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AUTOMORPHISMS OF TENSOR PRODUCTS OF IRRATIONAL ROTATION C^*-ALGEBRAS AND THE C^*-ALGEBRA OF COMPACT OPERATORS II

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1. PRELIMINARIES

First we will give a lemma about projections and semifinite, semicontinuous traces.

Let A be a (nonunital) C^*-algebra , tr a semifinite, semicontinuous trace on A and I the ideal of definition of tr. Let A^+ be the unitized C^*-algebra and I^+ the algebra obtained from I by adjoining the unit of A^+. In the same way as in Connes [4, Appendix 4] let tr^+ be a trace on I^+ which coincides with tr on I and takes the value 0 on $\mathbf{C} \subset I^+$.

LEMMA 1. *With the above notations for any projection $f \in A$, $f \in I$.*

Proof. Since tr is a semifinite, semicontinuous trace on A, by Connes [4, Appendix 3] there is a projection $\tilde{f} \in I^+$ such that $\|f - \tilde{f}\| < 1$. Thus there is a partial isometry $z \in A^+$ such that $zz^* = f$, $z^*z = \tilde{f}$. Hence $\tilde{f} = z^*fz$. Since $f \in A$ and $z \in A^+$, $\tilde{f} = z^*fz \in A$. Thus $\tilde{f} \in I$. Therefore we can see that

$$\mathrm{tr}(f) = tr(zz^*) = \mathrm{tr}(z^*z) = \mathrm{tr}(\tilde{f}) < \infty.$$

Hence $f \in I$. ∎

Let θ be an irrational number in [0, 1] and A_θ be the corresponding irrational rotation C^*-algebra. Let τ be the unique tracial state on A_θ and p be a Rieffel projection in A_θ with $\tau(p) = \theta$. Let $\mathbf{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space H and Tr be the canonical trace on $\mathbf{K}$

and let $\{e_{ij}\}_{i,j\in\mathbf{Z}}$ be matrix units of $\mathbf{K}$. Let $(A_\theta\otimes\mathbf{K})^+$ be the unitized C^*-algebra of $A_\theta\otimes\mathbf{K}$. Furthermore for any Hilbert space K let $\mathbf{B}(K)$ be the algebra of all bounded linear operators on K and for any C^*-algebra A let $M(A)$ be the double centralizer algebra of A.

In the previous paper [6] we obtained that if the topological stable rank of A_θ is equal to 1, for any automorphism α of $A_\theta\otimes\mathbf{K}$ with $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes\mathbf{K})$ there are an automorphism β of A_θ and a unitary element $w\in M(A_\theta\otimes\mathbf{K})$ such that $\alpha=\mathrm{Ad}(w)\circ\beta\otimes\mathrm{id}$. Furthermore in [10] Putnam showed that for any irrational number θ, A_θ has the topological stable rank 1. In the present paper we will show that if θ is not quadratic, for any automorphism α of $A_\theta\otimes\mathbf{K}$, $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes\mathbf{K})$ and that if θ is quadratic, there is an automorphism α of $A_\theta\otimes\mathbf{K}$ with $\alpha_*\neq\mathrm{id}$ on $K_0(A_\theta\otimes\mathbf{K})$.

2. AUTOMORPHISMS OF $A_\theta\otimes\mathbf{K}$ BY NON-QUADRATIC IRRATIONAL NUMBERS θ

We suppose that θ is an arbitrary irrational number in [0, 1]. Let α be an automorphism of $A_\theta\otimes\mathbf{K}$ and $q=\alpha(1\otimes e_{00})$. Let $\tilde{\varphi}$ be the monomorphism of $(1\otimes\otimes e_{00})(A_\theta\otimes\mathbf{K})(1\otimes e_{00})$ to $q(A_\theta\otimes\mathbf{K})q$ defined by

$$\tilde{\varphi}((1\otimes e_{00})x(1\otimes e_{00}))=\alpha((1\otimes e_{00})x(1\otimes e_{00}))$$

