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SIMILARITY, REDUCIBILITY AND APPROXIMATION OF THE COWEN-DOUGLAS OPERATORS

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ABSTRACT. An operator T on $\mathcal{H}$ is called strongly irreducible if T does not commute with any nontrivial idempotent operator. In this paper we obtain a characterization of the strongly irreducibility of Cowen-Douglas operators. For an analytic connected Cauchy domain and a positive integer n, we can find a strongly irreducible nice operator A in $\mathcal{B}_n(\Omega)$ — the class of Cowen-Douglas operators with index n. An operator A is called nice, if the commutant of either T or T^* is a strictly cyclic Abelian algebra. Finally, we obtain a characterization of operators which can be uniquely written as an algebraic direct sum of strongly irreducible nice operators.

KEYWORDS: *Strongly irreducible operator, nice operator, Cowen-Douglas operators.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ the collection of all bounded linear operators on $\mathcal{H}$.

For Ω a connected open subset of the complex plane $\mathbf{C}$ and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the set of operators B in $\mathcal{L}(\mathcal{H})$ satisfying

(a) $\Omega \subset \sigma(B) = \{w \in \mathbf{C} : B - w \text{ is not invertible}\}$,

(b) $\operatorname{ran}(B - w) = \mathcal{H}$ for every w in Ω,

(c) $\bigvee\limits_{w\in\Omega} \ker(B - w) = \mathcal{H}$ (spanning property) and

(d) $\dim\ker(B - w) = n$ for every $w \in \Omega$.

We call an operator in $\mathcal{B}_n(\Omega)$ a Cowen-Douglas operator.

M.J. Cowen and R.G. Douglas studied the operators in $\mathcal{B}_n(\Omega)$ from the view point of complex geometry. They proved that if f and $\tilde{f}$ are the curves in $\mathrm{Gr}\,(n,\mathcal{H})$ (the Grassmannian of n-dimensional subspaces of $\mathbf{C}^{2n}$) induced by T and $\tilde{T}$ in $\mathcal{B}_n(\Omega), T$ and $\tilde{T}$ are similar if and only if f and $\tilde{f}$ are similar. In Section 2, we reduce the problem of similarity of the operators in $\mathcal{B}_n(\Omega)$ to that of equivalence of two families of vectors by using operator theory rather than complex geometry.

In Section 3 and Section 4, we consider a general problem concerning reducibility and approximation of Cowen-Douglas operators, independently suggested by Domingo A. Herrero and Ze Jian Jiang. It can be described as follows.

An operator T on $\mathcal{H}$ is called *strongly irreducible* if T does not commute with any nontrivial idempotent operator. Otherwise, T is called *strongly reducible* (see [6], [8], [14]).

Strongly irreducible operators keep their strong irreducibility under similarity transformations; this is quite different from irreducible operators. In the past more than ten years, a lot of work on strongly irreducible operators has been done by the functional analysis seminar of Jilin University. D.A. Herrero, C.K. Fong and C.L. Jiang confirmed Ze Jian Jiang's Conjecture: A strongly irreducible operator is a rather suitable analogue of Jordan blocks in $\mathcal{L}(\mathcal{H})$ (see [7], [13], [15], [16]).

In this paper we obtain a characterization of the strongly irreducibility of Cowen-Douglas operators (see Section 2). In [9], L.J. Gray proved that a quasinilpotent operator can be approximated by "Jordan type" nilpotent operators which can be uniquely written as algebraic direct sum of countably many nilpotent Jordan blocks. Therefore, D.A. Herrero thought that the operators of the form $\lambda+\mu S, \mu \neq 0$, or their adjoint $(\lambda+\mu S)^*$, are more suitable as "Jordan blocks" of $\mathcal{L}(\mathcal{H})$ than strongly irreducible operators (S denotes the forward shift of multiplicity 1). But C.K. Fong and C.L. Jiang proved that the class of operators which are similar to a direct sum of finitely or countably many operators of the form $\lambda+\mu S$ or $(\lambda+\mu S)^*$ is not dense in $\mathcal{L}(\mathcal{H})$ (see [6], [17]).

Therefore, following Herrero's suggestion, we consider the strongly irreducible operators with a "nice" property.

An operator T in $\mathcal{L}(\mathcal{H})$ is called *nice*, if the commutant of either T or T^*, denoted by $\mathcal{A}'(T)$ and $\mathcal{A}'(T^*)$ respectively, is a strictly cyclic Abelian algebra. (Recall that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is called *strictly cyclic* if $\mathcal{A}x_0=\mathcal{H}$ for some x_0 in $\mathcal{H}$, where $\mathcal{A}x_0 \stackrel{\text{def}}{=} \{Ax_0, A\in\mathcal{A}\}$). It is obvious that each Jordan block is nice. In [11], D.A. Herrero gave an example of a nice operator T in $\mathcal{B}_1(\Omega)$. By Theorem 2.2 in [7], T is also strongly irreducible. In a private communication, D.A. Herrero posed the following question:

QUESTION H. For a connected Cauchy domain Ω and a positive integer n, can we find a strongly irreducible nice operator A in $\mathcal{B}_n(\Omega)$?

