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LEFT QUOTIENTS OF A C*-ALGEBRA, I : REPRESENTATION VIA VECTOR SECTIONS

NGAI-CHING WONG

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ABSTRACT. Let A be a C*-algebra, L a closed left ideal of A and p the closed projection related to L. We show that for an xp in $A^{**}p$ ($\cong A^{**}/L^{**}$) if $pAxp \subset pAp$ and $px^*xp \in pAp$ then $xp \in Ap$ ($\cong A/L$). The proof goes by interpreting elements of $A^{**}p$ (resp. Ap) as admissible (resp. continuous admissible) vector sections over the base space $F(p) = \{\varphi \in A^* : \varphi \geqslant 0, \varphi(p) = \|\varphi\| \leqslant 1\}$ in the notions developed by Diximier and Douady, Fell, and Tomita. We consider that our results complement both Kadison function representation and Takesaki duality theorem.

KEYWORDS: *C*-algebras, continuous fields of Hilbert spaces, left quotients, affine property.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 46L45.

1. INTRODUCTION

It is known that every closed left ideal L of a C*-algebra A is related to a closed projection p in the sense that $L = A^{**}(1-p) \cap A$ (and thus $L^{**} = A^{**}(1-p)$). Moreover, A/L (resp. A^{**}/L^{**}) is isometrically isomorphic to Ap (resp. $A^{**}p$) as Banach spaces ([11], [14], [2]). Here, a projection p in A^{**} is said to be *closed* if the face $F(p) = \{\varphi \in Q(A) : \varphi(p) = \|\varphi\|\}$ of the weak* compact convex set $Q(A) = \{\varphi \in A^* : \varphi \geqslant 0, \|\varphi\| \leqslant 1\}$ is closed (cf. [3]).

For each φ in $F(p)$, $L_\varphi = \{a \in A : \varphi(a^*a) = 0\}$ is a closed left ideal of A, and $L = \bigcap_{\varphi \in F(p)} L_\varphi$. This gives a natural embedding

$$A/L \hookrightarrow \prod_{\varphi \in F(p)} A/L_\varphi, \qquad a + L \longmapsto (a + L_\varphi)_{\varphi \in F(p)}.$$

Let H_φ be the completion of the pre-Hilbert space A/L_φ with respect to the inner product $\langle a+L_\varphi, b+L_\varphi\rangle_\varphi := \varphi(b^*a)$ for each φ in $F(p)$ (i.e. the GNS construction for φ). In this way, $A/L \cong Ap$ is embedded into the field of Hilbert spaces $(F(p), \{H_\varphi\}_\varphi)$. This also induces an embedding of $A^{**}/L^{**} \cong A^{**}p$ into $(F(p), \{H_\varphi\}_\varphi)$, since we can identify H_φ with the GNS Hilbert space for φ when φ is regarded as a positive functional on A^{**} and the GNS representation of A^{**} extends that of A.

By a result of Brown ([7], 3.5) pAp (resp. $pA^{**}p$) is isometrically order isomorphic to the Banach space of all continuous (resp. bounded) affine functions on $F(p)$ which vanish at zero. In particular, for every xp in Ap the scalar maps $\varphi \longmapsto \varphi(pa^*xp) = \varphi(a^*x), \forall a \in A$, and $\varphi \longmapsto \varphi(px^*xp) = \varphi(x^*x)$ are continuous on $F(p)$. In this paper, we proved that if xp in $A^{**}p$ satisfies conditions that the scalar maps $\varphi \longmapsto \varphi(a^*x), \forall a \in A$, and $\varphi \longmapsto \varphi(x^*x)$ are continuous on $F(p)$ then $xp \in Ap$. In other words, for xp in $A^{**}p$, $pAxp \subset pAp$ and $px^*xp \in pAp$ imply $xp \in Ap$.

We establish the above result by first looking for an admissibility condition characterizing those vector sections of the field of Hilbert spaces $(F(p), \{H_\varphi\}_\varphi)$ arising from elements of $A^{**}p$ (Theorem 2.6). Then, following ideas of Fell ([12]) and Diximier and Douady ([9]), we implement a continuous structure $\Gamma(Ap)$ of $(F(p), \{H_\varphi\}_\varphi)$ in which all vector sections arising from elements of Ap are continuous. Finally, we prove that continuous admissible vector sections of $(F(p), \{H_\varphi\}_\varphi, \Gamma(Ap))$ are exactly those arising from elements of Ap (Theorem 3.2). And this is translated to the result just mentioned above (Corollary 4.1).

The way we look at elements of $A^{**}p$ and Ap as admissible vector sections and continuous admissible vector sections over the compact convex set $F(p)$ suggests some interesting questions and results. For example, it is natural to ask for an xp in $A^{**}p$ if the continuity of the scalar maps $\varphi \longmapsto \varphi(a^*x), \forall a \in A$, and $\varphi \longmapsto \varphi(x^*x)$ on the extreme boundary $F(p) \cap (P(A) \cup \{0\})$ of $F(p)$ can imply $xp \in Ap$, where $P(A)$ is the pure state space of A. In [22], we prove that such an xp has a continuous atomic part in many cases, i.e. there is an ap in Ap such that $zxp = zap$, where z is the maximal atomic projection of A. Even when $p = 1$, this is new and supplements results of Shultz ([17]) and Brown ([8]), which say that for an x in A^{**} if $\varphi \longmapsto \varphi(x)$ is uniformly continuous on $P(A) \cup \{0\}$ then $zx \in zA$. On the other hand, following the plan of Tomita ([20]) and using ideas of Rieffel ([15]), we represent bounded Banach space operators on $A/L \cong Ap$ as fields of bounded Hilbert space operators in the context of $(F(p), \{H_\varphi\}_\varphi, \Gamma(Ap))$. Many ideas of Tomita about the theory of left regular representation of A on A/L can thus be implemented in this context (see [23]).

When $p = 1$, one can easily find the origin of our theory from Kadison function representation (see Section 3) and Takesaki duality theorem [18], [6], [5] (see Section 5). However, the results of Kadison and Takesaki are not ready to apply to left quotients Ap ($\cong A/L$) if $p \neq 1$ (i.e. $L \neq \{0\}$). To extend these classical tools to the general case of $p \neq 1$ as shown in this paper, Tomita ([19], [20]) indicates us a way to set up our theory and Akemann ([1], [2], [3]), Diximier and Douady ([9]), Effros ([11]), Fell ([12]), and Prosser ([14]) provide us the basic machinery.

