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AFD FACTOR OF TYPE III_0 WITH MANY ISOMORPHIC INDEX 3 SUBFACTORS

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ABSTRACT. There exists an AFD factor M of type III_0 with uncountably many non-conjugate subfactors N satisfying (i) M, N have the same flow of weights, and (ii) $M \supseteqq N$ has principal graph A_5.

KEYWORDS: *Subfactors, factors of type* III_0.

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0. INTRODUCTION

Let $M \supseteqq N$ be a factor-subfactor pair with a normal conditional expectation E. We consider such a pair with $\mathrm{Ind}\, E = 3$ ([6], [8], [14]) whose principal graph in A_5 (see [1]). When M is the AFD II_1 (or II_∞) factor, only one such pair (up to conjugacy) exists ([15], [17]), that is, $M \supseteqq N$ is conjugate to $R_0 \rtimes S_3 \supseteqq R_0 \rtimes S_2$. (Here, R_0 is the hyperfinite II_1-factor.) If M is the AFD factor of type III_1, then $M \supseteqq N$ splits into an inclusion of II_1-factors ([5]), that is, $M \supseteqq N$ is conjugate to $A \otimes M \supseteqq B \otimes M$ with an inclusion $A \supseteqq B$ of II_1-factors (which is conjugate to $R_0 \rtimes S_3 \supseteqq R_0 \rtimes S_2$). When M is of type III_λ, $\lambda \neq 1$, N is not necessarily isomorphic (as a factor) to M so that a situation is slightly different. In this article, let us further assume that M and N have the same flow of weights (see Remark 1 for the precise meaning). Under this assumption the AFD type III_λ $(0 < \lambda < 1)$ factor contains exactly two subfactors N_1, N_2 ([12]): $M \supseteqq N_1$ splits into an inclusion of II_1-factors while $M \supseteqq N_2$ does not. (See [10], [12], [13] for other classification results in the type III set-up.)

In the above mentioned cases subfactors (of the specified form) are thus quite rigid objects. The purpose of the present article is to point out that we have a completely different situation for factors of type III_0. Namely, we will show that an AFD type III_0 factor M generally (probably always) contains uncountably many subfactors N such that

(i) M, N have the same flow of weights, and

(ii) $M \supseteqq N$ has principal graphs A_5.

After collecting some standard results in Section 1, we will construct a model inclusion $M \supseteqq N$ of the required type in Section 2 by making use of a certain S_3-action. This action arises from a two-to-one ergodic extension of a given ergodic flow (X, F_t), and the flows of weights of M and N turn out to be exactly (X, F_t). In Section 3 the above two-to-one extension will be shown to be naturally reconstructed from an inclusion data of $M \supseteqq N$. Consequently, different extensions give rise to non-conjugate inclusions of factors (with the specified flow (X, F_t) of weights). Therefore, an ergodic flow with uncountably many non-isomorphic two-to-one ergodic extensions gives us the above mentioned AFD type III_0 factor.

1. PRELIMINARIES

Let $M \supseteqq N$ be a factor-subfactor pair, and $E : M \to N$ be a normal conditional expectation with $\operatorname{Ind} E = 3$ ([6], [8], [14]). Hence the principal graph for $M \supseteqq N$ is either D_4 or A_5 (see [1]). Throughout the article (unless otherwise is stated) we assume that the principal graph is A_5.

Choose and fix a faithful state $\varphi \in N_*^+$. We consider the inclusion

$$\widetilde{M} = M \rtimes_{\sigma^{\varphi \circ E}} \mathbf{R} \supseteqq \widetilde{N} = N \rtimes_{\sigma^{\varphi}} \mathbf{R}$$

of von Neumann algebras of type II_∞. Connes' Radon-Nikodym theorem guarantees that $\widetilde{M} \supseteqq \widetilde{N}$ does not depend upon the choice of φ. The expectation E is lifted to $\widehat{E} : \widetilde{M} \to \widetilde{N}$ ($\widehat{E}(\pi_\sigma(x)) = \pi_\sigma(E(x))$, $x \in M$, and $\widehat{E}(\lambda(t)) = \lambda(t), t \in \mathbf{R}$). This expectation $\widehat{E}$ comes from the canonical trace on $\widetilde{M}$ and commutes with the dual action $\{\theta_t\}_{t \in \mathbf{R}}$ (see [11] for details).

In this article we further require that the centers $Z(\widetilde{M})$ and $Z(\widetilde{N})$ coincide. Therefore, M and N have the same flow of weights.

REMARK 1. Even if $Z(\widetilde{M}) \neq Z(\widetilde{N})$, it might happen that the flow of weights of M and that of N are isomorphic as the examples in [4] show. However, "the same flow of weights" will mean $Z(\widetilde{M}) = Z(\widetilde{N})$ in what follows (i.e., $M = \mathcal{A} \supseteqq \mathcal{B} = N$ in the sense of [10]).

