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MAXIMAL ABELIAN AND SINGULAR SUBALGEBRAS IN $L(F_N)$

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INTRODUCTION

Let M be a type II_1 factor, tr its unique normalized trace, and let $A \subseteq M$ be a maximal abelian von Neumann subalgebra (M.A.S.A.). Let $\mathcal{N}_M(A) = \{u \in \in M \,|\, u \text{ unitary}, uAu^* = A\}$ be the normalizer of A in M and let $B = \mathcal{N}_M(A)''$ be the von Neumann subalgebra generated by $\mathcal{N}_M(A)$ in M. A is called *singular* if $B = A$ and A is called *regular* (or *Cartan*) if $B = M$. For singular M.A.S.A.'s see [4], [6], [8], [9], [13], [14], and for regular M.A.S.A.'s see [3] and [7].

The aim of this paper is to find certain singular M.A.S.A.'s in $L(F_N)$, the von Neumann algebra generated by the left regular representation of the free nonabelian group with N generators, say $a_1, \ldots, a_N$. Examples of singular M.A.S.A.'s in $L(F_N)$ are given in [6] and [11]. These algebras are generated by the convolutors a_i, $1 \leqslant i \leqslant \leqslant N$, in [6], or by $a_1 + \cdots + a_N + a_1^{-1} + \cdots + a_N^{-1}$ in [11]. Our examples are generated by convolutors of the form

$$(1) \qquad a_1^{r_1} + \cdots + a_N^{r_N} + a_1^{-r_1} + \cdots + a_N^{-r_N}, \quad \text{where } r_1, \ldots, r_N \in \mathbf{N} \setminus \{0\}.$$

In Theorem 4 of Section 1, developing the technique of Pytlic [10], we prove that even more general convolutors in $L(F_N)$ generate M.A.S.A.'s. Then, in Theorem 18 of Section 2, we prove the singularity for convolutors given by (1), using a result of Popa [7] about the Pukanszky invariant.

Let $\|x\|_2 = \mathrm{tr}(x^*x)^{\frac{1}{2}}$ be the Hilbert norm given by tr on M, let $L^2(M, \mathrm{tr})$ be the completion of M with respect to this norm, so that M acts (in the standard way) on $L^2(M, \mathrm{tr})$. We denote the induced scalar product by $\langle \cdot | \cdot \rangle$. Let also $J : L^2(M, \mathrm{tr}) \to L^2(M, \mathrm{tr})$ be the canonical conjugation (given by $Jx = x^*$, for

$x \in M$) and let $\mathcal{A} = (A \vee JAJ)''$ be the (abelian) von Neumann subalgebra generated in $B(L^2(M,\tau))$ by A and JAJ.

Denote by p_ξ the cyclic projection corresponding to $\xi \in L^2(M, \mathrm{tr})$ in $\mathcal{A}'$ (i.e. p_ξ is the projection on $\overline{\mathcal{A}\xi}^{\|\cdot\|_2}$). Since A is a M.A.S.A., p_1 is a central projection of $\mathcal{A}'$ (see Lemma 3.1 in [7]).

The Pukanszky invariant of A is the type of homogenity of the (type I) algebra $\mathcal{A}'$. This invariant was also considered by Ambrose-Singer.

Now, by Popa's result (Corollary 3.2 in [7]), if $\mathcal{A}'(I - P_1)$ is homogeneous of type I_∞, then A is singular.

Developing the technique of Rădulescu [11], we prove that the Pukanszky invariant of the von Neumann algebra A generated by any convolutor given by (1) is infinite.

Note that by a result of Popa ([6], Theorem 6.1) our result implies that the singularity holds also for von Neumann algebras generated by convolutors of the form

$$a_1^{r_1} + \cdots + a_k^{r_k} + a_1^{-r_1} + \cdots + a_k^{-r_k}$$

where $2 \leqslant k \leqslant N$ and $r_1, \ldots, r_k \in \mathbb{N} \setminus \{0\}$.

One is led to conjecture that given a free family $\{v_1, \ldots, v_k\}_{k \geqslant 2}$ in F_N, the von Neumann algebra generated by the convolutor $v_1 + \cdots + v_k + v_1^{-1} + \cdots + v_k^{-1}$ is singular.

Finally, we mention that similar problems can be studied in free products of groups instead of free groups (see [1], where the singularity is proved for the radial algebra in some free products of groups).

1. MAXIMAL ABELIAN SUBALGEBRAS

Let $N \geqslant 2$ be an integer and let F_N be the free nonabelian group with N generators $a_1, \ldots, a_n$. Let λ be the left regular representation of F_N on the Hilbert space $\ell^2(F_N)$. Denote by $L(F_N)$ the von Neumann algebra generated by $\lambda(F_N)$ in $B(\ell^2(F_N))$. It is well known that $L(F_N)$ is a type II_1 factor that acts standardly on $\ell^2(F_N)$ and that with this identifications the norm given by tr coincides with the usual norm $\|\cdot\|_2$ on $\ell^2(F_N)$. For a set $X \subseteq \ell^2(F_N)$, $\overline{X}$ will stand for the closure of X in the norm $\|\cdot\|_2$.

By $\mathbb{C}[F_N]$ we denote the group ring of F_N over $\mathbb{C}$, and by 1 we denote the unity of F_N. (However, the identity operator will usually be denoted by I.) By means of the identification of M with a subspace of $L^2(M, \mathrm{tr})$, $\mathbb{C}[F_N]$ corresponds to the subspace of finitely supported functions from $\ell^2(F_N)$. (We shall use the same notation for the elements of F_N and for their image in $\mathbb{C}[F_N]$. If $f = \sum\limits_{w \in F_N} \alpha_w \cdot w$ is a function in $\ell^2(F_N)$, then $\operatorname{supp} f = \{w \in F_N \mid \alpha_w \neq 0\}$.)

Each $w \in F_N \setminus \{1\}$ can be written uniquely as a finite product of generators and their inverses, $w = a_{i_1}^{\alpha_1} \cdots a_{i_p}^{\alpha_p}$, such that $i_k \neq i_{k+1}$, $\alpha_k \in \mathbf{Z} \setminus \{0\}$. This is the reduced form of w. We define the beginning of w to be $l(w) = a_{i_1}^{\operatorname{sign} \alpha_1}$ and the end of w to be $r(w) = a_{i_p}^{\operatorname{sign} \alpha_p}$. The canonical length function $|\cdot|$ on F_N is defined by $|w| = |\alpha_1| + \cdots + |\alpha_p|$, and $|1| = 0$. If H is a free subgroup of F_N with given generators, we shall denote by $|\cdot|_H$ the corresponding length function on H. For $v_1, \ldots, v_k \in F_N$, $k \geqslant 2$, we say that the product $v_1 v_2 \ldots v_k$ is reduced if none of the factors is 1 and for all $i = 1, \ldots, k-1$, $r(v_i) \cdot l(v_{i+1}) \neq 1$. We say that the product $v_1 \ldots v_k$ is reduced $\operatorname{mod}\{v_i \mid i \in I\}$, where $I \subset \{1, \ldots, k\}$, if some v_i's, $i \in I$, may equal 1 and after deleting those equal to 1, we obtain a reduced product.