We would like to express our deep gratitude to Professor L.G. Brown for many valuable advices. This paper is based on the author's doctoral dissertation ([21]) under his supervision.

2. REPRESENT W*-ALGEBRAS VIA ADMISSIBLE VECTOR SECTIONS

Let M be a W*-algebra with predual M_* and $Q_*(M) = \{\varphi \in M_* : \varphi \geqslant 0, \|\varphi\| \leqslant 1\}$. Let p be a projection in M and $F(p) = \{\varphi \in Q_*(M) : \varphi(p) = \|\varphi\|\}$, the face of the convex set $Q_*(M)$ supported by p. For each φ in $F(p)$, the GNS construction yields a cyclic representation $(\pi_\varphi, H_\varphi, \omega_\varphi)$ of M. $\overline{\pi_\varphi(M)\omega_\varphi} = H_\varphi$ and $\varphi(x) = \langle \pi_\varphi(x)\omega_\varphi, \omega_\varphi\rangle_\varphi$, $\forall x \in M$, where $\langle \cdot,\cdot\rangle_\varphi$ is the inner product of the Hilbert space H_φ. Write $x\omega_\varphi$ for $\pi_\varphi(x)\omega_\varphi, \forall x \in M, \forall \varphi \in F(p)$. Note that $p\omega_\varphi = \omega_\varphi, \forall \varphi \in F(p)$. By convention, we set H_φ to be the zero dimensional Hilbert space when $\varphi = 0$. In this way, there is an embedding $Mp \hookrightarrow \prod_{\varphi\in F(p)} H_\varphi$ defined by $xp \longmapsto (x\omega_\varphi)_{\varphi\in F(p)}$. If we equip the range of this embedding with the ℓ^∞ norm then it is even an isometry as

$$\|xp\|^2 = \sup_{\varphi\in Q_*(M)} \varphi(px^*xp) = \sup_{\varphi\in F(p)} \varphi(x^*x) = \sup_{\varphi\in F(p)} \|x\omega_\varphi\|_\varphi^2 = \|(x\omega_\varphi)_{\varphi\in F(p)}\|_\infty^2.$$

We are going to classify those vector sections in $\prod_{\varphi\in F(p)} H_\varphi$ arising from this embedding. First, we observe that fibers H_φ in $\prod_{\varphi\in F(p)} H_\varphi$ are not independent of each other. The following definition is taken from Tomita ([19] with a slight modification).

DEFINITION 2.1. Let M be a W*-algebra. For each ψ in M_* and each φ in $F(p)$, we set

$$\|\psi\|_\varphi = \sup\{|\ \psi(x)\ | : x \in M \text{ and } \|x\omega_\varphi\|_\varphi = \varphi(x^*x)^{\frac{1}{2}} \leqslant 1\},$$

and $L^2(\varphi) = \{\psi \in M_* : \|\psi\|_\varphi < \infty\}$. We say that ψ is *observable at* φ if $\psi \in L^2(\varphi)$. It follows from the Riesz–Fréchet theorem that for each ψ in $L^2(\varphi)$ there is a unique $\omega_{\psi\varphi}$ in H_φ such that

$$\psi(x) = \langle x\omega_\varphi, \omega_{\psi\varphi}\rangle_\varphi, \quad \forall x \in M.$$

It can be verified that the map Λ defined by $\Lambda_\varphi(\psi) = \omega_{\psi\varphi}$ is a conjugate isometrical isomorphism from $L^2(\varphi)$ onto H_φ ([19]). The proof of the following lemma is left to the readers.

LEMMA 2.2. *For each ψ in $L^2(\varphi)$ and x in M, we have*

$$\Lambda_\varphi(x\psi) = x^*\Lambda_\varphi(\psi).$$

In other words,

$$\omega_{(x\psi)\varphi} = x^*\omega_{\psi\varphi},$$

where $x\psi$ in $MF(p)$ is defined by $x\psi(y) = \psi(xy), \forall y \in M$.

DEFINITION 2.3. For each ψ, φ in $F(p)$ with $0 \leqslant \psi \leqslant \lambda\varphi$ for some $\lambda > 0$, let

$$T_{\psi\varphi} : H_\varphi \longrightarrow H_\psi$$

be the linear map from H_φ into H_ψ sending $x\omega_\varphi$ to $x\omega_\psi$. Note that $\psi \in L^2(\varphi)$ by the Cauchy–Schwartz inequality. Moreover, $\|T_{\psi\varphi}\| \leqslant \lambda^{1/2}$ and $T^*_{\psi\varphi}(x\omega_\psi) = x\omega_{\psi\varphi} = \Lambda_\varphi(x^*\psi), \forall x \in M$.

DEFINITION 2.4. A vector section $f : \varphi \longmapsto f(\varphi) \in H_\varphi$ is said to be *admissible* over $F(p)$ if whenever $\psi, \varphi \in F(p)$ such that $0 \leqslant \psi \leqslant \varphi$,

$$T_{\psi\varphi}(f(\varphi)) = f(\psi).$$

f is said to be an *affine vector section* over $F(p)$ if the functional

$$\varphi \longmapsto \langle f(\varphi), x\omega_\varphi\rangle_\varphi$$

is affine on the convex set $F(p)$ for each x in M.

It is easy to see that whenever $0 \leqslant \psi \leqslant \varphi \leqslant \rho$ in $F(p)$, $T_{\psi\varphi}T_{\varphi\rho} = T_{\psi\rho}$. Moreover, for an admissible vector section f and φ, ψ in $F(p)$ such that $0 \leqslant \psi \leqslant \lambda\varphi$ for some $\lambda > 0$, we have $T_{\psi\varphi}f(\varphi) = f(\psi)$, too.

PROPOSITION 2.5. *Every admissible vector section $f = (f(\varphi))_\varphi$ over $F(p)$ is bounded, i.e.* $\|f\|_\infty = \sup_{\varphi \in F(p)} \|f(\varphi)\|_\varphi < \infty$.