for any $x\in A_\theta\otimes\mathbf{K}$. Then by easy computation $\tilde{\varphi}$ is an isomorphism of $(1\otimes\otimes e_{00})(A_\theta\otimes\mathbf{K})(1\otimes e_{00})$ onto $q(A_\theta\otimes\mathbf{K})q$. Since $(1\otimes e_{00})(A_\theta\otimes\mathbf{K})(1\otimes e_{00})$ is isomorphic to A_θ, so is $q(A_\theta\otimes\mathbf{K})q$. We denote by φ the isomorphism of A_θ onto $q(A_\theta\otimes\mathbf{K})q$ induced by $\tilde{\varphi}$.

By the definitions of τ and Tr, $\tau\otimes\mathrm{Tr}$ is a semifinite, semicontinuous trace on $A_\theta\otimes\mathbf{K}$. Let J be the ideal of definition of $\tau\otimes\mathrm{Tr}$ and J^+ be the algebra obtained from J by adjoining the unit of $(A_\theta\otimes\mathbf{K})^+$. In the same way as in Connes [4, Appendix 4] let $(\tau\otimes\mathrm{Tr})^+$ be a trace on J^+ which coincides with $\tau\otimes\mathrm{Tr}$ on J and takes the value 0 on $\mathbf{C}\subset J^+$. Let $(\tau\otimes\mathrm{Tr})_*$ be the additive map of $K_0(A_\theta\otimes\mathbf{K})$ to $\mathbf{R}$ induced by $(\tau\otimes\mathrm{Tr})^+$. We note that $q\in J$ by Lemma 1.

THEOREM 2. *With the above notations if θ is not quadratic, then for any automorphism α of $A_\theta\otimes\mathbf{K}$ we have $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes\mathbf{K})$.*

Proof. By Pimsner and Voiculescu [9] and Rieffel [11] we know that

$$K_0(A_\theta\otimes\mathbf{K})=\mathbf{Z}[1\otimes e_{00}]\ \oplus\ \mathbf{Z}[p\otimes e_{00}].$$

Since α_* is an automorphism of $K_0(A_\theta\otimes\mathbf{K})$, we can suppose that

$$\alpha_*([1\otimes e_{00}])=[q]=a[1\otimes e_{00}]+b[p\otimes e_{00}],$$

$$\alpha_*([p\otimes e_{00}]) = c[1\otimes e_{00}] + d[p\otimes e_{00}],$$

where $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$ or -1. Let $\tau_1 = \tau \otimes \mathrm{Tr} \circ \varphi$. Then τ_1 is a finite trace on A_θ since $q \in J$. Hence there is a positive number t such that $\tau_1 = t\tau$. Since $\tau_1(1) = t\tau(1)$,

$$t = (\tau\otimes\mathrm{Tr})(\varphi(1)) = (\tau\otimes\mathrm{Tr})(q) = (\tau\otimes\mathrm{Tr})_*([q]) =$$

$$= (\tau\otimes\mathrm{Tr})_*(a[1\otimes e_{00}]) + (\tau\otimes\mathrm{Tr})_*(b[p\otimes e_{00}]) = a + b\theta.$$

Hence $\tau_1 = (a + b\theta)\tau$. By the definition of φ we can see that $\varphi(p) = \alpha(p\otimes e_{00})$. Then since $\tau_1(p) = (a+b\theta)\tau(p)$,

$$(a+b\theta)\theta = (\tau\otimes\mathrm{Tr})(\varphi(p)) = (\tau\otimes\mathrm{Tr})(\alpha(p\otimes e_{00})) = (\tau\otimes\mathrm{Tr})_*(\alpha_*([p\otimes e_{00}])) =$$

$$= (\tau\otimes\mathrm{Tr})_*(c[1\otimes e_{00}] + d[p\otimes e_{00}]) = c + d\theta.$$

