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(±)-REGULAR FACTORIZATION OF TRANSFER FUNCTIONS AND PASSIVE SCATTERING SYSTEMS FOR CASCADE COUPLING

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ABSTRACT. The purpose of this paper is to study the new notions of (+)-regular and (−)-regular factorizations of contractive analytic operator functions, as well as their applications to the problems of minimality, controllability and observability for cascade coupling of conservative or passive scattering systems. We consider also cascade factorization of minimal, optimal passive scattering systems.

KEYWORDS: (+)*-regular factorization, cascade coupling, conservative systems, passive systems.*

AMS SUBJECT CLASSIFICATION: Primary 47A45, 47A68; Secondary 93B05, 93B07.

1. INTRODUCTION

1. It is well-known that qualitative properties of dynamic systems, as controllability, observability and minimality, in the general case, are not invariant for cascade coupling. In the theory of automatic controls, there are many investigations on controllability and observability for cascade coupling of finite dimensional systems, where the results are expressed in terms of matrix rank or of McMillan degree of rational matrix functions ([5], [9], [6]). We will see that it is difficult to develop these results for infinite dimensional systems.

The purpose of this paper is to study the so-called (+)-regular and (−)-regular factorizations for analytic functions in the unit disk $\mathbf{D} = \{z : |z| < 1\}$, whose values are contractive operators from a Hilbert space U to a Hilbert space

V. An underlying theme is the minimality, controllability and observability of infinite dimensional systems for cascade coupling of two controllable or observable passive scattering systems. The results about ($\pm$)-regularity are obtained in terms of observable and controllable subspaces. We obtain criteria for minimality for cascade coupling and we consider also cascade factorizations of minimal, optimal passive scattering systems.

2. Let us consider a linear discrete stationary dynamic system $\alpha = (X, U, V; A, B, C, D)$ of the form

$$x_{n+1} = Ax_n + Bu_n$$

$$v_n = Cx_n + Du_n$$

$$x_n \in X, u_n \in U, v_n \in V.$$

The state space X, the input space U and the output space V are separable Hilbert spaces, the operators $A : X \to X, B : U \to X, C : X \to V, D : U \to V$ are linear bounded.

The operator function:

$$\theta z = D + zC(I - zA)^{-1}B$$

is called a *transfer function* of system α.

The subspaces

$$X^c_\alpha = \bigvee_{n=0}^{\infty} A^n BU, \quad X^o_\alpha = \bigvee_{n=0}^{\infty} A^{*n} CV$$

are called respectively *controllable* and *observable subspaces* of α. The system α is said to be *controllable* (resp. *observable*, *minimal*, *simple*) if $X^c_\alpha = X$ (resp. $X^o_\alpha = X, X^c_\alpha = X = X^o_\alpha, X^c_\alpha \vee X^o_\alpha = X$).

The system α is said to be a *passive scattering system* ([2]) (p.s.s.), if the operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus U \to X \oplus V$$

is contractive. If this operator is unitary, then the system α is said to be a *conservative scattering system* (c.s.s.).

It is well known ([1], [2], [3]) that the transfer function of a p.s.s. α belongs to the class $B(U, V)$ of all analytic functions in unit disk $\mathbf{D}$, whose values are contractive operators from U to V. It is also known ([4]) that a simple c.s.s. is determined by its transfer function uniquely up to unitary equivalence.

In this paper we shall use the following functional model of Sz.-Nagy and Foiaş for simple c.s.s. ([4]), constructed from a given transfer function $\Theta(z) \in B(U,V)$.

Let

$$X = [L_2^+(V) \oplus \overline{\Delta L_2(U)}] \ominus \{\Theta w \oplus \Delta w : w \in L_2^+(U)\} \tag{1}$$

$$A(\varphi \oplus \psi) = e^{-it}(\varphi(e^{it} - \varphi(0)) \oplus e^{-it}\psi(e^{it}) \tag{2}$$

$$Bu = e^{-it}(\Theta(e^{it}) - \Theta(0))u \oplus e^{-it}\Delta(e^{it})u \tag{3}$$

$$C(\varphi \oplus \psi) = \varphi(0), \quad Du = \Theta(0)u \tag{4}$$

where

$$\Delta \equiv \Delta(e^{it}) = (I - \Theta(e^{it})^*\Theta(e^{it}))^{\frac{1}{2}}$$

and $L_2^+(U)$ stands for the Hardy space of elements $f \in L_2(U)$ whose k-th Fourier coefficient $\hat{f}(k) = 0$ for all $k < 0$.

It is not difficult to show that the following lemma holds.

LEMMA 1. *For the model system* (1)-(4) *we have*

$$X \ominus X_\alpha^c = \{(\varphi \oplus \psi) : \varphi \oplus \psi \in X, \Theta^*\varphi + \Delta\psi = 0\}$$

$$X \ominus X_\alpha^o = \{(0 \oplus \psi) : (0 \oplus \psi) \in X\}.$$

DEFINITION 1. Let $\alpha_k = (X_k, U_k, V_k; A_k, B_k, C_k, D_k)$, $k = 1, 2$, be linear systems satisfying $V_1 = U_2$. Then the following system $\alpha = (X_1 \oplus X_2, U_1, V_2; A_1P_1 + A_2P_2 + B_2C_1P_1, B_1 + B_2D_1, D_2C_1P_1 + C_2P_2, D_2D_1)$ is called a *cascade coupling* of α_1, α_2 and it is written $\alpha = \alpha_2\alpha_1$.