Let

$$\widetilde{M} = \int_X^{\oplus} \widetilde{M}(\omega)\, d\omega \supseteqq \widetilde{N} = \int_X^{\oplus} \widetilde{N}(\omega)\, d\omega$$

be the (joint) central decomposition $(Z(\widetilde{M})) = Z(\widetilde{N}) \cong L^{\infty}(X))$. From this we obtain the (measurable) field $\{\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)\}_{\omega}$ of pairs of II_{∞}-factors. Since $(\widehat{E})^{-1}(1) = 3$ ([11], 2.1) and $\widehat{E}$ comes from the canonical trace, we conclude that $[\widetilde{M}(\omega) : \widetilde{N}(\omega)] = 3$, the Jones index (for a.e. $\omega \in X$). This means that the principal graph of $\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)$ is either D_4 or A_5 (and a priori this depends on ω).

Let M_1 be the basic extension of $E : M \to N$. The compatibility between performing the basic extension and taking the crossed product (2.1, [11]) says that the crossed product $\widetilde{M_1}$ (of M_1 relative to the modular automorphisms of $\varphi(E(J_M E^{-1}(J_M \cdot J_M)J_M)))$ is the same as the basic extension of $\widehat{E} : \widetilde{M} \to \widetilde{N}$. Also, $e_N = \pi_{\sigma}(e_N)$ is exactly the Jones projection for $\widetilde{M} \supseteqq \widetilde{N}$ (see 2.1, [11]). It is straight-forward to see that $\widetilde{M_1}$ has the same center as $\widetilde{M}, \widetilde{N}$, and $\widetilde{M_1}(\omega)$ (defined analogously) is exactly the basic extension of $\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)$.

Consider the (obviously measurable) function:

$$\omega \in X \longmapsto \dim(\widetilde{M_1}(\omega) \cap \widetilde{N}'(\omega)).$$

This functions is invariant under the flow F_t (a point-map realization of θ_t restricted to $Z(\widetilde{M}) \cong L^{\infty}(X)$), $\widetilde{M}(F_t(\omega)) \supseteqq \widetilde{N}(F_t(\omega))$ being conjugate to $\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)$. The central ergodicity of the dual action thus guarantees that the above function is a constant function (a.e.) (values are either 2 or 3). Therefore $\{\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)\}_{\omega}$ gives us either D_4 (a.e.) or A_5 (a.e.).

DEFINITION 2. The above D_4 or A_5 will be referred to as *the type II principal graph of $M \supseteqq N$* (an inclusion of type III factors).

Notice that, if the principal graph ($= A_5$ in this article) and the type II principal graph of $M \supseteqq N$ are different, then $M \supseteqq N$ does not split into an inclusion of II_1-factors. In fact, if $A \supseteqq B$ is an inclusion of II_1-factors, then we would obviously get $(A \otimes M)^{\sim} = A \otimes \widetilde{M} \supseteqq (B \otimes M)^{\sim} = B \otimes \widetilde{M}$.

2. MODEL

We start from a given non-singular ergodic flow (X, F_t) (not isomorphic to ($\mathbf{R}$, translation with speed 1)). We will construct a factor-subfactor pair $M \supseteqq N$ such that

(i) M and N have the same flow of weights, which is exactly (X, F_t),

(ii) the principal graph of $M \supseteqq N$ is A_5,

(iii) the type II principal graph of $M \supseteqq N$ is D_4.

Let us choose and fix an ergodic flow $(\widetilde{X}, \widetilde{F}_t)$ which is a two-to-one extension of (X, F_t) : $\widetilde{X} = X \times \{1,2\}$ and $\widetilde{F}_t(\omega, i) = (F_t(\omega), \varphi_{\omega,t}(i))$ with the cocycle $\varphi : X \times \mathbf{R} \to S_2$ (see also the last paragraph in Section 2).

Let Q be the AFD (type III_λ, $\lambda \neq 1$) factor whose flow of weights is exactly $\widetilde{X}, \widetilde{F}_t)$ (see [3]). Let α be the unique (up to cocycle conjugacy, [20]) outer S_3-action on Q specified by

$$\begin{cases} N(\alpha) = \{1\}, \text{ i.e., } \alpha_g \text{ is not centrally trivial } (g \neq 1 \in S_3), \\ \operatorname{mod} \alpha_b = 1, \\ \operatorname{mod} \alpha_a \text{ is the map exchanging the two sheets of } \widetilde{X} \\ ((\operatorname{mod} \alpha_a)(\omega, i) = (\omega, i+1)). \end{cases}$$

Here, $a = (1,2)$ and $b = (1,2,3)$ are the obvious generators of S_3, the permutations on $\{1,2,3\}$, $(a^2 = b^3 = 1, aba = b^2)$.

With $S_2 = \{1, a\}$ we set

$$M = Q \rtimes_\alpha S_3 \supseteqq N = Q \rtimes_\alpha S_2.$$

The principal graph of $M \supseteqq N$ is obviously A_5 (see [1]). We will show that the requirements (i), (iii) are also satisfied.

