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ON GENERALIZED CUNTZ C^*-ALGEBRAS

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1. INTRODUCTION

For each positive integer $N \geqslant 2$, the Cuntz C^*-algebra $\mathcal{O}_N$ is defined as the C^*-algebra generated by N isometries, on a Hilbert space $\mathcal{H}$, whose ranges are mutually orthogonal and whose range projections sum to the identity operator 1. For $N = \infty$, $\mathcal{O}_\infty$ is defined as the C^*-algebra generated by a sequence of isometries with mutually orthogonal ranges. (It is not required that the sum of their range projection is 1.) The Toeplitz C^*-algebra, i.e. the C^*-algebra generated by a non-unitary isometry [4,5], may be regarded as a degenerate case of the Cuntz C^*-algebras. It was shown by Cuntz that $\mathcal{O}_N$ is canonically unique in the sense that it is independent of the choice of the N isometries that generate $\mathcal{O}_N$ [6].

In this paper we generalize Cuntz C^*-algebras as follows. Let G be a countable dense subgroup of the real line, and G^+ be the semigroup of positive elements of G. Let $V_1, \ldots, V_N$ be N orthogonal semigroups of isometries, i.e.

(i) $V_i(t)^* V_i(t) = 1$ and $V_i(s+t) = V_i(s)V_i(t)$ for every $1 \leqslant i \leqslant N$ and $s, t \in G^+$,

(ii) $V_j(t)^* V_i(t) = 0$ (equivalently, $V_i(t)\mathcal{H} \perp V_j(t)\mathcal{H}$) for every $1 \leqslant i, j \leqslant N$, $i \neq j$, $t \in G^+$.

Here $N = 2, 3, \ldots, \infty$ and by abusing notations, when $N = \infty$ we mean a *sequence* of semigroups. Note that we do not require the sum of their range projections be 1, even when N is finite. We consider $C^*(V_1, \ldots, V_N)$, the C^*-algebra generated by the isometries $V_i(t), 1 \leqslant i \leqslant N, t \in G^+$. The C^*-algebra generated by *one* semigroup of isometries is either a quotient of $C(\widehat{G})$ or the generalized Toeplitz C^*-algebra studied by Douglas [9]. If we remove the condition that G is dense, we end up with either the Cuntz C^*-algebras or their extensions by the compact operators [6]. If we let $G = \mathbb{R}$ instead, we basically have nothing since Arveson has shown that there are no *strongly continuous* semigroups of isometries that are mutually orthogonal in the above sense [2]. (Nevertheless, Averson has been able to construct C^*-algebras which can be regarded as "continuous time" analogues of Cuntz C^*-algebras [2]).

Throughout this paper, we will always assume that G is countable and dense. For such G, there always exist semigroups of isometries which have the desired properties. Let $\mathcal{D}$ be a Hilbert space of dimension $1, 2, \ldots$ or $\aleph_0$. For $t_1, \ldots, t_n \in G^+; i_1, \ldots, i_n \in$ $\in \{1, \ldots, N\}$ and $i_m \neq i_{m+1}$ for every m, let

$$\mathcal{D}_{i_1,\ldots,i_n}(t_1, \ldots, t_n) = \mathcal{D}$$

and

$$\mathcal{H} = \mathcal{D} \oplus \sum\nolimits^{\oplus} \mathcal{D}_{i_1,\ldots,i_n}(t_1, \ldots, t_n)$$

where the Hilbert space direct sum ranges over all such multi-indices $i_1, \ldots, i_n$ and time parameters $t_1, \ldots, t_n$. Define $V_i(t)$ on the summands of $\mathcal{H}$ as follows.

$$\xi \in \mathcal{D} \mapsto \xi \in \mathcal{D}_i(t),$$

$$\xi \in \mathcal{D}_{i_1,\ldots,i_n}(t_1, \ldots, t_n) \mapsto \xi \in \mathcal{D}_{i,i_1,\ldots,i_n}(t, t_1, \ldots, t_n) \quad \text{for } i_1 \neq i,$$

$$\xi \in \mathcal{D}_{i_1,\ldots,i_n}(t_1, \ldots, t_n) \mapsto \xi \in \mathcal{D}_{i_1,\ldots,i_n}(t + t_1, t_2, \ldots, t_n) \quad \text{for } i_1 = i.$$

It is straightforward to check that $V_1, \ldots, V_N$ are N orthogonal semigroups of isometries.

The C^*-algebras $C^*(V_1, \ldots, V_N)$ are special cases of a family of C^*-algebras considered by us in [7]. This family of C^*-algebras arises naturally from the Arveson-Powers-Robinson index theory of semigroups of endomorphisms of type I factors [1,7,8]. We briefly recall some definitions in [7].

A *discrete product system* E is a family of infinite dimensional separable Hilbert spaces $\{E(t) : t \in G^+\}$, on which there is defined a *tensoring* operation satisfying;

(i) For each $s, t \in G^+$, there is a bilinear map $(u, v) \in E(s) \times E(t) \mapsto uv \in$ $\in E(s + t)$. Moreover, $[E(s)E(t)] = E(s + t)$.

(ii) (Associativity) $(uv)w = u(vw)$ for every $u, v, w \in E$.

(iii) $\langle uv, u'v'\rangle = \langle u, u'\rangle\langle v, v'\rangle$ for every $u, u' \in E(s)$ and $v, v' \in E(t)$.

note that (i) and (iii) imply that the map $u \otimes v \in E(s) \otimes E(t) \mapsto uv \in E(s + t)$ extends to a unitary operator from $E(s) \otimes E(t)$ onto $E(s + t)$.

A *representation* of a discrete product system E is a map $\varphi : E \to B(\mathcal{H})$ satisfying

(i) $\varphi(v)^*\varphi(u) = \langle u, v\rangle 1$ for every $u, v \in E(t)$ and $t \in G^+$ and

(ii) $\varphi(u)\varphi(v) = \varphi(uv)$ for $u, v \in E$.