It is well known, that if $\alpha = \alpha_2\alpha_1$ then $\Theta_\alpha(z) = \Theta_{\alpha_2}(z)\Theta_{\alpha_1}(z)$ and if α_1, α_2 are c.s.s. (resp. p.s.s.) then $\alpha = \alpha_2\alpha_1$ is also a c.s.s. (resp. p.s.s.).

LEMMA 2. *Let* α_1, α_2 *be simple c.s.s., constructed by* (1)–(4), *and* $\alpha = \alpha_2\alpha_1$. *Then we have*

$$X \ominus X_\alpha^o = \{(\varphi_1 \oplus \psi_1) \oplus (\varphi_2 \oplus \psi_2) \in X_1 \oplus X_2 : \varphi_2 + \Theta_2\varphi_1 = 0\}$$

$$X \ominus X_\alpha^c = \{(\varphi_1 \oplus \psi_1) \oplus (\varphi_2 \oplus \psi_2) \in X_1 \oplus X_2 : \varsigma_1 + \Theta_1^*\varsigma_2 = 0, \varsigma_k = \Theta_k^*\varphi_k + \Delta_k\psi_k, k = 1, 2\}.$$

2. (+)-REGULAR FACTORIZATION FOR TRANSFER FUNCTION OF CONSERVATIVE SCATTERING SYSTEMS

1. We introduce the new notion of (+)-regular factorization, which is very important for further considerations.

Let $\Theta(z) = \Theta_2(z)\Theta_1(z)$ be a factorization of the contractive operator function $\Theta(z) \in B(U, V)$, where $\Theta_k(z) \in B(U_k, V_k), k = 1, 2, U_1 = U, V_1 = U_2, V_2 = V$.

DEFINITION 2. The factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is said to be (+)-*regular*, if

$$\overline{\{\Delta_2\Theta_1 h \oplus \Delta_1 h : h \in L_2^+(U)\}} = \overline{\Delta_2 L_2^+(U_2)} \oplus \overline{\Delta_1 L_2^+(U_1)}\ .$$

The concept of (+)-regular factorization has been suggested to the author by the concept of regular factorization, introduced by Sz.-Nagy and Foiaş ([8]), and differs from the last by replacing the spaces $L_2(U_1), L_2(U_2)$ by $L_2^+(U_1)$ and $L_2^+(U_2)$.

The (+)-regularity of the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is equivalent to the following: the operator $z^+ : \Delta h \to \Delta_2\Theta_1 h \oplus \Delta_1 h, h \in L_2^+(U_1)$, after continuous extension, will be a unitary operator from $\overline{\Delta L_2^+(U)}$ to $\overline{\Delta_2 L_2^+(U_2)} \oplus \overline{\Delta_1 L_2^+(U_1)}$.

PROPOSITION 1. *If the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is (+)-regular, then it is also regular (in the sense of ([8]))*.

Proof. By a similar way as in the proof of Proposition VII.3.1 in ([8]), it is not difficult to show that if the factorization is (+)-regular then for all $t \in [0, 2\pi]$ we have

$$\overline{\{\Delta_2(e^{it})\Theta_1(e^{it})u \oplus \Delta_1(e^{it})u : u \in U\}} = \overline{\Delta_2(e^{it})U_2 \oplus \Delta_1(e^{it})U_1}$$

whence it follows that the factorization is regular. ∎

2. From Definition 2 it is easy to obtain

LEMMA 3. *The factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is (+)-regular if and only if the following inclusion*

$$\Theta_1^*\Delta_2 f_2 + \Delta_1 f_1 \in L_2^-(U_1)$$

where $f_k \in \overline{\Delta_k L_2^+(U_k)}, k = 1, 2$, is possible in the unique case: $f_1 = 0, f_2 = 0$.

THEOREM 1. *In order that the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ be (+)-regular, it is necessary and sufficient that for arbitrary simple c.s.s. α_1, α_2 with $\Theta_{\alpha_k}(z) = \Theta_k(z), k = 1, 2$ we have*

$$X_\alpha^o = X_{\alpha_1}^o \oplus X_{\alpha_2}^o \quad (\alpha = \alpha_2\alpha_1).$$

Proof. We see first that for $\alpha = \alpha_2\alpha_1$ we always have the following inclusion

$$X^o_\alpha \subset X^o_{\alpha_1} \oplus X^o_{\alpha_2}$$

and secondly that it is enough to prove the theorem for model systems.

