homomorphism of the algebra $O(U)$ of analytic functions into the algebra of bounded operators on X extending the natural calculus for polynomials of n variables, with U an open set in $\mathbb{C}^n$ containing $\sigma(a)$. The image of $\varphi \in O(U)$ under this homomorphism is denoted by $\varphi(a)$.

An important property of the Taylor functional calculus is the following spectral mapping theorem [8].

Let $f\colon U \to \mathbb{C}^m$ be an analytic map generated by a m-tuple of analytic functions and $f(a)$ be the corresponding m-tuple of operators. Then

$$\sigma\left(f(a)\right) = f\left(\sigma(a)\right).$$

Remark that the spectral mapping theorem is not trivial in the case of a polynomial map.

Another definition of the joint spectrum was given in the paper of R. E. Harte [3]. In contrast to Taylor's definition, Harte's one makes sense for arbitrary (not necessary commuting) operator families. Here is one of the versions of Harte's definition of joint spectrum.

The left joint spectrum of the operator family a is the set $\tau(a)$ of $\lambda \in \mathbb{C}^n$ for which there exists a sequence of unit vectors $\{x_k\} \subset X$ with

$$\lim_{k\to\infty} \|(a_i - \lambda_i)\, x_k\| = 0, \quad i = 1, \dots, n;$$

the Harte joint spectrum of a is the set

$$\sigma_H(a) = \tau(a) \cup \tau(a^*).$$

Note that for a commutative a, $\sigma_H(a) \subseteq \sigma(a)$; besides, $\lambda \in \tau(a)$ if and only if the operator α_{n-1} of the complex $\mathrm{Kos}\,(a - \lambda, X)$ is injective and has closed range, and $\lambda \in \tau\,(a^*)$ if and only if the operator α_0 of $\mathrm{Kos}\,(a - \lambda, X)$ is surjective. (See the first section for exact definitions of Koszul complex in a more general case).

No analytic functional calculus is known for the Harte spectrum but for a commutative a and a polynomial map p the spectral mapping theorem is valid:

$$\sigma_H\,(p(a)) = p\,(\sigma_H(a))\,. \tag{1}$$

This result was proved in [1], [6] and in Harte's paper [3] for another version of joint spectrum*).

In [4] Harte proved the spectral mapping theorem for operator families which he called quasi-commutative. A family a is quasi-commutative if and only if

$$[a_i, [a_j, a_k]] = 0, \quad i, j, k = 1, \dots, n$$

with $[u, v] = uv - vu$ for operators u, v. The equality (1) is proved in [4] for the version of Harte joined spectrum mentioned in the footnote and for polynomial maps with non-commutative variables.

It is clear that any quasi-commutative operator family generates nilpotent Lie algebra. A far generalization of Harte's result was proved by Ju. V. Turovskiĭ in [10]: equality (1) was proved for infinite families a generating nilpotent Lie algebras, for infinite families p of limits of rational maps and for all versions of the Harte spectrum.

In the paper [9], devoted to non-commutative functional calculus, J. L. Taylor in fact defined a joint spectrum of arbitrary operator family in terms of associated Lie algebra. We'll try to develop this approach here. We'll study the Taylor joint

*) λ belongs to this spectrum if and only if $a - \lambda$ generates a proper left or right ideal in the algebra of all bounded operators. For a Hilbert space X, this spectrum is equal to $\sigma_H(a)$.

spectrum $\sigma(a)$ under the assumption that the Lie algebra generated by a is a nilpotent one.

It is a consequence of [10] that in this case $\sigma_H(a) \subset \sigma(a)$. The main result of the paper is the spectral mapping theorem for finite families of non-commutative polynomials p:

$$\sigma(p(a)) = p(\sigma(a))$$

proved under the condition that the Lie algebra generated by $p(a)$ is finite-dimensional (and, as a consequence, it is nilpotent). Obviously, the concept of joint spectrum we'll study here has good sense only for operator families generating nilpotent Lie algebras.

The structure of the paper is the following.

In the first section we recall the construction of the Koszul complex for module over a Lie algebra and present homological facts, most of them known.

In the second section we present two definitions of joint spectrum and prove their equivalence for operator families generating nilpotent Lie algebras. The first one is more convenient to formulate and the second one is used in the proof of the main result.

In the third section we present a dual cohomological definition of the joint spectrum and prove its equivalence to the definitions of the second section, again in the nilpotent case. As a consequence, we prove the coincidence of $\sigma(a)$ and $\sigma(a^*)$.

In the fourth section we prove that if a generates nilpotent Lie algebra and $p(a)$ generates finite dimensional one, then $p(a)$ really generates a nilpotent Lie algebra.

In the fifth section the main result is proved.

The results of this paper were repeatedly discussed with Ju. V. Turovskiĭ at all stages of the work. In fact, the work was joint at an early stage. In particular, Ju. V. Turovskiĭ proved the projection property of joint spectrum (Consequence 5.5) and suggested Definition 2.1. The author is deeply grateful to Ju. V. Turovskiĭ for this help. The author wishes to thank V. P. Palamodov for the help in proving Proposition 2.5, and A. Ja. Khelemskiĭ, V. P. Palamodov, A. I. Shtern, V. S. Shul'man and D. P. Zhelobenko for useful discussions.

2. PRELIMINARIES

In this section we present some necessary facts from homological algebra.

Let X be a complex vector space, and let $\mathcal{L}(X)$ be the set of all linear operators on X. For a Banach space X, let $\mathcal{L}(X)$ be the set of all bounded operators. Let E be a complex Lie algebra, and let X be an E-module. This means, by definition, that a

Lie algebra homomorphism

$$\rho: E \to \mathcal{L}(X)$$

is given. For $u \in E$ and $x \in X$ we shall write ux instead of $\rho(u)x$ when ρ is obvious.

Denote by $\wedge E$ the exterior algebra generated by E, and by $\wedge^p E$ the p-th exterior space of E. An element

$$u_1 \wedge \cdots \wedge u_p \in \wedge^p E$$

will be often denoted by $\underline{u}$. Consider the chain Koszul complex Kos(E, ρ, X) (or simply Kos(E, X)) generated by the E-module X:

$$0 \leftarrow X \overset{\alpha}{\leftarrow} X \otimes E \overset{\alpha}{\leftarrow} \cdots \overset{\alpha}{\leftarrow} X \otimes \wedge^p E \overset{\alpha}{\leftarrow} \cdots$$

with $\alpha(x \otimes \underline{u}) = \sum_{i=1}^{p} (-1)^{i-1} \rho(u_i) x \otimes \overset{i}{\widehat{\underline{u}}} + \sum_{i<j} (-1)^{i+j-1} x \otimes [u_i, u_j] \wedge \overset{i,j}{\widehat{\underline{u}}}$ (cf. [9]; the notations $\overset{i}{\wedge}$ and $\overset{i,j}{\wedge}$ mean the omission of u_i and u_i, u_j respectively; we omit also indices in notations of boundary operators α).

The homology spaces of Kos(E, X) are denoted by $H_p(E, X)$, $p = 0, 1, \ldots$.

Let I be a Lie ideal of E. The space $X \otimes \wedge I$ is an E-module with respect to Lie algebra homomorphism $E \ni u \mapsto \Theta_u \in \mathcal{L}(X \otimes \wedge I)$:

$$\Theta_u(x \otimes \underline{v}) = \rho(u)x \otimes \underline{v} + \sum_{i=1}^{p} (-1)^{i-1} x \otimes [u, v_i] \wedge \overset{i}{\widehat{\underline{v}}}$$

with $x \in X$, $\underline{v} = v_1 \wedge \ldots \wedge v_p \in \wedge^p I$.

Obviously, X is an I-module, and the associated complex Kos(I, X) is subcomplex of Kos(E, X). Boundary operators of Kos(I, X) are also denoted by α.

The next lemma is proved by direct computation.

LEMMA 1.1. *The next diagram is commutative:*

$$\begin{array}{ccc} X \otimes \wedge^{p-1} I & \overset{\alpha}{\longleftarrow} & X \otimes \wedge^p I \\ \downarrow{\scriptstyle \Theta_u} & & \downarrow{\scriptstyle \Theta_u} \\ X \otimes \wedge^{p-1} I & \overset{\alpha}{\longleftarrow} & X \otimes \wedge^p I \end{array}$$

So the operators Θ_u define an endomorphism of the complex Kos(I, X). The relative operators on $H_p(I, X)$ are also denoted by Θ_u.

Let in particular $I = E$.

LEMMA 1.2. *Θ_u are zero operators on $H_p(E, X)$.*

Proof. Introduce the operators

$$i_u: X \otimes \wedge^p E \to X \otimes \wedge^{p+1} E, \quad i_u(x \otimes \underline{v}) = x \otimes u \wedge \underline{v}.$$

Direct computation proves that $i_u\alpha + \alpha i_u = \Theta_u$. Now let $\alpha h = 0$. Then $\Theta_u h = \alpha i_u h$. Lemma is proved.

REMARK 1.3. The operators Θ_u, i_u and lemmas 1.1, 1.2 are in fact known. They are homological analogues of the relative operators and their properties considered in [2] for cohomologies of Lie algebras.