Let V be a family in $F_N \setminus \{1\}$, with at least two elements, which satisfies:

$$\text{for any } v, w \in V^{\pm}, \text{ if } vw \neq 1 \text{ then } vw \text{ is reduced} \tag{2}$$

where $V^{\pm} := V \cup V^{-}$, $V^{-} := \{v^{-1} \mid v \in V\}$. We denote by χ_1 the convolutor $\sum_{v \in V}(v + v^{-1})$, and by H the subgroup generated by V (which is a free group with basis V!).

Let A be the von Neumann algebra generated by χ_1 in $L(F_N)$. We shall prove that A is a M.A.S.A. in $L(F_N)$.

In the sequel, $f \cdot g$ will stand for the convolution of f and g, for any $f, g \in \ell^2(F_N)$.

Lemma 1. *Let $v, v' \in V^{\pm}$ and $w \in F_N$ be such that vwv' is reduced* $\operatorname{mod}\{w\}$. *Then, for any $\varepsilon > 0$, there exists a function $\varphi \in \ell^2(F_N)$ such that:*

$$\|(\chi_1 \cdot \varphi - \varphi \cdot \chi_1) - (vw - wv')\|_2 < \varepsilon.$$

Proof. We use the construction of Pytlik (see Lemma 4.1 from [10]), but we border vwv' with words of equal H-length from H insteed of using words of equal length from F_N. ∎

Lemma 2. *Let $v, v' \in V^{\pm}$ and $w \in F_N \setminus \{1\}$ be such that $v'' \cdot w$ is reduced for all $v'' \in V^{\pm} \setminus \{v\}$ and wv'' is reduced for all $v'' \in V^{\pm} \setminus \{v'\}$. Then, for any $\varepsilon > 0$, there exists a function $\varphi \in \ell^2(F_N)$ such that:*

$$\|(\chi_1 \cdot \varphi - \varphi \cdot \chi_1) - (vw - wv')\|_2 < \varepsilon.$$

Proof. $\chi_1 \cdot w - w \cdot \chi_1 = vw - wv' + \sum_{\substack{v'' \neq v \\ v'' \in V^{\pm}}} v''w - \sum_{\substack{v''' \neq v' \\ v''' \in V^{\pm}}} wv'''.$

From the two sums above, we can form $2 \cdot \mathrm{card}\, V - 1$ pairs $v'' \cdot w - w \cdot v'''$ with $v''wv'''$ reduced. Applying for each of them Lemma 1, we find functions $\varphi_{v'',v'''} \in \ell^2(F_N)$ such that:

$$\|(\chi_1 \cdot \varphi_{v'',v'''} - \varphi_{v'',v'''} \cdot \chi_1) - (v''w - wv''')\|_2 < \frac{\varepsilon}{2\mathrm{card}\, V - 1}$$

Hence, for $\varphi = w - \sum_{(v'',v''')} \varphi_{v'',v'''}$ one has the desired conclusion. ■

Lemma 3. *Let $g \in \ell^2(F_N)$ be such that $g \cdot \chi_1 = \chi_1 \cdot g$.*

a) If $x, y \in H$ and $|x|_H = |y|_H$, then $g(x) = g(y)$.

b) If $x \notin H$, then $g(x) = 0$.

Proof. a) Use Lemma 1 above and the arguments of Lemma 4.1 and Proposition 4.2 in [10].

b) By the uniqueness of the reduced form, there exists $h \in H$, $w \in F_N$ such that $x = hw$ is reduced mod$\{h\}$, and $|h|_H$ is maximal. If $h \neq 1$, write $h = \overline{v}_1 \cdots \overline{v}_k$ in reduced form, $\overline{v}_i \in V^{\pm}$, and choose $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_k \in V^{\pm}$ such that $w\overline{w}_1, w\overline{w}_1\overline{w}_2, \ldots, w\overline{w}_1\overline{w}_2 \cdots \overline{w}_k$ are reduced. (If $u \in F_N \setminus \{1\}$, there is an element $v \in V^{\pm}$ such that uv is reduced: indeed, condition (2) implies that the map $l : V^{\pm} \to \{a_i^{\pm 1} \mid 1 \leqslant i \leqslant N\}$ is one-to-one, and card $V^{\pm} \geqslant 4$.) Now, from Lemma 1 and the argument of Proposition 4.2 in [10] it follows that:

$$g(x) = g(\overline{v}_2 \cdots \overline{v}_k w \overline{w}_1) = \cdots = g(w\overline{w}_1 \cdots \overline{w}_k).$$

So, we can assume that $h = 1$.

The following cases appear:

I: vx is reduced for every $v \in V^{\pm}$;

II: there exists $v_0 \in V^{\pm}$ such that $v_0 x$ is not reduced (and then this v_0 is unique, due to (2)).

Case I: There exist $\overline{v}_1, \overline{v}_2 \in V^{\pm}$ such that $\overline{v}_1 x, x\overline{v}_2$ are reduced and $\overline{v}_1 \neq \overline{v}_2^{-1}$. Then

$$g(x) = g(\overline{v}_1 x \overline{v}_2) = g(x\overline{v}_1\overline{v}_2),$$

where we have used Lemma 2 (with $w = x\overline{v}_2$, $v = \overline{v}_1$, $v' = \overline{v}_2^{-1}$) for the first equality, Lemma 1 (with $w = x\overline{v}_2$, $v = \overline{v}_1$, $v' = \overline{v}_1$) for the second equality and the argument of Proposition 4.2 from [10] for both.

Since $|x| < |x\overline{v}_2\overline{v}_1|$, and $x\overline{v}_2\overline{v}_1$ is also in case I, we can iterate this trick to find an infinite set of distinct elements y from F_N such that $g(y) = g(x)$. But $g \in \ell^2(F_N)$, therefore $g(x) = 0$.

Case II: There exist $\overline{v}_1, \overline{v}_2 \in V^{\pm}$ such that $x\overline{v}_1\overline{v}_2$ is reduced and $r(v_0 x\overline{v}_1) = r(\overline{v}_1)$. Then

$$g(x) = g(v_0 x \overline{v}_1) = g(x\overline{v}_1\overline{v}_2),$$

where we have used Lemma 2 with $w = x\overline{v}_1$, $v = v_0$, $v' = \overline{v}_1^{-1}$ for the first equality and with $w = v_0 x \overline{v}_1 \overline{v}_2$, $v = v_0^{-1}$, $v' = \overline{v}_2^{-1}$ for the second equality (here Lemma 2 can be applied because $l(v_0 x\overline{v}_1\overline{v}_2) = l(v_0)$ and $r(v_0 x \overline{v}_1 \overline{v}_2) = r(\overline{v}_2)$), and the argument of Proposition 4.2 from [10].