Thus we obtain that

$$b\theta^2 + (a-d)\theta - c = 0.$$

Since θ is not quadratic and is irrational, $a = d$ and $b = c = 0$. Moreover since $ad - bc = 1$ or -1 and $(\tau\otimes\mathrm{Tr})(q) = a + b\theta > 0$, $a = d = 1$. ■

Let $\mathrm{Aut}(A_\theta)$ (resp. $\mathrm{Aut}(A_\theta\otimes\mathbf{K})$) be the group of all automorphisms of A_θ (resp. $A_\theta\otimes\mathbf{K}$) and $\mathrm{Int}(A_\theta)$ be the normal subgroup of $\mathrm{Aut}(A_\theta)$ of inner automorphisms of A_θ. For any unitary element $w \in M(A_\theta\otimes\mathbf{K})$ let $\mathrm{Ad}(w)$ be the automorphism of $A_\theta\otimes\mathbf{K}$ defined by $\mathrm{Ad}(w)(x) = wxw^*$ where $x \in A_\theta\otimes\mathbf{K}$. We call $\mathrm{Ad}(w)$ a generalized inner automorphism of $A_\theta\otimes\mathbf{K}$ and let $\mathrm{Int}(A_\theta\otimes\mathbf{K})$ be the group of generalized inner automorphisms of $A_\theta\otimes\mathbf{K}$. Clearly it is a normal subgroup of $\mathrm{Aut}(A_\theta\otimes\mathbf{K})$. Let $\mathrm{Out}(A_\theta) = \mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$ and $\mathrm{Out}(A_\theta\otimes\mathbf{K}) = \mathrm{Aut}(A_\theta\otimes\mathbf{K})/\mathrm{Int}(A_\theta\otimes\mathbf{K})$. For any $\beta \in \mathrm{Aut}(A_\theta)$ (resp. $\alpha \in \mathrm{Aut}(A_\theta\otimes\mathbf{K})$) we denote by $[\beta]$ (resp. $[\alpha]$) the class of β (resp. α) in $\mathrm{Out}(A_\theta)$ (resp. $\mathrm{Out}(A_\theta\otimes\mathbf{K})$). Furthermore let Φ be the homomorphism of $\mathrm{Out}(A_\theta)$ to $\mathrm{Out}(A_\theta\otimes\mathbf{K})$ defined for any $\beta\in\mathrm{Aut}(A_\theta)$ by $\Phi([\beta]) = [\beta\otimes\mathrm{id}]$.

REMARK. By Putnam [10] we see that for any irrational number θ, $\mathrm{tsr}(A_\theta) = 1$ where $\mathrm{tsr}(A_\theta)$ denotes the topological stable rank of A_θ. Hence by [6, Theorem 5] for any $\alpha \in \mathrm{Aut}(A_\theta\otimes\mathbf{K})$ with $\alpha_* = \mathrm{id}$ on $K_0(A_\theta\otimes\mathbf{K})$, there are a unitary element $w \in M(A_\theta\otimes\mathbf{K})$ and a $\beta\in\mathrm{Aut}(A_\theta)$ such that

$$\alpha = \mathrm{Ad}(w)\circ\beta\otimes\mathrm{id}.$$

COROLLARY 3. *With the above notations if θ is not quadratic, then Φ is an isomorphism of* $\mathrm{Out}(A_\theta)$ *onto* $\mathrm{Out}(A_\theta\otimes\mathbf{K})$.

Proof. First we will show that Φ is surjective. Since θ is not quadratic, $\alpha_* = \mathrm{id}$ on $K_0(A_\theta \otimes \mathbf{K})$ for any $\alpha \in \mathrm{Aut}(A_\theta \otimes \mathbf{K})$ by Theorem 2. Thus by the above remark there are a unitary element $w \in M(A_\theta \otimes \mathbf{K})$ and a $\beta \in \mathrm{Aut}(A_\theta)$ such that

$$\alpha = \mathrm{Ad}(w) \circ \beta \otimes \mathrm{id}.$$

Hence

$$\Phi([\beta]) = [\beta \otimes \mathrm{id}] = [\mathrm{Ad}(w) \circ \beta \otimes \mathrm{id}] = [\alpha].$$

