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OPERATOR PROPERTIES OF MODULE MAPS

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ABSTRACT. The aim of this paper is to study operator theoretical properties of module maps, for example, the spectral theory or closed range results. Developing spectral theory in Banach algebra $\mathcal{C}(\Omega, B(X))$, where Ω is a compact Hausdorff space and X is a Banach space, or in other Banach algebras of $B(X)$-valued functions is a part of this program.

KEYWORDS: *Module maps, Banach algebras, spectral theory.*

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INTRODUCTION

Throughout this paper Ω is a compact Hausdorff space and X is a Banach space. In this paper we study module maps acting on certain modules of X-valued functions on Ω . To give a specific example, let $\mathcal{C} \equiv \mathcal{C}(\Omega, X)$ be the space of all continuous X-valued maps on Ω with the sup-norm

$$\|f\|_\Omega \equiv \sup\{\|f(\omega)\| : \omega \in \Omega\} \qquad (f \in \mathcal{C})$$

Let $C(\Omega)$ denote the usual Banach algebra of all continuous complex-valued functions on Ω. Then $\mathcal{C}$ is a module over $C(\Omega)$ where for all $\omega \in \Omega$

$$(gf)(\omega) = g(\omega)f(\omega) \qquad (g \in C(\Omega),\ f \in \mathcal{C}).$$

A linear map T with domain $D(T) \subseteq \mathcal{C}$, taking values in $\mathcal{C}$ is a module map if $D(T)$ is a submodule of $\mathcal{C}$ and

$$T(gf) = gT(f) \qquad (f \in D(T),\ g \in C(\Omega)).$$

We denote the space of all everywhere defined continuous module maps of $\mathcal{C}$ into $\mathcal{C}$ by $M(\mathcal{C})$. Now let $B(X)$ be the algebra of all bounded linear operators on X. If $T = \{T(\omega)\} \in \mathcal{C}(\Omega, B(X))$, then T determines a module map in $M(\mathcal{C})$ by setting for $f \in \mathcal{C}$

$$T(f)(\omega) = T(\omega)(f(\omega)) \qquad (\omega \in \Omega).$$

The Banach algebra of operator-valued functions $\mathcal{C}(\Omega, B(X))$ is an important subalgebra of $M(\mathcal{C})$. We also consider some interesting modules of holomorphic X--valued functions and the module maps on these spaces.

The aim of this paper is to study the operator theoretical properties of module maps, for example, spectral theory or closed range results. Developing spectral theory in the Banach algebra $\mathcal{C}(\Omega, B(X))$ and other Banach algebras of $B(X)$--valued functions is a part of this program.

There is ample motivation for studying module maps. Such maps are determined by families of operators, and families of operators play a role in many different parts of operator theory, for example, pencils of operators [6], perturbation theory of operators [10], and equations depending on a parameter ([12], Chapter 8, [6]).

1. MODULE MAPS

In this section we consider some basic properties of module maps on certain modules of X-valued functions on Ω. The notation needed is as listed below:

The space Ω.	The module.	The algebra.
A general compact Hausdorff space.	$\mathcal{C} = \mathcal{C}(\Omega, X)$	$C = C(\Omega)$
The closed unit disk in C.	$\mathcal{A} = \mathcal{A}(\Omega, X)$	$A = A(\Omega)$
The closed unit disk in C.	$\mathcal{A}_1 = \mathcal{A}_1(\Omega, X)$	$A_1 = A_1(\Omega)$
A compact interval in R.	$\mathcal{C}^n = \mathcal{C}^n(\Omega, X)$	$C^n = C^n(\Omega)$.

Here $\mathcal{A}(\Omega, X)$ is the space of all $f \in \mathcal{C}(\Omega, X)$ such that f is holomorphic on the interior of Ω; $\mathcal{A}_1(\Omega, X)$ is the space of all $f \in \mathcal{A}(\Omega, X)$ such that there exists a sequence $\{x_n\}_{n \geqslant 0} \subseteq X$ with $\sum_{n=0}^{\infty} \|x_n\|$ convergent such that $f(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$ for all $\omega \in \Omega$; $\mathcal{C}^n(\Omega, X)$ is the space of all $f \in \mathcal{C}(\Omega, X)$ such that f is n-times continuously differentiable on Ω. The corresponding algebras are the commutative Banach algebras which result in the case $X = C$. The complete norms involved are:

$$\|f\|_\Omega = \sup\{\|f(\omega)\| : \omega \in \Omega\} \quad \text{for} \quad f \in \mathcal{A};$$

$$\|f\|_1 = \sum_{n=0}^{\infty} \|x_n\| \quad \text{where} \quad f(\omega) = \sum_{n=0}^{\infty} x_n \omega^n \quad \text{for} \quad f \in \mathcal{A}_1;$$

$$\|f\|_{\Omega,n} = \sum_{k=0}^{\infty} (k!)^{-1} \|f^{(k)}\|_{\Omega} \text{ for } f \in C^n (\text{here } f^{(k)} \text{ denotes the } k\text{-th derivative of } f).$$

NOTE 1.1. We assume throughout that the notation $\mathcal{A}(\Omega, X)$ and $\mathcal{A}_1(\Omega, X)$ implies automatically that Ω is the closed unit disk. In fact, the results proved concerning these modules (or algebras, when X is an algebra) hold when Ω is any closed disk. We illustrate this in the case of $\mathcal{A}_1$.

Fix $\lambda_0 \in C$, $R > 0$, and let $\Gamma = \{\lambda \in C : |\lambda - \lambda_0| \leqslant R\}$. Let $\mathcal{A}_1(\Gamma, X)$ be the space of all functions $f : \Gamma \to X$ such that $\exists \{x_k\} \subseteq X$ with

$$f(\omega) = \sum_{k=0}^{\infty} x_k (\omega - \lambda_0)^k \quad (\omega \in \Gamma),$$

and

$$\|f\|_1' = \sum_{k=0}^{\infty} \|x_k\| R^k < \infty.$$

Define $\varphi : \Omega \to \Gamma$ (Ω the closed unit disk) by $\varphi(\omega) = R\omega + \lambda_0$. Define $W : \mathcal{A}_1(\Gamma, X) \to \mathcal{A}_1(\Omega, X)$ for $f \in \mathcal{A}_1(\Gamma, X)$ by

$$W(f)(\omega) \equiv f(\varphi(\omega)) \qquad (\omega \in \Omega).$$

If $f(\omega) = \sum_{k=0}^{\infty} x_k (\omega - \lambda_0)^k$, $\sum_{k=0}^{\infty} \|x_k\| R^k < \infty$, then

$$W(f)(\omega) = \sum_{k=0}^{\infty} (x_k R^k) \omega^k \qquad (\omega \in \Omega),$$

so $W(f) \in \mathcal{A}_1(\Omega, X)$ with

$$\|W(f)\|_1 = \sum_{k=0}^{\infty} \|x_k\| R^k = \|f\|_1'.$$

Thus, W is a linear isometry mapping $\mathcal{A}_1(\Gamma, X)$ onto $\mathcal{A}_1(\Omega, X)$. Results concerning $\mathcal{A}_1(\Omega, X)$ can be transferred to $\mathcal{A}_1(\Gamma, X)$ using W and W^{-1}.

At times we consider general modules $\mathcal{M}$ which are Banach modules over a commutative Banach algebra D. In this general case we call $T \in B(\mathcal{M})$ (the bounded linear maps on $\mathcal{M}$) a module map if

$$T(gf) = gT(f) \qquad (f \in \mathcal{M}, \; g \in D).$$

Let $M(\mathcal{M})$ denote the algebra of all module maps in $B(\mathcal{M})$.

PROPOSITION 1.2.
(i) *$M(\mathcal{M})$ is strongly closed in $B(\mathcal{M})$.*
(ii) *$M(\mathcal{M})$ is inverse closed in $B(\mathcal{M})$.*

Proof. To prove (i) assume $\{T_n\} \subseteq M(\mathcal{M})$ and $T_n \to T \in B(\mathcal{M})$ strongly on $\mathcal{M}$. For $f \in \mathcal{M}$, $g \in D$,

$$T(gf) \leftarrow T_n(gf) = gT_n(f) \to gT(f).$$

To verify (ii), assume $T \in M(\mathcal{M})$ and $T^{-1} \in B(\mathcal{M})$. For $f \in \mathcal{M}$ and $g \in D$, let $h = T^{-1}(f)$. Then $T(h) = f$ and $T(gh) = gT(h) = gf$. Therefore $T^{-1}(gf) = gh = gT^{-1}(f)$. It follows that $T^{-1} \in M(\mathcal{M})$. ∎

In general, module maps are determined by a family of operators, $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$. Any such family determines a map defined on functions from Ω into X in the following canonical way.

DEFINITION 1.3. Let $\{T(\omega)\}_{\omega\in\Omega}$ be a family of operators in $B(X)$. For $f : \Omega \to X$ define

$$T(f)(\omega) = T(\omega)(f(\omega)) \qquad (\omega \in \Omega).$$

In what follows we always use this definition to determine how a family of operators acts on a X-valued function.

THEOREM 1.4. *Let $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$. This family determines a module map on* (a) $\mathcal{C}(\Omega, X)$*;* (b) $\mathcal{C}^n(\Omega, X)$*;* (c) $\mathcal{A}(\Omega, X)$*; or* (d) $\mathcal{A}_1(\Omega, X)$*, if and only if, for every $x \in X$, $T(\omega)x$ is in the space in* (a), (b), (c), *or* (d), *respectively.*

Proof. The "only if" direction is immediate since for every $x \in X$, the constant function

$$c_x(\omega) = x \qquad (\omega \in \Omega)$$

is in all four of the spaces listed.