In [14], Z. J. Jiang proved the following fact:

THEOREM J. *Let A be a selfadjoint operator with the point spectrum $\sigma_p(A)$ empty. Then A cannot be expressed as an algebraic direct sum of finitely or countably many strongly irreducible operators.*

Based on the above theorem and Gray's result (see [9]), Z. J. Jiang posed the following conjecture:

CONJECTURE J. Let $\mathcal{B}$ be the set of operators which can be uniquely expressed as an algebraic direct sum of strongly irreducible operators. Then $\mathcal{B}$ is dense in $\mathcal{L}(\mathcal{H})$.

In Section 3 and Section 4 we shall give an affirmative answer to the above Question and Conjecture and obtain a characterization of operators which can be uniquely written as an algebraic direct sum of strongly irreducible nice operators.

2. SIMILARITY OF COWEN-DOUGLAS OPERATORS

For an operator B in $\mathcal{B}_n(\Omega)$, we denote by (E_B, π) the sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_B = \{(w, x) \mid w \in \Omega, x \in \ker(B - w); \pi(w, x) = w\}.$$

Subin in [18] proved that E_B is a complex bundle over Ω. By Proposition 1.1 in [4], we know that, for B in $\mathcal{B}_n(\Omega)$, the mapping $w \to \ker(B - w)$ defines a holomorphic Hermitian vector bundle over Ω.

PROPOSITION A. (see [4]). *For λ_0 in Ω and B in $\mathcal{B}_n(\Omega)$, let P be the orthogonal projection onto $\ker(\lambda_0 - B)$ while A in $\mathcal{L}(\mathcal{H})$ satisfies $A(B - \lambda_0) = I - P$ and $(B - \lambda_0)A = I$. Then A and B determine holomorphic $\mathcal{H}$-valued functions $T_B(\lambda) = \{e_j^B(\lambda)\}_{j=1}^n$ defined on some neighborhood Δ of λ_0 such that $\{e_1^B(\lambda), e_2^B(\lambda), \ldots, e_n^B(\lambda)\}$ forms a basis of $\ker(B - \lambda)$ for $\lambda \in \Delta$.*

The following proposition is due to Cowen and Douglas. Now we give it a new constructive proof.

PROPOSITION 2.1. *Let B be in $B_n(\Omega)$ and D a connected open subset of Ω. Then* $\bigvee_{\lambda\in D} \ker(\lambda - B) = \mathcal{H}$.

Proof. Let M be a normal operator on $\mathcal{H}$. Assume that the spectrum $\sigma(M)$ of M satisfies $\sigma(M) = \{\overline{\Omega\setminus \overline{D}}\}$. Then $\sigma(M) = \sigma_{\text{lre}}(M) \stackrel{\text{def}}{=} \{\lambda : M - \lambda \text{ is not semi-Fredholm}\}$ and $\sigma_{\text{lre}}(B\oplus M) = (\overline{\Omega\setminus \overline{D}})$. Hence, $D \subset \rho_F(B\oplus M) \stackrel{\text{def}}{=} \{\lambda; B\oplus M - \lambda \text{ is Fredholm}\}$.

Let

$$\mathcal{H}_r = \bigvee_{\lambda\in D} \ker(\lambda - B\oplus M).$$

Then $(B\oplus M)|\mathcal{H}_r = B|\mathcal{H}_r = B_r$. Let $\mathcal{H}_r^\perp$ be the orthogonal complement of $\mathcal{H}_r$ in $\mathcal{H}$. Then $\mathcal{H}\oplus\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_r^\perp \oplus \mathcal{H}$, so that

$$B\oplus M = \begin{bmatrix} B_r & * & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{matrix} \mathcal{H}_r \\ \mathcal{H}_r^\perp \\ \mathcal{H} \end{matrix}.$$