The next lemma is a consequence of the main lemma of homological algebra on the exactness of the long sequence of homology spaces for short exact sequence of complexes (see [7] for the analogue of this lemma).

LEMMA 1.4. *Let*

$$0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$$

be an exact sequence of E-modules.

Then the exactness of any two of the complexes $\mathrm{Kos}\,(E, X), \mathrm{Kos}\,(E, Y)$ *and* $\mathrm{Kos}\,(E, Z)$ *implies the exactness of the third one.*

Let $(\mathcal{X}, \alpha)$:

$$0 \leftarrow X_0 \overset{\alpha}{\leftarrow} X_1 \overset{\alpha}{\leftarrow} \cdots \overset{\alpha}{\leftarrow} X_p \leftarrow \cdots$$

be a chain complex of vector spaces, and a sequence of operators $\beta \in \mathcal{L}(X_p)$, $p = 0, 1, \ldots$, that defines an endomorphism of $(\mathcal{X}, \alpha)$. The cone of β is the following complex $\mathrm{Con}\,((\mathcal{X}, \alpha), \beta)$:

$$0 \leftarrow X_0 \overset{\gamma}{\leftarrow} X_1 \oplus X_0 \overset{\gamma}{\leftarrow} \cdots \overset{\gamma}{\leftarrow} X_{p+1} \oplus X_p \overset{\gamma}{\leftarrow} \cdots$$

with $\gamma(x, y) = (\alpha x + \beta y, -\alpha x)$ for $(x, y) \in X_{p+1} \oplus X_p$.

The next lemma is a particular case of Lemma 5.2 below. We wish to formulate it here to show how the construction of the cone is associated with $\mathrm{Kos}\,(E, X)$.

LEMMA 1.5. *Let I be an ideal of Lie algebra E of codimension one, $u \in E \backslash I$ and X an E-module. Then*

$$\mathrm{Kos}\,(E, X) = \mathrm{Con}\,(\mathrm{Kos}\,(I, X), \Theta_u).$$

Lemma 1.5 may be used in the proof of the projection property of the joint spectrum of a family of operators in a Banach space generating a nilpotent Lie algebra. This is a particular case of the main result of this paper (see Consequence 5.5 below).

The following result of Z. Slodkowski [5] plays an important role in proving results of the spectral mapping theorem type. It is convenient for us to formulate it in terms of the cone of a continuous endomorphism β of a complex of Banach spaces $\mathcal{X}$. Note that for $\lambda \in \mathbb{C}$ it is natural to define the endomorphism $\beta - \lambda$ of $\mathcal{X}$.

LEMMA 1.6. [5] *If the complex of Banach spaces $\mathcal{X}$ is not exact and β is a continuous endomorphism then, for a certain $\lambda \in \mathbb{C}$, $\mathrm{Con}(\mathcal{X}, \beta - \lambda)$ is not exact.*

Consider now the notion of bicomplex and its totalization, and formulate the necessary statements.

Let X_{ij}, $i, j = 0, 1, \ldots$ be vector spaces and let $\alpha: X_{i+1,j} \to X_{i,j}$, $\beta: X_{i,j+1} \to X_{i,j}$ be linear operators such that the following diagram

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\beta} & & \\
0 & \longleftarrow & X_{0,j} & \xleftarrow{\alpha} & X_{1,j} & \xleftarrow{\alpha} & \cdots & \xleftarrow{\alpha} & X_{i,j} & \xleftarrow{\alpha} & \cdots \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\beta} & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\beta} & & \\
0 & \longleftarrow & X_{0,1} & \xleftarrow{\alpha} & X_{1,1} & \xleftarrow{\alpha} & \cdots & \xleftarrow{\alpha} & X_{i,1} & \xleftarrow{\alpha} & \cdots \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\beta} & & \\
0 & \longleftarrow & X_{0,0} & \xleftarrow{\alpha} & X_{1,0} & \xleftarrow{\alpha} & \cdots & \xleftarrow{\alpha} & X_{i,0} & \xleftarrow{\alpha} & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}$$

is commutative and all rows and columns are complexes. In this case the system of spaces $X_{i,j}$ and operators α, β is said to be a bicomplex. Denote this bicomplex by $\mathcal{X}$, its j-th row and i-th column by $(\mathcal{X}_j, \alpha)$ and $(\mathcal{X}_i, \beta)$ respectively. The homology spaces of rows $H_i(\mathcal{X}_j, \alpha)$ and operators generated by operators β form under fixed i the complex $(H_i(\mathcal{X}_j, \alpha), \beta)$ and analogously homology spaces of columns and operators generated by α form the complex $(H_j(\mathcal{X}_i, \beta), \alpha)$ under a fixed j. Denote by $H_j(H_i(\mathcal{X}_j, \alpha), \beta)$ and $H_i(H_j(\mathcal{X}_i, \beta), \alpha)$ the homology spaces of these complexes.

The totalization of the bicomplex $\mathcal{X}$ is the following complex $\mathrm{Tot}(\mathcal{X})$:

$$0 \leftarrow T_0 \xleftarrow{\gamma} T_1 \xleftarrow{\gamma} \cdots \xleftarrow{\gamma} T_p \xleftarrow{\gamma} \cdots$$

with $T_p = \underset{i+j=p}{\otimes} X_{i,j}$, $\gamma x = \alpha x + (-1)^i \beta x$ for $x \in X_{i,j}$.

It is easy to see that the cone of an endomorphism of a complex is a particular case of totalization.

The following lemma is well known.

LEMMA 1.7. *If all homology spaces $H_j(H_i(\mathcal{X}, \alpha), \beta)$ or all $H_i(H_j(\mathcal{X}, \beta), \alpha)$ are trivial then the complex $\mathrm{Tot}(\mathcal{X})$ is exact.*

The condition of the next lemma is hold for bicomplexes appearing in this paper.

LEMMA 1.8. *Let the triviality of* $H_i(\mathcal{X}_0,\alpha)$ *implies the triviality of all* $H_i(\mathcal{X}_j,\alpha)$, $j = 1,2,\ldots$. *If the row* $(\mathcal{X}_0,\alpha)$ *is not exact and the operators* $\beta\colon H_i(\mathcal{X}_1,\alpha) \to H_i(\mathcal{X}_0,\alpha)$ *are trivial then at least one column of the bicomplex is not exact.*

Proof. Let $H_k(\mathcal{X}_0,\alpha) \neq 0$ and $H_i(\mathcal{X}_0,\alpha) = 0$ for $i < k$. Let all columns of the bicomplex be exact. Denote by Ker and Im respectively kernel and image of a linear operator. For $x \in X_{k,0}$, $x \in \operatorname{Ker}\alpha\backslash\operatorname{Im}\alpha$ we construct, using the exactness of columns, the sequence of elements

$$x_{k-j,j+1} \in X_{k-1,j+1}, \quad j = 0,1,\ldots,k$$

for which the following equalities are valid:

$$x = \beta x_{k,1};$$

$$\alpha x_{k,1} = \beta x_{k-1,2};$$

$$\cdots$$

$$\alpha x_{1,k} = \beta x_{0,k+1}.$$

The triviality of $H_i(\mathcal{X}_0,\alpha)$ implies the triviality of $H_i(\mathcal{X}_j,\alpha)$ hence we construct the sequence

$$y_{k-1+j,j+1} \in X_{k-1+j,j+1}, \quad j = k, k-1,\ldots,1$$

for which the following equalities are valid:

$$x_{0,k+1} = \alpha y_{1,k+1};$$

$$x_{1,k} - \beta y_{1,k+1} = \alpha y_{2,k};$$

$$\cdots$$

$$x_{k-1,2} - \beta y_{k-1,3} = \alpha y_{k,2};$$

$$\alpha(x_{k,1} - \beta y_{k,2}) = 0.$$

(It is clear how to find $y_{1,k+1}$; after that:

$$\alpha(x_{1,k} - \beta y_{1,k+1}) = \alpha x_{1,k} - \alpha\beta y_{1,k+1} = \beta x_{0,k+1} - \beta\alpha y_{1,k+1} =$$

$$= \beta x_{0,k+1} - \beta x_{0,k+1} = 0 \Rightarrow x_{1,k} - \beta y_{1,k+1} = \alpha y_{2,k}.)$$

Now use the triviality of the operator $\beta\colon H_k(\mathcal{X}_1,\alpha) \to H_k(\mathcal{X}_0,\alpha)$. We have

$$x = \beta x_{k,1} = \beta(x_{k,1} - \beta y_{k,2}) \in \operatorname{Im}\alpha$$

because of $x_{k,1} - \beta y_{k,2} \in \operatorname{Ker} \alpha$. This contradiction shows that the columns of the bicomplex are not exact. The Lemma is proved.

2. THE JOINT SPECTRUM OF OPERATOR FAMILIES

In this section we give two definitions of the joint spectrum of finite operator families and prove their equivalence in the case when the Lie algebra generated by the operator family is nilpotent.