Let $C^*(\varphi(E))$ be the C^*-algebra generated by the range of φ. Theorem 2.2 in [7] states that this C^*-algebra does not depend on the particular representation of E. More precisely, if $\varphi_1 : E \to \mathcal{B}(\mathcal{H}_1)$ and $\varphi_2 : E \to \mathcal{B}(\mathcal{H}_2)$ are two representations then the map $\varphi_1(u) \mapsto \varphi_2(u)$ extends to a *-isomorphism from $C^*(\varphi_1(E))$ onto $C^*(\varphi_2(E))$. We denote the isomorphism class of $C^*(\varphi(E))$ by $\mathcal{O}_E(G)$. This is a separable simple C^*--algebra whose non-degenerate representations correspond bijectively to representations of E [7, Corollary 2.22 and the preceeding remarks].

By a *word of length* t, we mean a product of the form

$$V_{i_1}(t_1)V_{i_2}(t_2)\cdots V_{i_n}(t_n)$$

where $t_1 + t_2 + \cdots + t_n = t$. A *reduced word* is one in which terms of the form $V_i(s)V_i(t)$ have been simplified to $V_i(s+t)$. For $t \in G^+$, we define $E_N(t)$ to be the closed linear span of all words of length t. It is easy to show that E_N, with the usual operator multiplication and with inner product defined by

$$\langle T_1, T_2\rangle 1 = T_2^* T_1,$$

is a discrete product system. Moreover, the collection of all reduced words of length t forms an orthonormal basis for $E_N(t)$. In particular the reduced words satisfy

$$V_{j_m}(t'_m)^* \cdots V_{j_1}(t'_1)^* V_{i_1}(t_1)\cdots V_{i_n}(t_n) = \begin{cases} 1 & \text{if } m = n;\ i_1 = j_1, \ldots;\ t_1 = t'_1, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding C^*-algebra $\mathcal{O}_{E_N}(G)$, i.e. the C^*-algebra generated by the range of the identity representation of E_N, is the same as the C^*-algebra generated by the semigroups $V_1, \ldots, V_N$. It follows that $C^*(V_1, \ldots, V_N)$ has all the properties that $\mathcal{O}_{E_N}(G)$ has. In particular, it is simple [7, Corollary 2.22]. It is clear that E_N depends only on N, not on the particular semigroups $V_1, \ldots, V_N$. More precisely, let $V'_1, \ldots, V'_N$ be another set of orthogonal semigroups of isometries and form the corresponding discrete product system E'_N. Recall from [7] that an isomorphism between E_N and E'_N is a bijection which preserves the tensoring operation and which restricts to a unitary operator from the "fiber" $E_N(t)$ onto the "fiber" $E'_N(t)$. It is easy to show that the map $V_i(t) \mapsto V'_i(t)$ extends to an isomorphism between E_N and E'_N. Consequently, this map extends to a *-isomorphism between the corresponding C^*-algebras [7, Theorem 2.2].

We summarize the above discussion in

THEOREM 1.1. *Let $V_1, \ldots, V_N$ and $V'_1, \ldots, V'_N$ be two sets of orthogonal semigroups of isometries. Then there is a unique *-isomorphism from $C^*(V_1, \ldots, V_N)$ onto $C^*(V'_1, \ldots, V'_N)$ that takes each isometry $V_i(t)$ to the corresponding isometry $V'_i(t)$. Consequently, $C^*(V_1, \ldots, V_N)$ is simple.*

In a related direction, Price [12] studied the C^*-algebra generated by two semigroups of isometries satisfying the "commutation relation" $V_1(t)^*V_2(t) = e^{-\lambda t}1$. It follows from a construction similar to the one above that Price's C^*-algebra belongs to the family of C^*-algebras $\mathcal{O}_E(G)$. Hence many of our results also hold for Price's C^*-algebra.

Some results of this paper are taken from the author's Ph. D. Thesis written under the direction of Professor William Arveson at the University of California at Berkeley. We thank him for his guidance, and for sending us a preprint of [2]. We also want to thank Professor Geoffrey Price for some helpful correspondences.

2. QUASI-FREE AUTOMORPHISMS

In [10], Evans considered quasi-free automorphisms of the Cuntz C^*-algebras. These automorphisms are defined similarly to those of the CAR algebra. In [7], the author defined quasi-free automorphisms of $\mathcal{O}_E(G)$ as follows. For every automorphism μ of E, there is a unique *-automorphism α_μ of $\mathcal{O}_E(G)$ satisfying $\alpha_\mu(\varphi(u)) = \varphi(\mu(u))$. It is shown in [7, Theorem 5.1] that, except for the identity automorphism, every α_μ is an outer automorphism. The group $\{\alpha_\mu : \mu \in Aut(E)\}$ is called the group of quasi-free automorphisms. It is isomorphic to the group $Aut(E)$ of all automorphisms of E. We will compute this group for $C^*(V_1, \ldots, V_N)$.

Recall from [7] that a *unit* of a discrete product system E is a cross section $u = \{u(t)\}_{t\in G^+}$ satisfying

(i) $u(t) \in E(t)$ for every $t \in G^+$,

(ii) $u(t) \neq 0$ for some, hence for every $t \in G^+$, and

(iii) $u(s)u(t) = u(s+t)$ for every $s, t \in G^+$.

In addition, if $\|u(t)\| = 1$ for every $t \in G^+$ then u is said to be *normalized*. For example, the semigroups $V_1, \ldots, V_N$ are normalized units of E_N. Conversely, we have

PROPOSITION 2.1. *Every unit of E_N is of the form fV_i, where f is a homomorphism from G^+ into the multiplicative group of non-zero complex numbers.*

Proof. Let W be a unit. We claim that W cannot be perpendicular to *all* the V_i's (definition: $W \perp V_i$ if $W(t) \perp V_i(t)$ for every $t \in G^+$). Otherwise, by taking inner product with an arbitrary reduced word of lengh t, we have

$$\langle W(t), V_{i_1}(t_1)\cdots V_{i_n}(t_n)\rangle = \langle W(t_1), V_{i_1}(t_1)\rangle \cdots \langle W(t_n), V_{i_n}(t_n)\rangle = 0,$$

which implies that $W(t) = 0$, a contradiction.