Let us choose an S_3-invariant faithful state $\psi \in W_*^+$. Let $\widetilde{\alpha}$ be the canonical extension ([2]) of α to $\widetilde{Q} = Q \rtimes_{\sigma^\psi} \mathbf{R}(\widetilde{\alpha}_g(\pi(x)) = \pi(\alpha_g(x)), x \in Q$, and $\widetilde{\alpha}_g(\lambda(t)) = \lambda(t)$, $t \in \mathbf{R}$). The following facts are standard ([2], [7], [19]):

(a) $\widetilde{M} \supseteqq \widetilde{N}$ (that is $(Q \rtimes_\alpha S_3)^\sim \supseteqq (Q \rtimes_\alpha S_2)^\sim$) is conjugate to $\widetilde{Q} \rtimes_{\widetilde{\alpha}} S_3 \supseteqq \widetilde{Q} \rtimes_{\widetilde{\alpha}} S_2$,

(b) the dual action on $\widetilde{Q}$ is the restriction of that on $\widetilde{M}$,

(c) the dual action on $\widetilde{M}$ acts trivially to the generators correspondings to S_3.

The centers of $\widetilde{M}, \widetilde{N}$ are the fixed point algebras of $Z(\widetilde{Q})$ under the (mod) actions of S_3 and S_2 respectively ([7], [19]). Hence, in the present set-up, we get

$$Z(\widetilde{M}) = Z(\widetilde{N}) = Z(\widetilde{Q})_{\operatorname{mod} \alpha_a} \cong L^\infty(X),$$

and the requirement (i) is indeed satisfied.

Let us decompose $\widetilde{Q}$ over $Z(\widetilde{Q})_{\mathrm{mod}\,\alpha_a} = L^\infty(X)$:

$$\widetilde{Q} = \int_X^{\oplus} \widetilde{Q}(\omega)\,\mathrm{d}\omega,$$

$$\widetilde{Q}(\omega) = \left\{ \begin{bmatrix} x & \\ & y \end{bmatrix} : x, y \in R_{01} \right\}.$$

Here, R_{01} is the *AFD* factor of type II_∞, and the first (resp., second) component in $\widetilde{Q}(\omega) = R_{01} \oplus R_{01}$ corresponds to the fiber algebra above $(\omega, 1) \in \widetilde{X} = X \times \{1, 2\}$ (resp., $(\omega, 2)$). Since $\mathrm{mod}\,\alpha_a$ exchanges the two sheets, $\widetilde{\alpha}_a$ exchanges R_{01} above $(\omega, 1)$ and that above $(\omega, 2)$. By rearranging the algebras above one of the two sheets, we may and do assume that $\widetilde{\alpha}_a$ looks like

$$\widetilde{\alpha}_a \left(\int_X^{\oplus} \begin{bmatrix} x(\omega) & \\ & y(\omega) \end{bmatrix} \mathrm{d}\omega \right) = \int_X^{\oplus} \begin{bmatrix} y(\omega) & \\ & x(\omega) \end{bmatrix} \mathrm{d}\omega.$$

On the other hand, $\mathrm{mod}\,\alpha_b = 1$ means that $\widetilde{\alpha}_b$ is a field of period 3 automorphisms (of factor components in the central decomposition of $\widetilde{Q}$). Since $\widetilde{\alpha}_a \widetilde{\alpha}_b \widetilde{\alpha}_a = \widetilde{\alpha}_{b^2}$, we conclude that

$$\widetilde{\alpha}_b \left(\int_X^{\oplus} \begin{bmatrix} x(\omega) & \\ & y(\omega) \end{bmatrix} \mathrm{d}\omega \right) = \int_X^{\oplus} \begin{bmatrix} \sigma_\omega(x(\omega)) & \\ & \sigma_\omega^2(y(\omega)) \end{bmatrix} \mathrm{d}\omega,$$

where σ_ω is a period 3 automorphism of R_{01}.

Hence, above each $\omega \in X$, the fiber algebras look like

$$(1) \qquad \begin{cases} \widetilde{M}(\omega) = \widetilde{Q}(\omega) \rtimes_\alpha S_3 \supseteq \widetilde{N}(\omega) = \widetilde{Q}(\omega) \rtimes_\alpha S_2, \\ \widetilde{Q}(\omega) = R_{01} \oplus R_{01}, \\ \alpha_a \left(\begin{bmatrix} x & \\ & y \end{bmatrix} \right) = \begin{bmatrix} y & \\ & x \end{bmatrix}, \ \alpha_b \left(\begin{bmatrix} x & \\ & y \end{bmatrix} \right) = \left(\begin{bmatrix} \sigma(x) & \\ & \sigma^2(y) \end{bmatrix} \right). \end{cases}$$

For simplicity let us agree to drop the subscript $\omega (\in X)$. Since α (S_3-action on $\widetilde{Q}(\omega) = R_{01} \oplus R_{01}$) is free and centrally ergodic, $\widetilde{M}(\omega) \supseteq \widetilde{N}(\omega)$ is indeed a factor-subfactor pair.

