J. OPERATOR THEORY
30(1993), 69–75

ON COMMUTANTS OF SUBNORMAL OPERATORS

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0. INTRODUCTION

If T is a bounded linear operator on a complex Hilbert space $\mathcal{H}$, let $\mathcal{W}(T)$ denote the weak closure of the polynomials in T and let $\{T\}'$ be the commutant of T. If $\mathcal{H}$ is finite dimensional then it is well known that these conditions are equivalent:

i) T is cyclic;

ii) $\mathcal{W}(T) = \{T\}'$;

iii) $\{T\}'$ is abelian.

Each of these conditions could be taken as a definition of "T has multiplicity one".

If $\mathcal{H}$ is infinite dimensional (and separable), then of course (ii) still implies (iii), but all other implications are false. In fact, if $\mu(\mathcal{A})$ denotes the cyclic multiplicity of an algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ (i.e., $\mu(\mathcal{A}) = \inf\{n : \exists f_1, f_2, \dots, f_n \in \mathcal{H}$ so $\mathcal{A}f_1 + \dots + \mathcal{A}f_n$is dense in$\mathcal{H}\}$), then there are operators T so that $\mathcal{W}(T) = \{T\}'$ but $\mu(\mathcal{W}(T)) = \infty$. See [4]. Herrero says that T has a tiny commutant if $\mathcal{W}(T) = \{T\}'$. T above can be taken to be a certain forward shift with operator weights ([4]). Further, if T is the backward shift of infinite multiplicity, it is an old result of Sarason that $\mu(\mathcal{W}(T)) = 1$, while $\{T\}'$ is highly nonabelian. See also [8], [12].

Motivated by multiplicity theory for normal operators, it is natural to seek a corresponding theory for subnormal operators.

Thus let S be a subnormal operator with minimal normal extension N. (Write $N = \operatorname{mne} S$.) We first need a notation of multiplicity one for subnormals. To the conditions (i), (ii), and (iii), we add

(iv) N is cyclic.

We review some known results in this setting. If S is cyclic, then $S \cong S_\nu$ for some compactly supported measure ν in the plane ([1], Cor. 5.3]). Here S_ν denotes multiplication by z on $P^2(\nu)$, the closure in $L^2(\nu)$ of the polynomials. Also mne $S_\nu = N_\nu$ = multiplication by z on $L^2(\nu)$.

Yoshino ([13], ([1, Cor. 5.4])) showed that $\{S_\nu\}' = \{M_\varphi : \varphi \in P^2(\nu) \bigcap L^\infty(\nu)\}$. Also ([1, Cor. 12.10]), $\mathcal{W}(S_\nu) = \{M_\varphi : \varphi \in P^\infty(\nu)\}$. Thus, $\{S_\nu\}'$ is abelian but $\mathcal{W}(S_\nu) \neq \{S_\nu\}'$ in general.

If S is a subnormal bilateral weighted shift which is not invertible, then $\mathcal{W}(S) = \{S\}'$ ([1, Prop. 8.22]), but S is not cyclic ([3], [1, Prop. 8.23]). In fact, $\mu(\mathcal{W}(S)) = 2$ ([6, Th. 5]). Also N is cyclic in this case.

Motivated by these and other examples, J. McCarthy raised the following question ([7, Conjecture, p.267]).

Question 1. If S is subnormal and $\mathcal{W}(S) = \{S\}'$, must mne S be cyclic?

In addition we consider the related question.

Question 2. If S is subnormal, is $\mu(\{S\}') \leqslant 2$?

We note that many operators, including all normal operators, have cyclic commutants ([5]). As noted earlier, ([4]), there are operator weighted shifts T so that $\mu(\{T\}') = \infty$. For S subnormal and not cyclic, very little seems to be known about $\{S\}'$. All of the known previous examples satisfy $\mu(\{S\}') \leqslant 2$.

The main result of this note is the construction of examples which settle both questions in the negative. For each n, $1 \leqslant n \leqslant \infty$, there is a (pure) subnormal operator S_n so that $\mathcal{W}(S_n) = \{S_n\}'$ and so $\mu(\mathcal{W}(S_n)) = \mu(\mathcal{W}(\text{mne}\, S_n)) = n$.