Now assume $T(\omega)x \in \mathcal{C} = \mathcal{C}(\Omega, X)$ for all $x \in X$. By the uniform boundedness principle $\exists\, M > 0$ such that

$$\|T(\omega)\| \leqslant M \quad \text{for all} \quad \omega \in \Omega.$$

Let $\{\omega_\lambda\}_{\lambda\in\Lambda}$ be a net in Ω with $\lim\limits_{\Lambda} \omega_\lambda = \omega_\circ$. Fix $f \in \mathcal{C}$.

$$\begin{aligned}\|T(f)(\omega_\lambda) - T(f)(\omega_\circ)\| &= \|T(\omega_\lambda)(f(\omega_\lambda)) - T(\omega)_\circ(f(\omega_\circ))\| \\ &\leqslant \|T(\omega_\lambda)(f(\omega_\lambda) - f(\omega_\circ))\| + \|(T(\omega_\lambda) - T(\omega_\circ))(f(\omega_\circ))\| \\ &\leqslant M\|f(\omega_\lambda) - f(\omega_\circ)\| + \|(T(\omega_\lambda) - T(\omega_\circ))(f(\omega_\circ))\|.\end{aligned}$$

Thus, $T(f) \in \mathcal{C}$.

Next, assume that $T(\omega)x \in \mathcal{A} = \mathcal{A}(\Omega, X)$ for all $x \in X$. Fix $f \in \mathcal{A}$. By the argument above, $T(f) \in \mathcal{C}$. Let ω be a given point in the interior of Ω. For any γ such that $\omega + \gamma \in \mathrm{int}(\Omega)$, $\gamma \neq 0$,

$$\frac{T(\omega+\gamma)(f(\omega+\gamma)) - T(\omega)(f(\omega))}{\gamma} = T(\omega+\gamma)\left(\frac{f(\omega+\gamma)-f(\omega)}{\gamma}\right) + \left(\frac{T(\omega+\gamma)-T(\omega)}{\gamma}\right)(f(\omega)).$$

Thus, letting $\gamma \to 0$, we have

$$[T(\omega)(f(\omega))]' = T(\omega)(f'(\omega)) + [T(\omega)(y)]', \quad \text{where} \quad y = f(\omega).$$

Therefore $T(f) \in \mathcal{A}$.

Similar arguments prove that when $T(\omega)x \in \mathcal{C}^n = \mathcal{C}^n(\Omega, X)$ for all x, then $T(f) \in \mathcal{C}^n$ whenever $f \in \mathcal{C}^n$. Assume that for each $x \in X$, $T(\omega)x \in \mathcal{A}_1 = \mathcal{A}_1(\Omega, X)$. Fix $f \in \mathcal{A}_1$. We shall prove that $T(f) \in \mathcal{A}_1$. The argument is fairly long. By hypothesis, for each $x \in X$ there exists a sequence $\{A_k(x)\}_{k \geqslant 0} \subseteq X$ with

$$T(\omega)x = \sum_{k=0}^{\infty} A_k(x)\omega^k \quad (\omega \in \Omega), \quad \text{and} \quad \sum_{k=0}^{\infty} \|A_k(x)\| < \infty.$$

Using the uniqueness of the coefficients in the power series expansion, it is easy to check that A_k is linear map on X for all $k \geqslant 0$. Since $\{T(\omega)\}_{\omega \in \Omega}$ is pointwise bounded on X, as before, the uniform boundedness principle implies $\exists\, M > 0$ with $\|T(\omega)\| \leqslant M$ for all $\omega \in \Omega$.

Now we prove

$$A_k \in B(X) \quad \text{for all} \quad k \geqslant 0. \tag{1.1}$$

Fix the parametrized circle $\gamma(t) = 1/2\, \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$. Fix $x \in X$. For all $\omega \in \Omega$ with $\omega \neq 0$,

$$(T(\omega)x)\omega^{-n-1} = A_n(x)\omega^{-1} + g(\omega) \quad \text{where} \quad g(\omega) = \sum_{k=0, k\neq n}^{\infty} A_k(x)\omega^{k-n-1}.$$

Now $g(\omega)$ has an antiderivative $G(\omega)$ for $\omega \in \mathrm{int}(\Omega) \setminus \{0\}$ by direct computation (term-by-term integration). Thus, $\int_\gamma g(\omega)\, \mathrm{d}\omega = 0$, and therefore,

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma (T(\omega)x)\omega^{-n-1}\, \mathrm{d}\omega = \frac{1}{2\pi \mathrm{i}} \int_\gamma A_n(x)\omega^{-1}\, \mathrm{d}\omega = A_n(x).$$

Then

$$\|A_n(x)\| = \left\| \frac{1}{2\pi i} \int_\gamma T(\omega) x \omega^{-n-1} \, d\omega \right\| \leqslant M\|x\|2^n.$$

This proves (1.1).

The next step is to show

$$\text{(1.2)} \qquad \exists J > 0 \text{ such that for all } x \in X, \sum_{k=0}^{\infty} \|A_k(x)\| \leqslant J\|x\|.$$

To prove (1.2), consider the linear map $W : X \to \mathcal{A}_1$ given by $W(x) = \sum_{k=0}^{\infty} A_k(x)\omega^k$. We show W is closed, and thus, bounded. Suppose $\{x_n\} \subseteq X$, $x_0 \in X$, $\|x_n - x_0\| \to 0$, and $\|W(x_n) - g\|_1 \to 0$, where $g(\omega) = \sum_{k=0}^{\infty} b_k \omega^k \in \mathcal{A}_1$. Using (1.1), for each fixed $k \geqslant 0$, $\|A_k(x_n) - A_k(x_0)\| \to 0$. By assumption

$$\sum_{j=0}^{\infty} \|A_j(x_n) - b_j\| \to 0,$$

so certainly, $\|A_k(x_n) - b_k\| \to 0$ for all k. It follows that $A_k(x_0) = b_k$, $k \geqslant 0$. This proves $W(x_0) = \sum_{k=0}^{\infty} A_k(x_0)\omega_k = g$. Therefore (1.2) holds.

Finally, assume $f \in \mathcal{A}_1$, $f(\omega) = \sum_{k=0}^{\infty} d_k \omega^k$, $\sum_{k=0}^{\infty} \|d_k\| < \infty$. Applying (1.2) we have,

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \|A_k(d_j)\| \right) \leqslant \sum_{j=0}^{\infty} J\|d_j\| < \infty.$$

Now

$$T(\omega)(f(\omega)) = \sum_{k=0}^{\infty} A_k(f(\omega))\omega^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} A_k(d_j)\omega^j \right) \omega^k \quad \text{for each } \omega.$$

This power series is in $\mathcal{A}_1$ by the previous calculation. ∎

We state the next result as a corollary of Theorem 1.4, although it could be proved by direct computation.

COROLLARY 1.5. *Assume that* $\{T(\omega)\}_{\omega\in\Omega}$ *is a family in* (a) $\mathcal{C}(\Omega, B(X))$; (b) $\mathcal{C}^n(\Omega, B(X))$; (c) $\mathcal{A}(\Omega, B(X))$; *or* (d) $\mathcal{A}_1(\Omega, B(X))$. *Then* $\{T(\omega)\}_{\omega\in\Omega}$ *determines a module map (via Definition* 1.3*) on* (a) $\mathcal{C}(\Omega, X)$; (b) $\mathcal{C}^n(\Omega, X)$; (c) $\mathcal{A}(\Omega, X)$; *or* (d) $\mathcal{A}_1(\Omega, X)$, *respectively.*

The proof of this corollary is clear, as using Theorem 1.4, it is enough to check that $T(\omega)x$ is in the module indicated for $x \in X$.

As a second application of Theorem 1.4, we have a result on extensions of module maps.

COROLLARY 1.6. *Let $\mathfrak{M}$ be any one of the four basic modules. Assume that $\mathfrak{N}$ is a subspace of $\mathfrak{M}$ which contains the constant functions c_x for all $x \in X$. If $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$ determines a linear map $T : \mathfrak{N} \to \mathfrak{M}$ (as in Definition 1.3) then T has an extension to a module map on $\mathfrak{M}$.*

The proof of Corollary 1.6 is straightforward. Thus, for example, an operator $T \in M(\mathcal{A}(\Omega, X))$ that is determined by a family $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$, has a unique extension to a map $\overline{T} \in M(\mathcal{C}(\Omega, X))$.

Next we show that every $T \in M(\mathcal{C}(\Omega, X))$ is determined by a family $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$. Here, and throughout this paper, we adopt the useful notation from [13], p. 40: for $g \in C(\Omega)$, V an open subset of Ω, $g \prec V$ means $0 \leqslant g \leqslant 1$ on Ω and $g \equiv 0$ on V^c.

COROLLARY 1.7. *Assume $T \in M(\mathcal{C})$, where $\mathcal{C} = \mathcal{C}(\Omega, X)$. Then $\exists\, \{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$ such that for all $f \in \mathcal{C}$*

$$T(f)(\omega) = T(\omega)(f(\omega)) \qquad (\omega \in \Omega).$$

Proof. For $\omega \in \Omega$ and $x \in X$, define

(1.3) $\quad T(\omega)x = T(f)(\omega)$ where f is any function in $\mathcal{C}$ with $f(\omega) = x$.

It will follow from (1.3) that

$$T(f)(\omega) = T(\omega)(f(\omega)) \qquad (\omega \in \Omega,\ f \in \mathcal{C}).$$

We need to check that the definition is well-defined. For this it suffices to show that when $h \in \mathcal{C}$ and $h(\omega) = 0$, then $T(h)(\omega) = 0$. For $\varepsilon > 0$ arbitrary let

$$U_\varepsilon = \{\gamma \in \Omega : \|h(\gamma)\| < \varepsilon\}.$$

Choose $g_\varepsilon \in C(\Omega)$ with $g_\varepsilon(\omega) = 1$ and $g_\varepsilon \prec U_\varepsilon$. Then $\|g_\varepsilon h\|_\Omega \to 0$ as $\varepsilon \to 0^+$. Therefore

$$T(h)(\omega) = g_\varepsilon(\omega)T(h)(\omega) = T(g_\varepsilon h)(\omega) \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Thus, $T(h)(\omega) = 0$.