Since $|x| < |x\overline{v}_1\overline{v}_2|$ and $x\overline{v}_1\overline{v}_2$ is also in case II, we can again iterate this trick and obtain $g(x) = 0$ too. ∎

THEOREM 4. *A is a M.A.S.A. in $L(F_N)$.*

Proof. It is an easy consequence of Lemma 3 (see Theorem 4.3 in [10]). ∎

2. SINGULAR SUBALGEBRAS

In this paragraph we take V to be:

$$V = \{a_1^{r_1}, a_2^{r_2}, \dots, a_N^{r_N}\}, \quad \text{where} \quad r_i \in \mathbb{N} \setminus \{0\}.$$

It is clear that V satisfies (2).

For γ_1 and A as above, let $\mathcal{A} = A \vee JAJ$. We shall prove that $\mathcal{A}'$ is of homogeneous type I_∞ on $I - p_1$.

DEFINITION. We say that $w \in F_N$ is a *core* if $w = 1$ or if the reduced form of w neither begins nor ends with a word from H (i.e., if $w = h_1 v h_2$ is reduced mod$\{h_1, v, h_2\}$ with $h_1, h_2 \in H$, then $h_1 = h_2 = 1$). We put the cores in families of related cores. These families are of the following three type:

$$C_0 = \{1\} \tag{3}$$

$$C(i,j,\alpha,\beta,w) = \{a_i^{\alpha} w a_j^{\beta}, a_i^{\alpha - r_i} w a_j^{\beta}, a_i^{\alpha} w a_j^{\beta - r_j}, a_i^{\alpha - r_i} w a_j^{\beta - r_j}\} \tag{4}$$

where $1 \leqslant i, j \leqslant N$ are such that $r_i \geqslant 2$, $r_j \geqslant 2$, $1 \leqslant \alpha \leqslant r_i - 1$, $1 \leqslant \beta \leqslant r_j - 1$ and $w \in F_N$ is such that $l(w) \neq a_i^{\pm 1}$, $r(w) \neq a_j^{\pm 1}$ and $w \neq 1$ if $i = j$.

$$C(i,\alpha) = \{a_i^{\alpha}, a_i^{\alpha - r_i}\} \tag{5}$$

where $1 \leqslant i \leqslant N$ is such that $r_i \geqslant 2$ and $1 \leqslant \alpha \leqslant r_i - 1$.

We remark that if we denote by $\mathcal{C}$ the colection of the families of cores described above, then:

(6) $\mathcal{C}$ is a partition of the set of cores;

(7) $\operatorname{span}\{H \cdot C \cdot H\}$ is orthogonal to $\operatorname{span}\{H \cdot C' \cdot H\}$ for every $C, C' \in \mathcal{C},\ C \neq C'$;

(8) $$\bigoplus_{C \in \mathcal{C}} \overline{\operatorname{span}}\{H \cdot C \cdot H\} = l^2(F_N);$$

(9) $\overline{\operatorname{span}}\{H \cdot C \cdot H\}$ is an invariant subspace for $\lambda(\chi_1)$ and $J\lambda(\chi_1)J$ for every $C \in \mathcal{C}$.

The core of type (3) was studied by Rădulescu, [11]. One has:

THEOREM 5. (see [11]). *The space* $\overline{\operatorname{span}} H = \overline{\operatorname{span}} H \cdot C_0 \cdot H$ *is a sum of orthogonal cyclic projections* $p_1 + \sum_{\xi \in I_0} p_\xi$, *where* $I_0 \subset l^2(F_N)$ *is an infinite family of norm one vectors and* p_ξ *is equivalent to* p_η *for any* $\xi, \eta \in I_0$.

In order to prove that the homogenity type of $\mathcal{A}'$ on $I - p_1$ is I_∞, it is sufficient to show that the central support of p_ξ, $\xi \in I_0$, is $I - p_1$. Indeed:

LEMMA 6. *Let* M *be a von Neumann algebra of type I, let* $\{e_n\}_{n \geqslant 1}$ *be an infinite family of mutually equivalent and orthogonal projections and let* $q = z(e_1)$ *be their central support. Then the reduced algebra* M_q *is of homogeneous type* I_∞.

Proof. By contradiction, let $0 \neq p \leqslant q$ be a central projection such that M_p is of homogeneous type I_k, with $k < \infty$. Then $\{pe_n\}_{n \geqslant 1} \subset M_p$ is an infinite family of nonzero (because $p \leqslant z(e_n)$) mutually orthogonal and equivalent projections. This is a contradiction with M_p being of homogeneous type I_k, with $k < \infty$ (see [12], Section 4). ∎

In fact, we show that for each family of cores $C \in \mathcal{C} \setminus \{C_0\}$, there exists a family of norm one vectors $I(C) \subset l^2(F_N)$ such that:

$$\overline{\operatorname{span}}\{H \cdot C \cdot H\} = \bigvee_{\xi \in I(C)} p_\xi,$$

and p_ξ is equivalent to p_ω for any $\xi \in I(C)$. Here ω is a fixed vector from I_0, which will be described below. By (7), (8) and the result quoted in Theorem 5, this implies that the central support of p_ω is $I - p_1$.

If $\mathcal{H} \subset \mathcal{K}$ are Hilbert spaces, we shall denote the orthogonal projection on $\mathcal{H}$ by $\operatorname{proj}_{\mathcal{H}}^{\mathcal{K}}$.

Let us fix some $C \in \mathcal{C}$. All the notations we introduce from now on depend on C, even if it does not appear explicitly. The family C we are refering to will always be clear.

For any integers $k, k', k'' \geqslant 0$ define:

$$V_{k',k''} := \operatorname{span}\{\overline{w} = w_1 \cdot w \cdot w_2 \,|\, \overline{w} \text{ is reduced, } w \in C,\ w_1, w_2 \in H,$$
$$|w_1|_H = k',\ |w_2|_H = k''\};$$
$$V_k := \operatorname{span}\{V_{k',k''} \,|\, k' + k'' = k\};$$
q_k is the orthogonal projection on $V_k \subset l^2(F_N)$;
$$S_k := \operatorname{span}\{q_k(\chi_1 \cdot w),\ q_k(w \cdot \chi_1) \,|\, w \in V_{k-1}\};$$
$$L_k := q_k \cdot \lambda(\chi_1) \cdot q_k, \quad R_k := q_k \cdot J\lambda(\chi_1)J \cdot q_k.$$

We introduce the convolutors

$$\chi_n := \begin{cases} \sum\limits_{w \in H,\ |w|_H = n} w, & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n < 0. \end{cases}$$

For $\gamma \in V_k$ $(k \geqslant 0)$ and $n, m \in \mathbf{Z}$, define

$$\gamma_{n,m} := \begin{cases} q_{n+m+k}(\chi_n \cdot \gamma \cdot \chi_m) & \text{if } n, m \geqslant 0, \\ 0 & \text{if } n < 0 \text{ or } m > 0. \end{cases}$$