Applying Apostol's triangular representation theorem (see [12], Theorem 3.38), it is easy to see that

$$D \subset \rho_{S-F}(B\oplus M)\cap \rho(B_0\oplus M)\cap \rho_r(B_r), \text{ and } \sigma_{\text{p}}(B_r^*) = \emptyset,$$

where we denote by $\rho_r(\cdot)$ the right resolvent domain of the operator (see [12, p.43]). Since $D \subset \rho(B_0 \oplus M)$, we deduce that $D \subset \rho(B_0)$. $\Omega \subset \rho_r(B_0)$ is completely apparent from $\Omega \subset \rho_r(B)$. It is easy to see from $D \subset \rho(B_0)$ that $\Omega \subset \rho(B_0)$. This shows that $\Omega \subset \sigma_{\text{p}}(B_1)$ because $\Omega \subset \sigma_{\text{p}}(B)$. Since $\ker(\lambda - B_r) = \ker(\lambda - B)$ for all λ in Ω, we get that $\bigvee_{\lambda\in D} \ker(\lambda - B) = \mathcal{H}$ and complete the proof. ∎

PROPOSITION 2.2. *Let B be in $B_n(\Omega)$ and $\{\lambda_k\}$ a sequence of complex numbers in Ω satisfying $\lim_{k\to\infty} \lambda_k = \lambda_1$. Then*

$$\bigvee_{k=1}^{\infty} \ker(\lambda_k - B) = \mathcal{H}.$$

Proof. Without loss of generality, we may assume that $\lambda_1 = 0$. Let $\mathcal{H}_1 = \bigvee_{k=1}^{\infty} \ker(\lambda_k - B)$. Then $\mathcal{H}_1$ is invariant under B, so that

$$B = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix},$$

where $B_1 = B|\mathcal{H}_1$. Since $0 \in \rho_r(B)$, we see that $0 \in \rho_r(B_2)$. Since $\dim\ker B = n$, we deduce that there exists an orthogonal projection P onto $\ker B$ with $\operatorname{ran} P = \ker B$ and A in $\mathcal{L}(\mathcal{H})$ such that $AB = I - P$. By a straightforward computation, we see that there exists A_1 on $\mathcal{H}_1$ such that $A_1B_1 = I_{\mathcal{H}_1} + K$, where K is an operator with finite rank. Hence $0 \in \rho_F(B_1)$ and $\operatorname{ran} B_1$ is closed. By using Lemma 3.14 in [12], we see that, for x_1 in $\ker B$, there exist $x_k \in \ker(\lambda_k - B), k > 1$, such that $\lim_{k\to\infty} x_k = x_1$. Hence $\operatorname{ran} B_1$ is dense in $\mathcal{H}_1$ and $0 \in \rho_r(B_1)$. Since B_1 is onto, it is easy to see that B_2 is injective (otherwise, $\dim\ker B \geqslant n+1$). By $0 \in \rho_r(B_2)$, we deduce that $0 \in \rho(B_2)$. Since $\rho(B_2)$ is an open set, we see that there exists an open subset D of Ω such that $0 \in D \subset \Omega \cap \rho(B_2)$. So $D \subset \sigma_p(B_1)$ and $\ker(\lambda - B_1) = \ker(\lambda - B)$ for λ in D. By Proposition 2.1, we complete the proof of Proposition 2.2. ∎

From Proposition 2.2 and its proof, we immediately obtain that

PROPOSITION 2.3. *For B in $\mathcal{B}_n(\Omega)$, $\bigvee_{\lambda\in\Omega} \ker(\lambda - B) = \mathcal{H}$ is equivalent to $\bigvee_{k=1}^{\infty} \ker(\lambda - B)^k = \mathcal{H}$, for some λ in Ω (see [4]).*

DEFINITION 2.4. Let $\{x_n\}_{n\geqslant1}$ and $\{y_n\}_{n\geqslant1}$ be two sequences of vectors in $\mathcal{H}$. We call $\{x_n\}$ and $\{y_n\}$ *equivalent* if there exist two positive numbers M_1 and M_2 such that

$$M_1\Big\|\sum_{j=1}^{k} a_j y_j\Big\| \leqslant \Big\|\sum_{j=1}^{k} a_j x_j\Big\| \leqslant M_2\Big\|\sum_{j=1}^{k} a_j y_j\Big\|$$

for any integer k and any complex numbers $a_1, a_2, \ldots, a_k$.

THEOREM 2.5. *Suppose that B_1 and B_2 are in $B_n(\Omega)$ and $\{\lambda_k\}_{k\geqslant0}$ is a sequence of complex numbers in Ω satisfying $\lim_{k\to\infty}\lambda_k = \lambda_0$. Let $\Gamma_{B_1(\lambda)}$ and $\Gamma_{B_2(\lambda)}$ be given by Proposition A. Then B_1 and B_2 are similar if and only if $\{e_j^{B_1}(\lambda_k); k = 0,1,2,\ldots, j = 1,2,\ldots,n\}$ and $\{e_j^{B_2}(\lambda_k); k = 0,1,2,\ldots, j = 1,2,\ldots,n\}$ are equivalent.*

Proof. "$\Longleftarrow$" Let X be defined by $Xe_j^{B_1}(\lambda_k) = e_j^{B_2}(\lambda_k), k = 0,1,2,\ldots, j = 1,2,\ldots,n$. By Proposition 1.2 and the equivalence property, it is not difficult to deduce that X is invertible and $XB_1 = B_2X$.