Proof. Assume the contrary and choose φ_n in $F(p)$ such that

$$\|f(\varphi_n)\|_{\varphi_n} > 2^n, n = 1, 2, \ldots .$$

Set

$$\varphi = \sum_n \frac{1}{2^n}\varphi_n$$

in $F(p)$. Since $0 \leqslant \varphi_n \leqslant 2^n\varphi$, $T_{\varphi_n\varphi}$ in $B(H_\varphi, H_{\varphi_n})$ exists and $\|T_{\varphi_n\varphi}\|^2 \leqslant 2^n$. Therefore,

$$\|f(\varphi_n)\|^2_{\varphi_n} = \|T_{\varphi_n\varphi} f(\varphi)\|^2_{\varphi_n} \leqslant 2^n \|f(\varphi)\|^2_\varphi, n = 1, 2, \ldots .$$

Hence

$$\|f(\varphi)\|^2_\varphi \geqslant \frac{1}{2^n}\|f(\varphi_n)\|^2_{\varphi_n} \geqslant \frac{1}{2^n} 2^{2n} = 2^n, n = 1, 2, \ldots,$$

a contradiction. ∎

Here is our main result:

THEOREM 2.6. *Mp is isometrically isomorphic to the Banach space of all admissible vector sections in $\prod_{\varphi\in F(p)} H_\varphi$ equipped with the norm $\|\cdot\|_\infty$. A vector section in $\prod_{\varphi\in F(p)} H_\varphi$ is admissible if and only if it is bounded and affine.*

We shall present the proof of Theorem 2.6 in two parts. It is trivial that each element of Mp defines an admissible vector section over $F(p)$ in the manner described at the start of this section. For the converse, we shall associate to each admissible vector section $f = (f(\varphi))_{\varphi\in F(p)}$ a bounded linear functional $\tilde{f}$ of the predual $\{\varphi(\cdot p)_{|Mp} : \varphi \in M_*\}$ of Mp, which can be identified with $MF(p) = \{x\,\varphi(\cdot) = \varphi(x\,\cdot) : x \in M, \varphi \in F(p)\}$ (cf. [11]).

LEMMA 2.7. *Let $f = (f(\varphi))_\varphi$ be an admissible vector section over $F(p)$.*
(a) *If ϕ in M_* is observable at both φ and ψ in $F(p)$ then*

$$\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi = \langle f(\psi), \omega_{\phi\psi}\rangle_\psi .$$

(b) *If $\phi \in M_*$ and $\varphi, \psi \in F(p)$ such that*

$$\phi = x^*\varphi = y^*\psi$$

for some x, y in M then

$$\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi = \langle f(\varphi), x\omega_\varphi\rangle_\varphi = \langle f(\psi), y\omega_\psi\rangle_\psi = \langle f(\psi), \omega_{\phi\psi}\rangle_\psi .$$

Proof. (a) Let $\rho = \frac{\varphi+\psi}{2} \in F(p)$. By the admissibility of f, $T_{\varphi\rho}(f(\rho)) = f(\varphi)$ and $T_{\psi\rho}(f(\rho)) = f(\psi)$. It is easy to see that $T^*_{\varphi\rho}(\omega_{\phi\varphi}) = \omega_{\phi\rho}$ and $T^*_{\psi\rho}(\omega_{\phi\psi}) = \omega_{\phi\rho}$. Now

$$\begin{aligned}\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi &= \langle T_{\varphi\rho} f(\rho), \omega_{\phi\varphi}\rangle_\varphi = \left\langle f(\rho), T^*_{\varphi\rho}(\omega_{\phi\varphi})\right\rangle_\rho \\ &= \langle f(\rho), \omega_{\phi\rho}\rangle_\rho = \langle f(\psi), \omega_{\phi\psi}\rangle_\psi .\end{aligned}$$

(b) First, note that ϕ is observable at both φ and ψ and thus the asserted equalities make sense. Our assertion follows from Lemma 2.2 and (a) and the following observation

$$\omega_{\phi\varphi} = \Lambda_\varphi(\phi) = \Lambda_\varphi(x^*\varphi) = x\Lambda_\varphi(\varphi) = x\omega_\varphi$$

and

$$\omega_{\phi\psi} = \Lambda_\psi(\phi) = \Lambda_\psi(y^*\psi) = y\Lambda_\psi(\psi) = y\omega_\psi. \quad ∎$$

Note that $Mp = (MF(p))^*$. This suggests us to make the following

DEFINITION 2.8. Let $f = (f(\varphi))_\varphi$ be an admissible vector section over $F(p)$. Define for each ϕ in $MF(p)$,

$$\tilde{f}(\phi) = \langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi ,$$

where $\varphi \in F(p)$ and ϕ is observable at φ.

Clearly, $\tilde{f}(0) = 0$. For a non-zero ϕ in $MF(p)$, it follows from Lemma 2.7 (a) that the definition of $\tilde{f}(\phi)$ is independent of the choice of φ for which $\phi \in L^2(\varphi)$, and $\varphi = |\phi|/\|\phi\|$ is just a good choice, where $|\phi|$ is the absolute value of ϕ coming from the polar decomposition of ϕ (see, e.g. [13]). Moreover, if $\phi = x^*\varphi$ for some φ in $F(p)$ then by Lemma 2.7 (b),

$$\tilde{f}(\phi) = \langle f(\varphi), x\omega_\varphi\rangle_\varphi .$$

Proof of the first part of Theorem 2.6. The first task is to prove that $\tilde{f}$ is a bounded linear functional of $MF(p)$ for every admissible vector section $f = (f(\varphi))_\varphi$ over $F(p)$. To verify that $\tilde{f}$ is additive, let ρ, φ and ψ be elements of $MF(p)$ such that $\rho = \varphi + \psi$. In case $\varphi = \psi = 0$, it is plain that $\tilde{f}(\rho) = \tilde{f}(\varphi) + \tilde{f}(\psi)$. Suppose that not both φ and ψ are zero. [19] or [11] showed that

$$\||\rho|(x)|^2 \leqslant (\|\varphi\| + \|\psi\|)(|\varphi| + |\psi|)(x^*x), \quad \forall x \in M.$$

Hence, $|\rho| \in L^2(\tau)$, where $\tau = \frac{|\varphi|+|\psi|}{\|\varphi\|+\|\psi\|} \in F(p)$. Clearly, $|\varphi|, |\psi| \in L^2(\tau)$. As a result, ρ, φ and $\psi \in L^2(\tau)$. Now, $\tilde{f}(\rho) = \langle f(\tau), \omega_{\rho\tau}\rangle_\tau$, $\tilde{f}(\varphi) = \langle f(\tau), \omega_{\varphi\tau}\rangle_\tau$

and $\tilde{f}(\psi) = \langle f(\tau), \omega_{\psi\tau}\rangle_\tau$. The additivity of $\tilde{f}$ follows easily since, by uniqueness, $\omega_{\rho\tau} = \omega_{\varphi\tau} + \omega_{\psi\tau}$. By Lemma 2.2, $\omega_{(\lambda\phi)\varphi} = \bar{\lambda}\omega_{\phi\varphi}, \forall\lambda \in \mathbb{C}, \forall\varphi \in F(p), \forall\phi \in L^2(\varphi)$. Therefore, $\tilde{f}$ is a linear functional on $MF(p)$. For the boundedness of $\tilde{f}$, assume ϕ is a nonzero element in $MF(p)$ and $\varphi = \frac{|\phi|}{\|\phi\|}$ then $\phi \in L^2(\varphi)$ with $\|\phi\|_{L^2(\varphi)} \leqslant \|\phi\|$ and

$$|\tilde{f}(\phi)| = |\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi| \leqslant \|f(\varphi)\|_\varphi \|\omega_{\phi\varphi}\|_\varphi \leqslant \|f\|_\infty \|\phi\|_{L^2(\varphi)} \leqslant \|f\|_\infty \|\phi\|.$$