The following formulae are easy to check (see [2] or [11]):

$$(10) \qquad \begin{cases} \chi_1 \cdot \chi_1 = \chi_2 + 2N; \\ \chi_1 \cdot \chi_n = \chi_n \cdot \chi_1 = \chi_{n+1} + (2N-1)\chi_{n-1}, & \text{for } n \geqslant 2; \end{cases}$$

$$(11) \qquad \begin{cases} \chi_1 \cdot \gamma_{n,m} = \gamma_{n+1,m} + (2N-1)\gamma_{n-1,m}, & \text{for } n \geqslant 1,\ m \geqslant 0; \\ \gamma_{n,m} \cdot \chi_1 = \gamma_{n,m+1} + (2N-1)\gamma_{n,m-1}, & \text{for } n \geqslant 0,\ m \geqslant 1. \end{cases}$$

We can now describe the vector $\omega \in I_0$. It is chosen such that for any $n, m, p, q \geqslant \geqslant 0$ the following formulae hold:

$$(12) \qquad \begin{cases} \chi_n \cdot \omega \cdot \chi_m = \omega_{n,m} - \omega_{n-2,m} - \omega_{n,m-2} + \omega_{n-2,m-2}; \\ \omega_{n,m} = \sum\limits_{s,t \geqslant 0} \chi_{n-2s} \cdot \omega \cdot \chi_{m-2t}; \end{cases}$$

$$(13) \qquad \begin{cases} \text{for } (n,m) \neq (p,q),\ \omega_{n,m} \text{ is orthogonal to } \omega_{p,q}; \\ \|\omega_{n,m}\|^2 = (2N-1)^{n+m}. \end{cases}$$

(See [11], Lemma 2 (a), Lemma 3 (a) and the proof of Theorem 7; almost all the vectors from I_0 satisfy these formulae.)

Our aim is to show that each V_k, $k \geqslant 0$, can be covered with cyclic projections p_ξ, which are equivalent to p_ω (i.e. $q_k \leqslant \bigvee_\alpha p_{\xi_\alpha}$, with p_{ξ_α} equivalent to p_ω). The case $k = 0$ is made by ad-hoc methods and will be developed later, in Lemmas 13-17. For $k \geqslant 1$, we shall use an induction argument, which relies on Lemmas 7-12 and is described in the Remark following Lemma 12.

LEMMA 7. *Let $k \geqslant 1$. If $V_{k'}$, $0 \leqslant k' \leqslant k-1$, can be covered with projections which are equivalent to p_ω, then these projections cover S_k too.*

Proof. This follows from the fact that for any $w \in V_{k-1}$,

$$q_k(\chi_1 \cdot w) = \chi_1 \cdot w - \overline{w}$$

where $\overline{w} \in \operatorname{span}\{V_{k-2}, V_{k-1}\}$.

Therefore, in proving the induction step for $k \geqslant 1$, we have to deal only with the space $V_k \ominus S_k$. ■

LEMMA 8. *Let $k \geqslant 1$.*

(a) *If $\gamma \in V_k \ominus S_k$ and $s \geqslant 0$, then*

$$\chi_1 \cdot \gamma_{0,s} = \gamma_{1,s} + (L_k\gamma)_{0,s};$$

$$\gamma_{s,0} \cdot \chi_1 = \gamma_{s,1} + (R_k\gamma)_{s,0}.$$

(b) *$R_k \cdot L_k = L_k \cdot R_k = 0$, hence L_k and R_k commute.*

(c) *$V_k \ominus S_k$ is a reducing subspace for L_k and R_k.*

(d) *L_k^2 is the orthogonal projection on $V_k \ominus \operatorname{span}\{V_{s,t} \mid s+t = k,\ s \geqslant 1\}$. In particular, the spectra of $L_k, R_k, L_k|_{V_k \ominus S_k}$ and $R_k|_{V_k \ominus S_k}$ are contained in $\{-1, 0, 1\}$ (and all these values can appear).*

Proof. (a) Let $\eta \in V_{k'}$, with $k' \geqslant 0$ and let $s \geqslant 0$. Then

$$\begin{aligned}\langle \chi_1 \cdot \gamma_{0,s} \mid \eta\rangle &= \langle \gamma_{0,s} \mid \chi_1 \cdot \eta\rangle = \langle q_{k+s}(\gamma\chi_s) \mid \chi_1 \cdot \eta\rangle = \\ &= \langle \gamma \cdot \chi_s \mid q_{k+s}(\chi_1 \cdot \eta)\rangle = \langle \gamma \mid q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s\rangle = \\ &= \langle \gamma \mid q_k(q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s)\rangle.\end{aligned} \tag{14}$$

For $k' < k+s-1$, $q_{k+s}(\chi_1 \cdot \eta) = 0$, hence the last member of (14) is zero.

For $k' = k+s-1$, $q_k(q_{k+s}(\chi_1 \cdot \eta)\chi_s) = q_k(\chi_1 \cdot q_{k-1}(\eta \cdot \chi_s))$, as one can see comparing the length of the words which appear. Hence the last member of (14) is zero again.

For $k' = k+s$, $q_k(q_{k+s}(\chi_1 \cdot \eta) \cdot \chi_s) = q_k(\chi_1 \cdot q_k(\eta \cdot \chi_s))$, hence the last member of (14) becomes:

$$\begin{aligned}\langle \gamma \mid q_k(\chi_1 \cdot q_k(\eta \cdot \chi_s))\rangle &= \langle q_k(\chi_1 \cdot \gamma) \mid \eta \cdot \chi_s\rangle = \\ = \langle q_k(\chi_1 \cdot \gamma) \cdot \chi_s \mid \eta\rangle &= \langle q_{k+s}(q_k(\chi_1 \cdot \gamma)\chi_s) \mid \eta\rangle = \\ = \langle [q_k(\chi_1 \cdot \gamma)]_{0,s} \mid \eta\rangle &= \langle (L_k\gamma)_{0,s} \mid \eta\rangle.\end{aligned}$$

So $\chi_1 \cdot \gamma_{0,s} = \gamma_{1,s} + (L_k\gamma)_{0,s}$.

The second formula can be proved similarly.

(b), (d) It is enough to verify the formulae for $\gamma \in V_k \cap F_N$, for each type of family of cores (because $V_k \cap F_N$ is an orthogonal basis of V_k). For example, $L_k \big|_{V_{s,t}} = 0$ whenever, $s \geqslant 1$ and $s + t = k$.