Sufficiency. Let

$$\Theta_1^* \Delta_2 f_2 + \Delta_1 f_1 \in L_2^-(U_2), \quad \text{where} \quad f_k \in \overline{\Delta_k L_2^+(U_k)}, \quad k = 1, 2.$$

Denoting

$$\varphi_1 = P_{L_2^+(U_2)} \Delta_2 f_2, \quad \psi_1 = f_1 \tag{5}$$

$$\varphi_2 = -\Theta_2 \varphi_1, \quad \psi_2 = f_2 - \Delta_2 \varphi_1 \tag{6}$$

it is easy to see that

$$\varphi_k \in L_2^+(V_k), \quad \psi_k \in \overline{\Delta_k L_2(U_k)}, \quad k = 1, 2. \tag{7}$$

Since the element $\varphi_1^- = \Delta_2 f_2 - \varphi_1$ belongs to $L_2^-(V_1)$, and

$$\Theta_1^* \Delta_2 f_2 + \Delta_1 f_1 = \Theta_1^* \varphi_1 + \Delta_1 \psi_1 + \Theta_1^* \varphi_1^- \in L_2^-(U_1)$$

we obtain

$$\Theta_1 \varphi_1 + \Delta_1 \psi \in L_2^-(U_1). \tag{8}$$

Moreover we have

$$\Theta_2^* \varphi_2 + \Delta_2 \psi_2 = -\Theta_2^* \Theta_1 \varphi_1 + \Delta_2 (f_2 - \Delta_2 \varphi_1) = \varphi_1^- \in L_2^-(V_1). \tag{9}$$

From (7), (8) and (9), it follows

$$\varphi_k \oplus \psi_k \in X_k, \quad k = 1, 2. \tag{10}$$

Taking into account that $\psi_k \in \overline{L_2^+(U_k)}, k = 1, 2$ and Lemma 1 we see that

$$\varphi_k \oplus \psi_k \in X^o_{\alpha_k}, \quad k = 1, 2,$$

which, together with (6) and Lemma 2 give

$$(\varphi_1 \oplus \psi_1) \oplus (\varphi_2 \oplus \psi_2) \in [X^o_{\alpha_1} \oplus X^o_{\alpha_2}] \ominus X^o_\alpha.$$

From this we have

$$\varphi_k = 0, \quad \psi_k = 0, \quad k = 1, 2.$$

Thus, by virtue of Lemma 3, the factorization is (+)-regular.

Necessity. Let us consider any element

$$h = (\varphi_1 \oplus \psi_1) \oplus (\varphi_2 \oplus \psi_2) \in [X^o_{\alpha_1} \oplus X^o_{\alpha_2}] \ominus X^o_\alpha. \tag{11}$$

Denote

$$f_1 = \psi_1, \quad f_2 = \psi_2 + \Delta_2\varphi_1. \tag{12}$$

Since $(\varphi_k \oplus \psi_k) \in X^o_{\alpha_k}, k = 1, 2$, then from Lemma 1 it follows that

$$\psi_k \in \overline{\Delta_1 L^+_2(U_k)}, \quad k = 1, 2,$$

and hence, from (12) we have

$$f_k \in \overline{\Delta_k L^+_2(U_k)}, \quad k = 1, 2.$$

Since $h \perp X^o_\alpha$ we obtain $\varphi_2 + \Theta_2\varphi_1 = 0$, hence

$$\Theta^*_1\Delta_2 f_2 + \Delta_1 f_1 = [\Theta^*_1\varphi_1 + \Delta_1\psi_1] + \Theta^*_1[\Theta^*_2\varphi_2 + \Delta_2\psi_2] \in L^-_2(U_1).$$

Therefore we have

$$f_1 = 0, \quad f_2 = 0.$$

Further, since

$$0 = \Delta_2 f_2 = \Delta_2\varphi_2 + \Theta^*_2\varphi_2 + \varphi_1,$$

and

$$\varphi_1 \in L^+_2(V_1), \quad \Theta^*_2\varphi_2 + \Delta_2\psi_2 \in L^-_2(V_2),$$

we obtain $\varphi_1 = 0$, hence $\varphi_2 = 0$. Lemma 1 gives then

$$(\varphi_k \oplus \psi_k) \in X_k \ominus X^o_{\alpha_k}, \quad k = 1, 2.$$

From this and (11) follows $h = 0$. This completes the proof of the theorem. ∎

This theorem implies immediately the following criterion for observability of cascade coupling.

COROLLARY 1. *Let α_1, α_2 be observable c.s.s. having transfer functions $\Theta_1(z), \Theta_2(z)$ respectively. In order that the cascade coupling $\alpha = \alpha_2\alpha_1$ is observable it is necessary and sufficient that the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is (+)-regular.*

COROLLARY 2. *Let $\Theta(z) \in B(U,V)$ have the property that $\overline{\Delta L_2^+(U)} = \overline{\Delta L_2(U)}$. Then the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is (+)-regular if and only if this factorization is regular.*

Proof. Let α be a simple c.s.s. having transfer function $\Theta(z)$. Since $\overline{\Delta L_2^+(U)} = \overline{\Delta L_2(U)}$, from ([2]) we have α observable. If the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is regular, then the system α has a factorization $\alpha = \alpha_2\alpha_1$, where α_1, α_2 are observable c.s.s. with transfer functions $\Theta_1(z), \Theta_2(z)$ respectively ([4]). The rest follows from Proposition 1 and Corollary 1.