Let $a = (a_1, \ldots, a_n)$ be an operator family on the vector space X, $\lambda = (\lambda_1, \ldots$ $\ldots, \lambda_n) \in \mathbb{C}^n$, $a - \lambda = (a_1 - \lambda_1, \ldots, a_n - \lambda_n)$, $E(a)$ be the Lie algebra generated by a in the algebra $\mathcal{L}(X)$. We shall use a notation $\operatorname{Kos}(a, X)$ for the Koszul complex generated by the $E(a)$-module X. The following definition was suggested by Ju. V. Turovskiĭ.

DEFINITION 2.1. The joint spectrum of a is the set $\sigma(a)$ of those $\lambda \in \mathbb{C}^n$ for which the complex $\operatorname{Kos}(a - \lambda, X)$ is not exact.

Now let E be the complex Lie algebra with generators $e_1, \ldots, e_n$ and $\rho: E \to \mathcal{L}(X)$ the Lie algebra homomorphism with $\rho(e_i) = a_i$, $i = 1, \ldots n$. Obviously ρ defines the E-module structure on X.

If $\lambda \in \mathbb{C}^n$ is such n-tuple of complex numbers that the map $e_i \mapsto \lambda_i$, $i = 1, \ldots, n$ extends to the Lie algebra homomorphism $E \to \mathbb{C}$ then we say λ is admissible and consider the Lie algebra homomorphism $\rho_\lambda: E \to \mathcal{L}(X)$ with $\rho_\lambda(e_i) = a_i - \lambda_i$, $i = 1, \ldots, n$.

DEFINITION 2.2. The E-joint spectrum of the operator family $a = (a_1, \ldots, a_n)$ is the set $\sigma_E(a)$ of all admissible $\lambda \in \mathbb{C}^n$ for which the complex $\operatorname{Kos}(E, \rho_\lambda, X)$ is not exact.

The main result of this section is the following.

THEOREM 2.3. *If the Lie algebra E is nilpotent then $\sigma(a) = \sigma_E(a)$.*

REMARKS 2.4. 1°. E is finite dimensional as nilpotent Lie algebra with finite number of generators. On the other hand the Theorem 2.3 will be proved as a consequence of more general results on infinite dimensional nilpotent Lie algebras.

2°. Definition 2.2 is more convenient than definition 2.1 because for given a it is easier to take E than to describe $E(a)$. For example let $a = (a_1, a_2)$ be such pare of operators that $[a_1, [a_1, a_2]] = [a_2, [a_1, a_2]] = 0$. Then we can take E to be the Lie algebra with the basis e_1, e_2, f and equalities $[e_1, e_2] = f$, $[e_1, f] = [e_2, f] = 0$. The homomorphism ρ maps e_i into a_i, $i = 1, 2$. On the other hand $E(a)$ depends of the concrete properties of a: it can have any dimension from 1 to 3. Note that $E(a)$ and $E(a-\lambda)$ may be different Lie algebras while E is the same one for all admisssible λ.

Let E and F be Lie algebras, let $h: F \to E$ be a Lie algebra epimorphism, let X be an E-module; hence X is an F-module. Denote $G = \operatorname{Ker} h$.

PROPOSITION 2.5. *Let* $[G, F] = 0$ *.Then the complexes* $\operatorname{Kos}(E, X)$ *and* $\operatorname{Kos}(F, X)$ *are simultaneously exact.*

Proof. Denote by $\wedge^q G \wedge \wedge^p F$ the subspace $\wedge^{p+q} F$ generated by $g_1 \wedge \cdots \wedge g_q \wedge \wedge f_1 \wedge \cdots \wedge f_p$ with $g_1, \ldots, g_q \in G$. It is easy to check that the following sequence

$$0 \leftarrow \wedge^{q-1} G \otimes \wedge^p E \overset{\tilde{h}}{\leftarrow} \wedge^{q-1} G \wedge \wedge^p F \overset{i}{\leftarrow} \wedge^q G \wedge \wedge^{p-1} F \leftarrow 0 \tag{2.1}$$

with inclusion i and

$$\tilde{h}\left(\underline{g} \otimes f_1 \wedge \cdots \wedge f_p\right) = \underline{g} \otimes h(f_1) \wedge \cdots \wedge h(f_p)$$

is exact.

Consider the following diagram

$$\begin{array}{ccccccc}
0 \longleftarrow & X \otimes \wedge^{q-1} G \otimes \wedge^{p+1} E & \overset{\tilde{h}}{\longleftarrow} & X \otimes \wedge^{q-1} G \wedge \wedge^{p+1} F & \longleftarrow & X \otimes \wedge^q G \wedge \wedge^p F & \longleftarrow 0 \\
& \Big\downarrow{\scriptstyle \alpha \otimes 1} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha} & \\
0 \longleftarrow & X \otimes \wedge^{q-1} G \otimes \wedge^p E & \overset{\tilde{h}}{\longleftarrow} & X \otimes \wedge^{q-1} G \wedge \wedge^p F & \longleftarrow & X \otimes \wedge^q G \wedge \wedge^{p-1} F & \longleftarrow 0
\end{array}$$

Here the rows are tensor products of (2.1) and the identity on X. The first column is the tensor product of $\alpha: X \otimes \wedge^{p+q} E \to X \otimes \wedge^{p+q-1} E$ and the identity on $\wedge^{q-1} G$, and the second and the third one are the restrictions of $\alpha: X \otimes \wedge^{p+q} F \to X \otimes \wedge^{p+q-1} F$. (The conditions $GX = 0$ and $[G, F] = 0$ imply the correctness of this construction).

The commutativity of this diagram is also a consequence of the conditions $GX = = 0$ and $[G, F] = 0$.

Continue the columns at two sides and get the following short sequence of complexes:

$$0 \leftarrow \wedge^{q-1} G \otimes \operatorname{Kos}(E, X) \leftarrow \wedge^{q-1} G \wedge \operatorname{Kos}(F, X) \leftarrow \wedge^q G \wedge \operatorname{Kos}(F, X) \leftarrow 0.$$

Consider the associated long sequence of homology spaces for every $q \geqslant 1$. Denote for simplicity the homology spaces of complexes $\operatorname{Kos}(E, X)$, $\wedge^q G \wedge \operatorname{Kos}(F, X)$ by H_i, H_i^q $(q \geqslant 0)$ respectively.

It is clear that

$$H_i\left(\wedge^q G \otimes \operatorname{Kos}(E, X)\right) = H_i \otimes \wedge^q G.$$

We have the following sequences of homology spaces:

$$0 \leftarrow H_0 \leftarrow H_0^0 \leftarrow 0 \leftarrow H_1 \leftarrow H_1^0 \leftarrow H_0^1 \leftarrow H_2 \leftarrow H_2^0 \leftarrow H_1^1 \leftarrow H_3 \leftarrow \cdots;$$

$$0 \leftarrow H_0 \otimes G \leftarrow H_0^1 \leftarrow 0 \leftarrow H_1 \otimes G \leftarrow H_1^1 \leftarrow H_0^2 \leftarrow H_2 \otimes G \leftarrow H_2^1 \leftarrow H_1^2 \leftarrow$$
$$\leftarrow H_3 \otimes G \leftarrow \cdots;$$

$$\cdots$$

$$0 \leftarrow H_0 \otimes \Lambda^q G \leftarrow H_0^q \leftarrow 0 \leftarrow H_1 \otimes \Lambda^q G \leftarrow H_1^q \leftarrow H_0^{q+1} \leftarrow H_2 \otimes \Lambda^q G \leftarrow H_2^q \leftarrow H_1^{q+1} \leftarrow$$
$$\leftarrow H_3 \otimes \Lambda^q G \leftarrow \cdots;$$

$$\cdots$$

We have to show that $H_i = 0$, $i = 0, 1, \ldots$ implies $H_i^0 = 0$, $i = 0, 1, \ldots$ and viceversa. Let $H_i = 0$, $i = 0, 1, \ldots$. Then $H_0^q = 0$ and $H_i^q = H_{i-1}^q$ with $q = 0, 1, \ldots$, $i = 1, 2, \ldots$. By an obvious induction we get $H_i^0 = 0$.

Now, let $H_i^0 = 0$, $i = 0, 1, \ldots$. Then from the first sequence of homology spaces we have $H_0 = H_1 = 0$ and $H_i = H_{i-2}^1$, $i \geqslant 2$, and further from every sequence we get $H_0^q = 0$, $q \geqslant 1$. Hence $H_2 = 0$. Further $H_1^q = H_0^{q+1} = 0$, $q \geqslant 1$ implies $H_3 = 0$. Again by induction we get $H_i = 0$, $i = 0, 1, \ldots$. The proposition is proved.

PROPOSITION 2.6. *If the Lie algebra F is nilpotent then the complexes* Kos(E, X) *and* Kos(F, X) *are simultaneously exact.*

Proof. Consider the sequence of ideals of the Lie algebra F:

$$[F, G] = G_1,\ [F, G_1] = G_2, \ldots, [F, G_{n-1}] = G_n.$$

Using the identities $F/G = E$ and $(F/G_k)/(G_{k-1}/G_k) = F/G_{k-1}$ with $k \geqslant 1$, $G_0 = G$, consider the following exact sequences of Lie algebras:

$$0 \to G/G_1 \to F/G_1 \to E \to 0,$$

$$0 \to G_1/G_2 \to F/G_2 \to F/G_1 \to 0,$$

$$\cdots$$

$$0 \to G_{n-1}/G_n \to F/G_n \to F/G_{n-1} \to 0,$$

$$0 \to G_n \to F \to F/G_n \to 0.$$

Since X is an E-module, X becomes a F/G_k-module and because of $[F/G_k, G_{k-1}/G_k] = 0$ for every epimorphism $F/G_k \to F/G_{k-1}$ the assumption of Proposition 2.5 is fulfilled.