Consequently, $\tilde{f} \in (MF(p))^* = Mp$. When we consider $\tilde{f}$ as an element of Mp, for any φ in $F(p)$ and x in M we have

$$\left\langle \tilde{f}\omega_\varphi, x\omega_\varphi \right\rangle_\varphi = \varphi(x^*\tilde{f}) = x^*\varphi(\tilde{f}) = \tilde{f}(x^*\varphi) = \langle f(\varphi), x\omega_\varphi\rangle_\varphi .$$

This means that the vector section $(\tilde{f}\omega_\varphi)_\varphi$ is exactly the original f.

Conversely, since the embedding $Mp \hookrightarrow \prod_{\varphi\in F(p)} H_\varphi$ is an isometry with respect to the ℓ^∞ norm, we have an isometrical isomorphism Θ from Mp onto the Banach space of all admissible vector sections f over $F(p)$ such that $\Theta(\tilde{f}) = f$. ∎

We now proceed to prove the second part of Theorem 2.6. The following easy lemma is stated for reference.

LEMMA 2.9. *Let $(y_\varphi)_{\varphi\in F(p)}$ be an affine vector section over $F(p)$. If $0 \leqslant \lambda \leqslant 1$, $\varphi, \psi_1, \dots, \psi_n \in F(p)$ and $\varphi = \psi_1 + \dots + \psi_n$ then for every x in M we have*

$$\langle y_{\lambda\varphi}, x\omega_{\lambda\varphi}\rangle_{\lambda\varphi} = \lambda \langle y_\varphi, x\omega_\varphi\rangle_\varphi$$

and

$$\langle y_\varphi, x\omega_\varphi\rangle_\varphi = \langle y_{\psi_1}, x\omega_{\psi_1}\rangle_{\psi_1} + \dots + \langle y_{\psi_n}, x\omega_{\psi_n}\rangle_{\psi_n} .$$

To motivate the next step of the proof, we note that for yp in Mp and $0 \leqslant \psi \leqslant \varphi$ in $F(p)$ we always have, for all x in M,

$$\psi(x^*y) = \langle y\omega_\psi, x\omega_\psi\rangle_\psi = \langle y\omega_\varphi, x\omega_{\psi\varphi}\rangle_\varphi = \langle y\omega_\varphi, T^*_{\psi\varphi}(x\omega_\psi)\rangle_\varphi = \langle T_{\psi\varphi}(y\omega_\varphi), x\omega_\psi\rangle_\psi .$$

LEMMA 2.10. *Let $(y_\varphi)_{\varphi\in F(p)}$ be an affine vector section over $F(p)$. Assume φ, ψ in $F(p)$ satisfy that $0 \leqslant \psi \leqslant \varphi$. Suppose there is a projection P in the commutant $\pi_\varphi(M)'$ of $\pi_\varphi(M)$ in $B(H_\varphi)$ such that $\psi(x) = \langle x\omega_\varphi, P\omega_\varphi\rangle_\varphi, \forall x \in M$. We have*

$$\langle y_\psi, x\omega_\psi\rangle_\psi = \langle y_\varphi, x\omega_{\psi\varphi}\rangle_\varphi, \quad \forall x \in M.$$

Proof. Write $\varphi = \psi + \rho$, where ρ in $F(p)$ is defined by

$$\rho(x) = \langle x\omega_\varphi, (1-P)\omega_\varphi\rangle_\varphi, \quad \forall x \in M.$$

Define two isometries R from H_ψ into H_φ and S from H_ρ into H_φ by setting

$$R(x\omega_\psi) = P(x\omega_\varphi) \quad \text{and} \quad S(x\omega_\rho) = (1-P)(x\omega_\varphi), \quad \forall x \in M.$$

Note that $RH_\psi = PH_\varphi$ and $SH_\rho = (1-P)H_\varphi$. Observe that for all x in M,

$$\langle y_\psi, x\omega_\psi\rangle_\psi = \langle Ry_\psi, R(x\omega_\psi)\rangle_\varphi = \langle Ry_\psi, xP\omega_\varphi\rangle_\varphi$$

and

$$\langle y_\rho, x\omega_\rho\rangle_\rho = \langle Sy_\rho, S(x\omega_\rho)\rangle_\varphi = \langle Sy_\rho, x(1-P)\omega_\varphi\rangle_\varphi .$$

By Lemma 2.9, for every x in M

$$\begin{aligned}\langle y_\varphi, x\omega_\varphi\rangle_\varphi &= \langle y_\psi, x\omega_\psi\rangle_\psi + \langle y_\rho, x\omega_\rho\rangle_\rho \\ &= \langle Ry_\psi, xP\omega_\varphi\rangle_\varphi + \langle Sy_\rho, x(1-P)\omega_\varphi\rangle_\varphi \\ &= \langle Ry_\psi + Sy_\rho, x\omega_\varphi\rangle_\varphi ,\end{aligned}$$

since $(1-P)Ry_\psi = PSy_\rho = 0$. Consequently, $y_\varphi = Ry_\psi + Sy_\rho$ and thus $Py_\varphi = Ry_\psi$. It is clear that $P\omega_\varphi = \omega_{\psi\varphi}$. Hence

$$\langle y_\psi, x\omega_\psi\rangle_\psi = \langle Ry_\psi, R(x\omega_\psi)\rangle_\varphi = \langle Py_\varphi, xP\omega_\varphi\rangle_\varphi = \langle y_\varphi, x\omega_{\psi\varphi}\rangle_\varphi . \quad \blacksquare$$

Proof of the second part of Theorem 2.6. Let $(y_\varphi)_{\varphi\in F(p)}$ be a bounded affine vector section over $F(p)$. We prove that for every φ and ψ in $F(p)$ such that $0 \leqslant \psi \leqslant \varphi$,

$$\langle y_\psi, x\omega_\psi\rangle_\psi = \langle y_\varphi, x\omega_{\psi\varphi}\rangle_\varphi , \quad \forall x \in M.$$