3. (−)-REGULAR FACTORIZATION FOR TRANSFER FUNCTION OF CONSERVATIVE SCATTERING SYSTEM

1. Passing to dual system, using the duality of observability and controllability, we obtain similar results for controllable subspaces of cascade couplings.

THEOREM 2. *Let α_1, α_2 be simple c.s.s. and $\alpha = \alpha_2\alpha_1$. The factorization $\tilde{\Theta}_\alpha(z) = \tilde{\Theta}_{\alpha_1}(z)\tilde{\Theta}_{\alpha_2}(z)$ (where $\tilde{\Theta}(z) = \Theta^*(\bar{z})$) is (+)-regular if and only if*

$$X_\alpha^c = X_{\alpha_1}^c \oplus X_{\alpha_2}^c.$$

2. To study controllable subspaces, it is useful to introduce the dual notation of a (−)-regular factorization.

DEFINITION 3. A factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is said to be *(−)-regular* if

$$\overline{\{\Delta_{1*}\Theta_2^* h \oplus \Delta_{2*}h : h \in L_2^-(V_2)\}} = \overline{\Delta_{1*}L_2^-(V_1)} \oplus \overline{\Delta_{2*}L_2^-(V_2)},$$

where

$$\Delta_{k*} = (I - \Theta_k(e^{it}\Theta_k(e^{it})^*)^{\frac{1}{2}}.$$

Definition 3 is equivalent to the following: the operator

$$z^- : \Delta_* h \to \Delta_{1*}\Theta_2^* h \oplus \Delta_{2*}h, \quad h \in L_2^-(V_2),$$

after continuous extension, will be a unitary operator from $\overline{\Delta_* L_2^-(V)}$ to $\overline{\Delta_{1*}L_2^-(V_1)} \oplus \overline{\Delta_{2*}L_2^-(V_2)}$.

Notice that from (−)-regularity we obtain regularity.

PROPOSITION 2. *The factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is $(-)$-regular if and only if the factorization $\tilde{\Theta}(z) = \tilde{\Theta}_1(z)\tilde{\Theta}_2(z)$ is $(+)$-regular.*

Proof. Let the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ be $(-)$-regular and let

$$\tilde{\Theta}_2^*(e^{it})\tilde{\Delta}_1(e^{it})f_1(e^{it}) + \tilde{\Delta}_2(e^{it})f_2(e^{it}) \in L_2^-(V_2),$$

where

$$f_k = \overline{\tilde{\Delta}_k(e^{it})L_2^+(V_k)}, \quad k = 1, 2.$$

From this we obtain

$$\Theta_2(e^{it})\Delta_{1*}(e^{it})\hat{f}_1(e^{it}) + \Delta_{2*}(e^{it})\hat{f}_2(e^{it}) \in L_2^+(V_2),$$

where

$$\hat{f}_k(e^{it}) \equiv (e^{-it})f_k(e^{-it}) \in \overline{\Delta_{k*}(e^{it})L_2^-(V_2)}, \quad k = 1, 2.$$

Hence, by $(-)$-regularity, we have $\hat{f}_k = 0, k = 1, 2$ and hence $f_k = 0, k = 1, 2$. Thus, the factorization $\tilde{\Theta}(z) = \tilde{\Theta}_1(z)\tilde{\Theta}_2(z)$ is $(+)$-regular. The inverse statement is proved in a similar way. ∎

Thus, we can reformulate the Theorem 2 in the following form:

THEOREM 3. *In order that the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ be $(-)$-regular it is necessary and sufficient that for arbitrary simple c.s.s. α_1, α_2 with $\Theta_{\alpha_k}(z) = \Theta_k(z), k = 1, 2$ we have*

$$X^c_{\alpha_2,\alpha_1} = X^c_{\alpha_1} \oplus X^c_{\alpha_2}.$$

COROLLARY 3. *Suppose that a function $\Theta(z) \in B(U, V)$ has the following (equivalent) properties*

1. $\overline{\tilde{\Delta}L_2^+(V)} = \overline{\tilde{\Delta}L_2(V)}$.
2. $\overline{\Delta_* L_2^-(V)} = \overline{\Delta_* L_2(V)}$.
3. *The equality $\Theta^*\varphi + \Delta\psi = 0$, where $\varphi \in L_2^+(V), \psi \in \overline{\Delta L_2(U)}$, is possible only in the case $\varphi = 0, \psi = 0$.*

Then the $(-)$-regularity of the factorization $\Theta(z) = \Theta_2(z)\Theta_1(z)$ is equivalent to its regularity.

We obtain also a criterion for minimality of c.s.s. for cascade couplings.

THEOREM 4. *Let α_1, α_2 be minimal c.s.s.. Then the system $\alpha = \alpha_2\alpha_1$ is minimal if and only if the respective factorization $\Theta_\alpha(z) = \Theta_{\alpha_2}(z)\Theta_{\alpha_1}(z)$ is $(+)$-regular and $(-)$-regular.*

4. (±)-REGULAR FACTORIZATION AND PASSIVE SCATTERING SYSTEMS

In this section we shall apply the obtained results to study the cascade coupling of passive scattering systems (p.s.s.).