Hence the complexes

$$\mathrm{Kos}\,(E,X),\ \mathrm{Kos}\,(F/G_1,X)\,,\ldots,\mathrm{Kos}\,(F/G_n,X)\,,\ \mathrm{Kos}\,(F,X)$$

are simultaneously exact.

REMARK 2.7. The analogue of Propositions 2.5 and 2.6 may be proved under the assumptions $F = G \oplus E$, $[G, E] = 0$, and h is the projection of F onto E.

LEMMA 2.8. *Let E be Lie algebra with generators $e_1, \ldots, e_n$, and let $\rho: E \to \mathcal{L}(X)$ be a Lie algebra homomorphism with $\rho(e_i) = a_i$, $i = 1, \ldots, n$. If $\lambda \in \sigma(a)$, then λ is admissible for ρ.*

Proof. We shall show that if λ is not admissible for ρ then the Lie algebra $E(a-\lambda)$ contains the identity operator; hence, according to Lemma 1.2, $H_p\,(E(a-\lambda), X) = 0$ i.e. $\lambda \notin \sigma(a)$. Consider the factor space $E/[E,E]$ and let $\tilde{e}_1, \ldots, \tilde{e}_n$ be images of $e_1, \ldots, e_n$ under the natural projection $E \to E/[E,E]$. The vectors $\tilde{e}_1, \ldots, \tilde{e}_n$ may be linearly dependent. It is clear that λ is admissible if and only if the equality $\sum_{i=1}^{n} \alpha_i \tilde{e}_i = 0$ implies $\sum_{i=1}^{n} \alpha_i \lambda_i = 0$. Let λ be not admissible i.e. for certain $\alpha_1, \ldots, \alpha_n$

$$\sum_{i=1}^{n} \alpha_i \tilde{e}_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i \lambda_i = 1.$$

The first equality implies $\sum_{i=1}^{n} \alpha_i e_i = c \in [E, E]$. Hence

$$\sum_{i=1}^{n} \alpha_i a_i = \rho(c) \in [E(a), E(a)] = [E(a-\lambda), E(a-\lambda)].$$

So it is clear that

$$I = \rho(c) - \sum_{i=1}^{n} \alpha_i\,(a_i - \lambda_i) \in E(a-\lambda).$$

The lemma is proved.

The proof of Theorem 2.3 is a consequence of Proposition 2.6 and Lemma 2.8: if $\lambda \in \sigma_E(a)$ apply Proposition 2.6 to the epimorphism $\rho_\lambda: E \to E(a-\lambda)$; if $\lambda \in \sigma(a)$ then λ is admissible by Lemma 2.8 and again apply Proposition 2.6 to $\rho(\lambda)$.

REMARKS. 2.9. 1°. Theorem 2.3 is false without the assumption that E is a nilpotent Lie algebra. Let E be a Lie algebra with the generators e_1, e_2 and equality $[e_1, e_2] = e_2$, and let $\rho: E \to \mathcal{L}(X)$ be a Lie algebra homomorphism with $\rho(e_1) = a$, $\rho(e_2) = 0$. The associated Koszul complex $\mathrm{Kos}\,(E, X)$ is the following:

$$0 \leftarrow X \overset{\alpha_0}{\leftarrow} X \oplus X \overset{\alpha_1}{\leftarrow} X \leftarrow 0,$$

with $\alpha_0(x_1, x_2) = ax_1$, $\alpha_1 x = (0, (a+1)x)$. The complex $\operatorname{Kos}(a, X)$ is the following:

$$0 \leftarrow X \overset{a}{\leftarrow} X \leftarrow 0.$$

It is clear that the invertibility of a is not equivalent to the exactness of $\operatorname{Kos}(E, X)$. Hence the sets $\sigma(a, 0)$ and $\sigma_E(a, 0)$ do not coincide in general.

2°. It is easy to prove that in the case of a nilpotent Lie algebra $E(a)$, the spectrum $\sigma(a)$ includes the Harte spectrum $\sigma_H(a)$. Indeed, let $N = \dim E(a)$, and consider the boundary operator α_{N-1} of $\operatorname{Kos}(a, X)$. There exists a basis $u_1, \ldots, u_N$ of $E(a)$ such that $[u_i, u_{i+k}]$ belongs to the linear subspace generated by $u_{i+k+1}, \ldots, u_N$. We have $[u_i, u_j] \wedge u_1 \wedge \overset{i,j}{\widehat{\ldots}} \wedge u_N = 0$, and hence

$$\alpha_{N-1}(x \otimes u_1 \wedge \cdots \wedge u_N) = \sum_{i=1}^{N} (-1)^{i-1} u_i x \otimes u_1 \overset{i}{\widehat{\cdots}} \wedge u_N.$$

Thus $0 \in \tau(u_1, \ldots, u_N)$ if and only if the operator α_{N-1} is not a topological inclusion. It is clear that $0 \in \tau(a)$ if and only if $0 \in \tau(u_1, \ldots, u_N)$. Hence $\tau(a) \subseteq \sigma(a)$. It is proved analogously that $0 \in \tau(a^*)$ if and only if the boundary operator α_0 of $\operatorname{Kos}(a, X)$ is not surjective. Hence $\tau(a^*) \subseteq \sigma(a)$.

3. THE DUAL DEFINITION OF THE JOINT SPECTRUM

It is known [2] that the cochain complex and cohomology spaces are associated with a module over a Lie algebra. Hence we get a dual notion of joint spectrum. We'll show in this section that for a nilpotent Lie algebra the dual definition of the joint spectrum is equivalent to the original one. As a corollary, we prove that the joint spectrum of the operator family generating a nilpotent Lie algebra is equal to the joint spectrum of the adjoint operator family.

Let X be a vector space, let E be a Lie algebra and let $\rho: E \to \mathcal{L}(X)$ be Lie algebra homomorphism defining the structure of an E-module on X. Denote $C^p(E, X) = \mathcal{L}(\wedge^p E, X)$ and consider the operators

$$\rho: C^p(E, X) \to C^{p+1}(E, X),$$

$$(\delta f)(\underline{u}) = \sum_{i=1}^{p+1} (-1)^{i-1} \rho(u_i) f\left(\overset{i}{\widehat{\underline{u}}}\right) + \sum_{i<j} (-1)^{i+j} f\left([u_i, u_j] \wedge \overset{i,j}{\widehat{\underline{u}}}\right)$$

with $f \in C^p(E, X)$, $\underline{u} \in \wedge^p E$.

The spaces $C^p(E, X)$ and the operators δ form a coachain complex $\mathcal{C}(E, X)$.

PROPOSITION 3.1. *Let E be a finite-dimensional nilpotent Lie algebra. Then the complexes $\mathcal{C}(E,X)$ and $\operatorname{Kos}(E,X)$ are isomorphic.*

Proof. Let $e_1,\dots,e_n$ be a basis of E. Let E' be the dual space of E and let $e^1,\dots,e^n$ be the basis of E', dual to $e_1,\dots,e_n$. Denote by ∇ the operator from E' into $E'\wedge E'$ dual to the Lie brackets considered as operator from $E\wedge E$ into E. Obviously

$$\nabla e^k = \sum_{i<j} C^k_{i,j} e^i \wedge e^j$$

where $C^k_{i,j}$ are the structure constants of E. Further the complex $\mathcal{C}(E,X)$ is isomorphic to the complex $\Gamma(E,X)$ of spaces $X\otimes \wedge^p E'$ and coboundary operators

$$\gamma\colon X\otimes \wedge^p E' \to X\otimes \wedge^{p+1}E',$$

$$\gamma(x\otimes \overline{u}) = \sum_{i=1}^{n} \rho(e_i)\,x\otimes e_i\wedge \overline{u} + \sum_{\nu=1}^{p}(-1)^\nu x\otimes \nabla u^\nu \wedge \hat{\overline{u}}^{\nu}$$

with $\overline{u} = u^1\wedge\cdots\wedge u^p \in \wedge^p E'$. The isomorphism of complexes is realized by the following operators:

$$R\colon X\otimes \wedge^p E' \to C^p(E,X),\ R(x\otimes\varphi)(u) = \varphi(u)x$$

with $x\in X$, $u\in \wedge^p E$, $\varphi\in \wedge^p E' = (\wedge^p E)'$.

We have to show that $R\gamma = \delta R$. It is convenient here and below to represent the operators α,δ,γ as sums $\alpha_1+\alpha_2$, $\delta_1+\delta_2$, $\gamma_1+\gamma_2$ according to their definitions. We prove that $R\gamma_1 = \delta_1 R$.