Therefore Φ is surjective. Next we will show that Φ is injective. Let (π_τ, H_τ) be the cyclic representation of A_θ associated with τ. Since A_θ is simple, π_τ is faithful. Thus we can identify $A_\theta \otimes \mathbf{K}$ with $\pi_\tau(A_\theta) \otimes \mathbf{K}$. We suppose that $\Phi([\beta]) = [\mathrm{id}]$ where $\beta \in \mathrm{Aut}(A_\theta)$. Then there is a unitary element $w \in M(A_\theta \otimes \mathbf{K})$ such that $\beta \otimes \mathrm{id} = \mathrm{Ad}(w)$. Thus for any $X \in \mathbf{K}$

$$(\beta \otimes \mathrm{id})(1 \otimes X) = w(1 \otimes X)w^*,$$

i.e.,

$$(1 \otimes X)w = w(1 \otimes X).$$

Since $\mathbf{K}$ is strongly dense in $\mathbf{B}(H)$, for any $X \in \mathbf{B}(H)(1 \otimes X)w = w(1 \otimes X)$. Since $(\mathbf{C}1 \otimes \mathbf{B}(H))' = \mathbf{B}(H_\tau) \otimes \mathbf{C}1$, $w \in \mathbf{B}(H_\tau) \otimes \mathbf{C}1$. Let $w = z \otimes 1$ where z is a unitary element in $\mathbf{B}(H_\tau)$. Since $\beta \otimes \mathrm{id} = \mathrm{Ad}(z \otimes 1)$, for any $x \in A_\theta$ and $X \in \mathbf{K}$

$$(\beta(x) \otimes X)(z \otimes 1) = (z \otimes 1)(x \otimes X),$$

i.e.,

$$\beta(x)z \otimes X = zx \otimes X.$$

Since $z \otimes 1 \in M(A_\theta \otimes \mathbf{K})$, $\beta(x)z \otimes X$ and $zx \otimes X$ are in $A_\theta \otimes \mathbf{K}$. Thus $\beta(x)z = zx \in A_\theta$ for any $x \in A_\theta$. Therefore z is a unitary element in A_θ and $\beta(x) = zxz^*$. Hence $[\beta] = [\mathrm{id}]$ in $\mathrm{Out}(A_\theta)$. Thus Φ is injective. ∎

3. AUTOMORPHISMS OF $A_\theta \otimes \mathbf{K}$ BY QUADRATIC IRRATIONAL NUMBERS θ

Next we will show that for any quadratic irrational number θ there is an automorphism α of $A_\theta \otimes \mathbf{K}$ with $\alpha_* \neq \mathrm{id}$ on $K_0(A_\theta \otimes \mathbf{K})$. Let a and b be integers which generate $\mathbf{Z}$ such that $a + b\theta \neq 0$. We also assume that $b \neq 0$. Let $V_\theta(a, b : k)$ be the standard module defined in Rieffel [13] where k is a positive integer. For each positive integer n, $M_n(A_\theta)$ denotes the $n \times n$-matrix algebra over A_θ. Then we can extend the unique tracial state τ on A_θ to the unnormalized finite trace on $M_n(A_\theta)$.

We also denote it by τ. Since $V_\theta(a,b:k)$ is a finitely generated projective right A_θ-module, $V_\theta(a,b:k)$ corresponds to a projection in some $M_n(A_\theta)$. We also denote it by $V_\theta(a,b:k)$.

LEMMA 4. *With the above notations let q be a projection in $M_m(A_\theta)$ where m is a positive integer. Then $\tau(V_\theta(a,b:k))=\tau(q)$ if and only if $V_\theta(a,b:k)$ is isomorphic to qA_θ^m as a module.*

Proof. It is clear that $\tau(V_\theta(a,b:k))=\tau(q)$ if $V_\theta(a,b:k)$ is isomorphic to qA_θ^m. We suppose that $\tau(V_\theta(a,b:k))=\tau(q)$. Then by Rieffel [13, Corollary 2.5], $V_\theta(a,b:k)$ is isomorphic to qA_θ^m. ∎

We will give definitions and well-known facts on quadratic irrational numbers (see Lang [7]).