1. We consider first several simple propositions on dilation and projection of linear systems.

DEFINITION 4. A linear system $\alpha = (X, U, V; A, B, C, D)$ is said to be the *projection of the linear system* $\tilde{\alpha} = (\tilde{X}, \tilde{U}, \tilde{V}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ onto the subspace X if

$$X \subset \tilde{X}, \quad U = \tilde{U}, \quad V = \tilde{V} \tag{13}$$

$$A = P_X \tilde{A}|_X, \quad B = P_X \tilde{B}, \quad C = \tilde{C}|_X, \quad D = \tilde{D} \tag{14}$$

where P_X is the orthoprojection of $\tilde{X}$ onto X. In this case we shall write $\alpha = P_X \tilde{\alpha}$.

Let $\alpha = P_X \tilde{\alpha}$, and moreover, suppose, that there exist subspaces G, G_* of $\tilde{X}$ such that

$$\tilde{X} = G_* \oplus X \oplus G$$

$$\tilde{A}G \subset G, \quad \tilde{A}^* G_* \subset G_*, \quad \tilde{B}^* G_* = \{0\}, \quad \tilde{C}G = \{0\}$$

then the system $\tilde{\alpha}$ is called a *dilation* ([1], [2]) of the system α and written $\tilde{\alpha} = \operatorname{dil} \alpha$. In this case, it is easy to see that the transfer functions of α and $\tilde{\alpha}$ are equal in a certain neighborhood of 0. It is not difficult to check the next statements.

PROPOSITION 3. *We have*

$$(P_{X_2} \tilde{\alpha}_2)(P_{X_1} \tilde{\alpha}_1) = P_{X_1 \oplus X_2} \tilde{\alpha}_2 \tilde{\alpha}_1,$$

$$(\operatorname{dil} \alpha_2)(\operatorname{dil} \alpha_1) = \operatorname{dil} \alpha_2 \alpha_1.$$

PROPOSITION 4. *Let α be a dilation of α_1 and of α_2 with respective decompositions of X:*

$$X = G_{1*} \oplus X_1 \oplus G_1, \quad X = G_{2*} \oplus X_2 \oplus G_2.$$

If moreover $G_2 \subset G_1, G_{2*} \subset G_{1*}$ then α_2 is a dilation of α_1.

2. Let us consider a cascade factorization of a passive scattering system.

THEOREM 5. *Let α be a p.s.s. and X_2 be an invariant subspace of the main operator A of α. Then there exist p.s.s. α_1, α_2 having state spaces $X_1 = X \ominus X_2, X_2$ respectively and $\alpha = \alpha_2\alpha_1$.*

Proof. According to ([2]), for any p.s.s. α there exists a c.s.s. $\tilde{\alpha}$ being a dilation of α. Then the state space of $\tilde{\alpha}$ has the following decomposition

$$\tilde{X} = G_* \oplus X \oplus G,$$

where G is an invariant subspace of $\tilde{A}$, hence $X_2 \oplus G$ is also invariant for $\tilde{A}$. From this, it follows that there exist c.s.s. α_1, α_2 having state spaces $\tilde{X}_1 = G_* \oplus X_1, \tilde{X}_2 = X_2 \oplus G$ respectively and $\tilde{\alpha} = \tilde{\alpha}_2\tilde{\alpha}_1$. By Proposition 3 we obtain

$$\alpha = P_X\tilde{\alpha} = (P_{X_2}\tilde{\alpha}_2)(P_{X_1}\tilde{\alpha}_1).$$

The systems $\alpha_k = P_{X_k}\tilde{\alpha}_k, k = 1, 2$ are the searched p.s.s.

3. In this section we study optimal minimal p.s.s. in terms of (+)-regular and (−)-regular factorizations.

According to ([3]), a p.s.s. $\alpha = (X, U, V; A, B, C, D)$ is said to be *optimal* if for any p.s.s. $\alpha' = (X', U, V; A', B', C', D')$ having the same transfer function as $\alpha : \Theta_{\alpha'}(z) = \Theta_\alpha(z)$ in a certain neighborhood of 0, we have

$$\left\|\sum_{k=0}^{n} A^k B u_k\right\| \leqslant \left\|\sum_{k=0}^{n} A'^k B' u_k\right\| \qquad (\forall n, \forall u_k \in U) \ .$$

In ([1], [3], [2]) it is proved that any contractive analytic in $\mathbf{D}$ operator function is the transfer function of a certain optimal minimal p.s.s. and an optimal minimal p.s.s. is uniquely determined by its transfer function (up to unitary equivalence).

An observable p.s.s. α is said to be $*$-optimal if for any p.s.s. α' with $\Theta_{\alpha'}(z) = \Theta_\alpha(z)$ (in $\mathbf{D}$) we have

$$\left\|\sum_{k=0}^{n} A^k B u_k\right\| \geqslant \left\|\sum_{k=0}^{n} A'^k B' u_k\right\| \qquad (\forall n, \forall u_k \in U) \ .$$

A minimal $*$-optimal p.s.s. is uniquely determined by its transfer function (up to unitary equivalence).