Indeed,

$$(R\gamma_1)(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p}) = R\left(\sum_{i=1}^{n} \rho(e_i)\,x\otimes e^i\wedge e^{j_1}\wedge\cdots\wedge e^{j_p}\right)(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}).$$

The right part of this equality is equal to 0 if $\{j_1,\dots,j_p\}$ is not a subset of $\{k_1,\dots,k_{p+1}\}$, and is equal to $(-1)^{t-1}\rho(e_{k_t})x$ if $k_1=j_1,\dots,k_{t-1}=j_{t-1}$, $k_{t+1} = j_t,\dots,k_{p+1}=j_p$. Further

$$(\delta_1 R)(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p})\,(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}) =$$

$$= R(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p})\left(\sum_{i=1}^{p+1}(-1)^{i-1} e_{k_1}\wedge\overset{i}{\widehat{\cdots}}\wedge e_{k_{p+1}}\right)$$

is equal to the same thing.

We prove now that $R\gamma_2 = \delta_2 R$. Really,

$$(R\gamma_2)(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p})\,(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}) =$$

$$= R\left(\sum_{\nu=1}^{p}(-1)^{\nu}x\otimes\sum_{i<j}C_{i,j}^{j_\nu}e^{i}\wedge e^{j}\wedge e^{j_1}\wedge\overset{\nu}{\widehat{\cdots}}\wedge e^{j_p}\right)\left(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}\right).$$

The right part of this equality is equal to 0 if the set $\{j_1,\ldots,j_p\}$ contains more than one element which does not belong to $\{k_1,\ldots,k_{p+1}\}$. If there is precisely one element j_ν with this property, let k_s and k_t with $s<t$ which do not belong to $\{j_1,\ldots,j_p\}$. Then we get

$$(-1)^{\nu}C_{k_s,k_t}^{j_\nu}\left(e^{k_s}\wedge e^{k_t}\wedge e^{j_1}\wedge\overset{\nu}{\widehat{\cdots}}\wedge e^{j_p}\right)\left(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}\right)x.$$

If $\{j_1,\ldots,j_p\}\subset\{k_1,\ldots,k_{p+1}\}$ let $k_s\notin\{j_1,\ldots,j_p\}$ and we get

$$\sum_{\nu=1}^{p}(-1)^{\nu}C_{j_\nu,k_s}^{j_\nu}\left(e^{j_\nu}\wedge e^{k_s}\wedge\cdots\wedge e^{j_1}\right)\left(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}\right)x.$$

Now consider

$$(\delta_2 R)\left(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p}\right)\left(e_{k_1}\wedge\cdots\wedge e_{k_{p+1}}\right)=$$

$$= R(x\otimes e^{j_1}\wedge\cdots\wedge e^{j_p})\left(\sum_{i<j}(-1)^{i+j}\left[e_{k_i},e_{k_j}\right]\wedge e_{k_1}\overset{i,j}{\widehat{\cdots}}\wedge e_{k_{p+1}}\right). \tag{3.2}$$

We discuss two cases analoguously and prove the coincidence of the right parts of (3.1) and (3.2).

Now let E be a nilpotent Lie algebra. We prove the isomorphism of the complexes $\operatorname{Kos}(E,X)$ and $\Gamma(E,X)$. Since E is nilpotent, the basis $e_1,\ldots,e_n$ may be chosen so that $C_{i,j}^{k}=0$ for $k\leqslant\max\{i,j\}$. The following isomorphism of complexes $\operatorname{Kos}(E,X)$ and $\Gamma(E,X)$ is used in the commutative case in [8]:

$$\varphi\colon X\otimes\wedge^{p}E\to X\otimes\wedge^{n-p}E',$$

$$\varphi\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=(-1)^{j_1+\cdots+j_p+p}x\otimes e^{j_1'}\wedge\cdots\wedge e^{j_{n-p}'}$$

with $\{j_1',\ldots,j_{n-p}'\}$ the tuple of indices complementing $\{j_1,\ldots,j_p\}$ in $\{1,\ldots,n\}$. We have to show that $\varphi\alpha_1=\gamma_1\varphi$.

Note that

$$(\varphi\alpha_1)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=\varphi\left(\sum_{\nu=1}^{p}(-1)^{\nu-1}\rho(e_{j_\nu})x\otimes e_{j_1}\wedge\overset{\nu}{\widehat{\cdots}}\wedge e_{j_p}\right)=$$

$$=\sum_{\nu=1}^{p}(-1)^{\nu-1+j_1+\overset{\nu}{\widehat{\cdots}}+j_p+p-1}\rho(e_{j_\nu})x\otimes e^{j_1'}\wedge\cdots\wedge e^{j_\nu}\wedge\cdots\wedge e^{j_{n-p}'};$$

$$(\gamma_1\varphi)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=\gamma_1\left((-1)^{j_1+\cdots+j_p+p}x\otimes e^{j'_1}\wedge\cdots\wedge e^{j'_{n-p}}\right)=$$

$$=(-1)^{j_1+\cdots+j_p+p}\sum_{s=1}^{n}\rho(e_s)\,x\otimes e^s\wedge e^{j'_1}\wedge\cdots\wedge e^{j'_{n-p}}.$$

In fact in the last sum the index $s\in\{j_1,\dots,j_p\}$. Hence it is sufficient to verify that

$$(-1)^{\nu+j_1+\overset{\nu}{\widehat{\cdots}}+j_p+p}e^{j'_1}\wedge\cdots\wedge e^{j_\nu}\wedge\cdots\wedge e^{j'_{n-p}}=(-1)^{j_1+\cdots+j_p+p}e^{j_\nu}\wedge e^{j'_1}\wedge\cdots\wedge e^{j'_{n-p}}.$$

Indeed, let $j'_t<j_\nu<j'_{t+1}$. Then it is clear that $j_\nu=t+\nu$ and the equality is verified.

We show now that $\varphi\alpha_2=\gamma_2\varphi$.

$$(\varphi\alpha_2)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=\varphi\left(\sum_{k<m}(-1)^{k+m-1}x\otimes[e_{j_k},e_{j_m}]\wedge e_{j_1}\wedge\overset{k,m}{\widehat{\cdots}}\wedge e_{j_p}\right)=$$

$$=\varphi\left(\sum_{k<m}(-1)^{k+m-1}\sum_{\mu>j_m}C^{\mu}_{j_k,j_m}x\otimes e_\mu\wedge e_{j_1}\wedge\overset{k,m}{\widehat{\cdots}}\wedge e_{j_p}\right).$$

In fact in the last sum $\mu\in\{j'_\nu : j'_\nu>j_m\}$.

Let $j_s<j'_\nu<j_{s+1}$. Then

$$(\varphi\alpha_2)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=\sum_{k<m}\sum_{j'_\nu>j_m}(-1)^{k+m-1+s+j_1+\overset{k,m}{\widehat{\cdots}}+j_p+j'_\nu+p-1}\cdot$$

$$\cdot C^{j'_\nu}_{j_k,j_m}x\otimes e^{j'_1}\wedge\cdots\wedge e^{j_k}\wedge\cdots\wedge e^{j_m}\wedge\cdots\wedge e^{j'_{n-p}}.$$

Further

$$(\gamma_2\varphi)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=\gamma_2\left((-1)^{j_1+\cdots+j_p+p}x\otimes e^{j'_1}\wedge\cdots\wedge e^{j'_{n-p}}\right)=$$

$$=\sum_{\nu=1}^{n-p}(-1)^{\nu+j_1+\cdots+j_p+p}\sum_{j_k<j_m<j'_\nu}C^{j'_\nu}_{j_k,j_m}x\otimes e^{j_k}\wedge e^{j_m}\wedge e^{j'_1}\wedge\overset{\nu}{\widehat{\cdots}}\wedge e^{j'_{n-p}}.$$

Let $j'_t<j_k<j'_{t+1}$, $j'_r<j_m<j'_{r+1}$. Then

$$(\gamma_2\varphi)\left(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_p}\right)=$$

$$=\sum_{\nu=1}^{n-p}\sum_{j_k<j_m<j'_\nu}(-1)^{\nu+j_1+\cdots+j_p+p+t+r}C^{j'_\nu}_{j_k,j_m}x\otimes e^{j'_1}\wedge\cdots\wedge e^{j_k}\wedge\cdots\wedge e^{j_m}\wedge\overset{\nu}{\widehat{\cdots}}\wedge e^{j'_{n-p}}.$$

It remains only to compare the signs at every summand. It is sufficient to notice that $j'_\nu=\nu+s$, $j_k=k+t$, $j_m=m+r$. The proof is finished.

REMARK 3.2. Proposition 3.1 is false without the assumption that E is a nilpotent Lie algebra. For example, let E be Lie algebra with basis e_1, e_2 and equality $[e_1, e_2] = e_2$, X an E-module, and $\rho(e_i) = a_i$, $i = 1, 2$. It is easy to verify that the complexes $\operatorname{Kos}(E, X)$ and $\Gamma(E, X)$ (which is isomorphic to $\mathcal{C}(E, X)$) have the following structure:

$$0 \leftarrow X \xleftarrow{(a_1, a_2)} X \oplus X \xleftarrow{\binom{-a_2}{a_1+1}} X \leftarrow 0,$$

$$0 \leftarrow X \xleftarrow{(-a_2, a_1-1)} X \oplus X \xleftarrow{\binom{a_1}{a_2}} X \leftarrow 0.$$

It is clear that the exactness of one complex does not imply the exactness of the other one.