By the Radon–Nikodym theorem (see e.g. Sakai [16]), there is a T in $\pi_\varphi(M)'$, $0 \leqslant T \leqslant 1$, such that $\psi(x) = \langle x\omega_\varphi, T\omega_\varphi\rangle_\varphi , \forall x \in M$, i.e. $T\omega_\varphi = \omega_{\psi\varphi}$. By the spectral theorem for bounded self-adjoint Hilbert space operators, we can write

$$T = \int_0^1 \lambda \,\mathrm{d}E(\lambda),$$

where E is the projection-valued measure related to T. For $\varepsilon > 0$, there is a partition $\{\Delta_1, \ldots, \Delta_n\}$ of $[0,1]$ and $\lambda_1, \ldots, \lambda_n$ between 0 and 1 such that $0 \leqslant \sum \lambda_k E(\Delta_k) \leqslant T$ and $\|T - \sum \lambda_k E(\Delta_k)\| < \varepsilon$. Define ψ_k in $F(p)$ by

$$\psi_k(x) = \langle x\omega_\varphi, E(\Delta_k)\omega_\varphi\rangle_\varphi , \quad \forall x \in M, k = 1, \ldots, n.$$

It is equivalent to say that

$$E(\Delta_k)\omega_\varphi = \omega_{\psi_k\varphi}, \; k = 1, \ldots, n.$$

Since $E(\Delta_k) \in \pi_\varphi(M)'$, $k = 1, \dots, n$, we have, by Lemma 2.10,

$$\langle y_{\psi_k}, x\omega_{\psi_k}\rangle_{\psi_k} = \langle y_\varphi, x\omega_{\psi_k\varphi}\rangle_\varphi = \langle y_\varphi, xE(\Delta_k)\omega_\varphi\rangle_\varphi .$$

Let $\psi_0 = \sum \lambda_k \psi_k$. We have $0 \leqslant \psi_0 \leqslant \psi \leqslant \varphi$. Write $\psi = \psi_0 + \rho$. Note $\rho \in F(p)$ and

$$\begin{aligned} \|\rho\| = \|\psi\| - \|\psi_0\| &= \left\langle \omega_\varphi, (T - \sum \lambda_k E(\Delta_k))\omega_\varphi \right\rangle_\varphi \\ &\leqslant \|T - \sum \lambda_k E(\Delta_k)\| \, \|\varphi\| < \|\varphi\|\varepsilon. \end{aligned}$$

By Lemma 2.9,

$$\begin{aligned} \langle y_\psi, x\omega_\psi\rangle_\psi &= \langle y_{\psi_0}, x\omega_{\psi_0}\rangle_{\psi_0} + \langle y_\rho, x\omega_\rho\rangle_\rho \\ &= \sum_{k=1}^n \lambda_k \langle y_{\psi_k}, x\omega_{\psi_k}\rangle_{\psi_k} + \langle y_\rho, x\omega_\rho\rangle_\rho \\ &= \sum_{k=1}^n \lambda_k \langle y_\varphi, xE(\Delta_k)\omega_\varphi\rangle_\varphi + \langle y_\rho, x\omega_\rho\rangle_\rho \\ &= \left\langle y_\varphi, x\sum_{k=1}^n \lambda_k E(\Delta_k)\omega_\varphi \right\rangle_\varphi + \langle y_\rho, x\omega_\rho\rangle_\rho . \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\langle y_\psi, x\omega_\psi\rangle_\psi - \langle y_\varphi, x\omega_{\psi\varphi}\rangle_\varphi\right| \\ &\quad\leqslant \left| \left\langle y_\varphi, x\Big(T - \sum_{k=1}^n \lambda_k E(\Delta_k)\Big)\omega_\varphi \right\rangle_\varphi \right| + \left|\langle y_\rho, x\omega_\rho\rangle_\rho\right| \\ &\quad\leqslant \|y_\varphi\|_\varphi \, \|x\| \left\|T - \sum_{k=1}^n \lambda_k E(\Delta_k)\right\| \|\omega_\varphi\|_\varphi + \|y_\rho\|_\rho \, \|x\| \, \|\omega_\rho\|_\rho \\ &\quad< K\|x\| \, \|\varphi\|^{\frac12}\varepsilon + K\|x\| \, \|\varphi\|^{\frac12}\varepsilon^{\frac12}, \end{aligned}$$

where $K = \|y\|_\infty$ is the bound of y. Since ε is arbitrary, $\langle y_\psi, x\omega_\psi\rangle_\psi = \langle y_\varphi, x\omega_{\psi\varphi}\rangle_\varphi = \left\langle y_\varphi, T^*_{\psi\varphi}(x\omega_\psi)\right\rangle_\varphi$, $\forall x \in M$. In other words, $T_{\psi\varphi}y_\varphi = y_\psi$, as asserted. Thus $(y_\varphi)_\varphi$ is admissible. The fact that every admissible vector section is bounded and affine follows from the first part of the proof. ∎

3. REPRESENT C*-ALGEBRAS VIA CONTINUOUS ADMISSIBLE VECTOR SECTIONS

Let A be a C*-algebra and p the closed projection in A^{**} related to a closed left ideal L of A. Let $F(p) = \{\varphi \in A^* : \varphi \geqslant 0, \varphi(p) = \|\varphi\| \leqslant 1\}$. As a special case of Theorem 2.6, we have

THEOREM 3.1. *$A^{**}p$ ($\cong A^{**}/L^{**}$) is isometrically isomorphic to the Banach space of all admissible vector sections over $F(p)$, which consists exactly of all bounded affine vector sections over $F(p)$.*

It is natural to ask which admissible vector sections Ap contains. Analogous to the classical Kadison function representation (cf. [13]) one may guess Ap consists of all "continuous" affine vector sections over $F(p)$. The question is how we define continuity for the field $(F(p), \{H_\varphi\}_\varphi)$ of Hilbert spaces. Of course, all vector sections arising from Ap should be continuous.

Recall the notion of a continuous field of (complex) Hilbert spaces ([12], [9]). Let T be a Hausdorff space called the *base space*. For each t in T, let H_t be a (complex) Hilbert space, called the *fiber Hilbert space*. A *vector section* is a function x on T such that $x(t) \in H_t, \forall t \in T$. A *(full) continuous structure* for the field $(T, \{H_t\}_{t\in T})$ of Hilbert spaces is a linear space Γ of vector sections, called *continuous vector sections*, satisfying the conditions:

(i) $t \longmapsto \|x(t)\|_{H_t}$ is continuous on T for all x in Γ.