Any linear system $\alpha = (X, U, V; A, B, C, D)$ is the dilation of following minimal system $X^{co} = (X^{co}, U, V; A^{co}, B^{co}, C^{co}, D)$ where

$$X^{co} = \overline{P_{X^o}X^c}, \quad A^{co} = P_{X^{co}}A|X^{co}, \quad B^{co} = P_{X^{co}}B, \quad C^{co} = C|X^{co}.$$

Moreover, if α is a simple c.s.s. then α^{co} is an optimal minimal p.s.s..

Similarly, the simple c.s.s. $\alpha = (X, U, V; A, B, C, D)$ is also a dilation of the following minimal $*$-optimal p.s.s.

$$\alpha^{oc} = (X^{oc}, U, V; A^{oc}, B^{oc}, C^{oc}, D),$$

where

$$X^{oc} = \overline{P_{X^c}X^o}, \quad A^{oc} = P_{X^{oc}}A|X^{oc}, \quad B^{oc} = P_{X^{oc}}B, \quad C^{oc} = C|X^{oc}.$$

DEFINITION 5. ([2]) The systems $\alpha = (X, U, V; A, B, C, D), \alpha' = (X', U, V; A', B', C', D')$ are said to be *weakly similar* if there exists an invertible linear operator $T : X' \to X$ having dense domain $\mathcal{D}_T$ and dense range $\mathcal{R}_T$ such that

$$A'\mathcal{D}_T \subset \mathcal{D}_T, \quad A\mathcal{R}_T \subset \mathcal{R}, \quad B'U \subset \mathcal{D}_T \tag{15}$$

$$TA'|\mathcal{D}_T = AT, \quad TB' = B, \quad CT = C'|\mathcal{D}_T\ . \tag{16}$$

In this case we have

$$TA'^nB' = A^nB, \quad T^*(A^*)^nC^* = (A'^*)^nC'^n. \tag{17}$$

We see also that the weakly similar systems have the same transfer function.

THEOREM 6. *Let the system $\alpha = (X, U, V; A, B, C, D)$ be a minimal p.s.s.. The system α is optimal (resp. $*$-optimal) if and only if for every minimal p.s.s. $\alpha' = (X', U, V; A', B', C', D')$ having the same transfer function as α : $\Theta_{\alpha'}(z) = \Theta_\alpha(z)$, we have*

$$\left\|\sum_{k=0}^{n}(A^*)^kC^*v_k\right\| \geqslant \left\|\sum_{k=0}^{n}(A'^*)^kC'^*v_k\right\| \qquad (\forall n \in \mathbf{N}, v_n \in V) \tag{18}$$

Proof. Let us prove the theorem for optimality; the statement concerning $*$-optimality is proved in a similar way.

Since the minimal systems α, α' having the same transfer function are weakly similar ([2]), there exists an invertible operator T satisfying (15)–(17). From the definition of optimality and (17), it follows that $\|T\| \leqslant 1$ and hence $\|T^*\| \leqslant 1$. Thus, from (17) follows (18).

Conversely, let α'' be any p.s.s. with the same transfer function $\Theta_{\alpha''}(z) = \Theta_\alpha(z)$ as α. Putting $\alpha' = (\alpha'')^{co}$, α' is minimal and $\Theta_{\alpha'}(z) = \Theta_\alpha(z)$. Hence α, α'

are weakly similar, i.e. we have (17), from which it follows $\|T^*\| \leqslant 1$, and $\|T\| \leqslant 1$. Thus

$$\Big\|\sum_{k=0}^{n} A^k B u_k\Big\| \leqslant \Big\|\sum_{k=0}^{n} (A')^k B' u_k\Big\| .$$

Since

$$(A')^k B' = P_{X''^{co}} (A'')^k B'',$$

we obtain

$$\Big\|\sum_{k=0}^{n} A^k B u_k\Big\| \leqslant \Big\|\sum_{k=0}^{n} (A'')^k B'' u_k\Big\|. \quad \blacksquare$$

It is easy to prove the following

LEMMA 4. *Let the system $\alpha = (X, U, V; A, B, C, D)$ be controllable or observable and let W be a unitary operator from X onto itself such that*

$$A = WAW^{-1}, \quad B = WB, \quad C = CW^{-1}.$$

Then W is the identity.

THEOREM 7. *Let α be optimal (resp. $*$-optimal) minimal p.s.s., whose transfer function $\Theta_\alpha(z)$ has $(+)$-regular and $(-)$-regular factorization $\Theta_\alpha(z) = \Theta_2(z)\Theta_1(z)$. Then there exist unique optimal (resp. $*$-optimal) minimal p.s.s. α_1, α_2 such that*

$$\alpha = \alpha_2\alpha_1, \quad \Theta_{\alpha_k}(z) = \Theta_k(z), \quad k = 1, 2.$$

Proof. Obviously, we can suppose α being of model form

$$\alpha = \hat{\alpha}^{co} = P_{\hat{X}^{co}} \hat{\alpha},$$

where $\hat{\alpha}$ is a simple c.s.s. having transfer function $\Theta_{\hat{\alpha}}(z) = \Theta_\alpha(z)$.