Without loss of generality, assume that W is not perpendicular to V_1, i.e. $\langle W(t_0), V_1(t_0)\rangle \neq 0$ for some $t_0 \in G^+$. This implies that $\langle W(t), V_1(t)\rangle \neq 0$ for all $t \in$

$\in G^+$. Indeed, if $t < t_0$ then

$$\langle W(t_0), V_1(t_0)\rangle = \langle W(t), V_1(t)\rangle \langle W(t_0 - t), V_1(t_0 - t)\rangle,$$

which implies $\langle W(t), V_1(t)\rangle \neq 0$. If $t > t_0$ then there is a positive integer n such that $t < nt_0$. Since $\langle W(nt_0), V_1(nt_0)\rangle = \langle W(t_0), V_1(t_0)\rangle^n \neq 0$, the previous argument implies that $\langle W(t), V_1(t)\rangle \neq 0$.

Let $\widetilde{W}(t) = \dfrac{W(t)}{\langle W(t), V_1(t)\rangle}$. Then $\widetilde{W}$ is a unit satisfying $\langle \widetilde{W}(t), V_1(t)\rangle = 1$ for every $t \in G^+$. We claim that $\widetilde{W}$ is perpendicular to V_i, $2 \leqslant i \leqslant N$. We use the fact that in a Hilbert space, if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence and $\langle e, e_n\rangle$ does not depend on n then $\langle e, e_n\rangle = 0$ for every n. Fix $t \in G^+$. Since G is dense, the collection of all reduced words of the form $V_1(s)V_i(s)V_1(t-s)V_i(t-s)$, where $0 < s < t$, form an orthonormal sequence in $E_N(2t)$. We have

$$\langle \widetilde{W}(2t), V_1(s)V_i(s)V_1(t-s)V_i(t-s)\rangle =$$

$$= \langle \widetilde{W}(s), V_1(s)\rangle \langle \widetilde{W}(s), V_i(s)\rangle \langle \widetilde{W}(t-s), V_1(t-s)\rangle \langle \widetilde{W}(t-s), V_i(t-s)\rangle =$$

$$= \langle \widetilde{W}(t), V_i(t)\rangle$$

which is independent of s. Hence $\langle \widetilde{W}(t), V_i(t)\rangle = 0$ for $2 \leqslant i \leqslant N$ and $t \in G^+$.

Using an argument similar to the one at the beginning of the proof, we conclude that $\widetilde{W}(t)$ is perpendicular to all the words of lengh t which contains at least one *letter* of the form V_i, $2 \leqslant i \leqslant N$. It follows that $\widetilde{W}(t)$ must be a scalar multiple of $V_1(t)$, say $\widetilde{W}(t) = g(t)V_1(t)$. Since $\langle \widetilde{W}(t), V_1(t)\rangle = 1$, we have $g(t) = 1$ for every $t \in G^+$. Then $W(t) = f(t)V_1(t)$ where $f(t) = \langle W(t), V_1(t)\rangle$ is the promised homomorphism.■

We let $\mathcal{S}_N$ be the group of permutations of N objects. Define a group operation on $\widehat{G} \times \cdots \times \widehat{G} \times \mathcal{S}_N$ by

$$(f_1, \ldots, f_N, \sigma)(f'_1, \ldots, f'_N, \sigma) = (f_{\sigma'(1)}f'_1, \ldots, f_{\sigma'(N)}f'_N, \sigma\sigma').$$

THEOREM 2.2. *$Aut(E_N)$ is isomorphic to the group $\widehat{G} \times \cdots \times \widehat{G} \times \mathcal{S}_N$.*

Proof. The isomorphism is defined as follows. For $(f_1, \ldots, f_N, \sigma) \in \widehat{G} \times \cdots \times \widehat{G} \times \mathcal{S}_N$, define $\mu_{(f_1,\ldots,f_N,\sigma)} \in Aut(E_N)$ on reduced words by

$$V_{i_1}(t_1)\cdots V_{i_n}(t_n) \mapsto f_{i_1}(t_1)\cdots f_{i_n}(t_n)V_{\sigma(i_1)}(t_1)\cdots V_{\sigma(i_n)}(t_n).$$

To show onto, we use the previous proposition. Let μ be an automorphism of E_N. Then $\mu(V_i)$ must be normalized unit. So $\mu(V_i) = f_iV_{\sigma(i)}$ for some $f_i \in \widehat{G}$. We claim that σ is one-one and onto, hence a permutation. Assume $\sigma(i) = \sigma(j)$. Then

$$\mu(\overline{f_i(t)}V_i(t)) = V_{\sigma(i)}(t) = V_{\sigma(j)}(t) = \mu(\overline{f_j(t)}V_j(t)).$$

Since μ is one-one, $\overline{f_i(t)}V_i(t) = \overline{f_j(t)}V_j(t)$. This implies $i = j$ by the orthogonality property of the semigroups $V_1, \ldots, V_N$. So σ is one-one. Let $j \in \{1, \ldots, N\}$. Since μ^{-1} is also an automorphism, we have $\mu^{-1}(V_j) = g_j V_{\tau(j)}$ for some $g_j \in \widehat{G}$. This implies

$$V_j(t) = \mu(g_j(t)V_{\tau(j)}(t)) = g_j(t)f_{\tau(j)}(t)V_{\sigma\tau(j)}(t),$$

and hence $\sigma\tau(j) = j$. So σ is onto.