To check the requirement (iii), it suffices to show that $\dim(\widetilde{M}_1(\omega) \cap \widetilde{N}(\omega)') = 3$. The standard Hilbert space of $\widetilde{M}(\omega)$ is $L^2(R_{01} \oplus R_{01}) \otimes \ell^2(S_3) = (L^2(R_{01}) \oplus L^2(R_{01})) \otimes \ell^2(S_3)$. Therefore, an operator in $B(L^2(\widetilde{M}(\omega)))$ can be expressed as a 6×6-matrix with $B(L^2(R_{01}) \oplus L^2(R_{01}))$-entries, where the rows and columns are indexed by $(g, h) \in S_3 \times S_3$. The two generators $\pi(x)$ ($x \in R_{01} \oplus R_{01}$) and λ_k ($k \in S_3$) in $(R_{01} \oplus R_{01}) \rtimes_\alpha S_3$ are given by the following matrices:

$$(2) \qquad \pi(x)_{gh} = \begin{cases} \alpha_{g^{-1}}(x) & \text{if } g = h, \\ 0 & \text{otherwise}, \end{cases}$$

$$(3) \qquad (\lambda_k)_{gh} = \begin{cases} 1 & \text{if} \quad gh^{-1} = k, \\ 0 & \text{otherwise.} \end{cases}$$

In this representation, the Jones projection $e = [e_{gh}]$ for $\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)$ is

$$(4) \qquad e_{11} = e_{aa} = 1, \qquad \text{other } e_{gh}\text{'s are } 0.$$

The modular conjugation J (on $L^2(\widetilde{M}(\omega))$) is given by

$$J_{gh} = \begin{cases} J_0 u(g)^* & \text{if} \quad h = g^{-1} \\ 0 & \text{otherwise,} \end{cases}$$

where J_0 is the modular conjugation of $R_{01} \oplus R_{01}$ and $u(g)$ is the canonical implementation of $\alpha_g \in \mathrm{Aut}(R_{01} \oplus R_{01})$. The commutant $((R_{01} \oplus R_{01}) \rtimes_\alpha S_2)'$ (in $B(L^2(\widetilde{M}(\omega)))$) is given by

$$((R_{01} \oplus R_{01}) \rtimes_\alpha S_2)' = \Big\{ X = [X_{gh}] : \\ X_{gh} = n'_{gh} u(g^{-1}h) \in (R_{01} \oplus R_{01})' u(g^{-1}h), n'_{kg,kh} = n'_{gh} \quad \text{for} \quad k \in S_2 \Big\},$$

where $(R_{01} \oplus R_{01})'$ is the commutant of $R_{01} \oplus R_{01}$ acting on $L^2(R_{01} \oplus R_{01})$. Therefore, the basic extension $\widetilde{M}_1(\omega) = J((R_{01} \oplus R_{01}) \rtimes_\alpha S_2)' J$ turns out to be

$$\widetilde{M}_1(\omega) = \Big\{ X = [X_{gh}] : X_{gh} \in R_{01} \oplus R_{01} \\ X_{gk,hk} = \alpha_{k^{-1}}(X_{gh}) \quad \text{for} \quad k \in S_2 \Big\}.$$

All of the above computations are straightforward (and standard) so that the details are left to the reader.

We are now ready to compute $\widetilde{M}_1(\omega) \cap \widetilde{N}(\omega)'$. Assume that $X = [X_{gh}] \in \widetilde{M}_1(\omega)$ ($X_{gh} \in R_{01} \oplus R_{01}$, $X_{ga,ha} = \alpha_{a^{-1}}(X_{gh})$ for $a = (1,2) \in S_2$) commutes with an arbitrary element in $\widetilde{N}(\omega)$. The commutativity with $\pi(x)$ ($x \in R_{01} \oplus R_{01}$) (see (2)) forces that

$$\alpha_{g^{-1}}(x) X_{gh} = X_{gh} \alpha_{h^{-1}}(x).$$

When $g \neq h$, $\alpha_{h^{-1}g}$ is free and $X_{gh} = 0$. On the other hand, X_{gg} belongs to $Z(R_{01} \oplus R_{01}) = \mathbf{C} \oplus \mathbf{C}$. The commutativity with λ_a (see (3)) implies that

$$X_{ag,ag} = X_{gg} \qquad \text{for} \qquad a = (1,2) \in S_2.$$

We thus conclude that

$$\widetilde{M}_1(\omega) \cap \widetilde{N}(\omega)' = \Big\{ X = [X_{gh}] : X_{gh} = 0 \text{ for } g \neq h, \\ Y_g = X_{gg} \in Z(R_{01} \oplus R_{01}) = \mathbf{C} \oplus \mathbf{C} (g \in S_3) \\ \text{satisfy } Y_{ga} = \alpha_{a^{-1}}(Y_g), Y_{ag} = Y_g \Big\}.$$