Thus among the conditions (i)–(iv) for subnormal operators only the implications (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv) are valid. There remains the general question of what can be said in general about the commutant of a subnormal operator.

THE EXAMPLE

Fix a natural number n. We will construct a (pure) subnormal operator S with minimal normal extension N so that

$$n = \mu\{\mathcal{W}(S)\} = \mu(\mathcal{W}(N)) \text{ and } W(S) = \{S\}'.$$

We begin by choosing $n+1$ positive measure $\nu_0, \nu_1, \ldots, \nu_n$ supported in the annulus $\{z : \frac{1}{2} \leqslant |z| \leqslant 1\}$ with the following properties:

(a) Each ν_j is circularly symmetric. That is, $\mathrm{d}\nu_j(re^{i\theta}) = \mathrm{d}\rho_j(\nu)\mathrm{d}\theta$ for some positive measure ρ_j supported in $\left[\frac{1}{2}, 1\right]$;

(b) $\nu_0 \perp \nu_j$ for $1 \leqslant j \leqslant n$ and $\rho_j \sim \rho_l$ for $1 \leqslant j, l \leqslant n$. ($\perp$ denotes mutual singularity and $\sim$ denotes mutual absolute continuity);

(c) The unit circle is in the support of each ν_j;

(d) For each $j, 0 \leqslant j \leqslant n$, we have

$$\sup\left\{\frac{\|z^m\|_{L^2(\nu_j)}}{\sum\limits_{l\neq j}\|z^m\|_{L^2(\nu_l)}} : m = 0, 1, 2, \ldots\right\} = \infty.$$

Of course the crucial property is (d). We postpone the construction of the these measures until the end of the paper.

For a compactly supported measure ν in $\mathbb{C}$, let $P^2(\nu)$ be the $L^2(\nu)$-closure of the polynomials. Let N_ν be the normal operator of multiplication by z on $L^2(\nu)$ and let $S_\nu = N_\nu|P^2(\nu)$. Then S_ν is a cyclic subnormal operator and $N_\nu = \operatorname{mne} S_\nu$. (See [1].)

For each ν_j as above, S_{ν_j} is subnormal weighted shift ([1, Th. 8.16], [10, Prop. 25]). Also note that each $F \in P^2(\nu_j)$ is analytic on the open disk D ([1, Prop. 8.19]).

We consider an extension of property (d).

LEMMA. Fix j, $0 \leqslant j \leqslant n$, *and suppose* $F \in P^2(\nu_j)$, $F \neq 0$. *Then,*

$$\sup\left\{\frac{\|z^m F(z)\|_{L^2(\nu_j)}}{\sum\limits_{l\neq j}\|z^m\|_{L^2(\nu_l)}} : m = 0, 1, 2, \ldots\right\} = \infty.$$

Proof. Since $F \in P^2(\nu_j)$, F is analytic on D. Thus if $M(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} |F(re^{i\theta}|^2 d\theta$, then M is a positive increasing function of r. Thus

$$\|z^m F(z)\|^2_{L^2(\nu_j)} = \int |z^m F(z)|^2 d\nu_j = \int\limits_{\frac{1}{2}}^{1} r^{2m} M(r) d\rho_j \geqslant M(1/2)\|z^m\|_{L^2(\nu_j)}.$$

Now apply (d).

The Lemma implies that if F is analytic on D and if the mapping defined on polynomials by $p \mapsto Fp$ extends to a bounded linear operator from $P^2(\nu_l)$ to $P^2(\nu_j)$, $j \neq l$, then $F = 0$.

Let N be the normal operator on $\bigoplus\limits_0^n L^2(\nu_j)$ defined by $N = \bigoplus\limits_0^n N_{\nu_j}$. We will construct our example S by restricting N to a certain invariant subspace $\mathcal{H}$, where $\mathcal{H} \subset \bigoplus\limits_0^n P^2(\nu_j)$. It is instructive to see why $T = \bigoplus\limits_0^n S_{\nu_j}$ fails to be the desired example. Each S_{ν_j} is a (cyclic) weighted shift, so T is an operator weighted shift. The range of T has condimension $n+1$ and it is easy to see ([1, Cor. 12.10]) that $\mathcal{W}(T) = \{\varphi(T) = \varphi(S_{\nu_0}) \oplus \cdots \oplus \varphi(S_{\nu_n}) : \varphi \in H^\infty\}$ and that $\mu(\mathcal{W}(T)) = n+1$. We will show in Claim 2 to follow that $\mu(\mathcal{W}(N)) = n$. Finally, the Lemma can be used to show that $\{T\}'$ splits. That is,