"$\Longrightarrow$" Since B_1 and B_2 are similar and by Proposition 4.29 in [4], we see that $\Gamma_{B_1(\lambda)} = \Gamma_{B_2(\lambda)}$. Hence

$$X\{e_1^{B_1}(\lambda_k), e_2^{B_1}(\lambda_k), \ldots, e_n^{B_1}(\lambda_k)\} = \{e_1^{B_2}(\lambda_k), e_2^{B_2}(\lambda_k), \ldots, e_n^{B_2}(\lambda_k)\}$$

for $j = 0,1,2,\ldots$. Therefore the equivalence property is immediately obtained. ∎

Using the definition of a nice operator, we obtain immediately that

THEOREM 2.6. *If B_1 and B_2 are similar, then $B_1 \oplus B_2$ is not a nice operator.*

3. STRONGLY IRREDUCIBLE COWEN-DOUGLAS OPERATORS

In [7] and [12], C.K. Fong and C.L. Jiang proved that if B is in $\mathcal{B}_1(\Omega)$, then the operators quasisimilar to B are strongly irreducible. By Proposition 1.21 in [4], we see that $\mathcal{A}'(B)$ can be always identified with a subalgebra of $H^\infty(\Omega)$. This shows that $\mathcal{A}'(B)$ is commutative.

THEOREM 3.1. *Suppose that B is in $\mathcal{B}_n(\Omega)$, $\{\lambda_k\}_{k\geqslant 1}$ is a sequence of complex numbers in Ω satisfying $\lim\limits_{k\to\infty} \lambda_k = \lambda_1$ and $\Gamma_{B(\lambda)} = \{e_1(\lambda), \ldots, e_n(\lambda)\}$ is given by Proposition A. Then B is strongly reducible if and only if there exists an invertible operator X in $\mathcal{L}(\mathcal{H})$ such that*

$$X\Gamma_B(\lambda_k) = \{Xe_{j_1}(\lambda_k), \ldots, Xe_{j_m}(\lambda_k)\} \oplus \{Xe_{i_1}(\lambda_k), \ldots, Xe_{i_p}(\lambda_k)\},$$

where $m + p = n$; $k = 1, 2, \ldots$ and we denote by "$\oplus$" the orthogonal direct sum.

Proof. "$\Longrightarrow$" Assume that B is strongly reducible. Then there exists an operator T in $\mathcal{L}(\mathcal{H})$ with $T = T_1 \oplus T_2, T_1$ in $\mathcal{B}_m(\Omega)$ and T_2 in $\mathcal{B}_p(\Omega)$ such that B and T are similar, i.e., there exists an invertible operator X such that $XBX^{-1} = T_1 \oplus T_2$. It is clear that $\Gamma_T(\lambda) = \{Xe_1(\lambda), \ldots, Xe_n(\lambda)\}$ is a holomorphic $\mathcal{H}$-valued function determined by T. By Proposition 1.18 in [4], $\Gamma_T(\lambda)$ is reducible. Hence

$$\Gamma_T(\lambda_k) = \{Xe_{j_1}(\lambda_k), \ldots, Xe_{j_m}(\lambda_k)\} \oplus \{Xe_{i_1}(\lambda_k), \ldots, Xe_{i_p}(\lambda_k)\}.$$

"$\Longleftarrow$" Without loss of generality, we may asssume that $j_1 = 1, \ldots, j_m = m; i_1 = m+1, \ldots, i_p = n$. Then $\Gamma(\lambda) = \{Xe_j(\lambda); 1 \leqslant j \leqslant m\}$ determines an operator T in $\mathcal{B}_n(\Omega), \Gamma_1(\lambda) = \{Xe_j(\lambda); 1 \leqslant j \leqslant m\}$ and $\Gamma_2(\lambda) = \{Xe_j(\lambda), m+1 \leqslant j \leqslant n\}$ determine T_1 in $\mathcal{B}_m(\Omega)$ and T_2 in $\mathcal{B}_p(\Omega)$ respectively (see [4], Section 1). It is clear that $XBX^{-1} = T$. To show that B is strongly reducible, it is enough to show that T is reducible. Hence, we need only to show that $\bigvee\limits_{k\geqslant 1} \{Xe_1(\lambda_k), \ldots, Xe_m(\lambda_k)\}$ and $\bigvee\limits_{k\geqslant 1} \{Xe_{m+1}(\lambda_k), \ldots, Xe_n(\lambda_k)\}$ are orthogonal since the span of these two subspaces is $\mathcal{H}$ by Proposition 1.2. We have $(Xe_i(\lambda), Xe_j(\lambda)) = 0$ for $1 \leqslant i \leqslant m < j \leqslant n$, and differentiating with respect to λ, we deduce that

$$\frac{\partial}{\partial \lambda}(Xe_i(\lambda), Xe_j(\lambda)) = (Xe_i'(\lambda), Xe_j(\lambda)) = 0.$$