Let $\mathbf{Q}$ be the ring of rational numbers. If $\theta=x+y\sqrt{d}$ where $x,y\in\mathbf{Q}$ and $d\in\mathbf{N}$, then we define $\theta'=x-y\sqrt{d}$ and we call θ' the *conjugate* of θ. Let θ be a quadratic irrational number. We say that it is reduced if $\theta>1$ and $-1<\theta'<0$ where θ' is the conjugate of θ.

By Lang [7, Chap. I, Section 1, Theorems 1, 2, Corollary 1 and Chap. IV, Section 1, Theorems 2, 3] for any quadratic irrational number θ there are a fractional transformation $g=\begin{bmatrix} k & l \\ m & n\end{bmatrix}\in GL(2,\mathbf{Z})$ and a reduced quadratic irrational number θ_1 such that

$$\theta=g\theta_1=\frac{k\theta_1+l}{m\theta_1+n}.$$

Furthermore using Lang [7, Chap. I, Section 1, Theorems 1, 2, Corollary 1 and Chap. IV, Section 1, Theorem 3] again, we can see that for any reduced quadratic irrational number θ_1 there is a fractional transformation $h\in GL(2,\mathbf{Z})$ with $h\neq \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}$ such that $\theta_1=h\theta_1$. Hence since $\theta_1=g^{-1}\theta$, we obtain that

$$\theta=g\theta_1=gh\theta_1=ghg^{-1}\theta.$$

Since $h\neq \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}$, neither is ghg^{-1}. By the above arguments for any quadratic irrational number θ there is a fractional transformation $g\in GL(2,\mathbf{Z})$ with $g\neq \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}$ such that $\theta=g\theta$.

THEOREM 5. *Let θ be a quadratic irrational number in [0, 1]. Then there is an automorphism α of $A_\theta\otimes\mathbf{K}$ such that $\alpha_*\neq$ id on $K_0(A_\theta\otimes\mathbf{K})$.*

Proof. Since θ is quadratic, there is a fractional transformation $g=\begin{bmatrix} d & c \\ b & a\end{bmatrix}\in$

$GL(2,\mathbf{Z})$ with $g \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ such that

$$\theta = g\theta = \frac{c+d\theta}{a+b\theta}.$$

The conditions $g \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $a+b\theta \neq 0$ imply $b \neq 0$. And we may assume that $a+b\theta > 0$. By Rieffel [11] we can take a projection $q \in M_m(A_\theta)$ such that $\tau(q) = a+b\theta$ where m is a positive integer. Then $qM_m(A_\theta)q$ is strongly Morita equivalent to A_θ and qA_θ^m is the $qM_m(A_\theta)q - A_\theta$-equivalence bimodule. By Rieffel [13, Theorem 1.4], $\tau(V_\theta(a,b:1)) = a+b\theta$. Hence by Lemma 4 we obtain that qA_θ^m is isomorphic to $V_\theta(a,b:1)$. Thus by Rieffel [13, Theorem 1 and Corollary 2.6], $qM_m(A_\theta)q$ is isomorphic to A_η where $\eta = \dfrac{c+d\theta}{a+b\theta}$. However since $\theta = \dfrac{c+d\theta}{a+b\theta}$, $qM_m(A_\theta)q$ is isomorphic to A_θ. Let φ be an isomorphism of A_θ onto $qM_m(A_\theta)q$. Then $\varphi(1) = q$. We consider the isomorphism $\varphi \otimes \mathrm{id}$ of $A_\theta \otimes \mathbf{K}$ onto $(q \otimes 1)(M_m(A_\theta) \otimes \mathbf{K})(q \otimes 1)$. Since $M_m(A_\theta)$ is simple, q is a full projection. Thus by Brown [2, Lemma 2.5] there is a partial isometry $w \in M(M_m(A_\theta) \otimes \mathbf{K})$ with $w^*w = I_m \otimes 1$ and $ww^* = q \otimes 1$ where I_m is the unit element in $M_m(A_\theta)$. Then $\mathrm{Ad}(w^*)$ is an isomorphism of $(q \otimes 1)(M_m(A_\theta) \otimes \mathbf{K})(q \otimes 1)$ onto $M_m(A_\theta) \otimes \mathbf{K}$. Let ψ be an isomorphism of $M_m(A_\theta) \otimes \mathbf{K}$ onto $A_\theta \otimes \mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(M_m(A_\theta) \otimes \mathbf{K})$ onto $K_0(A_\theta \otimes \mathbf{K})$ and let α be the automorphism of $A_\theta \otimes \mathbf{K}$ defined by