$$\{T\}' = \bigoplus_0^n \{S_{\nu_j}\}' = \{\varphi_0(S_{\nu_0}) \oplus \ldots \oplus \varphi_n(S_{\nu_n}) : \varphi_0, \ldots, \varphi_n \in H^\infty\}.$$

Thus $\{T\}'$ is cyclic and properly contains $\mathcal{W}(T)$.

For $f \in \bigoplus_0^n P^2(\nu_j)$, write $f = \bigoplus_0^n f(j)$, where $f(j) \in P^2(\nu_j)$. This notation will allow us to avoid double subscripts. Now for $1 \leqslant k \leqslant n$, define $f_k \in \bigoplus_0^n P^2(\nu_j)$ by $f_k(j) = 1$ if $j = 0$ or k and $f_k(j) = 0$ otherwise. Let $\mathcal{M}_k$ be the cyclic invariant subspace for N generated by f_k. Note that $N|\mathcal{M}_k \cong S_{\nu_0+\nu_k}$. Let $\mathcal{H}$ be the closed span of $\{\mathcal{M}_k\}_1^n$ and let $S = N|\mathcal{H}$. S bears some resemblance to the "nonexample" T. However, while T is an orthogonal direct sum, the Lemma shows that each pair of "summands" $\mathcal{M}_j$ and $\mathcal{M}_l$ meet at angle zero. ■

CLAIM 1. $N = \operatorname{mne} S$.

Proof. For $1 \leqslant k \leqslant n$, let $\mathcal{N}_k$ be the reducing subspace for N generated by f_k. Then $N|\mathcal{N}_k \cong N_{\nu_0+\nu_k} \cong N_{\nu_0} \oplus N_{\nu_k}$, since $\nu_0 \perp \nu_k$. Thus $\mathcal{N}_k = L^2(\nu_0) \oplus L^2(\nu_k)$, viewed as a subspace of $\bigoplus_0^n L^2(\nu_j)$. Thus the span of $\{\mathcal{N}_k\}_1^n$ is $\bigoplus_0^n L^2(\nu_j)$, so that $N = \operatorname{mne} S$. ■

CLAIM 2. $n = \mu(\mathcal{W}(S)) = \mu(\mathcal{W}(N))$.

Proof. First note that for any subnormal operator S, $\mu(\mathcal{W}(S)) \geqslant \mu(\mathcal{W}(\operatorname{mne} S))$. Also by contruction of S, we see that $\mu(\mathcal{W}(S)) \leqslant n$. Since $\nu_1 \sim \nu_j$ for $1 \leqslant j \leqslant n$, multiplicity theory for normal operators shows that $N \cong N_{\nu_0} \oplus (\bigoplus_1^n N_{\nu_1})$. Thus $\mu(\mathcal{W}(N)) = n$. This establishes the claim. ■

CLAIM 3. $\mathcal{W}(S) = \{S\}'$.

Proof. Let $A \in \{S\}'$. For $1 \leqslant k \leqslant n$, let $F_k = Af_k$. Our goal is to show that there is a function $\varphi \in \mathcal{H}^\infty$ so that $F_k(j) = \varphi f_k(j)$ for all k and j.

For $m \geqslant 0$, $\|S^m f_k\|^2 = \|z^m\|^2_{L^2(\nu_0)} + \|z^m\|^2_{L^2(\nu_k)}$ while

$$\|AS^m f_k\|^2 = \|S^m F_k\|^2 \geqslant \|z^m F_k(j)\|^2_{L^2(\nu_j)}$$

for each j, $0 \leqslant j \leqslant n$. Since A is bounded, the Lemma implies that $F_k(j) = 0$ unless $j = 0$ or k.