Now let $T \in M(\mathcal{C})$, and let $\{T(\omega)\}_{\omega\in\Omega}$ be as defined in (1.3). Clearly $T(\omega)$ is a linear map on X, and for all $x \in X$,

$$\|T(\omega)(x)\| = \|T(\omega)(c_x(\omega))\| \leqslant \|T\|\,\|c_x\|_\Omega = \|T\|\,\|x\|.$$

Therefore $\|T(\omega\| \leqslant \|T\|$ for all $\omega \in \Omega$. This proves $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$. ∎

THEOREM 1.8. *Let $\mathfrak{M}$ be any one of the three modules $\mathcal{C}$, $\mathcal{C}^n$, or $\mathcal{A}$. Assume $T \in M(\mathfrak{M})$ is determined by a family $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$. Then T^{-1} exists in $M(\mathfrak{M})$ if and only if $T(\omega)^{-1}$ exists in $B(X)$ for all $\omega \in \Omega$ and $\exists\, J > 0$ such that $\|T(\omega)^{-1}\| \leqslant J$ for all $\omega \in \Omega$.*

Proof. First assume that $\mathfrak{M} = \mathcal{C}(\Omega, X)$, $T(\omega)^{-1}$ exists for all $\omega \in \Omega$, and $\|T(\omega)^{-1}\| \leqslant J$ for all ω. By Theorem 1.4 it suffices to show that $T(\omega)^{-1}x \in \mathcal{C}$ for any fixed $x \in X$. For $\gamma \in \Omega$, $x \in X$, γ and x fixed, we have for $\omega \in \Omega$

$$\begin{aligned} \|T(\omega)^{-1}x - T(\gamma)^{-1}x\| &= \|T(\omega)^{-1}(T(\gamma) - T(\omega))T(\gamma)^{-1}x\| \\ &\leqslant J\|(T(\gamma) - T(\omega))y\| \end{aligned} \tag{1.4}$$

where $y = T(\gamma)^{-1}x$. Since $\omega \to T(\omega)y$ is a continuous map on Ω, it follows that $T(\omega)^{-1}x \in \mathcal{C}$. Thus, $T^{-1} = \{T(\omega)^{-1}\}_{\omega\in\Omega}$ is a module map.

The proof for the other modules is similar, making use of the inequality in (1.4). ∎

It is an open question whether Theorem 1.8 holds for maps $T = \{T(\omega)\}_{\omega\in\Omega} \in M(\mathcal{A}_1)$. We have as a simple corollary of Theorem 1.8, the following result.

COROLLARY 1.9. *Let X be a Hilbert space. Let $\mathfrak{M}$ be any one of the three modules $\mathcal{C}, \mathcal{C}^n$, or $\mathcal{A}$. Assume $T \in M(\mathfrak{M})$ is determined by a family $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$ where $T(\omega) = T(\omega)^*$ for all $\omega \in \Omega$. Then $\sigma(T)$, the spectrum of T as an operator in $B(\mathfrak{M})$, consists of real numbers.*

Proof. Assume $\lambda = a + ib$ where $a, b \in \mathbf{R}$, $b \neq 0$. For each $\omega \in \Omega$, $(\lambda - T(\omega))^{-1}$ exists and $\|(\lambda - T(\omega))^{-1}\| \leqslant |b|^{-1}$. Then the result follows directly from Theorem 1.8. ∎

It is not difficult to find module maps which are not determined by a function in $\mathcal{C}(\Omega, B(X))$. For example, any family $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$ which is strongly continuous on X determines a module map on $\mathcal{C}(\Omega, X)$ by Theorem 1.4, but such a family need not be in $\mathcal{C}(\Omega, B(X))$. It is more interesting to produce a map $T \in M(\mathcal{C}(\Omega, X))$ such that $T(\omega)^{-1}$ exists for all $\omega \in \Omega$, but T is not invertible in $M(\mathcal{C})$. We construct an example next.

EXAMPLE 1.10. Let $X = C[0,1]$, and let Ω be $\overline{\mathbf{N}}$, the space of natural numbers $\overline{\mathbf{N}} = \{1, 2, 3, \ldots\}$ with $\{\infty\}$ adjoined (so $\overline{\mathbf{N}}$ is compact). Now we give an example of a module map $T \in M(\mathcal{C}(\Omega, X))$ with the property that $T(n)$ is invertible for all $n \in \overline{\mathbf{N}}$, but T is not invertible on $\mathcal{C}(\Omega, X)$. For all integers n, let $e_n(x) = e^{2\pi i n x}$. For $n \in \overline{\mathbf{N}}$ let P_n be the projection on $\mathcal{C}[0,1]$ given by

$$P_n(g) = \left(\int_0^1 g(x) e_{-n}(x)\,\mathrm{d}x\right) e_n \qquad (g \in X).$$

Let I be the identity operator on X. Define $T = \{T_n\}_{n \in \Omega}$ by

$$T_n = (1 + n^{-1})I - P_n, \quad 1 \leqslant n < \infty; \quad T_\infty = I.$$

Clearly T_n is invertible for all $n \in \overline{\mathbf{N}}$. From Fourier analysis we know that for any $g \in C[0,1]$, $P_n(g) \to 0$. Therefore as $n \to \infty$

$$T_n(g) = (1 + n^{-1})g - P_n(g) \to g = T_\infty(g).$$

It follows that $\{T_n\}$ is a module map on $\mathcal{C}$ by Theorem 1.4. Now $T_n(e_n) = (1 + n^{-1})e_n - P_n(e_n) = 1/n\ e_n$. Therefore $\|T_n^{-1}\| \geqslant n$ for $1 \leqslant n < \infty$. It follows that T is not invertible on $\mathcal{C}$.

2. QUOTIENT MODULES

The main aim of this section is to establish certain quotient module results which have interesting applications to the spectral theory of module maps. For example, if B is Banach algebra with unit and K is a closed ideal of B, then it is shown in Theorem 2.2 that $\mathcal{C}(\Omega, B)/\mathcal{C}(\Omega, K)$ is isomorphic to $\mathcal{C}(\Omega, B/K)$. From this it follows that if $T \in \mathcal{C}(\Omega, B)$ has the property that $T(\omega)$ is invertible in B module K for all $\omega \in \Omega$ if and only if T is invertible in $\mathcal{C}(\Omega, B)$ module the ideal $\mathcal{C}(\Omega, K)$. When $B = B(X)$ and $K = K(X)$, then this is an interesting fact concerning module maps $T \in \mathcal{C}(\Omega, B(X))$ having the property that $T(\omega)$ is a Fredholm operator for all $\omega \in \Omega$.

The quotient module results derived in this section are more general than is strictly necessary for aplications to study of module maps. However, these general results are of independent interest. The Banach algebra $\mathcal{C}(\Omega, B)$, where B is a general Banach algebra, has been widely studied, especially the ideal theory of this algebra; see for example [9], [15], and [16].

We start with a lemma which is well-known, including the short proof for convenience.

LEMMA 2.1. *Let W be a subset of a normed linear space $(Y, \|\cdot\|)$. Assume $T \in \mathcal{C}(\Omega, B(Y))$, and let $\varepsilon > 0$ be given. Suppose for each $\omega \in \Omega$, $\exists\, w_\omega \in W$ such that $\|T(\omega) - w_\omega\| < \varepsilon$. Then there exist $\{w_1, \ldots, w_n\} \subseteq W$ and $\{h_1, \ldots, h_n\} \subseteq C(\Omega)$ such that*

$$\left\| T(\omega) - \sum_{k=1}^{n} h_k(\omega) w_k \right\|_\Omega < \varepsilon.$$

Proof. For each $\gamma \in \Omega$, let $V_\gamma = \{\omega : \|T(\omega) - w_\omega\| < \varepsilon\}$. By hypothesis $\{V_\gamma : \gamma \in \Omega\}$ is an open cover for Ω. Let $\{V_{\gamma_1}, V_{\gamma_2}, \ldots, V_{\gamma_n}\}$ be a finite subcover. Set $w_k = w_{\gamma_k}$ for $1 \leqslant k \leqslant n$. Choose $\{h_1, \ldots, h_n\} \subseteq C(\Omega)$ such that $h_k \prec V_{\gamma_k}$ for all k and $\sum_{k=1}^{n} h_k \equiv 1$ on Ω ([13], Theorem 2.13). Set

$$R(\omega) = \sum_{k=1}^{n} h_k(\omega) w_k.$$

Then

$$T(\omega) - R(\omega) = \sum_{k=1}^{n} h_k(\omega)[T(\omega) - w_k].$$

Fix $\omega \in \Omega$, and set $A = \{k : \omega \in V_{\gamma_k}\}$. Then

$$T(\omega) - R(\omega) = \sum_{k \in A} h_k(\omega)[T(\omega) - w_k],$$

and therefore,

$$\|T(\omega) - R(\omega)\| \leqslant \sum_{k \in A} h_k(\omega)\|T(\omega) - w_k\| < \sum_{k \in A} h_k(\omega)\varepsilon \leqslant \varepsilon. \quad \blacksquare$$

Now we prove the first of our quotient module results.

THEOREM 2.2. *Let $(B, \|\cdot\|)$ be a Banach space, and let K be a closed subspace of B. Then $\mathcal{C}(\Omega, B)/\mathcal{C}(\Omega, K)$ is isometrically isomorphic to $\mathcal{C}(\Omega, B/K)$.*

Proof. For $b \in B$, let $[b] = b + K$ be the coset of b in B/K. Define $\varphi : \mathcal{C}(\Omega, B) \to \mathcal{C}(\Omega, B/K)$ by $\varphi(R(\omega)) = [R(\omega)]$, $\omega \in \Omega$. First we show:

(2.1) $\mathcal{R}(\varphi)$ is dense in $\mathcal{C}(\Omega, B/K)$.

