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BOOK REVIEW:

Pseudo-differential boundary value problems, conical singularities and asymptotics, **Bert-Wolfgang Schulze**, Mathematical Topics, vol. 4, Akademie Verlag, Berlin, 1994, 580 pag., ISBN 3-05-501597-5.

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AMS SUBJECT CLASSIFICATION: 35S15.

This is a second book of Bert-Wolfgang Schulze about pseudo-differential operators on manifolds with singularities. It contains two chapters. The first of them is a survey on pseudo-differential operators on cones and on manifolds with conical singularities. The proofs are omitted or only sketched. The second chapter contains a theory of pseudo-differential boundary value problems as a special case of an edge theory. This theory generalizes the Boutet de Monvel's calculus (from his paper "Boundary problem for pseudo-differential operators" appeared in 1971 in Acta Mathematica) and completes some results of Višik and of Eskin.

Roughly speaking, the basic idea is to construct algebras of pseudo-differential operators adapted to the problem we study. Sobolev spaces in which these operators are continuous are also introduced. Next the behaviour of the symbolic calculus with respect to adjunction, composition of two operators, push-forward by a diffeomorphism or conjugation by a power of the "normal variable" is investigated. It is defined a notion of ellipticity. The ellipticity must be equivalent with the Fredholm property of the operator. It is proved that an elliptic operator admits a parametrix. For simplicity, only the case of classical operators is taken into consideration.

We present now in detail the contents of each chapter.

In the first chapter, *Mellin pseudo-differential operators*, there are introduced various operator algebras on cones $X^\wedge = X \times \mathbf{R}_+$ (X a smooth compact manifold) and on manifolds $\mathbf{B}$ with boundary $\partial\mathbf{B} = X$. To each manifold with conical singularities B we can associate a manifold with boundary $\mathbf{B}$, called the stretched manifold belonging to B. The asymptotic behaviour on $X^\wedge$ when $t \to 0$ ($t \in \mathbf{R}_+$) is important for applications to manifolds with conical singularities. But B. Schulze also considers the behaviour when t goes to infinity. Let us remark that the Mellin transform M is adapted for the study of operators on $\mathbf{R}_+$ (M is an isomorphism from $L^2(\mathbf{R}_+)$ on $L^2(\Gamma_{\frac{1}{2}})$, $\Gamma_{\frac{1}{2}} = \{z \in \mathbf{C}; \operatorname{Re} z = \frac{1}{2}\}$) and for the study of operators of Fuchs type ($M\left(-t\frac{\partial}{\partial t}\right) = zM$).

This chapter contains four sections:

1.1. Spaces with discrete asymptotics. The Sobolev spaces constructed in this section allow the description of the asymptotic behaviour of the solutions of elliptic equations in the neighbourhood of conical singularities. The regularizing operators in the present context are the so called Green operators (which contain the Green operators of Boutet de Monvel). This section ends with a first approach to the theory of Mellin symbols and Mellin operators. The Mellin transform plays now (for the variable t) the some role as the Fourier transform does in the case of ordinary pseudo-differential operators. For example, if a is a symbol on $\mathbf{R}_+ \times \mathbf{R}_+ \times \Gamma_{\frac{1}{2}}$ then we can define an operator on $\mathbf{R}_+$ by the formula

$$\operatorname{op}_M(a)u = M^{-1}_{z\to t} M_{t'\to z}\, a(t,t',z)u(t').$$

1.2. The cone algebra with discrete asymptotics. This is an algebra of pseudo-differential operators which contains the operators of the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t) \left(-t\frac{\partial}{\partial t}\right)^j$$

(and their parametrices if they are elliptic). The principal symbol of such operators has two components: an interior symbol, which is the ordinary symbol of a pseudo-differential operator and a conormal symbol defined through the Mellin transform. The two symbols satisfy some compatibility relation. This "double" principal symbols allows us to introduce a notion of ellipticity equivalent with the Fredholm property. In the case of operators on $X \times \mathbf{R}_+$ a third symbol, the exit symbol, is needed.

1.3. Mellin pseudo-differential operators with $L^\mu(X)$-valued symbols. In this section it is described a larger class of operators, first on $X^\wedge$ and next on $\mathbf{B}$. For this we need the concept of $L^\mu(X)$-valued symbol. The kernel of such operators in studied. The operators are defined by means of an amplitude which depends on t and t'. But we can find a complete symbol (which depends only on t) of such an operator.

If $\widetilde{A}(\widetilde{x}, D_{\widetilde{X}}) = \sum_{|\alpha|\leqslant\mu} a_\alpha(\widetilde{x}) D^\alpha_{\widetilde{X}}$ is a differential operator on $\mathbf{R}^{n+1}$, it can be represented in polar coordinates in the form $A = t^{-\mu}\mathrm{op}_M(h)$, where $h(t,z)$ is a symbol with values in $L^\mu(S^n)$. It is proved that every pseudo-differential operator on $\mathbf{R}^{n+1}$ admits such a representation, modulo a smoothing operator. That is, it is established a Mellin convention.

1.4. The cone algebra with continuous asymptotics. The continuous asymptotic types are useful for the boundary problems. In this section the results of the previous sections are extended to this case.

The second chapter, *Pseudo-differential boundary value problems,* has two sections:

2.1. Pseudo-differential operators on the half-axis. If $a(\tau)$ is a symbol on $\mathbf{R}$ then we can define an operator on $\mathbf{R}_+$ by the formula $\mathrm{op}_\psi(a) = r^+\mathrm{op}\,(a)e^+$, where $\mathrm{op}\,(a)$ is the ordinary pseudo-differential operator on $\mathbf{R}$ of symbol a, r^+ is the restriction operator from $\mathbf{R}$ to $\mathbf{R}_+$ and e^+ is the adjoint of r^+. In the first two subsections it is proved that these operators can be represented as Mellin pseudo-differential operators. Symbols of the form $a(t,\tau)$ which do not depend on t for t large are also considered. In this way it is established a link with previous works of Višik, Eskin, Rempel and Schulze. Next it is described a Mellin convention in a parameter dependent form (the parameters are $(y,\eta) \in \Omega \times \mathbf{R}^q$). This convention is the main ingredient in the boundary symbolic calculus. We need also a calculus with operator valued amplitude functions.

A boundary symbol is a matrix of operators of the form

$$\begin{pmatrix} a+m+g & k \\ b & q \end{pmatrix} (y, y', \eta),$$

where k (the trace), b (a potential amplitude function), q (an ordinary amplitude function) are finite dimensional, while a (a pseudo-differential operator), m (a smoothing Mellin operator) and g (a Green operator) operate on $\mathbf{R}_+$. In the case of a "classical" boundary value problem: $Au = f$ on $\overset{\circ}{\mathbf{B}}$, $Tu = \varphi$ on ∂B, the parametrix will have a symbol which has the transmission property. But for a mixed elliptic problem we have to use also symbols with the transmission property violated. These more general symbols are studied by B. Schulze. By specializing his theory we can recover the symbolic calculus with transmission property of Boutet de Monvel's algebra.

This section ends with a study of operator $\mathrm{op}_\psi(a)$ under the aspect of the cone algebra with discrete asymptotics and with some remarks on transmission problems.

2.2. Boundary value problems. The symbolic calculus developed in the previous section is applied to boundary value problems.

After each section the author comments the notions he introduces and proposes possible developments of the theory. In fact, this book is intended as a part of a program of a calculus of pseudo-differential operators on manifolds with singularities.

The book is addressed to all those interested in partial differential equations and related fields.

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