Similarly we have

$$\frac{\partial^Q}{\partial\lambda^Q}(Xe_i(\lambda), Xe_j(\lambda)) = (Xe_i^{(Q)}(\lambda), Xe_j(\lambda)) = 0$$

for $Q = 1, 2, \ldots$. Therefore, there exists a $\delta > 0$ such that

$$Xe_i(\lambda_k) = \sum_{Q=1}^{m} \frac{Xe_i^{(Q)}(\lambda_1)}{Q!}(\lambda_k - \lambda_1)^Q$$

for $|\lambda_k - \lambda_1| < \delta$. Hence, $(Xe_i(\lambda_k), Xe_j(\lambda_1)) = 0$ for $|\lambda_k - \lambda_1| < \delta$. Since $(Xe_i(\lambda_1), Xe_j(\lambda_k)) = \overline{(Xe_i(\lambda_k), Xe_j(\lambda_1))} = 0$, similarly, we have $(Xe_j(\lambda_l), Xe_j(\lambda_k)) = 0$ for $|\lambda_l - \lambda_1| < \delta$ and $|\lambda_k - \lambda_1| < \delta$. Using Proposition 1.2 again, we complete the proof of the Theorem 3.1. ∎

For B in $\mathcal{B}_n(\Omega)$ we have a unique decomposition theorem similar to L.J. Gray's result [9]. In fact, we obtain a stronger result:

THEOREM 3.2. *If $B \in \mathcal{L}(\mathcal{H})$ can be written as an algebraic direct sum (denoted by $\dot{+}$) of countably many strongly irreducible operators and $\mathcal{A}'(B)$ is commutative, then B can be uniquely written in the following form:*

$$B = \dot{+}_{j=1}^{k} B_j, \quad k \leqslant +\infty,$$

where B_j ($j = 1, 2, \ldots, k$) are strongly irreducible operators and each B_j is uniquely determined by B.

Proof. It is enough to show that if p_1 and p_2 are idempotent operators in $\mathcal{A}'(B)$ such that $B|p_1\mathcal{H}$ and $B|p_2\mathcal{H}$ are strongly irreducible, then either $p_1 = p_2$ or $p_1 \cdot p_2 = 0$.

Assume that $p_1 \neq p_2$ and $p_1 \cdot p_2 \neq 0$, then we can deduce that $p_1 \cdot p_2$ is an idempotent operator which is commutes with $B|p_1\mathcal{H}$ and $B|p_2\mathcal{H}$ respectively. It is clear that $p_1 p_2$ either in $\mathcal{L}(p_1\mathcal{H})$ or in $\mathcal{L}(p_2\mathcal{H})$ is nontrivial. This contradicts to our assumption that $B|p_1\mathcal{H}$ and $B|p_2\mathcal{H}$ are strongly irreducible. ∎

PROPOSITION 3.3. *Let $\mathcal{A}^\alpha(T)$ be the weak closure of rational functions of T with poles outside $\sigma(T)$ for T in $\mathcal{L}(\mathcal{H})$. If $\mathcal{A}^\alpha(T)$ is strictly cyclic, then T is strongly irreducible if and only if $\sigma(T)$ is connected.*

Proof. Let $\mathcal{U}$ be the maximal ideal space of $\mathcal{A}^\alpha(T)$. Since $\mathcal{A}^\alpha(T)$ is strictly cyclic, it is obvious that $\mathcal{A}^\alpha(T) = \mathcal{A}'(T) = \mathcal{A}''(T)$ (the double commutant of T). Following Shilov's idempotent theorem (see [11]), each idempotent operator in $\mathcal{A}^\alpha(T)$ is a characteristic function in $C_{\mathcal{U}}$ where $C_{\mathcal{U}}$ is the space of all continuous functions on $\mathcal{U}$. Hence, it is enough to show that $\mathcal{U}$ is homeomorphic to $\sigma(T)$. By the proof of Corollary 2.36 in [5], we can complete the proof. ∎