Fix $T \in \mathcal{C}(\Omega, B/K)$, and let $\varepsilon > 0$ be arbitrary. For each $\omega \in \Omega$, $\exists\,[b_\omega] \in B/K$ such that $T(\omega) = [b_\omega]$ so the hypotheses of Lemma 2.1 hold (with $W = B/K$). Applying the Lemma 2.1, we have for some collections $\{b_1, \ldots, b_n\} \subseteq B$ and $\{h_1, \ldots, h_n\} \subseteq$

$C(\Omega)$, $\left\|T(\omega) - \sum_{k=1}^{n} h_k(\omega)[b_k]\right\|_\Omega < \varepsilon$. Setting $R(\omega) = \sum_{k=1}^{n} h_k(\omega)b_k \in C(\Omega, B)$, we have

$$\varphi(R(\omega)) = \sum_{k=1}^{n} h_k(\omega)[b_k].$$

This proves (2.1).

Note that $\ker(\varphi) = C(\Omega, K)$. Define $\tilde{\varphi} : C(\Omega, B)/C(\Omega, K) \to C(\Omega, B/K)$ by $\tilde{\varphi}([R]) = \varphi(R)$ where $[R]$ is the coset $R + C(\Omega, K)$ in $C(\Omega, B)/C(\Omega, K)$. It is easy to check that

$$\|\tilde{\varphi}([R])\|_\Omega \leqslant \|[R]\|_\Omega \qquad (R \in C(\Omega, B)),$$

so $\|\tilde{\varphi}\| \leqslant 1$. We verify the opposite inequality. Fix $T \in C(\Omega, B)$. In Lemma 2.1 take $W = K$ and $\varepsilon = \|\tilde{\varphi}([T])\|_\Omega + \delta$ where $\delta > 0$ is arbitrary. Note that for every $\omega \in \Omega$, $\exists\, K_\omega \in K$ such that $\|T(\omega) - K_\omega\| < \varepsilon$. Applying the Lemma 2.1, $\exists\,\{K_1, \ldots, K_n\} \subseteq K$ and $\exists\,\{h_1, \ldots, h_n\} \subseteq C(\Omega)$ with $\|T(\omega) - J(\omega)\|_\Omega < \varepsilon$ where

$$J(\omega) = \sum_{k=1}^{n} h_k(\omega)K_k \in C(\Omega, K).$$

Thus,

$$\|\tilde{\varphi}([T])\|_\Omega + \delta = \varepsilon \geqslant \|T - J\|_\Omega \geqslant \|[T]\|_\Omega.$$

Since $\delta > 0$ was arbitrary, it follows that $\|\tilde{\varphi}\| \geqslant 1$. This proves that $\tilde{\varphi}$ is an isometry and, as a consequence, has closed range. This combined with (2.1) proves $\mathcal{R}(\tilde{\varphi}) = C(\Omega, B/K)$. ∎

Before proving the quotient module result for C^n, we need a preliminary result. In what follows, $\Omega = [a, b]$.

PROPOSITION 2.3. *Let $T \in C^n(\Omega, X)$, and let W be a subspace of X. Suppose $\gamma > 0$ is fixed, and for each $\omega \in \Omega$, $\exists\, w_\omega \in W$ such that*

$$\|T^{(n)}(\omega) - w_\omega\| < \gamma.$$

Then $\exists\,\{f_1, \ldots, f_m\} \subseteq C^n(\Omega)$ and $\{w_1, \ldots, w_m\} \subseteq W$ such that

$$\left\|T(\omega) - \sum_{k=1}^{m} f_k(\omega)w_k\right\|_{n,\Omega} < \left[\sum_{k=0}^{\infty}(k!)^{-1}(b-a)^k\right]\gamma.$$

Proof. Since $T(n) \in C(\Omega, X)$, we can apply Lemma 2.1, and as a result, $\exists\,\{g_{n,1}, \ldots, g_{n,j}\} \subseteq C(\Omega)$, and $\exists\,\{w_1, \ldots, w_j\} \subseteq W$ such that

$$\left\|T^{(n)}(\omega) - \sum_{k=1}^{j} g_{n,k}(\omega)w_k\right\|_\Omega < \gamma.$$

Let $g_{n-1,k}(\omega) = \int_a^\omega g_{n,k}(u)\,du$ for $1 \leqslant k \leqslant j$, and set $g_{n-1,j+1}(\omega) \equiv 1$ and $w_{j+1} = T^{(n-1)}(a)$. Then

$$T^{(n-1)}(\omega) = \int_a^\omega T^{(n)}(u)\,du + w_{j+1},$$

and for all $\omega \in \Omega$,

$$\left\| T^{(n-1)}(\omega) - \sum_{k=1}^{j+1} g_{n-1,k}(\omega) w_k \right\| \leqslant \int \left\| T^{(n)}(u) - \sum_{k=1}^{j} g_{n,k}(u) w_k \right\| du \leqslant \gamma(b-a).$$

Again, set $g_{n-2,k}(\omega) = \int_a^\omega g_{n-1,k}(u)\,du$ for $1 \leqslant k \leqslant j+1$, $g_{n-2,j+2}(\omega) \equiv 1$, $w_{j+2} = T^{(n-2)}(a)$. The same argument as given above shows

$$\left\| T^{(n-2)}(\omega) - \sum_{k=1}^{j+2} g_{n-2,k}(\omega) w_k \right\|_\Omega \leqslant (b-a)^2\gamma.$$

Clearly, we can repeat this argument applied to $T^{(n-p)}$, $1 \leqslant p \leqslant n$, where at the p-th stage, $g_{n-p,k}(\omega) = \int_a^\omega g_{n-p+1,k}(u)\,du$ for $1 \leqslant k \leqslant j+p-1$, $g_{n-p,j+p}(\omega) \equiv 1$, $w_{j+p} = T^{(n-p)}(a)$. Having completed this construction, set $f_k = g_{0,k}$, $1 \leqslant k \leqslant m = n+j$. Let $F(\omega) = \sum_{k=1}^{m} f_k(\omega) w_k$. As we have proved, for $0 \leqslant k \leqslant n$, $\|T^{(n-k)} - F^{(n-k)}\|_\Omega \leqslant (b-a)^k\gamma$. Therefore

$$\|T - F\|_{\Omega,n} < \left[\sum_{k=0}^{\infty} (k!)^{-1} (b-a)^k \right] \gamma. \quad ∎$$

Now we prove the quotient module result for C^n.

THEOREM 2.4. *Let Ω be a fixed interval, $\Omega = [a,b]$. Let $(B, \|\cdot\|)$ be a Banach space, and assume that K is a closed subspace of B. Then $C^n(\Omega,B)/C^n(\Omega,K)$ is bicontinuously isomorphic to $C^n(\Omega, B/K)$.*

Proof. For $b \in B$, let $[b] = b + K \in B/K$. Define $\varphi : C^n(\Omega,B) \to C^n(\Omega,B/K)$ by $\varphi(R(\omega)) = [R(\omega)]$, $\omega \in \Omega$. First we prove:

(2.2) $\mathcal{R}(\varphi)$ is dense in $C^n(\Omega, B/K)$.

Fix $T \in \mathcal{C}^n(\Omega, B/K)$. For each $\omega \in \Omega$, choose $b_\omega \in B$ such that $T^{(n)}(\omega) = [b_\omega]$. Now in Proposition 2.3 take $W = B/K$ and $\gamma > 0$ arbitrary. Applying that proposition, we have $\exists\{f_1, \dots, f_m\} \subseteq \mathcal{C}^n(\Omega)$ and $\{b_1, \dots, b_m\} \subseteq B$ such that

$$\left\| T(\omega) - \sum_{k=1}^{m} f_k(\omega)[b_k] \right\|_{\Omega,n} < \left[\sum_{k=0}^{\infty} (k!)^{-1}(b-a)^k \right] \gamma.$$

Set $R(\omega) = \sum_{k=1}^{m} f_k(\omega) b_k$, so that

$$\varphi(R(\omega)) = \sum_{k=1}^{m} f_k(\omega)[b_k].$$

Since $\gamma > 0$ is arbitrary, this proves (2.2).

Clearly, $\ker(\varphi) = \mathcal{C}^n(\Omega, K)$. Define

$$\tilde{\varphi} : \mathcal{C}^n(\Omega, B)/\mathcal{C}^n(\Omega, K) \to \mathcal{C}^n(\Omega, B/K)$$

by $\tilde{\varphi}(\langle R \rangle) = \varphi(R)$ where $\langle R \rangle$ is the coset $R + \mathcal{C}^n(\Omega, K)$ in $\mathcal{C}^n(\Omega, B)/\mathcal{C}^n(\Omega, K)$. Now $\tilde{\varphi}$ is continuous. We verify that $\tilde{\varphi}$ has closed range, which when combined with (2.2), proves that $\tilde{\varphi}$ maps onto $\mathcal{C}^n(\Omega, B/K)$.

Fix $T \in \mathcal{C}^n(\Omega, B)$. We prove $\exists M > 0$ independent of T such that

$$\|\langle T \rangle\|_{\Omega,n} \leqslant M \|\tilde{\varphi}(\langle T \rangle)\|_{\Omega,n}. \tag{2.3}$$

This inequality implies that $\tilde{\varphi}$ has closed range. Let $\delta > 0$ be arbitrary. For each $\omega \in \Omega$, $\exists K_\omega \in K$ such that

$$(n!)^{-1} \left\| T^{(n)}(\omega) - K_\omega \right\| \leqslant (n!)^{-1} \left\| [T^{(n)}(\omega)] \right\| + \delta \leqslant \|[T]\|_{\Omega,n} + \delta = \left\| \tilde{\varphi}(\langle T \rangle) \right\|_{\Omega,n} + \delta.$$

Apply Proposition 2.3 to $T^{(n)}$ with $W = K$ and

$$\gamma = n! \left(\left\| \tilde{\varphi}(\langle T \rangle) \right\|_{\Omega,n} + \delta \right).$$

Thus, $\exists\{f_1, \dots, f_m\} \subseteq \mathcal{C}^n(\Omega)$ and $\exists\{K_1, \dots, K_m\} \subseteq K$ such that for some constant $M_0 > 0$, $\left\| T(\omega) - \sum_{k=1}^{m} f_k(\omega) K_k \right\|_{\Omega,n} \leqslant M_0 \gamma$. Set $J = \sum_{k=1}^{m} f_k(\omega) K_k \in \mathcal{C}(\Omega, K)$. The norm of $\langle T \rangle$ in $\mathcal{C}^n(\Omega, B)/\mathcal{C}^n(\Omega, K)$ is

$$\|\langle T \rangle\|_{\Omega,n} = \inf\{\|T - L\|_{\Omega,n} : L \in \mathcal{C}^n(\Omega, K)\}.$$

Therefore we have shown

$$\|\langle T \rangle\|_{\Omega,n} \leqslant \|T - J\|_{\Omega,n} \leqslant M_0 \gamma.$$

Since $\delta > 0$ is arbitrary we have proved

$$\|\langle T \rangle\|_{\Omega,n} \leqslant M \|\tilde{\varphi}(\langle T \rangle)\|_{\Omega,n}$$

for some constant $M > 0$. This completes the proof of (2.3), and thus, the proof of the theorem. ∎

Because of the way in which the norm on $\mathcal{A}_1$ is defined, the quotient module result in this case is quite elementary. The details are as follows.