To show one-one, assume $\mu_{(f_1,\ldots,f_N,\sigma)} = \mu_{(f'_1,\ldots,f'_N,\sigma')}$. These automorphisms take $V_i(t)$ onto $f_i(t)V_{\sigma(i)}(t)$ and $f'_i(t)V_{\sigma'(i)}(t)$, respectively. Hence we must have $\sigma(i) = \sigma'(i)$ and $f_i(t) = f'_i(t)$ for every i, t. ■

3. THE CROSSED PRODUCT CONSTRUCTION

In this section, we show that if E has a unit then $\mathcal{O}_E(G)$ is a full, hereditary subalgebra of a crossed product of G by an AF-algebra. In particular, this property holds for $C^*(V_1, \ldots, V_N)$ since E_N has a unit. For simplicity of notations, we now write E_t and u_t instead of $E(t)$ and $u(t)$.

Recall [7, Definition 2.6] that for a representation $\varphi : E \to \mathcal{B}(\mathcal{H})$, the *even algebra* $\mathcal{F}_E(G)$ is defined by

$$\mathcal{F}_E(G) = \overline{\text{span}}^{\|\cdot\|}\{\varphi(E_t)\varphi(E_t)^* : t \in G^+ \cup \{0\}\}$$

where we define $\varphi(E_0) = \mathbf{C}1$ for convenience. This algebra plays an important role in the proof of simplicity of the algebra $\mathcal{O}_E(G)$ [7].

An alternative way to describe the even algebra is as follows. Let $G^+ \cup \{0\}$ be directed by the usual ordering of the real line. For $s, t \in G^+ \cup \{0\}, s \leqslant t$, define $\alpha_{ts} : \mathcal{B}(E_s) \to \mathcal{B}(E_t)(E_0 = \mathbf{C})$ as follows. If $s = 0$, $\alpha_{ts}(\lambda) = \lambda 1_t$, where 1_t is the identity operator on E_t. If $s = t$, α_{ss} is the identity map. If $0 < s < t$, $\alpha_{ts}(X) = X \otimes 1_{t-s}$ for $X \in \mathcal{B}(E_s)$, where we have identified $E_t = E_s \otimes E_{t-s}$ so that $\mathcal{B}(E_t) = \mathcal{B}(E_s)\overline{\otimes}\mathcal{B}(E_{t-s})$. It follows from the associativity of the operation on E that $\alpha_{ts}\alpha_{sr} = \alpha_{tr}$ for $r \leqslant s \leqslant t$. Thus we can define B_∞ to be the C^*-algebra inductive limit $\varinjlim \mathcal{B}(E_t)$. Define $\widetilde{\mathcal{F}}_E(G)$ to be the C^*-subalgebra of $\mathcal{B}_\infty$ generated by the collection of compact operators $\{\mathcal{K}(E_t) : t \in G^+ \cup \{0\}\}$. Proposition 2.10 of [7] states that $\widetilde{\mathcal{F}}_E(G)$ is canonically isomorphic to the even algebra. The isomorphism takes the rank-one operator $\langle \cdot, v\rangle u$ in $\mathcal{K}(E_t)$ to the operator $\varphi(u)\varphi(v)^*$ in $\varphi(E_t)\varphi(E_t)^*$.

Now let E be a discrete product system with normalized unit v. Define $e_t = \langle \cdot, v_t\rangle v_t$, a rank one projection in $\mathcal{B}(E_t)$. Note that $e_s \otimes e_t = e_{s+t}$. Indeed let $u \in E_s$, $w \in E_t$. Then

$$(e_s \otimes e_t)(u \otimes w) = \langle u, v_s\rangle v_s \otimes \langle w, v_t\rangle v_t = \langle uw, v_s v_t\rangle v_s v_t =$$

$$= \langle uw, v_{s+t}\rangle v_{s+t} = e_{s+t}(uw).$$

For each $x \in G$, let $\mathcal{B}_x = \widetilde{\mathcal{F}}_E(G)$. If $x \leqslant y$, we define the embedding $\beta_{xy} : \mathcal{B}_y \to \mathcal{B}_x$ by

$$\beta_{xy}(K) = e_{y-x} \otimes K \in \mathcal{K}(E_{y-x+t})$$

for $K \in \mathcal{K}(E_t)$, $t \in G^+ \cup \{0\}$. Then β_{xy} is a non-unital isometric *-homomorphism.

We claim that the maps β_{xy}'s are coherent, i.e. if $x \leqslant y \leqslant z$ then $\beta_{xy}\beta_{yz} = \beta_{xz}$. Let $K \in \mathcal{K}(E_t)$. Then

$$(\beta_{xy}\beta_{yz})(K) = e_{y-x} \otimes e_{z-y} \otimes K = e_{z-x} \otimes K = \beta_{xz}(K).$$

Thus we can define $\mathcal{B}$ to be the C^*-algebra inductive limit $\lim_{x \to -\infty} \mathcal{B}_x$. Note that $\mathcal{B}$ is non-unital and is an AF-algebra since each $\mathcal{B}_x$ is an AF-algebra [7, Proposition 2.12].

It is convenient to write the elements of $\mathcal{B}_x$, when embedded in $\mathcal{B}$, *formally* as $e_{(-\infty,x]} \otimes K \otimes 1_{(x+t,\infty)}$ where $K \in \mathcal{K}(E_t)$, $t \in G^+$. The identity of $\mathcal{B}_x$ is *formally* written as $P_x = e_{(-\infty,x]} \otimes 1_{(x,\infty)}$. And e_t is *formally* written as $e_{(x,x+t]}$ for any $x \in G$.

For $y \in G$, define $\alpha_y \in Aut(\mathcal{B})$ by

$$\alpha_y(e_{(-\infty,x]} \otimes K \otimes 1_{(x+t,\infty)}) = e_{(-\infty,x+y]} \otimes K \otimes 1_{(x+y+t,\infty)}$$

for $K \in \mathcal{K}(E_t)$, $t \in G^+ \cup \{0\}$, $x \in G$. That is, G acts on $\mathcal{B}$ by translation.