(ii) $\{x(t) : x \in \Gamma\}$ is norm dense in H_t for all t in T.

(iii) Let x be a vector section; if for any t in T and $\varepsilon > 0$ there exists an a in Γ such that $\|x(t) - a(t)\| < \varepsilon$ throughout a neighborhood of t then $x \in \Gamma$.

The triple $(T, \{H_t\}_t, \Gamma)$ is called *a continuous field of Hilbert spaces.*

A linear space X of vector sections which satisfies conditions (i) and (ii) defines a continuous structure $\Gamma(X)$, which is the set of all vector sections x satisfying the condition that

(iii)′ For any t in T and $\varepsilon > 0$ there exists an a in X such that $\|x(t) - a(t)\| < \varepsilon$ throughout a neighborhood of t.

It is easy to see that for a vector section $x = (x(t))_{t\in T}$, x is continuous (that is $x \in \Gamma(X)$) if and only if

1. $t \longmapsto \langle x(t), x(t)\rangle_{H_t}$ is continuous on T, and
2. $t \longmapsto \langle x(t), y(t)\rangle_{H_t}$ is continuous on T, $\forall y \in X$.

A vector section x is said to be *bounded* if $\|x\|_\infty = \sup_{t\in T} \|x(t)\|_{H_t} < \infty$. x is said to be *weakly continuous* in $(T, \{H_t\}_t, \Gamma(X))$ if the scalar function $t \longmapsto \langle x(t), y(t)\rangle_{H_t}$ is continuous on T for every y in $\Gamma(X)$. If x is bounded then $\Gamma(X)$ can be replaced by X in the above condition. A weakly continuous vector section x is continuous if and only if $t \longmapsto \langle x(t), x(t)\rangle_{H_t}$ is continuous on T (cf. [10]).

Now, let us point out that if $ap \in Ap$ then $\varphi \longmapsto \|a\omega_\varphi\|_\varphi$ is continuous on $F(p)$. It is also clear that $A\omega_\varphi$ is norm dense in H_φ for each φ in $F(p)$. Therefore, the set X of vector sections arising from Ap defines a continuous structure $\Gamma(X)$ which we shall henceforth write as $\Gamma(Ap)$. A vector section $(x_\varphi)_{\varphi\in F(p)}$ in $(F(p), \{H_\varphi\}_\varphi, \Gamma(Ap))$ is continuous if and only if for any $\varepsilon > 0$ and φ in $F(p)$ there exist an a in A and a neighborhood V_φ of φ in $F(p)$ such that

$$\|x_\psi - a\omega_\psi\|_\psi < \varepsilon, \quad \forall\psi \in V_\varphi.$$

In this context, a bounded vector section $(x_\varphi)_{\varphi\in F(p)}$ is *weakly continuous* if $\varphi \longmapsto \langle x_\varphi, a\omega_\varphi\rangle_\varphi$ is continuous on $F(p), \forall a \in A$. A weakly continuous vector section $(x_\varphi)_{\varphi\in F(p)}$ is *continuous* if $\varphi \longmapsto \langle x_\varphi, x_\varphi\rangle_\varphi$ is continuous on $F(p)$. Moreover, a vector section $(x_\varphi)_{\varphi\in F(p)}$ is continuous if and only if $\varphi \longmapsto \langle x_\varphi, y_\varphi\rangle_\varphi$ is continuous on $F(p)$ for all weakly continuous vector sections $(y_\varphi)_{\varphi\in F(p)}$. In fact, $(x_\varphi)_{\varphi\in F(p)}$ itself must be weakly continuous in this case, and thus $\varphi \longmapsto \langle x_\varphi, x_\varphi\rangle_\varphi$ is continuous on $F(p)$, too. It is plain that continuous vector sections need not arise from elements of Ap. However, we have

THEOREM 3.2. *Ap ($\cong A/L$) is isometrically isomorphic to the Banach space of all continuous admissible (= continuous and affine) vector sections of the continuous field of Hilbert spaces $(F(p), \{H_\varphi\}_\varphi, \Gamma(A))$.*

Proof. We adopt the notations used in the last section with M replaced by A^{**}. Let $f = (f(\varphi))_\varphi$ be a continuous admissible vector section over $F(p)$. In view of Theorem 2.6, it suffices to show that whenever $\phi_\lambda \longrightarrow \phi$ in the weak* topology of the polar $L^\circ = (A/L)^*$ of L in A^*, $\tilde{f}(\phi_\lambda) \longrightarrow \tilde{f}(\phi)$. By the Krein–Smulian theorem, we need only to check this for bounded nets. So assume $\|\phi_\lambda\| \leqslant 1$. Note that $L^\circ = \{\psi \in A^* : \psi = \psi(\cdot p)\}$ and hence if $\psi \in L^\circ$ and $\|\psi\| \leqslant 1$ then $|\psi| \in F(p)$. Since $F(p)$ is weak* compact, there is a subnet ϕ_k of ϕ_λ such that $\varphi_k = |\phi_k|$ converges to an element φ of $F(p)$ in the weak* topology (note that φ is not necessarily $|\phi|$, see e.g. [11]). Now for any a in A the inequalities

$$|\phi_k(a)|^2 \leqslant \|\varphi_k\|\varphi_k(a^*a), \quad \forall k,$$

imply

$$|\phi(a)|^2 \leqslant K\varphi(a^*a).$$

Here $K = \sup_k \|\varphi_k\| \leqslant 1$. Therefore, ϕ is observable at φ and thus

$$\tilde{f}(\phi) = \langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi .$$