THEOREM 2.5. *Let $(B, \|\cdot\|)$ be a Banach space, and let K be a closed subspace of B. Let Ω be the closed unit disk in C. Then $\mathcal{A}_1(\Omega, B/K)$ is bicontinuously isomorphic to $\mathcal{A}_1(\Omega, B)/\mathcal{A}_1(\Omega, K)$.*

Proof. Let $\pi : B \to B/K$ be the usual quotient map. For convenience we use the notation $[b] = \pi(b)$ for $b \in B$. For $T \in \mathcal{A}_1(\Omega, B)$, $T(\lambda) = \sum_{k=0}^{\infty} b_k\lambda^k$, let $\varphi(T) \in \mathcal{A}_1(\Omega, B/K)$ be given by,

$$\varphi(T) = \sum_{k=0}^{\infty}[b_k]\lambda^k.$$

Clearly $\varphi : \mathcal{A}_1(\Omega, B) \to \mathcal{A}_1(\Omega, B/K)$ is continuous and $\ker(\varphi) = \mathcal{A}_1(\Omega, K)$. We show that φ is surjective. Assume

$$R(\lambda) = \sum_{k=0}^{\infty} d_k\lambda^k \in \mathcal{A}_1(\Omega, B/K)$$

$\{d_k\} \subseteq B/K$. Choose $\{c_k\} \subseteq B$ such that $[c_k] = d_k$ for all k. For each k choose $p_k \in K$ such that

$$\|c_k - p_k\| \leqslant \|[c_k]\| + 2^{-k}.$$

Set $b_k = c_k - p_k$ for all k, so $\sum_{k=0}^{\infty} \|b_k\| < \infty$. Letting $T(\lambda) = \sum_{k=0}^{\infty} b_k\lambda^k$, we have $T \in \mathcal{A}_1(\Omega, B)$, and $\varphi(T) = R$. Thus, φ is surjective and induces a bicontinuous algebra isomorphism between $\mathcal{A}_1(\Omega, B)/\mathcal{A}_1(\Omega, K)$ and $\mathcal{A}_1(\Omega, B/K)$. ∎

We do not know if a quotient result of the type given in Theorems 2.2, 2.4, and 2.5, holds for $\mathcal{A}(\Omega, B)$.

Now we consider an application of these quotient module results to Banach algebras of vector-valued functions. Other applications will appear in later sections. Fix B a Banach algebra with unit. Let $\mathcal{B}$ be any one of the Banach algebras $\mathcal{C}(\Omega, B)$, $\mathcal{C}^n(\Omega, B)$, $\mathcal{A}(\Omega, B)$, or $\mathcal{A}_1(\Omega, B)$. Then as noted in Section 1, $T \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if $T(\omega)$ is invertible in B for all $\omega \in \Omega$. Now let R be a radical Banach algebra, and let R_1 denote R with an identity adjoined. Then $\mathcal{C}(\Omega, R)$ is an ideal in $\mathcal{C}(\Omega, R_1)$, and a similar statement holds for each of the algebras of the type considered above. If $T \in \mathcal{C}(\Omega, R)$, then since $T(\omega) \in R$ for all ω, we have $\lambda - T(\omega)$ is invertible in R_1 for all $\lambda \neq 0$. Thus, $\lambda - T$ is invertible in $\mathcal{C}(\Omega, R_1)$ for all $\lambda \neq 0$. It follows that the spectrum of T in $\mathcal{C}(\Omega, R)$ is zero. Since this holds for an arbitrary $T \in \mathcal{C}(\Omega, R)$, this algebra is a radical algebra. The same argument works for all the Banach algebras of R-valued functions under consideration. This leads to:

PROPOSITION 2.6. *Let R be a radical Banach algebra. Then $\mathcal{C}(\Omega, R)$, $\mathcal{C}^n(\Omega, R)$, $\mathcal{A}(\Omega, R)$, and $\mathcal{A}_1(\Omega, R)$ are radical algebras.*

For K a closed ideal in a Banach algebra B, $kh(K)$ is the intersection of all primitive ideals of B which contain K. From the definition it follows that $kh(K)/K$ is a radical Banach algebra.

PROPOSITION 2.7. *Let B be a Banach algebra with unit 1, and let K be a closed ideal in B. Set $I = kh(K)$. For $T \in \mathcal{C}(\Omega, B)$, T is invertible in $\mathcal{C}(\Omega, B)$ module $\mathcal{C}(\Omega, K)$ if and only if T is invertible in $\mathcal{C}(\Omega, B)$ module $\mathcal{C}(\Omega, I)$.*

The same result holds for the algebras $\mathcal{C}^n(\Omega, B)$ and $\mathcal{A}_1(\Omega, B)$.

Proof. By Theorem 2.2 and Proposition 2.6,

$$\mathcal{C}(\Omega, I)/\mathcal{C}(\Omega, K) \approx \mathcal{C}(\Omega, I/K),$$

and this quotient algebra is a radical algebra. It follows that

$$\mathcal{C}(\Omega, I) \subseteq kh(\mathcal{C}((\Omega, K)).$$

Using this, the result is a consequence of a standard Banach algebra fact ([4], [BA] 2.4). ∎

3. SPECTRA

In this section we study the spectral theory of a map $T \in \mathcal{C}(\Omega, B(X))$. Although we state the results and do the proofs for this particular case, many of the same results hold when $T \in \mathcal{C}^n(\Omega, B(X))$ or $T \in \mathcal{A}_1(\Omega, B(X))$. We note this in the statements of the theorems, but since the proofs in these cases are substantially the same, they are omitted. A map $T \in \mathcal{C} = \mathcal{C}(\Omega, B(X))$ has a number of interesting spectra. First note that T is also in the Banach algebras $M = M(\mathcal{C}(\Omega, X))$ and $B(\mathcal{C}(\Omega, X))$. Let $\sigma_{\mathcal{C}}^{\ell}(T)$, $\sigma_{\mathcal{C}}^{r}(T)$, and $\sigma_{\mathcal{C}}(T)$ denote the left, right, and usual spectrum of T relative to the Banach algebra $\mathcal{C}$. To denote the corresponding spectra of T relative to M, we replace $\mathcal{C}$ by M, and relative to $B(\mathcal{C}(\Omega, X))$, we use no subscript on σ. By Proposition 1.2, $\sigma_M(T) = \sigma(T)$, and an easy computation (or (3.1) below) shows $\sigma_{\mathcal{C}}(T) = \sigma(T)$. Concerning right or left spectra, we use a result of S. Bochner and R. Phillips ([5], Theorem 3) which applies in our situation as follows.

(3.1). Let B be a Banach algebra with unit. If $T \in \mathcal{C}(\Omega, B)[\mathcal{C}^n(\Omega, B), \mathcal{A}_1(\Omega, B)]$ and $T(\omega)$ has a left inverse in B for all $\omega \in \Omega$, then T has a left inverse in $\mathcal{C}(\Omega, B)[\mathcal{C}^n(\Omega, B), \mathcal{A}_1(\Omega, B)]$. The same result holds for "right" in place of "left".

Set $\mathcal{K} = \mathcal{C}(\Omega, K(X))$, and define $\omega_{\mathcal{C}}(T)$ to be the spectrum of the coset $T + \mathcal{K}$ in the Banch algebra $\mathcal{C}/\mathcal{K}$. The set $\omega_{\mathcal{C}}(T)$ is an essential-type spectrum and is related to $\{\omega(T(\gamma)) : \gamma \in \Omega\}$ where $\omega(T(\gamma))$ is the usual Fredholm spectrum of $T(\gamma)$ in $B(X)(\omega(T(\gamma))$ is the spectrum of $T(\gamma) + K(X)$ in the Calkin algebra, $B(X)/K(X))$. We use obvious extensions of all this notation for the "left" and "right" cases. The following result elucidates the relationships between these various spectra.

THEOREM 3.2. *Assume* $T \in \mathcal{C}(\Omega, B(X))$.

(i) $\sigma_{\mathcal{C}}(T) = \sigma_M(T) = \sigma(T)$.

(ii) $\sigma_{\mathcal{C}}^{\ell}(T) = \sigma_M^{\ell}(T), \sigma_{\mathcal{C}}^{r}(T) = \sigma_M^{r}(T)$.

(iii) $\sigma_{\mathcal{C}}(T) = \bigcup\{\sigma(T(\omega)) : \omega \in \Omega\}$;

$\sigma_{\mathcal{C}}^{\ell}(T) = \bigcup\{\sigma^{\ell}(T(\omega)) : \omega \in \Omega\}$;

$\sigma_{\mathcal{C}}^{r}(T) = \bigcup\{\sigma^{r}(T(\omega)) : \omega \in \Omega\}$.