Now we can prove the statement on the spectrum of adjoint operator family.

PROPOSITION 3.3. *Let $a = (a_1, \ldots, a_n)$ be an operator family on the Banach space X generating a nilpotent Lie algebra. Then $\sigma(a) = \sigma(a^*)$, where $a^* = (a_1^*, \ldots, a_n^*)$ is the adjoint operator family.*

Proof. Let E be the Lie algebra generated by a. Consider the complex $\left(Kos(E, X)\right)^*$ adjoint to $\operatorname{Kos}(E, X)$. There is a natural isomorphism between $(\operatorname{Kos}(E, X))^*$ and $\mathcal{C}(E, X^*)$. The exactness of a complex of Banach spaces is equivalent to the exactness of its adjoint [5]. Hence the exactness of $\operatorname{Kos}(E, X)$ is equivalent to the exactness of $\mathcal{C}(E, X^*)$. According to Proposition 3.1, the exactness of $\mathcal{C}(E, X^*)$ is equivalent to the exactness of $\operatorname{Kos}(E, X^*)$.

4. ON LIE ALGEBRAS GENERATED BY OPERATOR POLYNOMIALS

Let $a = (a_1, \ldots, a_n)$ be an operator family in a vector space X generating a nilpotent Lie algebra E. Let $p(a) = (p_1(a), \ldots, p_m(a))$ be a family of polynomials of a generating a finite-dimensional Lie algebra $F \subset \mathcal{L}(X)$. Our aim is to show that in fact F is a nilpotent Lie algebra.

Denote by $U(E)$ the universal enveloping algebra of E. It is clear that F is the image of a certain Lie subalgebra of $U(E)$. Hence F is isomorphic to a certain Lie factor-algebra of a Lie subalgebra of $U(E)$. Since F is finite-dimensional, its nilpotence is a consequence of the following proposition.

Denote $A_1 = U(E)$, $A_{k+1} = [A_1, A_k]$, $k \geqslant 1$.

PROPOSITION 4.1. *Let E be a nilpotent Lie algebra. Then*

$$\bigcap_{k=1}^{\infty} A_k = \{0\}.$$

Proof. Let $e_1, \ldots, e_n$ be such a basis of E with the associated structure constants $C_{ij}^k = 0$ for $k \leqslant \max\{i,j\}$. We introduce the notion of order for a basis element of $U(E)$. Let $u = e_1^{k_1} \cdots e_n^{k_n}$. The order of U is defined as

$$\operatorname{ord} u = \sum_{i=1}^{n} k_i 2^{i-1}.$$

According to Poincarè-Birkhoff-Witt theorem, the elements $e_1^{k_1} \ldots e_n^{k_n}$ form a basis of $U(E)$. Denote by J_k the linear subspaces of $U(E)$ generated by the basis elements of order $\geqslant k$. (It will be clear that J_k are ideals of $U(E)$. Obviously, $\bigcap_{k=1}^{\infty} J_k = \{0\}$.) We show that

$$J_k \cdot J_m \subseteq J_{k+m}.$$

Indeed, consider the product of two basis elements $u \cdot v$ with

$$u = e_1^{k_1} \cdots e_n^{k_n}, \quad v = e_1^{m_1} \cdots e_n^{m_n}.$$

Then

$$u \cdot v = e_1^{k_1+m_1} \cdots e_n^{k_n+m_n} + w$$

where w is obtained from a series of transformations of the products $e_i e_j$ in $e_1^{k_1} \ldots e_n^{k_n} \cdot e_1^{m_1} \ldots e_n^{m_n}$ with $i > j$ into

$$e_j e_i + \sum_{k=i+1}^{n} C_{i,j}^k e_k.$$

It is clear that after every such transformation the number

$$\sum_{i=1}^{n} k_i' 2^{i-1}$$

becomes larger for every monomial summand of uv, where k_i' is the number of occurences of e_i in the monomial. So

$$\operatorname{ord}\left(e_1^{k_1+m_1} \cdots e_n^{k_n+m_n}\right) = \operatorname{ord} u + \operatorname{ord} v$$

and w is decomposed into basis elements of order larger then $\operatorname{ord} u + \operatorname{ord} v$. Hence $A_k \subset J_k$ and $\bigcap_{k=1}^{\infty} A_k \subset \bigcap_{k=1}^{\infty} J_k = \{0\}$.

The proposition is proved.

Consequence 4.2. Let $a = (a_1, \ldots a_n)$ be an operator family on a vector space X generating a nilpotent Lie algebra, and let $p(a) = (p_1(a), \ldots, p_m(a))$ be a family of polynomials of a. If $p(a)$ generates a finite-dimensional Lie algebra, then this Lie algebra is nilpotent.

We give here a simple condition for the nilpotence of Lie algebra generated by operator polynomials.

PROPOSITION 4.3. *Let the operator family $a = (a_1, \dots, a_n)$ generate a nilpotent Lie algebra, and assume that the center of this Lie algebra consists of nilpotent operators. Then every family of polynomials of a generates a nilpotent Lie algebra.*

Proof. Since the center of $E(a)$ consists of nilpotent operators, we have $b_1 \cdots b_N = = 0$ for a certain N and any $b_1, \dots, b_N$ form the center of $E(a)$. The nilpotence of $E(a)$ implies that for certain m and any polynomials $p_1(a), \dots, p_m(a)$,

$$[p_1(a), \dots, [p_{m-1}(a), p_m(a)] \dots]$$

belongs to the ideal of the algebra of all polynomials of a generated by the center of $E(a)$. Hence for any polynomials $p_1(a), \dots, p_{mN}(a)$,

$$[p_1(a), \dots, [p_{mN-1}(a), p_{mN}(a)] \dots] = 0.$$

5. PROOF OF THE MAIN RESULT

The main result of this work is the following.

THEOREM 5.1. *Let $a = (a_1, \dots, a_n)$ be a family of linear bounded operators on the Banach space X generating a nilpotent Lie algebra, and let $p = (p_1, \dots, p_m)$ be a family of polynomials of n variables. Assume that the Lie algebra $E(p(a))$ generated by $p(a)$ is finite-dimensional. Then $E(p(a))$ is nilpotent and the following equality is valid:*

$$\sigma(p(a)) = p(\sigma(a)).$$

In order to prove Theorem 5.1 we introduce the following construction of a bicomplex.

Let E be a Lie algebra, let $\tilde{F}$ be Lie subalgebra of the universal enveloping algebra $U(E)$, let I be an ideal of E, and let X be an E-module. Hence $X \otimes \wedge I$ also has the structure of an E-module. The Lie algebra homomorphism $E \to \mathcal{L}(X \otimes \wedge E)$ generates the homomorphism $U(E) \to \mathcal{L}(X \otimes \wedge E)$. The image of $\tilde{F}$ under this homomorphism will be denoted by F. So $X \otimes \wedge I$ is an F-module.

We introduce the bicomplex $\mathcal{B}(I, F, X)$. Its rows are the complexes $\operatorname{Kos}(I, X) \otimes$

$\otimes \Lambda^q F$, its columns are the complexes $\mathrm{Kos}\,(F, X \otimes \Lambda^p I)$:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & \vdots & \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & & \downarrow{\scriptstyle \beta} & \\
0 & \longleftarrow & X \otimes \Lambda^q F & \overset{\alpha}{\longleftarrow} & X \otimes I \otimes \Lambda^q F & \overset{\alpha}{\longleftarrow} \cdots \overset{\alpha}{\longleftarrow} & & X \otimes \Lambda^p I \otimes \Lambda^q F & \overset{\alpha}{\longleftarrow} \cdots \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & & \downarrow{\scriptstyle \beta} & \\
 & & \vdots & & \vdots & & & \vdots & \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & & \downarrow{\scriptstyle \beta} & \\
0 & \longleftarrow & X \otimes F & \overset{\alpha}{\longleftarrow} & X \otimes I \otimes F & \overset{\alpha}{\longleftarrow} \cdots \overset{\alpha}{\longleftarrow} & & X \otimes \Lambda^p I \otimes F & \overset{\alpha}{\longleftarrow} \cdots \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} & & & \downarrow{\scriptstyle \beta} & \\
0 & \longleftarrow & X & \overset{\alpha}{\longleftarrow} & X \otimes I & \overset{\alpha}{\longleftarrow} \cdots \overset{\alpha}{\longleftarrow} & & X \otimes \Lambda^p I & \overset{\alpha}{\longleftarrow} \cdots \\
 & & \downarrow & & \downarrow & & & \downarrow & \\
 & & 0 & & 0 & & & 0 &
\end{array}$$

The commutativity of this diagram is a consequence of Lemma 1.1, since the elements of F are polynomials of operators Θ_u with $u \in E$ acting on $X \otimes I$, and every operator Θ_u commutes with the boundary of the complex $\mathrm{Kos}\,(I, X)$.

The totalization of $\mathcal{B}(I, F, X)$ is denoted by $\mathrm{Tot}\,(I, F, X)$.