Recall that the C^*-subalgebra $\mathcal{A}_1$ is *full* in $\mathcal{A}_2$ if the closed two-sided ideal generated by $\mathcal{A}_1$ is equal to $\mathcal{A}_2$. It is *hereditary* if $0 \leqslant x \leqslant y$ and $y \in \mathcal{A}_1$ imply $x \in \mathcal{A}_1$. If p is a projection in $\mathcal{A}_2$, then $p\mathcal{A}_2p$ is hereditary. In fact, it is the smallest hereditary subalgebra of $\mathcal{A}_2$ containing p.

THEOREM 3.1. *Let E be a discrete product system with a normalized unit v. Then there exists a C^*-dynamical system $(\mathcal{B}, G, \alpha)$, with $\mathcal{B}$ an AF-algebra, such that $\mathcal{O}_E(G)$ is a full, hereditary subalgebra of $G \times_\alpha \mathcal{B}$.*

Proof. Represent $G \times_\alpha \mathcal{B}$ faithfully on $\mathcal{H}$ so that

(i) there is a group of unitaries $\{U_x\}_{x \in G}$ on $\mathcal{H}$ satisfying $\alpha_x(B) = U_x B U_x^*$ for every $x \in G$, $B \in \mathcal{B}$, and

(ii) the finite sums of the form $\sum_x B_x U_x$, where $B_x \in \mathcal{B}$, form a dense $*$-subalgebra of $G \times_\alpha \mathcal{B}$.

Define $\varphi : E \to \mathcal{B}(P_0\mathcal{H})$ by

$$\varphi(u) = (e_{(-\infty,0]} \otimes \langle \cdot, v_t\rangle u \otimes 1_{(t,\infty)})(P_0 U_t P_0)$$

for $u \in E_t$.

Claim 1. $\varphi(v)^*\varphi(u) = \langle u, v\rangle P_0$ for $u, v \in E_t$.

$$\varphi(v)^*\varphi(u) = (P_0U_t^*P_0)(e_{(-\infty,0]}\otimes\langle\cdot, v\rangle v_t\otimes 1_{(t,\infty)})(e_{(-\infty,0]}\otimes\langle\cdot, v_t\rangle u\otimes 1_{(t,\infty)})(P_0U_tP_0) =$$

$$= P_0U_t^*(e_{(-\infty,0]} \otimes \langle u, v\rangle\langle\cdot, v_t\rangle v_t \otimes 1_{(t,\infty)})U_tP_0 =$$

$$= \langle u, v\rangle P_0\alpha_{-t}(e_{(-\infty,0]} \otimes e_{(0,t]} \otimes 1_{(t,\infty)})P_0 =$$

$$= \langle u, v\rangle P_0\alpha_{-t}(e_{(-\infty,t]} \otimes 1_{(t,\infty)})P_0 =$$

$$= \langle u, v\rangle P_0(e_{(-\infty,0]} \otimes 1_{(0,\infty)})P_0 = \langle u, v\rangle P_0.$$

Claim 2. $\varphi(uv) = \varphi(u)\varphi(v)$ for $u \in E_s, v \in E_t$.

$$\varphi(u)\varphi(v) = (e_{(-\infty,0]}\otimes\langle\cdot, v_s\rangle u\otimes 1_{(s,\infty)})(P_0U_sP_0)(e_{(-\infty,0]}\otimes\langle\cdot, v_t\rangle v\otimes 1_{(t,\infty)})(P_0U_tP_0) =$$

$$= (e_{(-\infty,0]} \otimes \langle\cdot, v_s\rangle u \otimes 1_{(s,\infty)})\alpha_s(e_{(-\infty,0]} \otimes \langle\cdot, v_t\rangle v \otimes 1_{(t,\infty)})U_{s+t}P_0 =$$

$$= (e_{(-\infty,0]} \otimes \langle\cdot, v_s\rangle u \otimes 1_{(s,\infty)})(e_{(-\infty,s]} \otimes \langle\cdot, v_t\rangle v \otimes 1_{(s+t,\infty)})U_{s+t}P_0 =$$

$$= (e_{(-\infty,0]} \otimes \langle\cdot, v_s\rangle u \otimes 1_{(s,\infty)})(e_{(-\infty,0]} \otimes (e_{(0,s]} \otimes \langle\cdot, v_t\rangle v) \otimes 1_{(s+t,\infty)})U_{s+t}P_0 =$$

$$= (e_{(-\infty,0]} \otimes (\langle\cdot, v_s\rangle u \otimes \langle\cdot, v_t\rangle v) \otimes 1_{(s+t,\infty)})U_{s+t}P_0 =$$

$$= (e_{(-\infty,0]} \otimes \langle\cdot, v_{s+t}\rangle uv \otimes 1_{(s+t,\infty)})(P_0U_{s+t}P_0) = \varphi(uv).$$

We conclude that φ is a representation of E on $P_0\mathcal{H}$. Thus we can form $\mathcal{O}_E(G)$, the C^*-algebra generated by the range of φ.

Claim 3. $\mathcal{F}_E(G) = \mathcal{B}_0$. Let $u, v \in E_t$. Then

$$\varphi(u)\varphi(v)^* = (e_{(-\infty,0]}\otimes\langle\cdot, v_t\rangle u\otimes 1_{(t,\infty)}(P_0U_tP_0)(P_0U_t^*P_0)(e_{(-\infty,0]}\otimes\langle\cdot, v\rangle v_t\otimes 1_{(t,\infty)}) =$$

$$= U_t\alpha_{-t}(e_{(-\infty,0]} \otimes \langle\cdot, v_t\rangle u \otimes 1_{(t,\infty)})P_0\alpha_{-t}(e_{(-\infty,0]} \otimes \langle\cdot, v\rangle v_t \otimes 1_{(t,\infty)})U_t^* =$$

$$= U_t(e_{(-\infty,-t]} \otimes \langle\cdot, v_t\rangle u \otimes 1_{(0,\infty)})$$

$$(e_{(-\infty,-t]} \otimes \langle\cdot, v_t\rangle v_t \otimes 1_{(0,\infty)})(e_{(-\infty,-t]} \otimes \langle\cdot, v\rangle v_t \otimes 1_{(0,\infty)})U_t^* =$$

$$= U_t(e_{(-\infty,-t]} \otimes \langle\cdot, v\rangle u \otimes 1_{(0,\infty)})U_t^* =$$

$$= \alpha_t(e_{(-\infty,-t]} \otimes \langle\cdot, v\rangle u \otimes 1_{(0,\infty)}) = e_{(-\infty,0]} \otimes \langle\cdot, v\rangle u \otimes 1_{(t,\infty)}.$$