Based on (1) (and by looking at the double coset spaces $S_2 \setminus S_3/S_2$), we know that $\widetilde{M_1}(\omega) \cap \widetilde{N}(\omega)'$ consists of diagonal matrices (with entries $\{Y_g\}_{g\in S_3}$ as above) satisfying

$$\begin{cases} Y_1 = Y_a = \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix}, \\ Y_b = Y_{ab} = \begin{bmatrix} \beta & \\ & \gamma \end{bmatrix}, \\ Y_{b^2} = Y_{ab^2} = \begin{bmatrix} \gamma & \\ & \beta \end{bmatrix}, \end{cases}$$

for some scalars α, β, γ. Therefore, $\widetilde{M_1}(\omega) \cap \widetilde{N}(\omega)'$ is a 3-dimensional abelian algebra, and the type II principal graph is indeed D_4.

THEOREM 3. *Let $M \supseteqq N$ be the factor-subfactor pair constructed from a two-to-one ergodic extension $(\widetilde{X}, \widetilde{F}_t)$ of (X, F_t). Then the inclusion $M \supseteqq N$ satisfies*

(i) *$M \supseteqq N$ have the same flow of weights, which is exactly (X, F_t),*

(ii) *the principal graph of $M \supseteqq N$ is A_5,*

(iii) *the type II principal graph of $M \supseteqq N$ is D_4.*

Assume that $(\widetilde{X}, \widetilde{F}_t)$ and $(\widetilde{X}^1, \widetilde{F}_t^1)$ are isomorphic (as extensions) two-to-one ergodic extensions of (X, F_t). Or more generally, assume that $(\widetilde{X}, \widetilde{F}_t) \to (X, F_t)$ and $(\widetilde{X}^1, \widetilde{F}_t^1) \to (X^1, F_t^1)$ are isomorphic (as extensions) two-to-one ergodic extensions (see [18] for example), that is, there exists a non-singular isomorphism $\psi : X \to X^1$ with $F_t^1 \circ \psi = \psi \circ F_t$ and a measurable map $\beta : X \to S_2$ (relabeling of sheets) satisfying

$$\varphi^1_{\psi(\omega),t} = \beta_{F_t(\omega)} \circ \varphi_{\omega,t} \circ \beta_\omega^{-1}.$$

Here, φ, φ^1 are the relevant cocycles (see the beginning of Section 2). In this case, $\widetilde{\psi} : (\omega, i) \mapsto (\psi(\omega), \beta_\omega(i))$ is a non-singular (relative to the obvious measures on $\widetilde{X}, \widetilde{X}^1$) isomorphism intertwining $\widetilde{F}_t$ and $\widetilde{F}_t^1$, and we have $\pi^1 \circ \widetilde{\psi} = \psi \circ \pi$ (where $\pi : \widetilde{X} \to X$, $\pi^1 : \widetilde{X}^1 \to X^1$ are the natural projection maps). This $\widetilde{\psi}$ obviously sends $\bmod \alpha$ to $\bmod \alpha^1$. Therefore, the two actions α, α^1 are cocycle conjugate ([20]), and the resulting inclusions $M \supseteqq N$ and $M^1 \supseteqq N^1$ are conjugate.

LEMMA 4. *The following three conditions are equivalent:*

(i) *$(\widetilde{X}, \widetilde{F}_t) \to (X, F_t)$ and $(\widetilde{X}^1, \widetilde{F}^1) \to (X^1, F_t^1)$ are isomorphic extensions in the sense described above.*

(ii) *There exist non-singular isomorphisms $\psi : X \to X^1$ and $\widetilde{\psi} : \widetilde{X} \to \widetilde{X}^1$ such that $F_t^1 \circ \psi = \psi \circ F_t$, $\widetilde{F}_t^1 \circ \widetilde{\psi} = \widetilde{\psi} \circ \widetilde{F}_t$ and $\pi^1 \circ \widetilde{\psi} = \psi \circ \pi$.*

(iii) *Via π, π^1 let us consider $L^\infty(X)$ and $L^\infty(X^1)$ as subalgebras of $L^\infty(\widetilde{X})$ and $L^\infty(\widetilde{X}^1)$ respectively. Let $\theta_t, \widetilde{\theta}_t, \theta_t^1, \widetilde{\theta}_t^1$ be the $\mathbf{R}$-actions (on the relevant algebras) induced by $F_t, \widetilde{F}_t, F_t^1, \widetilde{F}_t^1$ respectively. Then there exists a (von Neumann* *algebra) isomorphism* $\Lambda : L^\infty(\widetilde{X}) \to L^\infty(\widetilde{X}^1)$ *such that* Λ *intertwines* $\widetilde{\theta}_t$ *and* $\widetilde{\theta}_t, \Lambda$ *sends* $L^\infty(X)$ *onto* $L^\infty(X^1)$, *and* $\Lambda|_{L^\infty(X)}$ *intertwines* θ_t *and* θ_t^1.