The following remarks are necessary for the proof. For every p the bicomplex $\mathcal{B}(I, F, X)$ generates the complex with the q-th space

$$H_p\,(\mathrm{Kos}\,(I, X) \otimes \Lambda^q F) = H_p(I, X) \otimes \Lambda^q F$$

and the boundary operators induced by the operators β (also denoted by β).

It is clear that this complex may be identified with $\mathrm{Kos}\,(F, H_p(I, X))$. Now let $I = E$. What can we say about $\mathrm{Kos}\,(F, H_p(E, X))$? According to Lemma 1.2, Θ_u with $u \in E$ acts on $H_p(E, X)$ as the zero operator. Hence $f \in F$ acts on $H_p(E, X)$ as a scalar operator equal to the free member of the polynomial f. It is easy to see (use again Lemma 1.2) that $\mathrm{Kos}\,(F, H_p(E, X))$ is exact if and only if F contains a polynomial $f\,(\Theta_{u_1}, \ldots, \Theta_{u_k})$ with $f(0, \ldots, 0) \neq 0$.

More generally, let $\lambda\colon E \to \mathbb{C}$ be a homomorphism of Lie algebras, and let $\rho\colon E \to$ $\to \mathcal{L}(X)$ be a Lie algebra homomorphism defining an E-module structure on X. Then X has another E-module structure defined by the homomorphism $\rho_\lambda = \rho - \lambda\colon E \to \mathcal{L}(X)$.

Consider the bicomplex $\mathcal{B}_\lambda(I, F, X)$ with the rows $\mathrm{Kos}\,(I, \rho_\lambda, X) \otimes \Lambda^q F$ and the columns $\mathrm{Kos}\,(F, X \otimes \Lambda^p I)$ as in the bicomplex $\mathcal{B}(I, F, X)$. Its totalization is denoted by $\mathrm{Tot}_\lambda(I,F,X)$. It is clear that the complex $\mathrm{Kos}(F, H_p(E,\rho_\lambda,X))$ is exact if and only if for a certain $f(\Theta_{u_1},\dots,\Theta_{u_k}) \in F$, $f(\lambda(u_1),\dots,\lambda(u_k)) \neq 0$. Lemma 1.8 implies that this condition is sufficient for the exactness of $\mathrm{Tot}_\lambda(E,F,X)$.

The next lemma establishes the connection between $\mathrm{Tot}(I,F,X)$ and $\mathrm{Tot}(J,F,X)$ in the case when I and J are ideals of E, $I \subset J$ and $\dim J/I = 1$.

LEMMA 5.2. *Let I and J be ideals of E, $I \subset J$, $\dim J/I = 1$, $c \in J \setminus I$. Then for a certain endomorphism δ_c of the complex $\mathrm{Tot}(I,F,X)$, one has*

$$\mathrm{Tot}(J,F,X) = \mathrm{Con}\,(\mathrm{Tot}(I,F,X),\delta_c).$$

Proof. Decompose $\wedge^p J$ into the direct sum:

$$\wedge^p J = \begin{array}{c}\wedge^p I\\ \oplus\\ c\wedge\wedge^{p-1}I\end{array}$$

The following operator

$$\left(X\otimes\wedge^{p-1}J\otimes\wedge^q F\right)\oplus\left(X\otimes\wedge^p I\otimes\wedge^{q-1}F\right)\xleftarrow{\alpha+(-1)^p\beta} X\otimes\wedge^p J\otimes\wedge^q F$$

which is a part of the complex $\mathrm{Tot}(J,F,X)$ can be represented by the following diagram:

$$\begin{array}{ccc}
\left(X\otimes\wedge^{p-1}I\otimes\wedge^q F\right)\oplus\left(X\otimes\wedge^p I\otimes\wedge^{q-1}F\right) & \xleftarrow{1} & X\otimes\wedge^p I\otimes\wedge^q F\\
\oplus & \overset{4}{\nwarrow}\ \overset{3}{\nwarrow} & \oplus\\
\left(X\otimes c\wedge\wedge^{p-2}I\otimes\wedge^q F\right)\oplus\left(X\otimes c\wedge\wedge^{p-1}I\otimes\wedge^{q-1}F\right) & \xleftarrow{2} & X\otimes c\wedge\wedge^{p-1}I\otimes\wedge^q F
\end{array}$$

We have to define the operator associated with every arrow. (Two other arrows with the origin in $X\otimes\wedge^p I\otimes\wedge^q F$ correspond to zero operators). It is clear that $\alpha+(-1)^p\beta$ transforms $X\otimes\wedge^p I\otimes\wedge^q F$ into $\left(X\otimes\wedge^{p-1}I\otimes\wedge^q F\right)\oplus\left(X\otimes\wedge^p I\otimes\wedge^{q-1}F\right)$ so the first arrow is $\alpha+(-1)^p\beta$.

Consider the action of $\alpha+(-1)^p\beta$ on $X\otimes c\wedge\wedge^{p-1}I\otimes\wedge^q F$:

$$\begin{aligned}
(\alpha+(-1)^p\beta)\,(x\otimes c\wedge\underline{u}\wedge\underline{v}) &= \rho(c)x\otimes\underline{u}\otimes\underline{v}+\\
&+\sum_{i=1}^{p-1}(-1)^i\rho(u_i)\,x\otimes c\wedge\overset{i}{\widehat{\underline{u}}}\otimes\underline{v}+\\
&+\sum_{i=1}^{p-1}(-1)^{i+1}x\otimes[c,u_i]\wedge\overset{i}{\widehat{\underline{u}}}\otimes\underline{v}+\\
&+\sum_{i<j}(-1)^{i+j-1}x\otimes[u_i,u_j]\wedge c\wedge\overset{i,j}{\widehat{\underline{u}}}\otimes\underline{v}+
\end{aligned}$$

$$+\sum_{k=1}^{q}(-1)^{p+k-1}v_k\,(x\otimes c\wedge \underline{u})\otimes \overset{k}{\widehat{\underline{v}}}+$$

$$+\sum_{k<l}(-1)^{p+k+l-1}x\otimes c\wedge \underline{u}\otimes [v_k, v_l]\wedge \overset{k,l}{\widehat{\underline{v}}}\,.$$

Note that for every $v\in F$, $v\,(x\otimes c\wedge \underline{u}) = c\wedge v\,(x\otimes \underline{u}) + w$ with $w\in X\otimes \wedge^p I$. We have:

$$(\alpha+(-1)^p\beta)\,(x\otimes c\wedge \underline{u}\wedge \underline{v}) = -c\wedge \left(\alpha+(-1)^{p-1}\beta\right)(x\otimes \underline{u}\otimes \underline{v})+$$

$$+\Theta_c\,(x\otimes \underline{u})\otimes \underline{v}+\omega\,(x\otimes \underline{u}\otimes \underline{v})\,.$$

So the second arrow is $-\left(\alpha+(-1)^{p-1}\beta\right)$, the third one is Θ_c and the fourth one is ω. At last it is clear that $c\wedge \wedge^p I$ is isomorphic to $\wedge^p I$. Hence the complex $\mathrm{Tot}\,(J,F,X)$ is isomorphic to $\mathrm{Con}\,(\mathrm{Tot}\,(I,F,X),\delta_c)$ with $\delta_c = \Theta_c+\omega$. The lemma is proved.

Next we have some remarks about the operator δ_c.

1°. Let $\lambda\colon E\to \mathbb{C}$ be a Lie algebra homomorphism. The operator δ_c is the sum of $\rho(c)\otimes 1$ and a certain operator defined only by commutators of c. Hence if we add a scalar summand to ρ, δ_c will also change by a scalar summand. So the following equality is valid:

$$\mathrm{Tot}_\lambda(J,F,X) = \mathrm{Con}\ (\mathrm{Tot}_\lambda(I,F,X),\delta_c-\lambda(c))\,.$$

2°. The operator δ_c on the direct sum $\bigoplus_{p+q=k} X\otimes\wedge^p I\otimes\wedge^q F$ has a block-triangular form with the operators Θ_c on the diagonal. Moreover if E is a nilpotent Lie algebra, Θ_c and δ_c are triangular operator-matrices with the operators $\rho(c)$ on the diagonal. In this case we have

$$\sigma\,(\delta_c) = \sigma\,(\rho(c))\,.$$

From this moment assume that E is nilpotent and finite-dimensional. Denote by F_0 the image of $\tilde{F}$ in $\mathcal{L}(X)$ under the homomorphism induced by ρ. Consequence 4.2 implies the equivalence of the properties of F_0 (and F) to be nilpotent and finite-dimensional. The following lemma states that if F_0 is finite-dimensional then F is also finite-dimensional.

LEMMA 5.3. *If the Lie algebra F_0 is finite-dimensional then the Lie algebra F is also finite-dimensional.*

Proof. Since E is a nilpotent Lie algebra, it is clear that for every $u\in E$ the operator Θ_u on $X\otimes\wedge E$ is the triangular operator matrix with the operator $\rho(u)$

on the diagonal. Hence for every polynomial $q \in \tilde{F}$ the corresponding operator on $X \otimes \wedge E$ is the triangular matrix with the operator $\rho(q)$ on the diagonal. For

$$q \in \tilde{F}^{(m)} = \left[\tilde{F}, \dots, \tilde{F}, \left[\tilde{F}, \tilde{F}\right] \dots\right]$$

with $F_0^{(m)} = 0$, it is clear that $\rho(q) = 0$. Hence $F^{(m)}$ is nilpotent and, as a consequence, a finite-dimensional Lie algebra. But the factor-space $F/F^{(m)}$ is also finite-dimensional. Hence F is a finite-dimensional Lie algebra. The lemma is proved.