(c) The inclusion $L_k(S_k) \subset S_k$ follows from:

$$L_k q_k(\chi_1 \cdot v) = 0, \quad \text{for } v \in V_{k-1},$$

$$L_k q_k(v \cdot \chi_1) = q_k(q_{k-1}(\chi_1 \cdot v) \cdot \chi_1), \quad \text{for } v \in V_{k-1}.$$

Since L_k is a selfadjoint operator, $V_k \ominus S_k$, is a reducing subspace for it. R_k can be dealt with analogously. ∎

From (b) and (c) of the previous Lemma each $V_k \ominus S_k$, $k \geqslant 1$, has a basis made from eigenvectors for both L_k and R_k. If $\gamma \in V_k \ominus S_k$ is an eigenvector such that $L_k\gamma = \lambda_1 \cdot \gamma$, $R_k\gamma = \lambda_2 \cdot \gamma$, then from (b) and (d) of Lemma 8:

$$(\lambda_1, \lambda_2) \in \{(\varepsilon, 0), (0,0), (0, \varepsilon) \,|\, \varepsilon \in \{-1, 1\}\}.$$

LEMMA 9. *Let $k \geqslant 0$, and let $\gamma \in V_k \ominus S_k$ be an eigenvector for both L_k and R_k, of norm one, with eigenvalues (λ_1, λ_2).*

(a) *If $(\lambda_1, \lambda_2) = (0,0)$, then:*

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} - (\gamma_{n,m-2} + \gamma_{n-2,m}) + \gamma_{n-2,m-2},$$

$$\gamma_{n,m} = \sum_{s,t \geqslant 0} \chi_{n-2s} \cdot \gamma \cdot \chi_{m-2t}, \quad \text{for } n, m \geqslant 0.$$

(b) *If $(\lambda_1, \lambda_2) = (\varepsilon, 0)$, $\varepsilon \in \{-1, 1\}$, then:*

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} + \varepsilon \cdot \gamma_{n-1,m} - \gamma_{n,m-2} - \varepsilon \cdot \gamma_{n-1,m-2},$$

$$\gamma_{n,m} = \sum_{s,t \geqslant 0} (-\varepsilon)^s \cdot \chi_{n-s} \cdot \gamma \cdot \chi_{m-2t}, \quad \text{for } n, m \geqslant 0.$$

(c) *If $(\lambda_1, \lambda_2) = (0, \varepsilon)$, $\varepsilon \in \{-1, 1\}$, then:*

$$\chi_n \cdot \gamma \cdot \chi_m = \gamma_{n,m} + \varepsilon \cdot \gamma_{n,m-1} - \gamma_{n-2,m} - \varepsilon \cdot \gamma_{n-2,m-1},$$

$$\gamma_{n,m} = \sum_{s,t \geqslant 0} (-\varepsilon)^t \cdot \chi_{n-2s} \cdot \gamma \cdot \chi_{m-t}, \quad n, m \geqslant 0.$$

Proof. In each of the cases (a), (b), (c), the second formula is an easy consequence of the first.

For the first formula we use (a) from Lemma 8 and induction based on formulae (10) and (11). For example, in case (a):

$$\chi_1\cdot\gamma\cdot\chi_1=\chi_1\cdot\gamma_{o,1}=\gamma_{1,1},$$
$$\chi_2\cdot\gamma\cdot\chi_1\overset{(10)}{=}\chi_1(\chi_1\cdot\gamma\cdot\chi_1)-2N\gamma\cdot\chi_1\overset{(11)}{=}$$
$$=\gamma_{2,1}+(2N-1)\gamma_{0,1}-2N\cdot\gamma_{0,1}=\gamma_{2,1}-\gamma_{0,1}.$$
■

LEMMA 10. *Let $k\geqslant 1$ and let γ be as in Lemma 9. Then $\gamma_{n,m}$ is orthogonal to $\gamma_{p,q}$ for $(n,m)\neq(p,q)$.*

Proof. Since $\gamma_{n,m}\in V_{k+n+m}$, the assertion is true for $n+m\neq p+q$. If $n+m=p+q$, then:

$$\langle\gamma_{n,m}\,|\,\gamma_{p,q}\rangle=\langle\gamma_{n,m}\,|\,q_{k+n+m}(\chi_p\cdot\gamma\cdot\chi_q)\rangle=\langle\chi_p\cdot\gamma_{n,m}\cdot\chi_q\,|\,\gamma\rangle.$$

But for every $m,n,p,q\geqslant 0$

$$\chi_p\cdot\gamma_{n,m}\cdot\chi_q\in\operatorname{span}\{\gamma_{s,t}\,\big|\,|n-s|\leqslant p,\ |m-t|\leqslant q\},$$

as one can see by induction using (10) and the fact that

$$\chi_1\cdot\gamma_{n,m}\in\operatorname{span}\{\gamma_{n-1,m};\gamma_{n,m};\gamma_{n+1,m}\},$$

which is clear from (11) and Lemma 8 (a). Hence, if $p<n$, then:

$$\chi_p\cdot\gamma_{n,m}\cdot\chi_q\in\operatorname{span}\{\gamma_{s,t}\,|\,s\geqslant n-p>0\}\subset V_k^{\perp}$$

and similarly for $p>n$, when $q<m$. ■

REMARK. The last two lemmas show that for γ as in Lemma 9, the family $\{\gamma_{n,m}\}_{n,m\geqslant 0}$ is an orthogonal basis for $\overline{\mathcal{A}\gamma}$.

LEMMA 11. *Let $k,m,n\geqslant 0$, $\gamma\in V_k$. Then:*

$$\|\gamma_{n,m}\|_2^2=(2N-1)^{n+m}\cdot\|\gamma\|_2^2.$$

Proof. Let $\gamma=\sum_{g\in V_k\cap F_N}\lambda_g\cdot g$. For $g,g'\in V_k\cap F_N$, $g\neq g'$, $\operatorname{supp}g_{n,m}\cap\operatorname{supp}g'_{n,m}=\emptyset$, hence $g_{n,m}$ is orthogonal to $g'_{n,m}$. Therefore it is enough to prove the formula for $\gamma\in V_k\cap F_N$. In this case, $\gamma_{n,m}$ is the characteristic function of its support, and the cardinal of the support is $(2N-1)^{n+m}$. ■

LEMMA 12. *Let γ be as in Lemma 9 and let $T_0:\mathcal{A}\gamma\to\mathcal{A}\omega$ be the linear mapping defined by $T_0(\chi_n\cdot\gamma\cdot\chi_m)=\chi_n\cdot\omega\cdot\chi_m$. Then T_0 is well defined and it extends to an* *invertible bounded linear operator $T : \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$. Since $T \in \mathcal{A}'$, one obtains that p_γ and p_ω are equivalent in $\mathcal{A}'$.*

Proof. The vectors $\{\chi_n\gamma\chi_m\}_{n,m\geqslant 0}$ are linearly independent because for any integer $k \geqslant 0$, $\{\chi_n\gamma\chi_m \mid n+m \leqslant k\}$ is a family of generators for $\text{span}\{\gamma_{n,m} \mid n+m \leqslant k\}$ (see Lemma 9) and has the same cardinality as the dimension of the space it generates.

If γ has eigenvalues (0,0), then $T_0\gamma_{n,m} = \omega_{n,m}$ (use Lemma 9 and (12)), hence T is unitary (due to (12), (13) and to Lemmas 10 and 11).