Proof. (ii) $\Leftrightarrow$ (iii) is obvious, and it suffices to show (ii) $\Rightarrow$ (i). Since (ii) $\Rightarrow$ (i) for $\mathbf{Z}$-actions is well-known, it suffices to reduce the situation to the $\mathbf{Z}$-action case.

Represent (X, F_t) under a ceiling function f on a base space X_0 with a base transformation T, i.e., $X = \{(u,s) : u \in X_0, 0 \leqq s < f(u)\}$ and $F_{f(u)}(u,0) = (Tu, 0)$ for example. We consider the sets $\pi^{-1}(X_0), \psi(X_0)$, and $(\pi^1)^{-1}(\psi(x_0)) = \widetilde{\psi}(\pi^{-1}(X_0))$, where $X_0 \,(= X_0 \times \{0\}) \subseteqq X$. Thanks to the intertwining properties in (ii), $(\widetilde{X}, \widetilde{F}_t), (X^1, F_t^1), (\widetilde{X}^1, \widetilde{F}_t^1)$ can be represented under the same ceiling function f in such a way that the relevant base spaces are exactly the above three sets respectively. Let $\widetilde{T}, T^1$, and $\widetilde{T}^1$ be the respective base transformations. From the construction, $\widetilde{T}$ (resp. $\widetilde{T}^1$) is an extension of T (resp. T^1), and $(\widetilde{T}, T), (\widetilde{T}^1, T^1)$ are isomorphic extensions. ∎

The technique in the above proof also shows that having a (two-to-one ergodic) extension $(\widetilde{X}, \widetilde{F}_t) \to (X, F_t)$ in the sense described above is the same as the following: flows $\widetilde{F}_t$ on $\widetilde{X}$ and F_t on X are given with the requirement $\pi \circ \widetilde{F}_t = F_t \circ \pi$.

3. MAIN RESULTS

Here, we further analyze the pair $M \supseteqq N$ constructed in the previous section. Although the flow (X, F_t) appears as the flow of weights of M (and of N), the two-to-one extension $(\widetilde{X}, \widetilde{F}_t)$ seems to have disappeared during the construction. Actually this is not the case. We will show that the two-to-one extension, $(\widetilde{X}, \widetilde{F}_t) \to (X, F_t)$, can be captured (in a very natural way) from an inclusion data of $M \supseteqq N$.

Recall $\widetilde{M}(\omega) \supseteqq \widetilde{N}(\omega)$ (in Section 2) described by (1). We set

$$V = \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right)\lambda_b + \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right)\lambda_{b^2} \in (R_{01} \oplus R_{01}) \rtimes_\alpha S_3 = \widetilde{M}(\omega),$$

where $(\pi = \pi_\alpha, \lambda)$ is the covariant representation of the S_3-action α on $R_{01} \oplus R_{01}$. The last equality in (1) and the covariance relation imply

$$\pi\left(\begin{bmatrix} x & \\ & y \end{bmatrix}\right)\lambda_a = \lambda_a \pi\left(\begin{bmatrix} y & \\ & x \end{bmatrix}\right); \qquad x, y \in R_{01}, \tag{5}$$

$$\pi\left(\begin{bmatrix} x & \\ & y \end{bmatrix}\right)\lambda_b = \lambda_b \pi\left(\begin{bmatrix} \sigma^2(x) & \\ & \sigma(y) \end{bmatrix}\right); \qquad x, y \in R_{01}. \tag{6}$$

(In particular, $\pi\left(\begin{bmatrix} x & \\ & y\end{bmatrix}\right)$ $(x, y \in \mathbf{C})$ commutes with λ_b). Thank to (6), one can easily observe that V is a unitary, $V^3 = 1$, and

$$V^2 = \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_b + \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_{b^2}. \tag{7}$$

We also have

$$\begin{cases} V\lambda_a V^* = \lambda_a, \\ V\pi\left(\begin{bmatrix} x & \\ & y\end{bmatrix}\right)V^* = \pi\left(\begin{bmatrix} \sigma(x) & \\ & \sigma(y)\end{bmatrix}\right); \qquad x, y \in R_{01}. \end{cases}$$

For example, we compute

$$\begin{aligned} V\lambda_a V^* &= \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_b\lambda_a\lambda_b^*\pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right) + \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_b\lambda_a\lambda_{b^2}^*\pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right) \\ &\quad + \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_{b^2}\lambda_a\lambda_b^*\pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right) \\ &\quad + \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_{b^2}\lambda_a\lambda_{b^2}^*\pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right) \\ &= \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_{b^2a} + \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_a \\ &\quad + \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_a + \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_{ba} \\ &\quad \text{(because of } bab^{-2} = b^2ab^{-1} = a \text{ and (5))} \\ &= \pi\left(\begin{bmatrix} 1 & \\ & 0\end{bmatrix}\right)\lambda_a + \pi\left(\begin{bmatrix} 0 & \\ & 1\end{bmatrix}\right)\lambda_a = \lambda_a. \end{aligned}$$

The second equality can be proved similarly.