The claim follows from the fact that $\mathcal{F}_E(G)$ is generated by the identity P_0 and elements of the form $\varphi(u)\varphi(v)^*$, while $\mathcal{B}_0$ is generated by P_0 and elements of the form $e_{(-\infty,0]} \otimes \langle\cdot, v\rangle u \otimes 1_{(t,\infty)}$.

Claim 4. $\mathcal{B}_0 = P_0\mathcal{B}P_0$.

Since $\mathcal{B}_0 = P_0\mathcal{B}_0P_0$, we need only to prove $\mathcal{B}_0 \supseteq P_0\mathcal{B}P_0$. But $\mathcal{B}$ is generated by the $\mathcal{B}_x$'s, which are increasing as x decreases to $-\infty$. Hence it is enough to show $\mathcal{B}_0 \supseteq P_0\mathcal{B}_xP_0$ for every $x < 0$. Let $K \in \mathcal{K}(E_t)$, $t \in G^+ \cup \{0\}$.

If $0 \leqslant t \leqslant -x$ then

$$P_0(e_{(-\infty,x]} \otimes K \otimes 1_{(x+t,\infty)})P_0 =$$

$$= (e_{(-\infty,x]} \otimes e_{(x,0]} \otimes 1_{(0,\infty)})(e_{(-\infty,x]} \otimes K \otimes 1_{(x+t,\infty)})(e_{(-\infty,x]} \otimes e_{(x,0]} \otimes 1_{(0,\infty)}) =$$

$$= e_{(-\infty,x]} \otimes (e_{(x,0]}(K \otimes 1_{-x-t})e_{(x,0]}) \otimes 1_{(0,\infty)} =$$

$$= \langle (K \otimes 1_{-x-t})v_{-x}, v_{-x}\rangle (e_{(-\infty,x]} \otimes e_{(x,0]} \otimes 1_{(0,\infty)} =$$

$$= \langle (K \otimes 1_{-x-t})v_{-x}, v_{-x}\rangle P_0 \in \mathcal{B}_0.$$

If $t > -x$, we may assume $K = K_1 \otimes K_2$ where $K_1 \in \mathcal{K}(E_{-x})$, $K_2 \in \mathcal{K}(E_{t+x})$ (since elements of the form $K_1 \otimes K_2$ generate $\mathcal{K}(E_t)$). Then

$$P_0(e_{(-\infty,x]} \otimes K \otimes 1_{(x+t,\infty)})P_0 =$$

$$= e_{(-\infty,x]} \otimes (e_{(x,0]}K_1e_{(x,0]} \otimes K_2) \otimes 1_{(x+t,\infty)} =$$

$$= \langle K_1v_{-x}, v_{-x}\rangle (e_{(-\infty,x]} \otimes (e_{(x,0]} \otimes K_2) \otimes 1_{(x+t,\infty)}) =$$

$$= \langle K_1v_{-x}, v_{-x}\rangle (e_{(-\infty,0]} \otimes K_2 \otimes 1_{(x+t,\infty)}) \in \mathcal{B}_0.$$

Claim 5. $\mathcal{O}_E(G) = P_0(G \times_\alpha \mathcal{B})P_0$, hence is hereditary.

The inclusion $\subseteq$ follows immediately from the definition of φ. To prove the other inclusion, it suffices to show $\mathcal{O}_E(G) \ni P_0BU_xP_0$ for every $B \in \mathcal{B}$, $x \in G$. Since $P_0BU_xP_0 = (P_0(U_{-x}B^*U_x)U_{-x}P_0)^* = (P_0\alpha_{-x}(B^*)U_{-x}P_0)^*$, we may assume $x \geqslant 0$. Then

$$U_xP_0 = U_xP_0U_x^*U_x = \alpha(P_0)U_x = P_xU_x = P_0P_xU_x =$$

$$= P_0U_xU_x^*P_xU_x = P_0U_x\alpha_{-x}(P_x) = P_0U_xP_0.$$

So $P_0BU_xP_0 = (P_0BP_0)(U_xP_0) \in \mathcal{F}_E(G)U_xP_0$, and we only need to show $U_xP_0 \in \mathcal{O}_E(G)$. Indeed if $x = 0$ then $U_xP_0 = P_0 =$ the identity of $\mathcal{O}_E(G)$. While if $x > 0$ then

$$\varphi(v_x) = (e_{(-\infty,0]} \otimes \langle\cdot, v_x\rangle v_x \otimes 1_{(x,\infty)})(P_0U_xP_0) =$$

$$= (e_{(-\infty,0]} \otimes e_{(0,x]} \otimes 1_{(x,\infty)})(U_xP_0) =$$

$$= P_x(U_xP_0) = U_x\alpha_{-x}(P_x)P_0 = U_xP_0P_0 = U_xP_0.$$

Claim 6. $\mathcal{O}_E(G)$ is full, i.e. the closed 2-sided ideal $\mathcal{I}$ generated by $\mathcal{O}_E(G)$ is equal to all of $G \times_\alpha \mathcal{B}$. Since $\{P_x\}_{x\to-\infty}$ is an approximate identity of $G \times_\alpha \mathcal{B}$, it suffices to show $\mathcal{I} \ni P_x$ or every $x \in G$. Indeed,

$$P_x = \alpha_x(P_0) = (U_x P_0)P_0(P_0 U_x^*) \in \mathcal{I}$$

because $P_0 \in \mathcal{O}_E(G)$ and $U_x P_0,\ P_0 U_x^* \in G \times_\alpha \mathcal{B}$. ∎