If γ has eigenvalues $(\varepsilon, 0)$, $\varepsilon \in \{-1, 1\}$, then:

$$T_0 \cdot \gamma_{n,m} = \omega_{n,m} - \varepsilon \cdot \omega_{n-1,m}$$

(use Lemma 9 and (12)). We define the operators $U_0 : \mathcal{A}\gamma \to \mathcal{A}\omega$ and $S_1^0 : \mathcal{A}\gamma \to \mathcal{A}\gamma$ given by $U_0\gamma_{n,m} = \omega_{n,m}$ and $S_1^0\gamma_{n,m} = \gamma_{n-1,m}$. U_0 clearly extends to a unitary $U : \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$. If we consider the unitary operator $W : \overline{\mathcal{A}\gamma} \to \ell^2(N) \otimes \ell^2(N)$ given by

$$W(\gamma_{n,m}) = (2N-1)^{-\frac{n+m}{2}}\delta_n \otimes \delta_m,$$

then

$$WS_1^0W^* = (2N-1)^{-\frac{1}{2}}S^* \otimes I,$$

where both sides are restricted to the space $W(\mathcal{A}\gamma)$ and S is the unilateral shift. Therefore S_1^0 has an extension $S_1 : \overline{\mathcal{A}\gamma} \to \overline{\mathcal{A}\omega}$ too, and $\|S_1\| = (2N-1)^{-\frac{1}{2}}$.

Since $T_0 = U(I - \varepsilon S_1)\big|_{\mathcal{A}\gamma}$, the conclusion follows.

The cases of eigenvalues $(0, \varepsilon)$, $\varepsilon \in \{-1, 1\}$ is similar. ∎

Remark. The induction step is now completely proved. For $k \geqslant 1$ we can choose a basis for $V_k \ominus S_k$ made from eigenvectors for both L_k and R_k (by Lemma 8), and Lemma 12 shows that the cyclic projections associated to the vectors of this basis are equivalent to p_ω. Hence, if we admit that $\text{span}\{V_{k'} \mid k' \leqslant k-1\}$ is covered by projections equivalent to p_ω, then Lemma 7 and the above arguments show that V_k can be covered by projections equivalent to p_ω too.

We shall study now the spaces V_0 for the families of cores $C \in \mathcal{C} \setminus \{C_0\}$.

Let $V_0 = \text{span}C(i, j, \alpha, \beta, w)$ (see (4)). Consider the vectors of norm one

$$\xi^{\varepsilon,\delta} = \frac{1}{2}(a_i^\alpha + \varepsilon \cdot a_i^{\alpha-r_i})w(a_j^\beta + \delta \cdot a_j^{\beta-r_j}),$$

where $\varepsilon, \delta \in \{-1, 1\}$. It is obvious that

$$\text{span}\{\xi^{\varepsilon,\delta} \mid \varepsilon, \delta \in \{-1, 1\}\} = V_0.$$

Fix now a pair $(\varepsilon, \delta) \in \{-1, 1\}^2$ and denote $\xi^{\varepsilon,\delta}$ by ξ. We shall prove that p_ξ is equivalent to p_ω.

LEMMA 13. (a)

$$\chi_n \cdot \xi \cdot \chi_m = \xi_{n,m} + \varepsilon \cdot \xi_{n-1,m} + \delta \cdot \xi_{n,m-1} + \varepsilon \cdot \delta \cdot \xi_{n-1,m-1},$$

$$\xi_{n,m} = \sum_{s,t \geqslant 0} (-\varepsilon)^s \cdot (-\delta)^t \chi_{n-s} \cdot \xi \cdot \chi_{m-t}, \quad \text{for } n, m \geqslant 0.$$

(b) *If $(m, n) \neq (p, q)$, where $m, n, p, q \in \mathbb{N}$, then $\xi_{n,m}$ is orthogonal to $\xi_{p,q}$, hence the family $\{\xi_{n,m}\}_{n,m\geqslant 0}$ is an orthogonal basis for $\overline{\mathcal{A}\xi}$.*

(c) $\|\xi_{n,m}\|^2 = (2N-1)^{n+m}$, *for $n, m \geqslant 0$.*

Proof. (a) Denote $\eta = a_i^\alpha + \varepsilon \cdot a_i^{\alpha - r_i}, \zeta = a_j^\beta + \delta a_j^{\beta - r_j}$. Since all the words which appear in the support of $\chi_p \cdot \eta$ end with a nonzero power of a_i (and similarly for $\zeta \cdot \chi_q$), one has

$$(\eta \cdot w \cdot \zeta)_{p,q} = \eta_{p,0} \cdot w \cdot \zeta_{0,q}, \quad \text{for } p, q \text{ integers.} \tag{15}$$

For $1 \leqslant k \leqslant r_i - 1$ one has

$$\chi_n \cdot a_i^k = (a_i^k)_{n,0} + (a_i^{k-r_i})_{n-1,0},$$

hence:

$$\chi_n \cdot \eta = \eta_{n,0} + \varepsilon \cdot \eta_{n-1,0}, \tag{16}$$

and similar formulae hold for $\zeta \cdot \chi_m$.

Now, (15) and (16) give the first formula. The second formula follows from the first.

(b) See the proof of Lemma 10.

(c) See Lemma 11. ■

LEMMA 14. *Consider the linear mapping $T_0 : \mathcal{A}\xi \to \mathcal{A}\omega$ defined by $T_0(\chi_n \cdot \gamma \cdot \cdot \chi_m) = \chi_n \cdot \omega \cdot \chi_m$. Then T_0 is well defined and it extends to an invertible bounded linear operator $T : \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\omega}$. Since $T \in \mathcal{A}'$, p_γ and p_ω are equivalent in $\mathcal{A}'$.*

Proof. That T_0 is well defined follows as in Lemma 12.

The formula of T_0 with respect to the basis $\{\xi_{n,m}\}_{n,m\geqslant 0}$ is

$$T_0 \cdot \xi_{n,m} = \omega_{n,m} - \varepsilon \cdot \omega_{n-1,m} - \delta \cdot \omega_{n,m-1} + \varepsilon \cdot \delta \cdot \omega_{n-1,m-1}.$$

Consider the operators $S_1, S_2 : \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\xi}$, $U : \overline{\mathcal{A}\xi} \to \overline{\mathcal{A}\omega}$ given by

$$S_1 \xi_{n,m} = \xi_{n-1,m}, \; S_2 \xi_{n,m} = \xi_{n,m-1}, \; U\xi_{n,m} = \omega_{n,m}, \quad \text{for } n, m \geqslant 0.$$

As in the proof of Lemma 12, U is unitary and $\|S_1\| = \|S_2\| = (2N-1)^{-\frac{1}{2}}$. Since

$$T_0 = U(I - \varepsilon S_1)(I - \delta S_2)\,\big|_{\mathcal{A}\xi},$$

the lemma holds. ∎

Take now $V_0 = \operatorname{span} C(i,\alpha)$ (see (5)). We introduce the vectors of norm one

$$\eta^\varepsilon = \frac{1}{\sqrt{2}}(a_i^\alpha + \varepsilon \cdot a_i^{\alpha - r_i}), \quad \text{where} \quad \varepsilon \in \{-1, 1\}.$$

It is obvious that

$$V_0 = \operatorname{span}\{\eta^\varepsilon \mid \varepsilon \in \{-1,1\}\}.$$

Fix $\varepsilon \in \{-1,1\}$ and denote η^ε by η. We shall prove that p_η is equivalent to p_ω, but in this case, instead of an invertible operator $T : \overline{\mathcal{A}\eta} \to \overline{\mathcal{A}\omega}$ from $\mathcal{A}'$, we have only a closed, densely defined injective operator with dense range, which is affiliated to $\mathcal{A}'$.