(iv) $\omega_{\mathcal{C}}(T) = \bigcup\{\omega(T(\gamma)) : \gamma \in \Omega\}$;

$\omega_{\mathcal{C}}^{\ell}(T) = \bigcup\{\omega^{\ell}(T(\gamma)) : \gamma \in \Omega\}$;

$\omega_{\mathcal{C}}^{r}(T) = \bigcup\{\omega^{r}(T(\gamma)) : \gamma \in \Omega\}$.

The analogous results hold when $T \in \mathcal{C}^n(\Omega, B(X))$ *and when* $T \in \mathcal{A}_1(\Omega, B(X))$.

Proof. (i) follows from Proposition 1.2 and (3.1). We prove that $\sigma_{\mathcal{C}}^{\ell}(T) = \sigma_M^{\ell}(T) = \bigcup\{\sigma^{\ell}(T(\omega)) : \omega \in \Omega\}$. The other assertions in (ii) and (iii) have similar proofs. First note that $\bigcup\{\sigma^{\ell}(T(\omega)) : \omega \in \Omega\} \subseteq \sigma_M^{\ell}(T) \subseteq \sigma_{\mathcal{C}}^{\ell}(T)$. Now if $\lambda - T(\omega)$ has a left inverse for all $\omega \in \Omega$, then $\lambda - T$ has a left inverse in $\mathcal{C}(\Omega, B(X))$ by (3.1). This fact proves that $\bigcup\{\sigma^{\ell}(T(\omega)) : \omega \in \Omega\} = \sigma_{\mathcal{C}}^{\ell}(T)$, so the result follows.

To prove (iv), first note that $\bigcup\{\omega(T(\gamma)) : \gamma \in \Omega\} \subseteq \omega_{\mathcal{C}}(T)$. Now suppose $\lambda - T(\gamma)$ is Fredholm on X for all $\gamma \in \Omega$. Then by Theorem 2.2, $\lambda - T$ has an inverse in $\mathcal{C}(\Omega, B(X))$ modulo $\mathcal{C}(\Omega, K(X))$. This proves the first equality in (iv), and the second two have similar proofs. ∎

For X a Banach space, let $F(X)$ be the ideal in $B(X)$, $F(X) = \{T \in B(X) : T$ has finite dimensional range$\}$. Also let $I(X) = kh(F(X))$. The operators in $I(X)$ are called inessential operators ([11], [4]). Often we use the abbreviated notation: $F = F(X)$; $\overline{F}$, the closure of F in the operator norm; $K = K(X)$; and $I = I(X)$. As is well-known, $F \subseteq \overline{F} \subseteq K \subseteq I$, and also, $kh(F) = kh(\overline{F}) = kh(K) = I$.

PROPOSITION 3.3. *Assume $T \in \mathcal{C}(\Omega, B(X))$. Then*

$$\omega_{\mathcal{C}}(T) = \{\lambda : \lambda - T \text{ is not invertible in } \mathcal{C}(\Omega, B(X)) \text{ modulo } \mathcal{C}(\Omega, \overline{F})\}$$
$$= \{\lambda : \lambda - T \text{ is not invertible in } \mathcal{C}(\Omega, B(X)) \text{ modulo } \mathcal{C}(\Omega, I)\}.$$

Proof. Let $\Delta = \{\lambda : \lambda - T$ is not invertible in $\mathcal{C}(\Omega, B(X))$ modulo $\mathcal{C}(\Omega, I)\}$. Since $K \subseteq I$, we have $\Delta \subseteq \omega_{\mathcal{C}}(T)$. Now $I = kh(K)$, so Proposition 2.7 applies. Thus, if $\lambda \notin \Delta$, then $\lambda - T$ is invertible in $\mathcal{C}(\Omega, B(X))$ modulo $\mathcal{C}(\Omega, I)$, and therefore, $\lambda - T$ is invertible in $\mathcal{C}(\Omega, B(X))$ modulo $\mathcal{C}(\Omega, K)$, i.e. $\lambda \notin \omega_{\mathcal{C}}(T)$. The proof of the other equality in the statement of the proposition is the same (since $I = kh(\overline{F})$). ∎

Now we combine (3.1) and Theorem 2.5 to derive the following result.

THEOREM 3.4. *Let B be a Banach algebra with unit and let K be a closed ideal of B. Assume $\Gamma = \{\lambda \in C : |\lambda - \lambda_o| \leqslant R\}$ and $T \in \mathcal{A}(\Gamma, B)$ has the property that $T(\omega) = \sum_{k=0}^{\infty} b_k(\omega - \lambda_o)^k$ where $\{b_k\} \subseteq B$ and $\sum_{k=0}^{\infty} \|b_k\| R^k < \infty$. Then $T(\omega)$ is left (right) invertible in B modulo K for all $\omega \in \Gamma$ if and only if T is left (right) invertible in $\mathcal{A}(\Gamma, B)$ modulo $\mathcal{A}(\Gamma, K)$.*

Proof. We may assume without loss of generality that Γ is the closed unit disk (see Note 1.1). Assume $T(\omega)$ is left invertible in B modulo K for all $\lambda \in \Gamma$. By hypothesis $T \in \mathcal{A}_1(\Gamma, B)$. Let $[b]$ denote the coset of $b \in B$ in B/K. We have $[T(\omega)]$ is left invertible in B/K for all $\omega \in \Gamma$, so by (3.1) $\varphi(T)$ is left invertible in $\mathcal{A}_1(\Gamma, B/K)$ (φ as in Theorem 2.5). By Theorem 2.5 it follows that T is left invertible in $\mathcal{A}_1(\Gamma, B)$ modulo $\mathcal{A}_1(\Gamma, K)$. This implies that T is invertible in $\mathcal{A}(\Gamma, B)$ modulo $\mathcal{A}(\Gamma, K)$. ∎

For S an operator on a Banach space, let $AP\sigma(S)$ denote the approximate point spectrum of S ([14], p. 282). The next result relates $AP\sigma(T)$ to $\{AP\sigma(T(\omega)) : \omega \in \Omega\}$ when $T \in M(\mathcal{C}(\Omega, X))$.

THEOREM 3.5. *Let $T \in M(\mathcal{C}(\Omega, X))$. Assume $\lambda \in AP\sigma(T(\omega))$ for some $\omega \in \Omega$. Then $\lambda \in AP\sigma(T)$.*

Proof. By hypothesis, $\exists \{x_n\} \subseteq X$, $\|x_n\| = 1$ for all n, such that

$$\|(\lambda - T(\omega))x_n\| \equiv \delta_n \to 0.$$

Let

$$V_n = \{\gamma \in \Omega : \|(\lambda - T(\gamma))x_n\| < 2\delta_n\}.$$

Choose $g_n \prec V_n$ such that $g_n(\omega) = 1$. Then $\|x_n g_n\|_\Omega = 1$, and for all $\gamma \in \Omega$

$$\|(\lambda - T)(\gamma)(x_n g_n(\gamma))\| = g_n(\gamma)\|(\lambda - T(\gamma))x_n\| \leqslant 2\delta_n.$$

Thus,

$$\|(\lambda - T)(x_n g_n)\|_\Omega \leqslant 2\delta_n \to 0. \quad \blacksquare$$

COROLLARY 3.6. *Assume $T \in M(\mathcal{C}(\Omega, X))$ and $AP\sigma(T(\omega)) = \sigma(T(\omega))$ for all $\omega \in \Omega$. Then $AP\sigma(T) = \sigma(T)$. This will hold if $T(\omega) \in K(X)$ for all $\omega \in \Omega$.*

There is a converse to Theorem 3.5 when $T \in \mathcal{C}(\Omega, B(X))$. We consider this result next.

PROPOSITION 3.7. *Assume $T \in \mathcal{C}(\Omega, B(X))$. If $\lambda \in AP\sigma(T)$, then $\exists \omega_o \in \Omega$ such that $\lambda \in AP\sigma(T(\omega_o))$.*

Proof. By hypothesis $\exists \{g_n\} \subseteq \mathcal{C}(\Omega, X)$ with $\|(\lambda - T)g_n\|_\Omega = \delta_n \to 0$, and $\|g_n\|_\Omega = 1$ for all n. For each n choose $\omega_n \in \Omega$ such that $\|g_n(\omega_n)\| = 1$, and set $x_n = g_n(\omega_n)$. Let ω_o be a limit point of $\{\omega_n : n \geqslant 1\}$. Define for $n \geqslant 1$

$$U_n = \{\gamma \in \Omega : \|T(\gamma) - T(\omega_o)\| < n^{-1}\}.$$

Since $\omega_o \in U_n$ for all n and ω_o is a limit point of $\{\omega_n\}_{\geqslant 1}$, there exists a subsequence $\{\omega_{n_k}\}_{k \geqslant 1}$ such that $\omega_{n_k} \in U_{n_k}$, $k \geqslant 1$. Then

$$\begin{aligned}\|(\lambda - T(\omega_o))x_{n_k}\| &\leqslant \|(T(\omega_o) - T(\omega_{n_k}))x_{n_k}\| + \|(\lambda - T(\omega_{n_k}))(g_{n_k}(\omega_{n_k}))\| \\ &\leqslant n_k^{-1} + \delta_{n_k} \to 0. \quad \blacksquare\end{aligned}$$

4. CLOSED RANGE

When S is a linear operator with domain and range in some linear space, we let $\mathfrak{N}(S)$ and $\mathcal{R}(S)$ denote the null space and range of S, respectively. The main question in this section is: under what conditions does a module map $T \in M(\mathcal{C}(\Omega, X))$ have closed range in $\mathcal{C}(\Omega, X)$?

Many of the results concerning module maps $T \in M(\mathcal{C})$ also apply to module maps $T \in M(\mathcal{A}(\Omega, X))$ which are determined by a family of operators $\{T(\omega)\}_{\omega \in \Omega} \subseteq B(X)$. For in this case by Corollary 1.6, T has a unique extension $\overline{T} \in M(\mathcal{C})$. If $\overline{T}$ has closed range in $\mathcal{C}$, then $\exists\, m > 0$ such that

$$\|\overline{T}(f)\|_\Omega \geqslant m\,\|f + \mathfrak{N}(\overline{T})\|_\Omega \qquad (f \in \mathcal{C}).$$

Here $\|f + \mathfrak{N}(\overline{T})\|_\Omega = \inf\{\|f - g\|_\Omega : g \in \mathfrak{N}(\overline{T})\}$. Thus, if $\mathfrak{N}(T) = \mathfrak{N}(\overline{T})$, we have

$$\|T(f)\|_\Omega \geqslant m\,\|f + \mathfrak{N}(T)\|_\Omega \qquad (f \in \mathcal{A}).$$

Therefore T has closed range in $\mathcal{A}$.