Recall that a C^*-algebra $\mathcal{A}_1$ is *nuclear* if for every C^*-algebra $\mathcal{A}_2$, there is exactly one C^*-norm on the algebric tensor product $\mathcal{A}_1 \odot \mathcal{A}_2$. The class of nuclear C^*-algebras includes AF-algebras and is closed under stable isomorphism and taking crossed product by amenable groups. A C^*-subalgebra $\mathcal{A}_1$ of a nuclear C^*-algebra $\mathcal{A}_2$ is not necessarily nuclear. However, if in addition, $\mathcal{A}_1$ and $\mathcal{A}_2$ have strictly positive elements and $\mathcal{A}_1$ is a full hereditary subalgebra of $\mathcal{A}_2$, then $\mathcal{A}_1$ is stably isomorphic to $\mathcal{A}_2$ [3], hence is also nuclear. Thus we have

COROLLARY 3.2. *If E has a unit then $\mathcal{O}_E(G)$ is nuclear.*

In particular, $C^*(V_1, \ldots, V_N)$ is nuclear.

4. K-THEORY

In this section, we compute the K-groups of $C^*(V_1, \ldots, V_N)$. More generally, we compute the K-groups of $\mathcal{O}_E(G)$ when E has a unit.

PROPOSITION 4.1. *$K_1(\widetilde{\mathcal{F}}_E(G)) = 0$ and $K_0(\widetilde{\mathcal{F}}_E(G)) = \sum^{\oplus}_{t\in G^+\cup\{0\}} \mathbf{Z}[e_t]_0$.*

Proof. We use [7,Lemma 2.9]. Let $t_1 < \cdots < t_n$. By induction and a straight forward application of the six-term exact sequence to the split exact sequence

$$0 \to \mathcal{K}(E_{t_n}) \to \mathbf{C}1 + \mathcal{K}(E_{t_1}) + \cdots + \mathcal{K}(E_{t_n}) \to \mathbf{C}1 + \mathcal{K}(E_{t_1}) + \cdots + \mathcal{K}(E_{t_{n-1}}) \to 0,$$

we have $K_1(\mathbf{C}1 + \mathcal{K}(E_{t_1}) + \cdots + \mathcal{K}(E_{t_n})) = 0$ and $K_0(\mathbf{C}1 + \mathcal{K}(E_{t_1}) + \cdots + \mathcal{K}(E_{t_n})) = = \mathbf{Z}[1]_0 \oplus \mathbf{Z}[e_{t_1}]_0 \oplus \cdots \oplus \mathbf{Z}[e_{t_n}]_0$. The assertion follows by taking inductive limits. ∎

COROLLARY 4.2. *$K_1(\mathcal{B}) = 0$ and $K_0(\mathcal{B}) = \sum^{\oplus}_{x\in G} \mathbf{Z}[P_x]_0$.*

THEOREM 4.3. *$K_1(\mathcal{O}_E(G)) = 0$ and $K_0(\mathcal{O}_E(G)) = \mathbf{Z}[1]_0$.*

Proof. First, suppose G is finitely generated so that $G = \theta_1\mathbf{Z} + \cdots + \theta_m\mathbf{Z}$ where $\theta_1, \ldots, \theta_m > 0$ are rationally independent. Apply the Pimsner-Voiculescu exact sequence to $\theta_m\mathbb{Z} \times_\alpha \mathcal{B}$, we have

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \xrightarrow{1-(\alpha_{-\theta_m})_*} & K_0(\mathcal{B}) & \xrightarrow{i_*} & K_0(\theta_m\mathbb{Z} \times_\alpha \mathcal{B}) \\ \uparrow & & & & \downarrow \\ K_1(\theta_m\mathbb{Z} \times_\alpha \mathcal{B}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{B}). \end{array}$$

Since $\alpha_{-\theta_m}(P_x) = P_{x-\theta_m}$, the map $1-(\alpha_{-\theta_m})_* : K_0(\mathcal{B}) \to K_0(\mathcal{B})$ is given by

$$1-(\alpha_{-\theta_m})_* : n = (n_x[P_x]_0)_{x\in G} \mapsto ((n_x - n_{x+\theta_m})[P_x]_0)_{x\in G}.$$

This implies $\mathrm{Ker}(1-(\alpha_{-\theta_m})_*) = 0$. Exactness at the two left corners and the fact $K_1(\mathcal{B}) = 0$ then implies $K_1(\theta_m\mathbb{Z} \times_\alpha \mathcal{B}) = 0$. Hence we have the exact sequence

$$0 \longrightarrow K_0(\mathcal{B}) \xrightarrow{1-(\alpha_{-\theta_m})_*} K_0(\mathcal{B}) \xrightarrow{i_*} K_0(\theta_m\mathbb{Z} \times_\alpha \mathcal{B}) \longrightarrow 0.$$

We claim that the range of $1-(\alpha_{-\theta_m})_*$ consists of all elements $l = (l_x[P_x]_0)_{x\in G}$ such that

$$\sum_{k\in\mathbb{Z}} l_{x-\theta_m k} = 0 \tag{4.1}$$

for every $x \in G$. Indeed if l is in the range, then there exists $n = (n_x[P_x]_0)_{x\in G}$ such that $l_x = n_x - n_{x+\theta_m}$ for every $x \in G$. Thus $\sum_k l_{x-\theta_m k} = \sum_k n_{x-\theta_m k} - \sum_k n_{x-\theta_m(k-1)} = 0$. Conversely suppose (4.1) holds. Define $n_x = l_x + l_{x+\theta_m} + l_{x+2\theta_m} + \cdots$ for every $x \in G$. Then $n_x - n_{x+\theta_m} = l_x$ for every $x \in G$, and so it remains to be shown that $n \in \sum_{x\in G}^{\oplus} \mathbb{Z}[P_x]_0$, i.e. that $n_x = 0$ for all but finitely many x's. Let $\mathcal{T} = \{s_1 < \cdots < s_n\}$ be the support of l. Then clearly (from the definition of n) $n_x = 0$ when $x > s_n$. When $x \leqslant s_1$ condition (4.1) implies $n_x = 0$. For $s_1 < x \leqslant s_n$, we have

$$\mathcal{T} \cap \{x, x+\theta_m, x+2\theta_m, \ldots\} = \emptyset$$

(in which case $n_x = 0$) for all except finitely many x's.