4. THE NICE PROPERTY AND THE APPROXIMATION THEOREM OF COWEN-DOUGLAS OPERATORS

Let Ω be an analytic connected Cauchy domain and let $\mathcal{L}^2(\partial\Omega)$ be the Hilbert space of complex functions on $\partial\Omega$ which are square integrable with respect to $(1/2\pi)$-times arc-length measure on $\partial\Omega$. $M_z(\partial\Omega)$ will stand for the operator of multiplication by z acting on $\mathcal{L}^2(\partial\Omega)$. $\mathcal{H}^2(\partial\Omega)$ is the subspace spanned by the rational functions with poles outsider $\overline{\Omega}$ in the norm $\|\cdot\|_1$ of $\mathcal{L}^2(\partial\Omega)$. Then $\mathcal{H}^2(\partial\Omega)$ is invariant under $M_z(\partial\Omega)$. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be a finite subset of $\mathbf{C}\setminus\overline{\Omega}$ having exactly one point in every component of this last set and let $W^{1,2}(\partial\Omega)$ be the Sobolev space consisting of all distributions μ on $\partial\Omega$ which have the distribution derivative in $\mathcal{L}^2(\partial\Omega)$ (with respect to arc lenght measure dm on $\partial\Omega$) with the norm of $\|\cdot\|_2$, where

$$\|f\|_2 = \left\{\int_{\partial\Omega} (|f(z)|^2 + |df/dm|^2)\, dm\right\}^{\frac{1}{2}}.$$

We denote by $\mathcal{A}_1$ the subspace of $W^{1,2}(\partial\Omega)$ spanned by the rational functions with poles in a subset of Λ. By the above definition, $W^{1,2}$ can be continuously imbedded into $\mathcal{L}^2(\partial\Omega)$. Hence, $\mathcal{A}_1$ can be continuously imbeded into $\mathcal{H}^2(\partial\Omega)$.

THEOREM 4.1. (see [11]). *Consider M_z as acting on $W^{1,2}$. Then $\mathcal{A}_1$ is invariant under M_z, and $A_z = M_z|\mathcal{A}_1$ satisfies:*

(i) *$\mathcal{A}''(A_z) = \mathcal{A}'(A_z) = \{M_g : g \in \mathcal{A}_1\}$ is a maximal Abelian strictly cyclic subalgebra of $\mathcal{L}(\mathcal{A})_1$,*

(ii) *$\sigma(A_z) = \overline{\Omega}$ and $\operatorname{ind}(\lambda - A_z) = -1, \ker(\lambda - A_z) = \{0\}$ for λ in Ω, where M_g stands for the operator of multiplication by g acting on $\mathcal{A}_1$.*

PROPOSITION 4.2. (see [6]). *$\mathcal{A}_z^*$ is in $\mathcal{B}_1(\overline{\Omega})$, where $\overline{\Omega} \overset{\text{def}}{=} \{\overline{\lambda}; \lambda \in \Omega\}$.*

THEOREM 4.3. *For each integer n, there exists a strongly irreducible nice operator B in $\mathcal{B}_n(\Omega)$.*

This theorem gives an affirmative answer to Question II for a connected analytic Cauchy domain.

Before proving this theorem we need the following result.

LEMMA 4.4. *Let δ_{A_z} be the mapping defined by*

$$\delta_{A_z}(Y) = A_z Y - Y A_z, \text{ for } Y \in \mathcal{L}(\mathcal{A}_1).$$

Then

$$\mathcal{A}'(A_z) \cap \operatorname{ran} \delta_{A_z} = \{0\}.$$

Proof. Clearly, $\ker \delta_{A_z} = \mathcal{A}'(A_z)$. Assume that there exist A and A_1 in $\mathcal{L}(\mathcal{A}_1)$ such that

(1) $A_z A_1 - A_1 A_z = A$,

(2) $A_z A = A A_z$.

If $A \neq 0$, then there exists g in $\mathcal{A}_1$ with $g \neq 0$ such that $A_z A_1 - A_1 A_z = M_g$. Let $e_0 = 1, e_n = \lambda^n, n = 1, 2, 3, \ldots, \lambda \in \Omega$ and $A_1(e_0) = h$. It is clear that h is in $\mathcal{A}_1$. By a simple calculation, we now get that

$$A_1(\lambda^n) = \lambda^n h(\lambda) + n\lambda^{n-1} g(\lambda).$$

It is clear that

$$\lim_{n\to\infty} \frac{\|A_1(\lambda^n)\|_1}{\|\lambda^n\|_1} = \infty$$

which would show that A_1 is an unbounded operator. This is a contradiction. ∎

Proof of Theorem 4.3. For $n = 1$, we take $B = A_z^*$. By Proposition 3.2 and Theorem 3.1, B is in $\mathcal{B}_1(\Omega)$ and B is a strongly irreducible nice operator.