A similar argument shows that $F_k(0) = F_l(0)$ for $1 \leqslant k \neq l \leqslant n$. In fact, let $g = f_k - f_l$. Then $g(k) = 1 \in P^2(\nu_k)$, $g(l) = -1 \in P^2(\nu_l)$, and $g(j) = 0$ if $j \neq k$ or l. Thus for $m \geqslant 0$, $\|S^m g\|^2 = \|z^m\|^2_{L^2(\nu_k)} + \|z^m\|^2_{L^2(\nu_l)}$, and $\|AS^m g\|^2 = \|S^m(F_k - F_l)\|^2 \geqslant \|z^m(F_k(0) - F_l(0)\|^2_{L^2(\nu_0)}$. Thus the Lemma shows $F_k(0) = F_l(0)$.

Next we show that for $1 \leqslant k \leqslant n$, we have $F_k(0) = F_k(k)$ as analytic functions on D. Since $F_k \in \mathcal{H}$, F_k is in the closure of $\mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_n$. Thus there are n sequences $\{p_m^{(1)}, \ldots, \{p_m^{(n)}\}\}$ of polynomials so that

$$\lim_{m\to\infty} \sum_{j=1}^{n} p_m^{(j)} = F_k(0) \quad \text{in } P^2(\nu_0),$$

$$\lim_{m\to\infty} p_m^{(k)} = F_k(k) \quad \text{in } P^2(\nu_k),$$

$$\lim_{m\to\infty} p_m^{(j)} = 0 \quad \text{in } P^2(\nu_k), \text{ for } 1 \leqslant j \leqslant n,\ j \neq k.$$

Since convergence in $P^2(\nu_j)$ implies uniform convergence on compact subsets of D [1, Prop. 8.19], we see that $\sum_{j=1}^{n} p_m^{(j)}$ converges pointwise on D to both $F_k(0)$ and $F_k(k)$.

We have shown that the $2n$ functions of form $F_k(0)$ or $F_k(k)$, $1 \leqslant k \leqslant n$, all coincide on D. Denote this function by φ. Since $\varphi \in P^2(\nu_j)$ for each j and since $\{z^m\}_0^\infty$ is an orthogonal basis for $P^2(\nu_j)$, the Taylor series of φ at 0 converges in $P^2(\nu_j)$ to φ. Thus $Af_k = F_k \in \mathcal{M}_k$, $1 \leqslant k \leqslant n$, and $\mathcal{M}_k$ is hyperinvariant for S. Now a standard argument [1, p.147] shows that $\varphi \in \mathcal{H}^\infty$.

We see that $A = \varphi(N)|\mathcal{H}$, $\varphi(N) \in \mathcal{W}(N)$ so $\varphi(N)|\mathcal{H} \in \mathcal{W}(N|\mathcal{H}) = \mathcal{W}(S)$, and $\{S\}' = \mathcal{W}(S)$. ∎

REMARK 1. It is easy to see that S is in fact an operator weighted shift. Let $\mathcal{N}$ be the span of $\{f_k\}_1^n$. Then the subspaces $\{S^m\mathcal{N}\}_0^\infty$ are pairwise orthogonal, $\mathcal{H} = \bigoplus_0^\infty S^m\mathcal{N}$, and S acts as subnormal operator weighted shift relative to this decomposition. While it is known [4], that operator weighted shifts can have tiny commutants, it is perhaps surprising that such shifts can be taken to be subnormal.

REMARK 2. These examples can be strenghtened slightly. If S_n denotes the example with cyclic multiplicity n, let

$$S = \bigoplus_1^\infty (2^{-n-1}S_n + 2^{-n+1}I).$$

Then ([2]) the spectra of summands of S are sufficiently disjoint that $\mathcal{W}(S) = \bigoplus_1^\infty \mathcal{W}(S_n) = \bigoplus_1^\infty \{S_n\}' = \{S\}'$. Also $\mu(\mathcal{W}(S)) = \mu(\mathcal{W}(\text{mne}\, S)) = \infty$.