Since the factorization $\Theta_{\hat{\alpha}}(z) = \Theta_\alpha(z) = \Theta_2(z)\Theta_1(z)$ is $(+)$-regular, then it is also regular, hence $\hat{\alpha}$ has a cascade factorization

$$\hat{\alpha} = \hat{\alpha}_2\hat{\alpha}_1,$$

where $\hat{\alpha}_1, \hat{\alpha}_2$ are simple c.s.s. with transfer functions

$$\Theta_{\hat{\alpha}_1}(z) = \Theta_1(z), \quad \Theta_{\hat{\alpha}_2}(z) = \Theta_2(z).$$

Moreover, from Theorems 1 and 3 we have

$$\hat{X}^{co} = \overline{P_{\hat{X}^o} \hat{X}^c} = \hat{X}_1^{co} \oplus \hat{X}_2^{co},$$

and hence, from Proposition 3 we obtain

$$P_{\hat{X}^{co}}\hat{\alpha} = (P_{\hat{X}_2^{co}}\hat{\alpha}_2)(P_{\hat{X}_1^{co}}\hat{\alpha}_1).$$

Thus

$$\alpha = \alpha_2\alpha_1,$$

where

$$\alpha_k = (P_{\hat{X}_k^{co}}\hat{\alpha}_k), \quad k = 1, 2,$$

are optimal minimal p.s.s..

To prove the part concerning $*$-optimality, it is enough to consider the model system $\hat{\alpha}^{oc}$ instead of $\hat{\alpha}^{co}$.

Let us prove uniqueness. Suppose that we have another factorization $\alpha = \alpha_2'\alpha_1'$ where α_1', α_2' are optimal minimal p.s.s. such that $\Theta_{\alpha_k'}(z) = \Theta_k(z), k = 1, 2$. Then α_k is unitarily equivalent to α_k' with a unitary transformation W_k. In this case, it is not difficult to see that the system $\alpha = \alpha_2\alpha_1$ is unitarily equivalent to the system $\alpha = \alpha_2'\alpha_1'$ with the unitary transformation $W_1 \oplus W_2$. Next, from Lemma 4 it follows that $\alpha_1 = \alpha_1', \alpha_2 = \alpha_2'$, which completes the proof. ∎

THEOREM 8. *Let α_1, α_2 be optimal (resp. $*$-optimal) minimal p.s.s. such that the factorization $\Theta(z) = \Theta_{\alpha_2}\Theta_{\alpha_1}(z)$ is $(+)$-regular and $(-)$-regular. Then the cascade coupling $\alpha = \alpha_2\alpha_1$ is an optimal (resp. $*$-optimal) minimal p.s.s..*

Proof. Suppose that the optimal minimal p.s.s. α_1, α_2 have model forms

$$\alpha_k = P_{\hat{X}_k^{co}}\hat{\alpha}_k, \quad k = 1, 2,$$

where $\hat{\alpha}_1, \hat{\alpha}_2$ are simple c.s.s. such that $\Theta_{\hat{\alpha}_k}(z) = \Theta_{\alpha_k}(z)$. By Proposition 3 we obtain

$$\alpha_2\alpha_1 = (P_{\hat{X}_2^{co}}\hat{\alpha}_2)(P_{\hat{X}_1^{co}}\hat{\alpha}_1) = P_{\hat{X}_1^{co}\oplus\hat{X}_2^{co}}\hat{\alpha}_2\hat{\alpha}_1 = P_{\hat{X}^{co}}\hat{\alpha}_2\hat{\alpha}_1.$$

Since the factorization is regular, then the system $\hat{\alpha}_2\hat{\alpha}_1$ is a simple c.s.s., hence $\alpha_2\alpha_1$ is an optimal minimal p.s.s..

The part of the theorem concerning $*$-optimality is proved in a similar way.

We shall see that a similar statement is also true for minimal (possibly not optimal) p.s.s.. For that purpose we consider several simple necessary results.

LEMMA 5. *Let the systems α_1, α_2 be weakly similar respectively to α_1', α_2', then $\alpha_2\alpha_1$ is weakly similar to $\alpha_2'\alpha_1'$.*

Proof. Suppose that α_k is weakly similar to α_k' by the linear transformation T_k $(k = 1, 2)$. Then it is easy to see that $\alpha_2\alpha_1$ is weakly similar to $\alpha_2'\alpha_1'$ by the linear transformation $T_1 \oplus T_2$. ∎

DEFINITION 6. If the transformation operator T in Definition 5 is bounded (T^{-1} may be unbounded), then the system α' is said to be a *quasiaffine transformation* of the system α.

PROPOSITION 5. *Let α' be a quasiaffine transformation of α. Then*

a) *The adjoint system α^* is a quasiaffine transformation of α'^*.*

b) *If α' is controllable, then α is controllable.*

c) *If α is observable, then α' is also observable.*

The proof of these statements follows from (17), using the boundeness of the transformation operator T.

LEMMA 6. *If α'_1, α'_2 are respectively quasiaffine transformations of α_1, α_2, then $\alpha'_2\alpha'_1$ is a quasiaffine transformation of $\alpha_2\alpha_1$.*

The proof of this lemma is similar to the proof of Lemma 5.