Thanks to the fact that F is a finite-dimensional space, $X \otimes \wedge^q F$ are Banach spaces and the application of Lemma 1.6 is possible. Moreover, the nilpotence of F and the natural epimorphism $F \to F_0$ imply that the complexes $\mathrm{Kos}(F_0, X)$ and $\mathrm{Kos}(F, X)$ are simultaneously exact. (cf. Proposition 2.6)

LEMMA 5.4. *If the complex $\mathrm{Kos}(F, X)$ is exact, then the complexes $\mathrm{Kos}(F, X \otimes \otimes \wedge^p E)$ are also exact.*

Proof. Analogously to the proof of Lemma 5.3, consider the operators on $X \otimes \wedge^p E$ as triangular matrices and conclude the statement from Lemma 1.4 by induction on the dimension of matrices.

Now we prove Theorem 5.1. Let E be a nilpotent Lie algebra with the generators $e_1, \dots, e_n$, and let $\rho: E \to \mathcal{L}(X)$ be a Lie algebra homomorphism with $\rho(e_i) = a_i$, $i = 1, \dots, n$. Let $p_1, \dots, p_m$ be polynomials of noncommuting variables, and let $\tilde{F}$ be the Lie subalgebra of the universal enveloping algebra $U(E)$ generated by the corresponding polynomials of $e_1, \dots, e_n$. The Lie algebras F and F_0 have been defined. Obviously F_0 is the Lie algebra generated by the family $p(a)$ of polynomials of a. According to Theorem 2.3, $\sigma(a) = \sigma_E(a)$.

Proof of the inclusion $p(\sigma(a)) \subseteq \sigma(p(a))$: It is enough to prove that $0 \in \sigma(a)$ and $p(0) = 0$ imply $0 \in \sigma(p(a))$. In other words, we have to prove that if $\mathrm{Kos}(E, X)$ is not exact and $p(0) = 0$ then $\mathrm{Kos}(F_0, X)$ is not exact. Consider the bicomplex $\mathcal{B}(E, F, X)$. The condition $p(0) = 0$ implies that the operator

$$\beta: H_k(E, X) \otimes F \to H_k(E, X)$$

is equal to zero (cf. the remarks to the definition of the bicomplex). Hence the assumptions of Lemma 1.8 are fulfilled (in particular, the property of the rows of the bicomplex is obvious). Lemma 1.8 implies that at least one of the columns $\mathrm{Kos}\left(F, X \otimes \wedge^k E\right)$ is not exact. Hence, according to Lemma 5.4 $\mathrm{Kos}(F, X)$ is not exact, and according to Theorem 2.3 $\mathrm{Kos}(F_0, X)$ is not exact too. So $0 \in \sigma(p(a))$.

Proof of the inclusion $\sigma(p(a)) \subseteq p(\sigma(a))$: We start from the reduction to the case when $e_1, \dots, e_n$ is such a basis of E that the relative structure constants $C_{i,j}^k = 0$

with $k \leqslant \max\{i,j\}$. Assume that the basis $e'_1, \ldots, e'_N$ has this property and $\rho(e'_i) = a'_i$, $i = 1, \ldots, N$, $a' = (a'_1, \ldots, a'_N)$. After a linear change of variables $e_1, \ldots, e_n$ in the polynomials $p_1, \ldots, p_m$, we get the family of polynomials $q = (q_1, \ldots, q_m)$ of variables $e'_1, \ldots, e'_N$. The following equalities will be proved:

$$\sigma(p(a)) = \sigma(q(a')) = q(\sigma(a')) = p(\sigma(a)).$$

Here the first equality is obvious, the second will be proved as the main result, and the third one is the consequence of the results of the second section. Indeed $\lambda \in \sigma(a) = \sigma_E(a)$ if and only if $\lambda_i = \tilde{\lambda}(e_i)$, $i = 1, \ldots, n$ with $\tilde{\lambda}$ a Lie algebra homomorphism: $E \to \mathbb{C}$, and the complex $\operatorname{Kos}(E, \rho_\lambda, X)$ is not exact. Analogously, $\lambda' \in \sigma(a')$ if and only if $\lambda'_i = \tilde{\lambda}(e'_i)$, $i = 1, \ldots, N$ and the complex $\operatorname{Kos}(E, \rho_\lambda, X)$ is not exact. It is clear that $q(\lambda') = p(\lambda)$ which proves the third equality.

So we consider $e_1, \ldots, e_n$ to be a basis of E with relative structure constants $C^k_{i,j} = 0$ with $k \leqslant \max\{i,j\}$ and prove the inclusion

$$\sigma(p(a)) \subseteq p(\sigma(a)).$$

It is enough to prove that $0 \in \sigma(p(a))$ implies $p(\lambda) = 0$ for a certain $\lambda \in \sigma(a)$. Denote by I_k the ideal of E generated by $e_k, \ldots, e_n$. Obviously

$$E = I_1 \supset I_2 \supset \cdots \supset I_n = \mathbb{C}e_n$$

and $\dim I_{k-1}/I_k = 1$.

We find $\lambda \in \sigma(a)$ with $p(\lambda) = 0$ in the following way. Since $0 \in \sigma(p(a))$, the complex $\operatorname{Kos}(F, X)$ is not exact. By Lemma 5.2

$$\operatorname{Tot}\,(I_n, F, X) = \operatorname{Con}\,(\operatorname{Kos}(F, X), \delta_{e_n}).$$

By Lemma 1.6 for a certain $\lambda_n \in \mathbb{C}$, the complex

$$\operatorname{Tot}_\lambda\,(I_n, F, X) = \operatorname{Con}\,(\operatorname{Kos}(F, X), \delta_{e_n} - \lambda_n)$$

is not exact.

Continuing this process by induction we get at the $(n-k+1)$-th step such $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ that the complex

$$\operatorname{Tot}_\lambda\,(I_k, F, X) = \operatorname{Con}\,(\operatorname{Tot}_\lambda\,(I_{k+1}, F, X), \delta_{e_k} - \lambda_k)$$

is not exact. So we get $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that the complex $\operatorname{Tot}_\lambda(E, F, X)$ is not exact.

To be sure that our construction is correct, we have to show that the map $e_i \mapsto$ $\mapsto \lambda_i$, $i = 1, \ldots, n$ is extended to a Lie algebra homomorphism $E \to \mathbb{C}$. Let $e_j, \ldots, e_n$ be a basis of $[E, E]$. Then the map $e_i \mapsto \lambda_i$ is extended to a Lie algebra homomorphism if and only if $\lambda_j = \cdots = \lambda_n = 0$. The fact that the complex

$$\mathrm{Con}\,\left(\mathrm{Tot}_\lambda\left(I_{k+1}, F, X\right), \delta_{e_k} - \lambda_k\right)$$

is not exact implies that $\lambda_k \in \sigma\left(\delta_{e_k} - \lambda_k\right) = \sigma\left(a_k\right)$. But $a_k \in [E(a), E(a)]$. Hence $\sigma\left(a_k\right) = \{0\}$ [10]. So $\lambda_k = \cdots = \lambda_n = 0$.

Lemma 1.8 implies that if the complex $\mathrm{Tot}_\lambda(E, F, X)$ is not exact, then one of the complexes

$$\mathrm{Kos}\,\left(F, H_k\left(E, \rho_\lambda, X\right)\right)$$

is not exact. Hence $p(\lambda) = 0$ (cf. the remarks to the definition of the bicomplex). Theorem 5.1 is completely proved.

The following property of joint spectrum is called the projection property. It is a consequence of Theorem 5.1. Ju. V. Turovskiĭ proved it as a consequence of Lemma 1.5 and Lemma 1.6.

CONSEQUENCE 5.5. Let a family $a = (a_1, \ldots, a_n)$ of linear bounded operators on the Banach space generate a nilpotent Lie algebra, $a' = (a_1, \ldots, a_{n-1})$, $\pi\colon \mathbb{C}^n \to$ $\to \mathbb{C}^{n-1}$ be the projection to the first $n-1$ coordinates. Then $\sigma(a') = \pi(\sigma(a))$.

Proposition 4.3 shows that the assumption of Theorem 5.1 is fulfilled if the center of $E(a)$ consists of nilpotent operators. It is also fulfilled in the following example.

EXAMPLE 5.6. Let a, b be bounded linear operators in a Banach space with

$$[a,[a,b]] = [b,[a,b]] = 0.$$

It is easy to check that ab commutes with ba.

Theorem 5.1 implies that

$$\sigma(ab, ba) = \{(\lambda\mu, \mu\lambda)\colon (\lambda, \mu) \in \sigma(a, b)\}.$$

So the spectrum of the commutative operator family (ab, ba) is calculated through the spectrum of a non-commutative (but generating a nilpotent Lie algebra) operator family (a, b).

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