$$\alpha = \psi \circ \mathrm{Ad}(w^*) \circ (\varphi \otimes \mathrm{id}).$$

Then $(\mathrm{Ad}(w^*) \circ (\varphi \otimes \mathrm{id}))(1 \otimes e_{00}) = w^*(q \otimes e_{00})w$ and in $K_0(M_m(A_\theta) \otimes \mathbf{K})$

$$[(\mathrm{Ad}(w^*) \circ (\varphi \otimes \mathrm{id}))(1 \otimes e_{00})] = [w^*(q \otimes e_{00})w] = [w^*(q \otimes e_{00})(q \otimes e_{00})w] =$$

$$= [(q \otimes e_{00})ww^*(q \otimes e_{00})] = [(q \otimes e_{00})(q \otimes 1)(q \otimes e_{00})] = [q \otimes e_{00}].$$

Since $\tau(q) = a+b\theta$,

$$[q \otimes e_{00}] = a[1 \otimes f_{11} \otimes e_{00}] + b[p \otimes f_{11} \otimes e_{00}]$$

in $K_0(M_m(A_\theta) \otimes \mathbf{K})$ where $\{f_{ij}\}_{i,j=1}^m$ are matrix units of $M_m(\mathbf{C})$ and we identify $M_m(A_\theta)$ with $A_\theta \otimes M_m(\mathbf{C})$. Hence $\alpha_* \neq \mathrm{id}$ on $K_0(A_\theta \otimes \mathbf{K})$ since $\psi_* = \mathrm{id}$ of $K_0(M_m(A_\theta) \otimes \mathbf{K})$ onto $K_0(A_\theta \otimes \mathbf{K})$ and $b \neq 0$. ∎

REMARK. Let $\mathbf{T}^2$ be the two-torus and $C(\mathbf{T}^2)$ the C^*-algebra of all continuous functions on $\mathbf{T}^2$. We identify $C(\mathbf{T}^2)$ with all countinuous functions f on $[0,1] \times$

$\times[0,1]$ with $f(s_1,0)=f(s_1,1)$ and $f(0,s_2)=f(1,s_2)$ for $s_1,s_2\in[0,1]$. Let $\tilde{u}$ and $\tilde{v}$ be the unitary elements in $C(\mathbf{T}^2)$ defined by

$$\tilde{u}(s_1,s_2)=\mathrm{e}^{2\pi i s_1},\quad \tilde{v}(s_1,s_2)=\mathrm{e}^{2\pi i s_2}.$$