Supposing $1 < n < \infty$, we define

$$B = \begin{bmatrix} A_z & I & & & \\ & A_z & I & & \\ & & A_z & \ddots & \\ & & & \ddots & I \\ & & & & A_z \end{bmatrix} \begin{matrix} \mathcal{A}_1 \\ \mathcal{A}_1 \\ \vdots \\ \vdots \\ \mathcal{A}_1 \end{matrix}.$$

Assume in addition that $T = (T_{i,j})_{i,j}^n$ is an $n \times n$ operator-valued matrix satisfying $TB = BT$; then

$$0 = BT - TB$$

$$= \begin{bmatrix} [A_z,T_{11}]+T_{21} & [A_z,T_{12}]+T_{22}-T_{11} & \cdots & [A_z,T_{1n}]+T_{2n}-T_{1,n-1} \\ [A_z,T_{21}]+T_{31} & [A_z,T_{22}]+T_{32}-T_{21} & \cdots & [A_z,T_{2n}]+T_{3n}-T_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ [A_z,T_{n-1,1}]+T_{n,1} & [A_z,T_{n-1,2}]+T_{n2}-T_{n-1,1} & \cdots & [A_z,T_{n-1,n}]+T_{n-1,n}-T_{n-1,n-1} \\ [A_z,T_{n,1}] & [A_z,T_{n,2}]-T_{n,1} & \cdots & [A_z,T_{nn}]-T_{n,n-1} \end{bmatrix},$$

where $[A_z, C] = A_zC - CA_z$ and, for $1 \leqslant i < n; 1 < j \leqslant n$, the (i,j)th-entry is equal to $[A_z, T_{ij}] + T_{i+1,j} - T_{i,j-1}$. The $(n-1)$th-entry indicates that $T_{n,1}$ is in $\mathcal{A}'(A_z)$ and the $(n,2)$th-entry shows that

$$T_{n,1} = [A_z, T_{n,2}] = \delta_{A_z}(T_{n,2}) \in \operatorname{ran} \delta_{A_z}.$$

Therefore $T_{n,1} = 0$ by Lemma 3.4. Now the $(n-1,1)$th-entry and $(n,2)$th--entry show that T_{n-1} and $T_{n,2}$ commute with A_z. By induction, we deduce that

$$T_{n,1} = T_{n-1,1} = \cdots = T_{2,1} = T_{n,2} = \cdots = T_{n,n-1} = 0$$

and $T_{1,1}$ and $T_{n,n}$ are in $\mathcal{A}'(A_z)$. Similarly, we infer that

$$T_{i,j} = 0 \quad \text{for} \quad 1 \leqslant j < i \leqslant n$$

and that $T_{ii}, i = 1, 2, \ldots, n$, are in $\mathcal{A}'(A_z)$. The (1,2)th-, (2,3)th-, ..., $(n-1,n)$th-entries show that $T_{11} = T_{22} = T_{33} = \cdots = T_{nn} = M_{g_0}$, where g_0 is in $\mathcal{A}_1$ and $T_{i,i+1}$ in $\mathcal{A}'(A_z)$ for $i = 1, 2, \ldots, n-1$. By induction, we deduce that $T_{i,i+k} \in \mathcal{A}'(A_z)$ and $T_{1,1+k} = T_{2,2+k} = \cdots = T_{n-k,n} = M_{g_k}$ for $1 < k < n$, where g_k is in $\mathcal{A}_1$. Hence supposing $T = (T_{i,j})_{i,j=1}^n$, we have that

$$T = \begin{bmatrix} M_{g_0} & M_{g_1} & M_{g_2} & \cdots & \cdots & M_{g_{n-1}} \\ & M_{g_0} & M_{g_1} & M_{g_2} & \cdots & M_{g_{n-2}} \\ & & & & \ddots & \vdots \\ & & & \ddots & & \\ & & & & & M_{g_1} \\ & & & \ddots & & M_{g_0} \end{bmatrix},$$

where $g_0, g_1, \ldots, g_{n-1}$ are in $\mathcal{A}_1$. Therefore $\mathcal{A}'(B)$ is strictly cyclic. Now we assume that T is an idempotent operator in $\mathcal{A}'(B)$. Since $T = T^2$ implies $T_{ii} = T_{ii}^2$ for all $1 \leqslant i \leqslant n$ and A_z is strongly irreducible, we deduce that $T_{11} = T_{22} = \cdots = T_{nn} = I$ or 0. Therefore $T = I$ or 0. It follows that B is strongly irreducible. Furthermore, B^* is a strongly irreducible nice operator. Since A_z^* is $\mathcal{B}_1(\Omega)$, we deduce that B^* belongs to $B_n(\Omega)$ by straightforward computation. This completes the proof of Theorem 4.3. ∎