Let $\varepsilon > 0$. Since f is a continuous vector section in $(F(p), \{H_\varphi\}_\varphi, \Gamma(Ap))$, there exist a neighborhood U_φ of φ in $F(p)$ and an a in A such that $\|f(\psi) - a\omega_\psi\|_\psi < \frac{\varepsilon}{3}$ in U_φ. Thus

$$\|f(\varphi) - a\omega_\varphi\|_\varphi < \frac{\varepsilon}{3}$$

and

$$\|f(\varphi_k) - a\omega_{\varphi_k}\|_{\varphi_k} < \frac{\varepsilon}{3},$$

eventually. Also,

$$|\phi(a) - \phi_k(a)| < \frac{\varepsilon}{3}$$

eventually. So for k sufficiently large,

$$\begin{aligned}
\left|\tilde{f}(\phi) - \tilde{f}(\phi_k)\right| &= \left|\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi - \langle f(\varphi_k), \omega_{\phi_k\varphi_k}\rangle_{\varphi_k}\right| \\
&\leqslant \left|\langle f(\varphi), \omega_{\phi\varphi}\rangle_\varphi - \langle a\omega_\varphi, \omega_{\phi\varphi}\rangle_\varphi\right| + \left|\langle a\omega_\varphi, \omega_{\phi\varphi}\rangle_\varphi - \langle a\omega_{\varphi_k}, \omega_{\phi_k\varphi_k}\rangle_{\varphi_k}\right| \\
&\quad + \left|\langle a\omega_{\varphi_k}, \omega_{\phi_k\varphi_k}\rangle_{\varphi_k} - \langle f(\varphi_k), \omega_{\phi_k\varphi_k}\rangle_{\varphi_k}\right| \\
&\leqslant \|f(\varphi) - a\omega_\varphi\|_\varphi + |\phi(a) - \phi_k(a)| + \|a\omega_{\varphi_k} - f(\varphi_k)\|_{\varphi_k} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Consequently, $\tilde{f}(\phi_k) \longrightarrow \tilde{f}(\phi)$. Since the same argument can be applied to any subnet of ϕ_λ, we have $\tilde{f}(\phi_\lambda) \longrightarrow \tilde{f}(\phi)$. Hence f defines an element in Ap, as asserted. ∎

4. CONTINUITY AND WEAK CONTINUITY

From now on, a continuous admissible (resp. admissible) vector section is considered as an element of Ap (resp. $A^{**}p$). Denote by $\mathcal{W}_p$ the family of all weakly continuous admissible vector sections over $F(p)$.

COROLLARY 4.1. *Let $xp \in A^{**}p$.*

(i) *$px^*xp \in pAp$ and $pa^*xp \in pAp$, $\forall ap \in Ap \Leftrightarrow xp \in Ap$.*

(ii) *$pa^*xp \in pAp$, $\forall ap \in Ap \Leftrightarrow xp \in \mathcal{W}_p$.*

(iii) *$pw^*xp \in pAp$, $\forall wp \in \mathcal{W}_p \Leftrightarrow xp \in Ap$.*

Proof. It follows from ([7], 3.5) that for x, y in A^{**}, $\varphi \longmapsto \langle x\omega_\varphi, y\omega_\varphi\rangle_\varphi = \varphi(y^*x)$ is continuous on $F(p)$ if and only if $py^*xp \in pAp$. Recalling the discussion of fields of Hilbert spaces in Section 3, we see that (ii) and (iii) are immediate whilst (i) is just a restatement of Theorem 3.2. ∎

In case $p = 1$, an admissible vector section xp is weakly continuous if and only if $x \in RM(A)$, the set of right multipliers of A (cf. [4]). In general, we have $RM(A)p \subseteq \mathcal{W}_p$. To investigate what $\mathcal{W}_p$ contains, we quote a result of Brown ([7], 3.9):

THEOREM 4.2. *Let A be a σ-unital C^*-algebra and p a closed projection in A^{**}. Let xp in $A^{**}p$ be such that $\|xp\| = 1$ and $Axp \subseteq Ap$. Then there is a right multiplier r of A in A^{**} such that $\|r\| = 1$ and $xp = rp$.*

COROLLARY 4.3. *If A is a σ-unital C^*-algebra and p is a closed, central projection in A^{**} then $\mathcal{W}_p = RM(A)p$.*

COROLLARY 4.4. *Let A be a σ-unital C^*-algebra and p a closed projection in A^{**}. For an xp in $\mathcal{W}_p$,*

$$xp \in RM(A)p \Leftrightarrow px^*Axp \subseteq pAp.$$

Proof. One direction is obvious. For the other one, we assume $xp \notin RM(A)p$. Then there is an a in A such that $axp \notin Ap$ by Theorem 4.2. Since axp is also a weakly continuous vector section, we must have $px^*a^*axp \notin pAp$. Hence, px^*Axp is not contained in pAp. ∎

COROLLARY 4.5. *Let A be a σ-unital C^*-algebra and p a closed projection in A^{**}.*

(i) *If $xp \in \mathcal{W}_p$ and $xp = pxp$ then $xp \in RM(A)p$.*

(ii) *If A is simple and $p \in M(A)$ then $\mathcal{W}_p = RM(A)p$.*

Proof. (i) We check the condition $px^*Axp \subseteq pAp$. In fact,

$$\begin{aligned} px^*Axp &= px^*Apxp \subseteq pApxp, \quad \text{since } xp \in \mathcal{W}_p, \\ &= pAxp \subseteq pAp, \qquad \text{again since } xp \in \mathcal{W}_p. \end{aligned}$$

(ii) Since ApA is an ideal of A and A is simple, either $ApA = \{0\}$ or the norm closure $\overline{ApA}$ of ApA coincides with A. But $ApA = \{0\}$ implies $p = 0$. The assertion becomes trivial in this case. So assume $\overline{ApA} = A$. Now if $xp \in \mathcal{W}_p$, we have

$$px^*Axp = px^*\overline{ApA}xp \subseteq \overline{pApAp} \subseteq pAp.$$

The proof is complete since it is always true that $RM(A)p \subseteq \mathcal{W}_p$. ∎

In the following we present an example to show that the conclusions of Corollary 4.5 can fail if the hypothesis in (i) or (ii) is not fulfilled.

EXAMPLE 4.6. Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Let p be the projection of H onto span $\{e_1, e_3, e_5, \ldots\}$ and $A = C^*(\mathcal{K}, 1-p)$, the C*-subalgebra of $B(H)$ generated by $\mathcal{K}$, the C*-subalgebra of all compact operators on H, and $1-p$. Then the separable (hence σ-unital) C*-algebra A is given by

$$A = \{T + \lambda(1-p) : T \in \mathcal{K}, \lambda \in \mathbf{C}\}$$

and A^{**} can be described as

$$A^{**} = B(H) \oplus \mathbf{C}(1-p).$$

When A^{**} is viewed in this way, the embedding of A into A^{**} is given by

$$T + \lambda(1-p) \longrightarrow (T + \lambda(1-p), \lambda(1-p)).$$

Identify p with $p \oplus 0$ in A^{**}. Then $p \in M(A)$. Note that A is not simple and thus Corollary 4.5 (ii) does not apply. It is easy to see that $Ap = \mathcal{K}p$, $pAp = p\mathcal{K}p$ and $\mathcal{W}_p = B(H)p$. On the other hand,

$$RM(A) = \{(K + \lambda(1-p) + pS, \lambda(1-p)) : K \in \mathcal{K}, S \in B(H) \text{ and } \lambda \in \mathbf{C}\}.$$