PROPOSITION 4.1. *Assume $T \in M(\mathcal{C}(\Omega, X))$, and $\mathcal{R}(T)$ is closed. Then for all $\omega \in \Omega$*

$$\mathfrak{N}(T(\omega)) = \{f(\omega) : f \in \mathfrak{N}(T)\}^{-}.$$

Proof. Clearly we have the inclusion

$$\{f(\omega) : f \in \mathfrak{N}(T)\}^{-} \subseteq \mathfrak{N}(T(\omega)).$$

Now fix ω, and suppose that $x \in \mathfrak{N}(T(\omega))$. For $n \geqslant 1$ let

$$V_n = \{\gamma \in \Omega : \|T(\gamma)x\| < n^{-1}\}.$$

Choose g_n with $g_n \prec V_n$ and $g_n(\omega) = 1$. Thus,

$$\|T(\omega)(xg_n(\omega))\| \leqslant n^{-1} \qquad (\omega \in \Omega).$$

It follows that $\|T(xg_n)\|_\Omega \to 0$. Since $\mathcal{R}(T)$ is closed, $\exists\, m > 0$ such that $\|T(f)\|_\Omega \geqslant m\|f + \mathfrak{N}(T)\|_\Omega$ for all f. Therefore $\exists\,\{f_n\} \subseteq \mathfrak{N}(T)$ with $\|xg_n - f_n\|_\Omega \to 0$. In particular, at ω we have $\|x - f_n(\omega)\| \to 0$. ∎

COROLLARY 4.2. *If $T \in M(\mathcal{C}(\Omega, X))$, $\mathcal{R}(T)$ is closed, and $\mathfrak{N}(T(\omega))$ is finite dimensional for some $\omega \in \Omega$, then $\mathfrak{N}(T(\omega)) = \{f(\omega) : f \in \mathfrak{N}(T)\}$.*

THEOREM 4.3. *Assume $T = \{T(\omega)\} \in M(\mathcal{C})$ has $\mathcal{R}(T)$ closed. Therefore $\exists\, m > 0$ such that*

$$\|T(f)\|_\Omega \geqslant m\,\|f + \mathfrak{N}(T)\|_\Omega \qquad (f \in \mathcal{C}(\Omega, X)).$$

Then for every $\omega \in \Omega$

$$\|T(\omega)y\| \geqslant m\,\|y + \mathfrak{N}(T(\omega))\| \qquad (y \in X),$$

and thus, $\mathcal{R}(T(\omega))$ is closed.

Proof. Suppose on the contrary that $\exists\, \omega \in \Omega$ and $\exists\, y \in X$ such that

$$\|T(\omega)y\| < m\,\|y + \mathfrak{N}(T(\omega))\|.$$

Let

$$V = \{\gamma \in \Omega : \|T(\gamma)y\| < m\,\|y + \mathfrak{N}(T(\omega))\|.$$

Choose h with $h \prec V$ and $h(\omega) = 1$. Then for all $\gamma \in \Omega$

$$\|T(\gamma)(yh(\gamma)\| = h(\gamma)\|T(\gamma)y\| < m\,\|y + \mathfrak{N}(T(\omega))\|.$$

Therefore

$$\|T(yh)\|_\Omega < m\,\|y + \mathfrak{N}(T(\omega))\|. \tag{4.1}$$

For any $f \in \mathfrak{N}(T)$

$$\|y + \mathfrak{N}(T(\omega))\| \leqslant \|y - f(\omega)\| = \|yh(\omega) - f(\omega)\| \leqslant \|yh - f\|_\Omega.$$

Therefore

$$\|y + \mathfrak{N}(T(\omega))\| \leqslant \|yh + \mathfrak{N}(T)\|_\Omega. \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\|T(yh)\|_\Omega < m\,\|yh + \mathfrak{N}(T)\|_\Omega,$$

contradicting the hypothesis of the theorem. ∎

Now we characterize when a module map on $\mathcal{C}(\Omega, X)$ has closed range.

THEOREM 4.4. *Assume $T = \{T(\omega)\} \in M(\mathcal{C})$. Then $\mathcal{R}(T)$ is closed in $\mathcal{C}$ if and only if T has the following two properties:*

(i) *There exists $m > 0$ such that for all $\omega \in \Omega$*

$$\|T(\omega)x\| \geqslant m\,\|x + \mathfrak{N}(T(\omega))\| \qquad (x \in X).$$

(ii) *For all $\omega \in \Omega$, $\mathfrak{N}(T(\omega)) = \{f(\omega) : f \in \mathfrak{N}(T)\}^-$.*

Proof. First assume $\mathcal{R}(T)$ is closed in $\mathcal{C}$. Then (i) and (ii) hold by Proposition 4.1 and Theorem 4.3.

Conversely, assume that T satisfies (i) and (ii). Then for all $\omega \in \Omega$ and all $f \in \mathcal{C}$, we have

$$\|T(f)(\omega)\| = \|T(\omega)(f(\omega))\| \geqslant m\,\|f(\omega) + \mathfrak{N}(T(\omega))\|.$$

Therefore

$$\|T(f)\|_\Omega \geqslant m\Sigma \tag{4.3}$$

where

$$\Sigma = \sup_{\omega\in\Omega} \|f(\omega) + \mathfrak{N}(T(\omega))\|.$$

We use property (ii) to verify that

$$\|f + \mathfrak{N}(T)\|_\Omega \leqslant \Sigma. \tag{4.4}$$

This inequality combined with (4.3) implies that $\mathcal{R}(T)$ is closed.

To prove (4.4), suppose $f \in C(\Omega, X)$, and let $\varepsilon > 0$ and $\omega \in \Omega$ be arbitrary. Choose $y \in \mathfrak{N}(T(\omega))$ such that

$$\|f(\omega) - y\| \leqslant \|f(\omega) + \mathfrak{N}(T(\omega))\| + \varepsilon.$$

By (ii), $\exists\, g \in \mathfrak{N}(T)$ with $\|y - g(\omega)\| < \varepsilon$. Therefore

$$\|f(\omega) - g(\omega)\| < \|f(\omega) - y\| + \varepsilon \leqslant \|f(\omega) + \mathfrak{N}(T(\omega))\| + 2\varepsilon \leqslant \Sigma + 2\varepsilon.$$

We have proved that for each $\omega \in \Omega$, $\exists\, g_\omega \in \mathfrak{N}(T)$ such that $\|f(\omega) - g_\omega(\omega)\| < \Sigma + 2\varepsilon$. Let

$$V_\omega = \{\gamma \in \Omega : \|f(\gamma) - g_\omega(\gamma)\| < \Sigma + 2\varepsilon\}.$$

Take a finite cover of Ω, $\{V_{\omega_1}, \ldots, V_{\omega_n}\}$. Choose $h_k \prec V_{\omega_k}$ for $1 \leqslant k \leqslant n$, such that $\sum_{k=1}^{n} h_k \equiv 1$ on Ω. Set

$$g \equiv \sum_{k=1}^{n} h_k g_{\omega_k} \in \mathfrak{N}(T).$$

Clearly, $\|f - g\|_\Omega < \Sigma + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (4.4) holds. ∎

COROLLARY 4.5. *Let $T \in M(C)$, and assume $T(\omega) \in \Psi^0(X)$ (Fredholm of index zero) for all $\omega \in \Omega$. The following are equivalent:*

(i) *T is invertible in $M(C)$;*

(ii) *$\mathfrak{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed;*

(iii) *$\mathcal{R}(T) = C$.*

Proof. First assume that $\mathcal{R}(T)$ is closed and $T(\omega)^{-1} \in B(X)$ for all $w \in \Omega$. By Theorem 4.4 (i), $\exists\, m > 0$ such that $\|T(\omega)x\| \geqslant m\|x\|$ for all $\omega \in \Omega$ and all $x \in X$. Therefore $\|T(\omega)^{-1}\| \leqslant m^{-1}$ for all $\omega \in \Omega$. This implies by Theorem 1.8 that (i) holds.

Now assume that (ii) holds. By Theorem 4.4 (ii), for all $\omega \in \Omega$:

$$\mathfrak{N}(T(\omega)) = \{f(\omega) : f \in \mathfrak{N}(T)\}^- = \{0\}.$$

Since $T(\omega) \in \Psi^0(X)$, we have $T(\omega)^{-1}$ exists for all $\omega \in \Omega$. Then the argument in the first paragraph implies that (i) holds.