The above facts show that V normalizes the II_∞-subfactor $\widetilde{N}(\omega) = (R_{01} \oplus R_{01}) \rtimes_\alpha S_2$ and gives us the $\mathbf{Z}_3$-action AdV.

LEMMA 5. *The 6×6-matrices representing $VeV^*, V^2e(V^2)^*$ (where e is the Jones projection, see (4)) are as follows:*

(i) *$VeV^* = p$ is a diagonal matrix, and its diagonal entries $\{p_g\}_{g \in S_3}$ are given by*

$$p_b = p_{ab} = \begin{bmatrix} 1 & \\ & 0\end{bmatrix}, \qquad p_{b^2} = p_{ab^2} = \begin{bmatrix} 0 & \\ & 1\end{bmatrix}, \qquad \textit{all other } p_g\textit{'s are } 0.$$

(ii) *$V^2e(V^2)^* = q$ is a diagonal matrix, and its diagonal entries $\{q_g\}_{g \in S_3}$ are given by*

$$q_b = q_{ab} = \begin{bmatrix} 0 & \\ & 1\end{bmatrix}, \qquad q_{b^2} = Q_{ab^2} = \begin{bmatrix} 1 & \\ & 0\end{bmatrix}, \qquad \textit{all other } q_g\textit{'s are } 0.$$

Proof. We will prove just (i). ((ii) can be proved similarly.) Let P_g be the "rank-1" projection whose (g,g)-component is 1. Notice the following facts:

$$\begin{cases} \lambda_h P_g \lambda_h^* = P_{hg} \quad (h, g \in S_2), \\ p_g \text{ commutes with } \pi\left(\begin{bmatrix} x & \\ & y \end{bmatrix}\right); \quad x, y \in R_{01}. \end{cases}$$

The first is easy, and the second follows from the fact that the two involved matrices are diagonal. Since $e = P_1 + P_a$ (see (4)), $VeV^* = VP_1V + VP_aV^*$ can be expressed as the linear combination (whose coefficients are $\pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right)$, $\pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right)$, or 0) of eight terms of the form: $\lambda_h P_g \lambda_{h'}^*$ ($g \in S_2 = \{1, a\}$, and $h, h' \in \{b, b^2\}$). Actually four of them vanish. For example, we have

$$\begin{aligned} \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) \lambda_b P_1 \lambda_{b^2}^* \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right) &= \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) P_b \lambda_{b^2} \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right) \\ &= \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right) P_b \lambda_{b^2} = 0, \end{aligned}$$

since $\pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right)$ commutes with P_b and λ_{b^2}. In this way we end up with

$$VeV^* = \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) P_b + \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right) P_{b^2} + \pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) P_{ab^2} + \pi\left(\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\right) P_{ab}.$$

Therefore (i) is proved. (Notice that, for example, the (ab^2, ab^2)-component of $\pi\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right)$ is $\alpha_{(ab^2)^{-1}}\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$ by (1) and (2).) ∎

Notice that p, q in Lemma 5 and the Jones projection e are the minimal projections in $\widetilde{M}_1(\omega) \cap \widetilde{N}(\omega)'$. (Recall the computation right before Theorem 2). Neither of V and V^2 commutes with e so that AdV and AdV^2 (restricted to $(R_{01} \oplus R_{01}) \rtimes_\alpha S_2$) are outer ([6]). We thus have shown $((R_{01} \oplus R_{01}) \rtimes_\alpha S_2) \rtimes_{AdV} \mathbf{Z}_3 = (R_{01} \oplus R_{01}) \rtimes_\alpha S_3$ (see [16] or [9]), which of course gives us an alternative proof for the requirement (ii) in Section 2.

So far we have been looking at the "fiber algebras" $\widetilde{M}(\omega) \supseteq \widetilde{N}(\omega)$ and have not paid attention on the dual action. We now go back to the global inclusion $\widetilde{M} \supseteq \widetilde{N}$. From now on the above unitary V, the Jones projection e, and the unitary λ_g ($g \in S_3$) (arising from the S_3-action on $\widetilde{Q}(\omega)$, see (1)) will be denoted by V_ω, e_ω, and $(\lambda_g)_\omega$ respectively. The Jones projection e_N for $M \supseteq N$ (and $\widetilde{M} \supseteq \widetilde{N}$, see Section 1) is obviously

$$e_N = \int_X^\oplus e_\omega \, d\omega.$$

As pointed out above, the minimal projections in the 3-dimensional abelian algebra $\widetilde{M}_1(\omega) \cap \widetilde{N}(\omega)'$ are

$$\left\{e_\omega, V_\omega e_\omega V_\omega^*, V_\omega^2 e_\omega (V_\omega^2)^*\right\}.$$