PROPOSITION 6. *Let α_1, α_2 be optimal (resp. $*$-optimal) minimal p.s.s. and let α'_1, α'_2 be minimal p.s.s. such that $\Theta'_{\alpha_k}(z) = \Theta_{\alpha_k}(z), k = 1, 2$. If the system $\alpha_2\alpha_1$ is observable (resp. controllable) then $\alpha'_2\alpha'_1$ is observable (resp. controllable).*

Proof. From the suppositions of Proposition 6 and from ([2]) it follows that α_k is weakly similar to α'_k $(k = 1, 2)$. Let

$$T_k(A'_k)^n B'_k = A_k^n B_k, \quad k = 1, 2 . \tag{19}$$

If α_k is optimal, then from (19) we have that α'_k is a quasiaffine transformation of α_k (since $\|T_k\| \leqslant 1$); hence, by Lemma 6, $\alpha'_2\alpha'_1$ is a quasiaffine transformation of $\alpha_2\alpha_1$. If the system $\alpha_2\alpha_1$ is observable then $\alpha'_2\alpha'_1$ is also observable (by Proposition 5).

The case of $*$-optimality is similarly considered. ∎

THEOREM 9. *Let α_1, α_2 be minimal p.s.s.. If the factorization $\Theta(z) = \Theta_{\alpha_2}(z)\Theta_{\alpha_1}(z)$ is $(+)$-regular and $(-)$-regular, then $\alpha_2\alpha_1$ is a minimal p.s.s..*

Proof. Let α'_1, α'_2 be optimal minimal p.s.s. such that

$$\Theta_{\alpha'_k}(z) = \Theta_{\alpha_k}(z), \quad k = 1, 2 .$$

By Theorem 8, the system $\alpha'_2\alpha'_1$ is optimal minimal. Thus it is observable, whence by Proposition 6 the system $\alpha_2\alpha_1$ is observable.

Similarly, considering $*$-optimal minimal p.s.s. α''_1, α''_2 with transfer function $\Theta_{\alpha''_k}(z) = \Theta_{\alpha_k}(z)$ we obtain that $\alpha_2\alpha_1$ is controllable. ∎

THEOREM 10. *Suppose that an optimal (resp. $*$-optimal) minimal p.s.s. α has a cascade coupling $\alpha = \alpha_2\alpha_1$ such that the factorization $\Theta_\alpha(z) = \Theta_{\alpha_2}(z)\Theta_{\alpha_1}(z)$ is $(+)$-regular and $(-)$-regular. Then the systems α_1, α_2 are optimal (resp. $*$-optimal) minimal.*

Proof. It is obvious to see that α_1, α_2 are minimal.

If α_1 is not optimal, then there exists a p.s.s. α_1' such that $\Theta_{\alpha_1'}(z) = \Theta_{\alpha_1}(z)$ and $\exists z_k : |z_k| < 1, u_k \in U$ and

$$\left\|\sum_{k=1}^{n}(I - z_kA_1)^{-1}B_1u_k\right\| > \left\|\sum_{k=1}^{n}(I - z_kA_1')^{-1}B_1'u_k\right\|.$$

Then for the system $\alpha' = \alpha_2\alpha_1'$ we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{n}(I - z_kA')^{-1}B'u_k\right\|^2 \\
&\quad= \left\|\sum_{k=1}^{n}(I - z_kA_1')^{-1}B_1'u_k\right\|^2 + \left\|\sum_{k=1}^{n}(I - z_kA_2)^{-1}B_2\Theta_1(z_k)u_k\right\|^2 \\
&\quad< \left\|\sum_{k=1}^{n}(I - z_kA_1)^{-1}B_1u_k\right\|^2 + \left\|\sum_{k=1}^{n}(I - z_kA_2)^{-1}B_2\Theta_1(z_k)u_k\right\|^2 \\
&\quad= \left\|\sum_{k=1}^{n}(I - z_kA)^{-1}Bu_k\right\|^2 .
\end{aligned}$$

This contradicts the optimality of α.

Suppose α_2 is not optimal. By Theorem 6, there exists a minimal p.s.s. α_2' such that $\Theta_{\alpha_2'}(z) = \Theta_{\alpha_2}(z)$ and $\exists z_k : |z_k| < 1, v_k \in V$ with

$$\left\|\sum_{k=1}^{n}(I - \bar{z}_kA_2'^{*})^{-1}C_2'^{*}v_k\right\| > \left\|\sum_{k=1}^{n}(I - \bar{z}_kA_2^{*})^{-1}C_2^{*}v_k\right\|.$$

Taking $\alpha'' = \alpha_2'\alpha_1$ we obtain

$$\left\|\sum_{k=1}^{n}(I - \bar{z}_kA_2''^{*})^{-1}C_2''^{*}v_k\right\| > \left\|\sum_{k=1}^{n}(I - \bar{z}_kA_2^{*})^{-1}C_2^{*}v_k\right\| .$$

By virtue of $(+)$-regularity and $(-)$-regularity of the factorization $\Theta_{\alpha''}(z) = \Theta_{\alpha_2'}(z)\Theta_{\alpha_1}(z)$, the system α'' is minimal (by Theorem 9), hence, by Theorem 6, it is not optimal. The theorem is proved. ∎

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