LEMMA 15. *Let* $\tau = \frac{1}{\sqrt{2}}(a_i^{\alpha + r_i} + \varepsilon \cdot a_i^{\alpha - 2r_i}) \in V_1$ *and introduce the notations:*

$$\overline{\eta}_{n,m} = \eta_{n,m} - \varepsilon \cdot \tau_{n-1,m-1};$$

$$\overline{\overline{\eta}}_{n,m} = \sum_{s \geqslant 0} \overline{\eta}_{n-s,m-s}.$$

Then:

(a) $\chi_n \cdot \eta \cdot \chi_m = \overline{\overline{\eta}}_{n,m} + \varepsilon(\overline{\overline{\eta}}_{n-1,m} + \overline{\overline{\eta}}_{n,m-1}) + \overline{\overline{\eta}}_{n-1,m-1}$,

$$\overline{\eta}_{n,m} = \chi_n \cdot \eta \cdot \chi_m + \sum_{s \geqslant 1} (-\varepsilon)^s \cdot (\chi_{n-s} \cdot \eta \cdot \chi_m + \chi_n \cdot \eta \cdot \chi_{m-s}).$$

(b) $\langle \eta_{n,m} \mid \eta_{p,q} \rangle = \begin{cases} 0, & \text{if } n+m \neq p+q, \\ 2(2N-1)^{n+m-|n-p|}, & \text{if } n+m = p+q. \end{cases}$

(c) $\{\overline{\eta}_{n,m}\}_{n,m \geqslant 0}$ *is a basis for* $\mathcal{A}\eta$.

(d) *If* $|(n+m)-(p+q)| \geqslant 2$, *then* $\overline{\eta}_{n,m}$ *is orthogonal to* $\overline{\eta}_{p,q}$.

Proof. (a) We prove succesively the following formulae (each of them has one similar for the right action of χ_1, which we do not write):

(i) $\chi_1 \cdot \eta_{0,0} = \eta_{1,0} + \varepsilon \cdot \eta_{0,0}$;

(ii) $\chi_1 \cdot \eta_{0,1} = \eta_{1,1} + \varepsilon \cdot \eta_{0,1} + \eta_{0,0} - \varepsilon \cdot \tau_{0,0}$;

(iii) $\chi_1 \cdot \tau_{0,0} = \tau_{1,0} + \eta_{0,0}$;

(iv) $\chi_1 \cdot \tau_{0,1} = \tau_{1,1} + \eta_{0,1}$;

(v) $\chi_1 \cdot \tau_{0,n} = \tau_{1,n} + \eta_{0,n}$, for $n \geqslant 0$;

(by induction, using (iv) and (11))

(vi) $\chi_1 \cdot \eta_{0,n} = \eta_{1,n} + \varepsilon \cdot \eta_{0,n} + \eta_{0,n-1} - \varepsilon \cdot \tau_{0,n-1}$, for $n \geqslant 0$;

(by induction, using (i), (ii), (iii) and (11)).

Using (v) and (vi) we can proceed as in Lemma 9, and we obtain the first formula. The second formula is a consequence of the first.

(b) The case $n + m \neq p + q$ is obvious. For $n + m = p + q$, since $\eta_{s,t}$ is the characteristic function of its support, we have to compute the cardinality of the intersection $\operatorname{supp} \eta_{n,m} \cap \operatorname{supp} \eta_{p,q}$. But:

$$\operatorname{supp} \eta_{s,t} = \operatorname{supp} (a_i^{\alpha})_{s,t} \cup \operatorname{supp} (a_i^{\alpha - r_i})_{s,t},$$

the union being disjoint, and:

$$\operatorname{supp} (a_i^{\alpha})_{n,m} \cap \operatorname{supp} (a_i^{\alpha - r_i})_{p,q} = \emptyset,$$
$$\operatorname{card}[\operatorname{supp} (a_i^{\alpha})_{n,m} \cap \operatorname{supp} (a_i^{\alpha})_{p,q}] = (2N-1)^{n+m-|n-p|},$$
$$\operatorname{card}[\operatorname{supp} (a_i^{\alpha - r_i})_{n,m} \cap \operatorname{supp} (a_i^{\alpha - r_i})_{p,q}] = (2N-1)^{n+m|n-p|}.$$

(c) The family $\{\overline{\eta}_{n,m}\}_{n,m \geqslant 0}$ is a system of generators for $\mathcal{A}\eta$ due to the relations from (a).

Assume that:

$$\sum_{k=0}^{K} \sum_{s=0}^{k} \lambda_{s,k-s} \cdot \overline{\eta}_{s,k-s} = 0, \quad \text{for } K \geqslant 0 \text{ and } \lambda_{s,t} \in \mathbb{C}.$$

We apply q_K, the projection on V_K, and we obtain:

$$\sum_{s=0}^{K} \lambda_{s,K-s} \cdot \eta_{s,K-s} = 0.$$

But the family $\{\eta_{s,k-s}\}_{0 \leqslant s \leqslant k}$ is linearly independent for any $k \geqslant 0$ (because $\det(\langle \eta_{s,k-s} \mid \eta_{t,k-t} \rangle)_{0 \leqslant s,\ t \leqslant k}$ is nonzero by [11], Lemma 5), hence $\lambda_{s,K-s} = 0$ for any $0 \leqslant s \leqslant K$. Therefore, an induction argument shows that the given family is linearly independent.

(d) Use that $\overline{\eta}_{n,m} \in V_{m+n} + V_{n+m-1}$. ■

LEMMA 16. *Consider the linear mapping $W_0 : \mathcal{A}\eta \to \mathcal{A}\omega$ defined by $W_0 \cdot \overline{\eta}_{n,m} = \omega_{n,m}$, for $n, m \geqslant 0$. Then W_0 is a well defined closable operator on $\overline{\mathcal{A}\eta}$ into $\overline{\mathcal{A}\omega}$, and its closure W is densely defined, injective with dense range.*

Proof. By Lemma 15, (c), and (12), (13), W_0 is bijective. We shall prove that both W_0 and W_0^{-1} are closable and then the lemma follows. (If W_0 is closable and bijective, then W is injective if and only if W_0^{-1} is closable).