We close this note with a construction of measures ν_j, $0 \leqslant j \leqslant n$ satisfying (a)-(d). Since $d\nu_j = d\rho_j d\theta$, it is enough to construct positive Borel measures ρ_j, $0 \leqslant j < n$, supported in $\left[\frac{1}{2}, 1\right]$ so that

(b)′ $\rho_0 \perp \rho_j$ for $1 \leqslant j \leqslant n$ and $\rho_j \sim \rho_l$ for $1 \leqslant j, l \leqslant n$;

(c)′ 1 is in the support of each ρ_j, $0 \leqslant j \leqslant n$;

(d)′ For each $j, 0 \leqslant j \leqslant n$, we have

$$\sup \left\{ \frac{\|t^m\|_{L^2(\rho_j)}}{\sum_{l \neq j} \|t^m\|_{L^2(\rho_l)}} : m = 0, 1, 2, \ldots \right\} = \infty.$$

Let δ_r denote point mass at r. Choose a positive sequence $\{a_k\}$ so that $a_k \geqslant k \sum_{k+1}^{\infty} a_j$ for each $k \geqslant 1$. For example, we can take $a_k = k^{-k}$. Each measure ρ_j will have form $\rho_j = \sum_{k=1}^{\infty} b_k(j)\delta_{r_k}$. We first specify the coefficients $b_k(j)$.

Let $b_k(0) = 0$ if k is even and $b_k(0) = a_k$ if k is odd. For each fixed j, $1 \leqslant j \leqslant n$, let $b_k(j) = 0$ if k is odd. For k even, let $b_k(j) = a_k$ if $\frac{k}{2} = j \pmod n$ and let $b_k(j) = \frac{a_k}{k}$ if $\frac{b}{2} \neq j \pmod n$. Thus $0 \leqslant b_k(j) \leqslant a_k$ for all j and k, so each ρ_j is a finite measure. The choice of the $b_k(j)$ shows (b)′ holds.

It remains to specify $\{r_k\}_1^{\infty}$. This sequence will be defined inductively, satisfying $r_k \to 1$. Thus (c)′ will hold. Note that $\|t^m\|^2_{L^2(\rho_j)} = \sum_1^{\infty} b_k(j) r_k^{2m}$. Thus for each k,

$$(*) \qquad b_k(j) r_k^{2m} \leqslant \|t^m\|^2_{L^2(\rho_j)} \leqslant \sum_1^{k-1} a_i r_i^{2m} + b_k(j) r_k^{2m} + \sum_{k+1}^{\infty} a_i.$$

Let $r_1 = \frac{1}{2}$. If $r_2, \ldots, r_{k-1}$ in $\left(\frac{1}{2}, 1\right)$ have been specified, we choose r_k as follows. First choose $m = m_k$ large enough that $\sum_1^{k-1} a_i r_i^{2m_k} < \frac{a_k}{k}$. Then choose r_k so $1 - \frac{1}{k} < r_k < 1$ and $r_k^{2m_k} > \frac{1}{2}$.

To see that (d)′ holds, note that if k is odd, then using $(*)$, $\|t^{m_k}\|^2_{L^2(\rho_0)} \geqslant \frac{a_k}{2}$, and for $1 \leqslant l \leqslant n$, $\|t^{m_k}\|^2_{L^2(\rho_l)} \leqslant \frac{a_k}{k} + 0 + \frac{a_k}{k}$. Thus

$$\frac{\|t^{m_k}\|_{L^2(\rho_0)}}{\sum_{l=1}^{n} \|t^{m_k}\|_{L^2(\rho_l)}} \geqslant \frac{\sqrt{k}}{2n}.$$

For j fixed, $1 \leqslant j \leqslant n$, and k even with $\frac{k}{2} = j \pmod n$, we have $\|t^{m_k}\|_{L^2(\rho_j)} \geqslant \frac{a_k}{2}$, and for $0 \leqslant l \leqslant n$, $l \neq j$, $\|t^{m_k}\|_{L^2(\rho_l)} \leqslant \frac{a_k}{k} + \frac{a_k}{k} + \frac{a_k}{k} = \frac{3a_k}{k}$. Thus

$$\frac{\|t^{m_k}\|_{L^2(\rho_j)}}{\sum\limits_{l \neq j} \|t^{m_k}\|_{L^2(\rho_l)}} \geqslant \frac{\sqrt{k}}{n\sqrt{6}}.$$

So (d)′ holds.

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Received April 5, 1991; revised June 28, 1991.