Hence $RM(A)p = \mathcal{K}p + pB(H)p$. It is clear that $\mathcal{W}_p \neq RM(A)p$. For example, if T is the unilateral shift, i.e. $Te_n = e_{n+1}, n = 1, 2, \ldots$ then $Tp \in \mathcal{W}_p$ but $Tp \notin RM(A)p$ (since $(1-p)Tp = Tp \notin Ap$). We also note that $Tp \neq pTp = 0$ and thus Corollary 4.5 (i) does not apply, either. ∎

5. COMPARISON WITH TAKESAKI DUALITY THEOREM

Let A be a C*-algebra. Let H be a Hilbert space of sufficiently large infinite dimension such that every cyclic representation of A is unitarily equivalent to a cyclic representation of A on H. Let p_π be the projection of H onto $H_\pi = \overline{\pi(A)H}^{\|\cdot\|}$ for each π in the set Rep (A, H) of all representations of A on H. For each partial isometry u in $B(H)$ and π in Rep (A, H) such that $u^*u \geqslant p_\pi$, we denote by π^u the representation $u\pi u^*$, i.e. $\pi^u(a) = u\pi(a)u^*, \forall a \in A$. We equip Rep (A, H) the point strong operator topology (PSOT):

$$\pi_\lambda \xrightarrow{\text{PSOT}} \pi \text{ in Rep}(A, H) \text{ if } \pi_\lambda(a)h \xrightarrow{\|\cdot\|} \pi(a)h \text{ in } H, \quad \forall a \in A, \forall h \in H.$$

DEFINITION 5.1. ([18], [6]). A function $T : \text{Rep}(A, H) \longmapsto B(H)$ is said to be a *TB-admissible operator field* if the following conditions are satisfied:

(TB_1) $\|T\| := \sup\{\|T(\pi)\| : \pi \in \mathrm{Rep}(A, H)\} < \infty$.

(TB_2) $T(\pi) = p_\pi T(\pi) = T(\pi)p_\pi, \forall \pi \in \mathrm{Rep}(A, H)$.

(TB_3) $T(\pi + \pi') = T(\pi) + T(\pi')$ whenever $\pi, \pi' \in \mathrm{Rep}(A, H)$ such that $H_\pi \perp H_{\pi'}$.

(TB_4) $T(\pi_u) = uT(\pi)u^*$ whenever $\pi \in \mathrm{Rep}(A, H)$ and u is a partial isometry in $B(H)$ such that $u^*u \geqslant p_\pi$.

In [6], Bichteler extended Takesaki duality theorem ([18]) for separable C*-algebras A to the general form:

THEOREM 5.2. *The set of all TB-admissible operator fields is isometrically isomorphic to A^{**} in the sense that for each TB-admissible operator field $T = (T(\pi))_\pi$ there is a t in A^{**} such that*

$$\pi(t) = T(\pi), \quad \forall \pi \in \mathrm{Rep}(A, H).$$

(Here π is understood to be (uniquely) extended to a $\sigma(A^{**}, A^*)$-continuous representation (again denoted by π) of A^{**} on H.) Moreover, $t \in A$ if and only if T is PSOT–SOT continuous in the sense that if $\pi_\lambda \xrightarrow{\text{PSOT}} \pi$ in $\mathrm{Rep}(A, H)$ *then $T(\pi_\lambda) \longrightarrow T(\pi)$ in $B(H)$ with the strong operator topology (SOT).*

A similar argument as in the proof of Proposition 2.5 gives

PROPOSITION 5.3. *Every function $T : \mathrm{Rep}(A, H) \longrightarrow B(H)$ which satisfies $(TB_2), (TB_3)$ and (TB_4) is TB-admissible. In other words, (TB_1) is redundant.*

DEFINITION 5.4. Let $\pi \in \mathrm{Rep}(A, H)$ and $h \in H$ with $\|h\| \leqslant 1$. Let φ in $Q(A)$ be defined by $\varphi := \langle \pi(\cdot)h, h\rangle_H$. We define an isometry $U^{\varphi}_{\pi,h}$ from H_φ into H by

$$U^{\varphi}_{\pi,h}(a\omega_\varphi) := \pi(a)h, \quad \forall a \in A,$$

where $a\omega_\varphi$ denotes $\pi_\varphi(a)\omega_\varphi$ in the GNS representation $(\pi_\varphi, H_\varphi, \omega_\varphi)$ induced by φ, as before.

Some lengthy computation and straightforward reasoning will bring us the following connection of Takesaki duality theorem and our representation theory developed in earlier sections in this paper.

THEOREM 5.5. ([21]). *There exists an isometrical isomorphism from the Banach space of all admissible vector sections $x = (x_\varphi)_\varphi$ over $Q(A)$ onto the Banach space of all TB-admissible operator fields $T = (T(\pi))_\pi$ such that the relation*

$$U^{\varphi}_{\pi,h} x_\varphi = T(\pi)h$$

is satisfied whenever $\varphi = \langle\pi(\cdot)h, h\rangle_H$ *for some* π *in* $\text{Rep}(A, H)$ *and* h *in* H *with* $\|h\| \leqslant 1$. *Moreover,* $T = (T(\pi))_\pi$ *is a continuous TB-admissible operator field if and only if* $x = (x_\varphi)_\varphi$ *is a continuous admissible vector section.*

Roughly speaking, Takesaki [18] represented x in A^{**} as a field of operators (matrices) $\pi(x)$'s and we represent x as a field of vectors (columns) $x\omega_\varphi$'s. The general version of our representation of $A^{**}p$ is to pay attention only on those columns $x\omega_\varphi$'s of the matrix $\pi(x)$ in the range of the closed projection p (i.e. φ is supported by p, or equivalently, $p\omega_\varphi = \omega_\varphi$). Moreover, xp comes from Ap if and only if xp has continuous coordinates $\varphi \longmapsto (x\omega_\varphi, a\omega_\varphi)_\varphi, \forall a \in A$, and a continuous norm $\varphi \longmapsto (x\omega_\varphi, x\omega_\varphi)_\varphi^{1/2}$ over $F(p)$. In this sense, our results extend Takesaki duality theorem.

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NGAI-CHING WONG
Department of Applied Mathematics
National Sun Yat-sen University
Kao-hsiung, 80424, Taiwan, R.O.C.

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