This choice of a basis determines the isomorphism

$$\widetilde{M}_1 \cap \widetilde{N}' \cong L^\infty(X \times \{0,1,2\}),$$

and the dual action θ_t (canonically extended to $\widetilde{M}_1$ with the requirement $\theta_t(e_N) = e_N$) gives us the three-to-one extension: $(X \times \{0,1,2\}, F_t') \to (X, F_t)$ ($\widehat{E} \circ \widehat{E}_1$ sends $\widetilde{M}_1 \cap \widetilde{N}'$ to $Z(\widetilde{N}) = L^\infty(X)$ and intertwines θ_t, where $E_1 : M_1 \to M$ is the dual expectation, see [8], [11]). On the other hand, we have already known

$$\begin{aligned}(\widetilde{M}_1 \cap \widetilde{N}')_\theta &= M_1 \cap N' \qquad \text{(Corollary 6, [11])},\\ &= \mathbf{C} \oplus \mathbf{C} \qquad \text{(principal graph} = A_5).\end{aligned}$$

This means that $(X \times \{0,1,2\}, F_t')$ has exactly two ergodic components.

Let $\{(\theta_t)_\omega\}$ be the family of isomorphisms ($(\theta_t)_\omega$ sends $\widetilde{M}_1(\omega)$ to $\widetilde{M}_1(F_t(\omega))$) determined by the dual action θ_t. Since $\theta_t(e_N) = e_N$ ($(\theta_t)_\omega e_\omega = e_{F_t(\omega)}$), the 0-th sheet in (8) (which is just a copy of (X, F_t)) is the "trivial" ergodic component corresponding to the Jones projection e_N. Deleting this trivial component, (from $(1 - e_N)(\widetilde{M}_1 \cap \widetilde{N}')$) we obtain the ergodic flow $\widetilde{F}_t$ (the restriction of F_t' to the non-trivial ergodic component) on $X \times \{1,2\}$, which is a two-to-one extension of (X, F_t).

We claim that this two-to-one extension is the one given at the beginning of Section 2. In fact, $\widetilde{F}_t$ in Section 2 sends $((\omega,1),(\omega,2))$ to $((F_t(\omega),1),(F_t(\omega),2))$ or $((F_t(\omega),2),(F_t(\omega),1))$. The property (c) in Section 2 shows $(\theta_t)_\omega((\lambda_b)_\omega) = (\lambda_b)_{F_t(\omega)}$. Thus, by (b) in Section 2 and (7) in Section 3, the first (resp. second) case means that $(\theta_t)_\omega$ sends V_ω to $V_{F_t(\omega)}$ (resp. to $V^2_{F_t(\omega)}$). Since $\theta_t(e_N) = e_N$, the claim is now obvious.

Summing up the arguments so far, we have proved the next theorem.

THEOREM 6. (i) *Let $M \supseteq N$ be the factor-subfactor pair constructed (in Section 2) from a two-to-one ergodic extension $(\widetilde{X}, \widetilde{F}_t)$ of (X, F_t). Then the given two-to-one ergodic extension is isomorphic (as an extension) to the one obtained from the abelian algebra $(1 - e_N)(\widetilde{M}_1 \cap \widetilde{N}')$ together with the dual action.*

(ii) *Let $(\widetilde{X}, \widetilde{F}_t)$ and $(\widetilde{X}^1, \widetilde{F}_t^1)$ be two-to-one ergodic extensions of (X, F_t) with $M \supseteq N$ and $M^1 \supseteq N^1$ respectively. These two inclusions of factors are conjugate if and only if the extensions are isomorphic (as extensions).*

In the case that (X, F_t) is periodic, there exists only one (up to isomorphism) two-to-one ergodic extension. This fact is related to Loi's result (see Section 0): The Powers factor M contains exactly one (up to conjugacy) non-splitting factor N_2 such that $M \supseteqq N_2$ has principal graph A_5 and N_2, M are isomorphic.

On the other hand, when M is of type III_0, (X, F_t) probably admits uncountably many (non-isomorphic) two-to-one ergodic extensions. Unfortunately the author is unable to determine if this statement is correct. However, in ergodic theory examples of ergodic transformations with uncountably many two-to-one ergodic extentions are in abundance (see p. 262, [18], for example). Therefore, by using a constant ceiling function, one obtains an ergodic flow with the similar property and gets the next result.

COROLLARY 7. *There exists an AFD type III_0 factor M that admits uncountably many non-conjugate subfactors N satisfying*

(i) *M, N have the same flow of weights,*
(ii) *the principal graph of $M \supseteqq N$ is A_5,*
(iii) *the type II principal graph of $M \supseteqq N$ is D_4.*

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