W_0 and W_0^{-1} would clearly be closable if $\{\overline{\eta}_{n,m}\}_{n,m\geqslant 0}$ were an orthogonal basis of $\mathcal{A}\eta$. Fortunately, our situation is not too far from this.

For $n \geqslant 1$, denote:

$$P_n = \operatorname{proj}^{\overline{\mathcal{A}\eta}}_{\operatorname{span}\{\overline{\eta}_{s,t}\,|\,s+t\leqslant n\}},$$
$$P'_n = \operatorname{proj}^{\overline{\mathcal{A}\eta}}_{\overline{\operatorname{span}\{\overline{\eta}_{s,t}\,|\,s+t\geqslant n\}}},$$
$$Q_n = \operatorname{proj}^{\overline{\mathcal{A}\omega}}_{\operatorname{span}\{\overline{\omega}_{s,t}\,|\,s+t\leqslant n\}},$$

and put I for the identity operator both on $\overline{\mathcal{A}\eta}$ and on $\overline{\mathcal{A}\omega}$.

It is not hard to see that P_n and P'_{n+2} are orthogonal projections and $I-P_n-P'_{n+2}$ is a finite dimensional projection.

Note that $P'_n(\mathcal{A}\eta) \subset \mathcal{A}\eta$ because for all $s,t \geqslant 0$,

$$P'_n(\overline{\eta}_{s,t}) = \begin{cases} 0, & \text{if } s+t \leqslant n-2,\\ \operatorname{proj}^{\overline{\mathcal{A}\eta}}_{\operatorname{span}\{\overline{\eta}_{p,q}\,|\,0\leqslant p,q,\ p+q=n\}}(\overline{\eta}_{s,t}), & \text{if } s+t = n-1,\\ \overline{\eta}_{s,t}, & \text{if } s+t \geqslant n.\end{cases}$$

In the following we shall consider all the operators restricted to $\mathcal{A}\eta$, respectively $\mathcal{A}\omega$.

It is easy to verify that

$$W_0 \cdot P_n = Q_n \cdot W_0 \cdot P_n, \quad Q_n \cdot W_0 \cdot P'_{n+1} = 0, \text{ hence } Q_n \cdot W_0 \cdot P'_{n+2} = 0.$$

Then

$$(17)\quad Q_n\cdot W_0 = Q_n\cdot W_0\cdot P_n + Q_n\cdot W_0\cdot P'_{n+2} + Q_n\cdot W_0(I-P_n-P'_{n+2}) = W_0\cdot P_n + X_n,$$

where $X_n = Q_n \cdot W_0(I-P_n-P'_{n+2})$ and W_0P_n are bounded operators.

Since $W_0^{-1}\cdot\omega_{p,q} = \eta_{p,q}$, one has:

$$W_0^{-1}Q_n = P_nW_0^{-1}Q_n,$$

$$P_n\cdot W_0^{-1}(I-Q_{n+1}) = 0,$$

hence

$$P_n\cdot W_0^{-1} = P_n\cdot W_0^{-1}Q_n + P_n\cdot W_0^{-1}(Q_{n+1}-Q_n) + P_n\cdot W_0^{-1}(I-Q_{n+1}) = W_0^{-1}\cdot Q_n + Y_n$$

where $Y_n = P_n\cdot W_0^{-1}(Q_{n+1}-Q_n)$ and $W_0^{-1}\cdot Q_n$ are again bounded operators.

We now can show that W_0 and W_0^{-1} are closable operators. The proof is given only for W_0, for W_0^{-1} the arguments being similar. Take $x_i \in \mathcal{A}\xi$ $(i \geqslant 0)$ such that

$\lim_{i\to\infty} x_i = 0$ and $y_i = W_0\, x_i \xrightarrow[i]{} y$. We have to show that $y = 0$. Fix $n \geqslant 1$. Then, by (17)

$$Q_n y_i = Q_n \cdot W_0 \cdot x_i = (W_0 P_n + X_n) x_i \xrightarrow[i]{} 0$$

because $W_0 \cdot P_n$ and X_n are bounded operators. But $y_i \xrightarrow[i]{} y$, hence

$$Q_n y = \lim_{i\to\infty} Q_n y_i = 0.$$

Since $Q_n \xrightarrow[n]{\text{s.o.}} I$, one has $y = 0$. ∎

LEMMA 17. *Consider the linear mapping $T_0 : \mathcal{A}\eta \to \mathcal{A}\omega$ given by $T_0(\chi_n \cdot \eta \cdot \chi_m) = \chi_n \cdot \omega \cdot \chi_m$. Then T_0 is a well defined closable operator. Its closure $T : \mathcal{D}_T \subset \overline{\mathcal{A}\eta} \to \overline{\mathcal{A}\omega}$ is densely defined, injective, with dense range. Since T is affiliated to $\mathcal{A}'$, p_η and p_ω are equivalent.*

Proof. That T_0 is well defined follows as in Lemma 12, using Lemma 15 (a) and (c). The formula of T_0 with respect to the basis $\{\overline{\eta}_{n,m}\}_{n,m} \geqslant 0$ of $\mathcal{A}\eta$ is:

$$T_0 \overline{\eta}_{n,m} = \omega_{n,m} - \varepsilon(\omega_{n-1,m} + \omega_{n,m-1}) + \varepsilon(\omega_{n-1,m-2} + \omega_{n-2,m-1}) - \omega_{n-2,m-2}.$$

Consider the operators $S_1, S_2 : \mathcal{A}\omega \to \mathcal{A}\omega$, $S_1\omega_{n,m} = \omega_{n-1,m}$, $S_2\omega_{n,m} = \omega_{n,m-1}$, and the operator $W_0 : \mathcal{A}\eta \to \mathcal{A}\omega$, $W_0\overline{\eta}_{n,m} = \omega_{n,m}$ considered in Lemma 16. Then $\|S_1\| = \|S_2\| = (2N-1)^{-\frac{1}{2}}$ as in Lemma 12, and:

$$T_0 = [I - \varepsilon(S_1 + S_2) + \varepsilon S_1 S_2 (S_1 + S_2) - S_1^2 S_2^2] \cdot W_0 \big|_{\mathcal{A}\eta} =$$

$$= (I - S_1 S_2)(I - \varepsilon S_1)(I - \varepsilon S_2) W_0 \big|_{\mathcal{A}\eta}.$$

Since $\|S_1\| = \|S_2\| < 1$, $(I - S_1S_2)(I - \varepsilon S_1)(I - \varepsilon S_2)$ is invertible and then Lemma 16 gives the desired conclusion. ∎

From the whole Section 2, one obtains:

THEOREM 18. *The von Neumann algebra $\mathcal{A}' = \{\lambda(\chi_1), J\lambda(\chi_1)J\}' \subset B(\ell^2(F_N))$ is of homogeneous type I_∞ on $I - p_1$. Consequently, the algebra $A = \{\lambda(\chi_1)\}'' \subset L(F_N)$ is singular.*

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