If the definition of $\mathcal{B}_n(\Omega)$ admits $n = +\infty$, we have that

THEOREM 4.5. (see [11]). *There exists a strongly irreducible nice operator B in $\mathcal{B}_\infty(\Omega)$.*

THEOREM 4.6. *Let T be a nice operator in $\mathcal{L}(\mathcal{H})$. Then T can be uniquely written as an algebraic direct sum of finitely or countably many strongly irreducible operators.*

Proof. Without loss of generality, suppose that $\mathcal{A}'(T)$ is strictly cyclic. We denote by $\mathcal{U}$ the maximal ideal space of $\mathcal{A}'(T)$. Since $\mathcal{U}$ is a compact Hausdorff space, it is easy to see that $\mathcal{U}$ has at most countably many components. Using Shilov's idempotent theorem, we deduce that there exist only l idempotent operators $p_i = X_{\Omega_i}, i = 1, 2, \ldots, l, 1 \leqslant l \leqslant +\infty$ such that $T|p_i\mathcal{H}$ is strongly irreducible and $\sum_{i=1}^{l} p_i = I$, where $\mathcal{U}$ consists of l compotents $\Omega_1, \Omega_2, \ldots, \Omega_l$ and X_{Ω_i} is the characteristic function of Ω_i, i.e., X_{Ω_i} is 1 in Ω_i and 0 in $\mathcal{U} \setminus \Omega_i$. Then $T = \dot{+}_{i=1}^{l} T|p_i\mathcal{H}$ is the unique strongly irreducible decomposition. ∎

THEOREM 4.7. *Suppose that T is in $\mathcal{L}(\mathcal{H})$ such that* $\operatorname{ind}(T - \lambda) \geqslant 0$, *for $\lambda \in \rho_{SF}(T)$. Then, for given $\varepsilon > 0$, there exists a nice operator T_ε satisfying:*

(1) $\|T - T_\varepsilon\| < \varepsilon$,

(2) *each strongly irreducible direct summand $T_{k\varepsilon}$ of T_ε belongs to $B_{n_k}(\Omega_k)$, $1 \leqslant n_k \leqslant \infty$.*

Proof. By Lemma 8 in [13], Theorem 6.2 in [12], Theorem 9.2 in [2], Theorem 3.3 and 3.5, we can complete the proof of Theorem 4.7. ∎

Applying Theorem 4.7 to T^*, we can easily derive the following consequence:

COROLLARY 4.8. *Let T be in $\mathcal{L}(\mathcal{H})$ such that* $\operatorname{ind}(T - \lambda) \leqslant 0$ *for λ in $\rho_{S-F}(T)$. For a given $\varepsilon > 0$, there exists a nice operator T_ε satisfying*

(a) $\|T_\varepsilon - T\| < \varepsilon$,

(b) *each strongly irreducible direct summand $T_{k\varepsilon}^*$ of T_ε^* belongs to $\mathcal{B}_{n_k}(\Omega_k)$, $1 \leqslant n_k \leqslant +\infty$.*

By combining Theorem 4.6 and 4.7, Corollary 4.8, Theorem 6.2 in [12] and the proof of the Theorem 4.6, we obtain that

THEOREM 4.9. *Suppose that $T \in \mathcal{L}(\mathcal{H})$. For a given $\varepsilon > 0$, there exist two nice operators $T_{1\varepsilon}$ and $T_{2\varepsilon}$ such that*

(a) $\|T - (T_{1\varepsilon} \dot{+} T_{2\varepsilon})\| < \varepsilon$,

(b) $\sigma_p(T_{1\varepsilon}) \bigcap \sigma_p(T_{2\varepsilon}) = \emptyset$,

(c) *each strongly irreducible direct summand $T_{k\varepsilon}$ of $(T_{1\varepsilon} \dot{+} T_{2\varepsilon})$ satisfies that either $T_{k\varepsilon}$ or $T_{k\varepsilon}^*$ is a Cowen-Douglas operator.*

The proof of Conjecture J. Let T be in $\mathcal{L}(\mathcal{H})$ and $T_\epsilon = T_{1\epsilon}\dot{+}T_{2\epsilon}$ be given by Theorem 4.9. By the conclusion (b) and (c) in Theorem 4.9, it is easy to deduce that $\mathcal{A}'(T_\epsilon) = \{A\dot{+}B :\in \mathcal{A}'(T_{1\epsilon}) \text{ and } B \in \mathcal{A}'(T_{2\epsilon})\}$ is commutative. By Theorem 3.2, we can affirm Conjecture J.

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