Now we claim that $K_0(\theta_m\mathbb{Z} \times_\alpha \mathcal{B})$ is isomorphic to the free abelian group $\sum_{y\in\theta_1\mathbb{Z}+\cdots+\theta_{m-1}\mathbb{Z}}^{\oplus} \mathbb{Z}_y$ where each $\mathbb{Z}_y = \mathbb{Z}$. Indeed the map

$$l = (l_x[P_x]_0)_{x\in\theta_1\mathbb{Z}+\cdots+\theta_m\mathbb{Z}} \mapsto \Big(\sum_{k\in\mathbb{Z}} l_{y-\theta_m k}\Big)_{y\in\theta_1\mathbb{Z}+\cdots+\theta_{m-1}\mathbb{Z}}$$

is a homomorphism from $K_0(\mathcal{B}) = \sum^{\oplus}_{x\in\theta_1\mathbf{Z}+\cdots+\theta_m\mathbf{Z}} \mathbf{Z}[P_x]_0$ onto $\sum^{\oplus}_{y\in\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}} \mathbf{Z}_y$ with kernel equal to the range of $1-(\alpha_{-\theta_m})_*$. Hence $K_0(\theta_m\mathbf{Z}\times_\alpha\mathcal{B}) \cong K_0(\mathcal{B})/\mathrm{Ran}(1-$ $-\alpha_{-\theta_m})_*) \cong \sum^{\oplus}_{y\in\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}} \mathbf{Z}_y$.

Next we claim that the set $\{[P_x]_0 : x\in\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}\}$ is a basis for $K_0(\theta_m\mathbf{Z}\times_\alpha \mathcal{B})$. Since $\{[P_x]_0 : x\in\theta_1\mathbf{Z}+\cdots+\theta_m\mathbf{Z}\}$ generates $K_0(\mathcal{B})$, and since i_* is onto, $\{[P_x]_0 : x\in\theta_1\mathbf{Z}+\cdots+\theta_m\mathbf{Z}\}$ also generates $K_0(\theta_m\mathbf{Z}\times_\alpha\mathcal{B})$. It is clear from (4.1) that if $x,y\in\theta_1\mathbf{Z}+\cdots+\theta_m\mathbf{Z}$, then $[P_x]_0$ and $[P_y]_0$ are equal in $K_0(\theta_m\times_\alpha\mathcal{B})$ if and only if $x-y\in\theta_m\mathbf{Z}$. Thus $\{[P_x]_0 : x\in\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}\}$ generates $K_0(\theta_m\mathbf{Z}\times_\alpha\mathcal{B})$. To show that it is a basis, assume

$$n_1[P_{x_1}]_0+\cdots+n_k[P_{x_k}]_0=0$$

in $K_0(\theta_m\mathbf{Z}\times_\alpha\mathcal{B})$, where $n_1,\ldots,n_k\in\mathbf{Z}$ and $x_1,\ldots,x_k$ are distinct elements of $\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}$. Then $n_1[P_{x_1}]_0+\cdots+n_k[P_{x_k}]_0$ is in the kernel of i_*, so in the range of $1-(\alpha_{-\theta_m})_*$. Again (4.1) easily implies $n_1=\cdots=n_k=0$.

We conclude that $K_0(\theta_m\mathbf{Z}\times_\alpha\mathcal{B}) = \sum^{\oplus}_{x\in\theta_1\mathbf{Z}+\cdots+\theta_{m-1}\mathbf{Z}} \mathbf{Z}[P_x]_0$. Repeat the above argument to get $K_0(\theta_{m-1}\mathbf{Z}\times_\alpha(\theta_m\mathbf{Z}\times_\alpha\mathcal{B})) = \sum^{\oplus}_{x\in\theta_1\mathbf{Z}+\cdots+\theta_{m-2}\mathbf{Z}} \mathbf{Z}[P_x]_0$. Since θ_{m-1} and θ_m are rationally independent, we have $\theta_{m-1}\mathbf{Z}\times_\alpha(\theta_m\mathbf{Z}\times_\alpha\mathcal{B}) = (\theta_{m-1}\mathbf{Z}+\theta_m\mathbf{Z}\times_\alpha\mathcal{B}$. Hence $K_0((\theta_{m-1}\mathbf{Z}+\theta_m\mathbf{Z})\times_\alpha\mathcal{B}) = \sum^{\oplus}_{x\in\theta_1\mathbf{Z}+\cdots+\theta_{m-2}\mathbf{Z}} \mathbf{Z}[P_x]_0$. Thus, after repeating the argument m times, we get $K_1(G\times_\alpha\mathcal{B})=0$ and $K_0(G\times_\alpha\mathcal{B})=\mathbf{Z}[P_0]_0$. By Theorem 3.1 and [11, Theorem 1.2], we get $K_1(\mathcal{O}_E(G))=0$ and $K_0(\mathcal{O}_E(G))=\mathbf{Z}[1]_0$.

Finally, for arbitrary G, write G as the union of an increasing sequence of finitely generated subgroups. The assertion of the theorem then follows by taking inductive limits of the K-groups. ∎

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