They generate $C(\mathbf{T}^2)$ and $K_1(C(\mathbf{T}^2))=\mathbf{Z}[\tilde{u}]\oplus\mathbf{Z}[\tilde{v}]$. Let σ be the action of $\mathbf{R}$ on $\mathbf{T}^2$ by translation at angle θ and we consider the action of $\mathbf{R}$ on $C(\mathbf{T}^2)$ induced by σ. We also denote it by σ. Then we can consider the crossed product $C(\mathbf{T}^2)\times_\sigma\mathbf{R}$ of $C(\mathbf{T}^2)$ by $\mathbf{R}$ and by Green [5] it is isomorphic to $A_\theta\otimes\mathbf{K}$. We identify them. In the same way as in Connes [4] for any σ-equivariant automorphism β of $C(\mathbf{T}^2)$ we can define an automorphism $\widehat{\beta}$ of $C(\mathbf{T}^2)\times_\sigma\mathbf{R}$. However if β is a σ-equivariant automorphism of $C(\mathbf{T}^2)$, $\beta\circ\sigma_t=\sigma_t\circ\beta$ for any $t\in\mathbf{R}$. Hence by the Fourier expansion on $\mathbf{T}^2$ we can see that there are real numbers η_1 and a η_2 such that

$$\beta(\tilde{u})=\mathrm{e}^{2\pi i\eta_1}\tilde{u},\quad \beta(\tilde{v})=\mathrm{e}^{2\pi i\eta_2}\tilde{v}.$$

Therefore if θ is quadratic, the automorphism α of $C(\mathbf{T}^2)\times_\sigma\mathbf{R}$ constructed in Theorem 5 can not be induced any σ-equivariant automorphism of $C(\mathbf{T}^2)$.

In fact we suppose that there is a σ-equivariant automorphism β of $C(\mathbf{T}^2)$ with $\alpha=\widehat{\beta}$. Then $\beta_*=\mathrm{id}$ on $K_1(C(\mathbf{T}^2))$. By Connes [4] there is the Thom isomorphism φ_σ^1 of $K_1(C(\mathbf{T}^2))$ onto $K_0(C(\mathbf{T}^2)\times_\sigma\mathbf{R})$ such that $\widehat{\beta}_*\circ\varphi_\sigma^1=\varphi_\sigma^1\circ\widehat{\beta}_*$. Hence we obtain that $\widehat{\beta}_*=\mathrm{id}$ on $K_0(C(\mathbf{T}^2)\times_\sigma\mathbf{R})$ since $\beta_*=\mathrm{id}$ on $K_1(C(\mathbf{T}^2))$. However this contradicts $\alpha_*\neq\mathrm{id}$ on $K_0(C(\mathbf{T}^2)\times_\sigma\mathbf{R}$.

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REFERENCES

1. BLACKADAR, B., *K-theory for operator algebras*, M.S.R.I. Publication, Springer-Verlag, 1986.
2. BROWN, L. G., Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pacific J. Math.*, **71**(1977), 335–348.
3. BROWN, L. G.; GREEN, P.; RIEFFEL, M. A., Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pacific J. Math.*, **71**(1977), 349–363.
4. CONNES, A., An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbf{R}$, *Adv. in Math.*, **39**(1981), 31–55.
5. GREEN, P., The structure of imprimitivity algebras, *J. Functional Analysis*, **36**(1980), 105–113.

6. KODAKA, K., Automorphisms of tensor products of irrational rotation C^*-algebras and the C^*-algebras of compact operators, *Tokyo J. Math.*, **13**(1990), 457–468.
7. LANG, S., *Introduction to diophantime approximations*, Addison-Wesley, 1966.
8. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, 1979.
9. PIMSNER, M. V.; VOICULESCU, D., Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras, *J. Operator Theory*, **4**(1980), 93–118.
10. PUTNAM, I. F., The invertible elements are dense in the irrational rotation C^*-algebras, *J. Reine. Angew. Math.*, **410**(1990), 160–166.
11. RIEFFEL, M. A., C^*-algebras associated with irrational rotations, *Pacific J. Math.*, **93**(1981), 415–429.
12. RIEFFEL, M. A., Dimension and stable rank in the K-Theory of C^*-algebras, *Proc. London Math. Soc.*, **46**(1983), 301–333.
13. RIEFFEL, M. A., The cancellation theorem for projective modules over irrational rotation C^*-algebras, *Proc. London Math. Soc.*, **47**(1983), 285–302.
14. TAKESAKI, M., *Theory of operator algebras I*, Springer-Verlag, 1979.

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