Assume (iii). For any $x \in X$, $\exists\, f \in C$ such that $T(f) = c_x$, so $T(\omega)(f(\omega)) = x$ for all $\omega \in \Omega$. It follows that $\mathcal{R}(T(\omega)) = X$ for all $\omega \in \Omega$. Since $T(\omega) \in \Psi^0(X)$, we have $T(\omega)^{-1}$ exists for all $\omega \in \Omega$. Then as before, (i) holds. ∎

PROPOSITION 4.6. *Assume $T \in M(C(\Omega, X))$ has the property: for each $\omega \in \Omega$, $\exists \Delta$ a compact subset of Ω with $\omega \in \text{int}(\Delta)$ such that T_Δ has closed range in $C(\Delta, X)$, where T_Δ denotes the restriction of T to Δ. Then $\mathcal{R}(T)$ is closed.*

Proof. Suppose $g \in \overline{\mathcal{R}(T)}$, and choose $\{u_n\} \subseteq C(\Omega, X)$ with $\|T(u_n) - g\|_\Omega \to 0$. For each $\omega \in \Omega$ choose Δ_ω a compact neighborhood of ω with the properties given in the statement of the proposition, so in particular, T_{Δ_ω} has closed range. Let $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$ be a finite subcollection of $\{\Delta_\omega : \omega \in \Omega\}$ such that $\Omega = \bigcup_{k=1}^{n} \text{int}(\Delta_k)$. For $1 \leqslant k \leqslant n$, choose $h_k \prec \text{int}(\Delta_k)$ with $1 \equiv \sum_{k=1}^{n} h_k$ on Ω. For each k, $\|T(u_n) - g\|_{\Delta_k} \to 0$, so by hypothesis, $\exists f_k \in C(\Delta_k, X)$ with $T(\omega)(f_k(\omega)) = g(\omega)$ for all $\omega \in \Delta_k$. Define $f \in C(\Omega, X)$ by

$$f(\omega) = \sum_{k=1}^{n} h_k(\omega) f_k(\omega) \qquad (\omega \in \Omega).$$

Note here that $h_k(\omega) f_k(\omega)$ is a continuous function on Ω if we set $h_k(\omega) f_k(\omega) = 0$ for $\omega \in \Delta_k^c$. For all $\omega \in \Omega$,

$$T(f)(\omega) = \sum_{k=1}^{n} T(h_k f_k)(\omega) = \sum_{k=1}^{n} h_k(\omega) T_{\Delta_k}(f_k)(\omega) = \sum_{k=1}^{n} h_k(\omega) g(\omega) = g(\omega)$$

Thus, $g \in \mathcal{R}(T)$. ∎

5. UNBOUNDED MODULE MAPS

Up to this point we have dealt only with the theory of bounded module maps. In this section we briefly consider unbounded module maps, indicating how the bounded case can give information concerning the unbounded case.

Let $\{S(\omega) : \omega \in \Omega\}$ be a family of closed operators in X. We denote the domain of $S(\omega)$ by D_ω. The family $\{S(\omega)\}$ determines a closed module map on any one of the four basic modules, denoted by $\mathfrak{M}$, as follows. Let

$$D(S) = \{f \in \mathfrak{M} : f(\omega) \in D_\omega \text{ for all } \omega \in \Omega, \text{ and } S(\omega)(f(\omega)) \in \mathfrak{M}\}.$$

For $f \in D(S)$, let

$$S(f) = S(\omega)(f(\omega)) \in \mathfrak{M}.$$

A routine computation verifies that S is a closed operator and a module map. We say S is affiliated with $M = M(\mathfrak{M})$ if for some λ_0, $(\lambda_0 - S)^{-1} \in M$. When S is affiliated with M, it is convenient, and involves no loss of generality, to assume that $S^{-1} \in M$.

THEOREM 5.1. *Assume $\{S(\omega) : \omega \in \Omega\}$ is a family of closed operators in X, and let S be defined as above. Assume $T = S^{-1} \in M$.*

(i) $\sigma(S) = \{\lambda^{-1} : \lambda \in \sigma_M(T),\ \lambda \neq 0\}$.

(ii) *For* $\lambda \neq 0$,

$$\mathcal{R}(\lambda - S) = \mathcal{R}(\lambda^{-1} - T).$$

(iii) *For* $\lambda \neq 0$,

$$\mathfrak{N}(\lambda - S) = T(\mathfrak{N}(\lambda^{-1} - T)).$$

Proof. All three statements follow from the equations below where $\lambda \neq 0$:

$$(\lambda - S)T = -\lambda(\lambda^{-1} - T),$$

and

$$T(\lambda - S)f = -\lambda(\lambda^{-1} - T)f \quad \text{for} \quad f \in D(S). \quad \blacksquare$$

As Theorem 5.1 indicates, when $T = S^{-1} \in M$, one can derive considerable information concerning S by studying T. For example, for $\lambda \neq 0$, $(\lambda - S)$ has closed range if and only if $\lambda^{-1} - T$ has closed range (in fact, these ranges are the same). Now we give some examples of operators $S = \{S(\omega)\}_{\omega \in \Omega}$ which are affiliated with $M(\mathfrak{M})$ for some modules $\mathfrak{M}$.

EXAMPLE 5.2. Let Ω be a fixed compact Hausdorff space. Let $X = C[a,b]$. Set $\mathcal{C} = \mathcal{C}(\Omega, X)$, and assume $N = \{N(\omega)\}_{\omega \in \Omega} \in M(\mathcal{C})$. Consider the family of equations:

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial t} = N(\omega)(y) + g(\omega, t); \\ y(\omega, a) = 0 \ \text{ for all } \ \omega \in \Omega. \end{array} \right. \tag{5.1}$$

Here $y(\omega, t)$ is an unknown function in $\mathcal{C}(\Omega, C^1[a,b])$ and $g \in \mathcal{C}$ is given. Let $D = \{f \in C^1[a,b] : f(a) = 0\}$, and define $S(\omega)$ on D by

$$S(\omega)(f) = f' - N(\omega)f.$$

Thus, (5.1) is the equation $S(y) = g$ where $S = \{S(\omega)\}_{\omega \in \Omega}$. Also S is affiliated with $M(\mathcal{C})$ since it has the inverse $S^{-1} = T \in M(\mathcal{C})$ given by

$$T(\omega)(g(\omega))(t) = \int_a^t e^{N(\omega)(t-s)} g(s, \omega)\, ds$$

where $g \in \mathcal{C}$, $t \in [a,b]$. It is straightforward to verify that $T \in M(\mathcal{C})$. Note that for each t, $e^{N(\omega)t} \in M(\mathcal{C})$, and that by Theorem 1.4 it is enough to check that for any $k \in C[a,b]$,

$$\int_a^t e^{-N(\omega)s} k(s)\, ds \in \mathcal{C}.$$

EXAMPLE 5.2. Let Ω be a fixed compact Hausdorff space, and let $X = L^2[a,b]$. Set $\mathcal{C} = \mathcal{C}(\Omega, X)$. Fix $h(\omega) \in C(\Omega)$ with $h(\omega) \geqslant 0$ for all $\omega \in \Omega$. Consider the family of equations:

$$\begin{cases} \frac{\partial y}{\partial t} = g(\omega, t); \\ y(\omega, a) + h(\omega) y(\omega, b) = 0. \end{cases} \tag{5.2}$$

Here $y(\omega,t)$ is an unknown function in $\mathcal{C}(\Omega, Y)$ where Y is the Banach space of absolutely continuous functions on $[a,b]$ which have derivative in L^2. The function $g \in \mathcal{C}$ is given. Let D_ω be the set of all $f \in Y$ such that $f(a) + h(\omega)f(b) = 0$. Set

$$S(\omega)f = f' \qquad (f \in D_\omega).$$

Thus, for $S = \{S(\omega)\}_{\omega\in\Omega}$, (5.2) is the corresponding equation $S(y) = g$. Now $S(\omega)^{-1} \equiv T(\omega)$ exists for each ω where $T(\omega)$ is the integral operator on $L^2[a,b]$:

$$T(\omega)(k) = (1 + h(\omega))^{-1} \left\{ \int_a^t k(s)\, ds - h(\omega) \int_t^b k(s)\, ds \right\} \qquad (k \in L^2).$$

It is clear that $T = \{T(\omega)\}_{\omega\in\Omega} \in \mathcal{C}(\Omega, B(X))$. Thus $S = \{S(\omega)\}_{\omega\in\Omega}$ is affiliated with $\mathcal{C}$.

EXAMPLE 5.3. Let Γ be a domain in the complex plane, and let $\{S(\gamma)\}_{\gamma\in\Gamma}$ be a family of closed operators on X with the properties:

(i) All of the operators $S(\gamma)$ have the same fixed domain $D \subseteq X$;

(ii) $S(\gamma)x$ is holomorphic on Γ for each $x \in D$.

A family $\{S(\gamma)\}_{\gamma\in\Gamma}$ with these properties is called holomorphic of type A; see [10], p. 375.

Now fix Ω a closed disk, $\Omega \subseteq \Gamma$. Assume $S(\omega)^{-1} \equiv T(\omega)$ exists for all $\omega \in \Omega$, $\{T(\omega)\}_{\omega\in\Omega} \subseteq B(X)$, and $\exists\, J > 0$ such that $\|T(\omega)\| \leqslant J$ for all $\omega \in \Omega$. Next we verify:

(iii) If $\{\omega_n\} \subseteq \Omega$, $\omega_n \to \omega_o$, and $\{x_n\} \subseteq X$, $x_n \to x_o$, then $T(\omega_n)x_n \to T(\omega_o)x_o$.

Computing, we have

$$\begin{aligned}\|T(\omega_n)x_n - T(\omega_\circ)x_\circ\| &\leqslant \|T(\omega_n)(x_n - x_\circ)\| + \|(T(\omega_n) - T(\omega_\circ))x_\circ\| \\ &\leqslant J\|x_n - x_\circ\| + \|T(\omega_n)(S(\omega_\circ) - S(\omega_n))(T(\omega_\circ)x_\circ)\| \\ &\leqslant J\|x_n - x_\circ\| + J\|(S(\omega_\circ) - S(\omega_n))T(\omega_\circ)x_\circ)\|.\end{aligned}$$

Then (iii) follows from this inequality and (ii).

Fix $\omega_\circ \in \operatorname{int}(\Omega)$ and assume $\{\omega_n\} \in \operatorname{int}(\Omega)$, $\omega_n \to \omega_\circ$, and $\omega_n \neq \omega_\circ$ for all n. For any $y \in X$, setting $z_\circ = T(\omega_\circ)y \in D$, we have

$$\frac{T(\omega_n)y - T(\omega_\circ)y}{\omega_n - \omega_\circ} = T(\omega_n)\left[\frac{S(\omega_\circ) - S(\omega_n)}{\omega_n - \omega_\circ}\right](z_\circ) \longrightarrow T(\omega_\circ)\left(-\frac{\mathrm{d}S(\omega)z_\circ}{\mathrm{d}\omega}\Big|_{\omega=\omega_\circ}\right).$$

Here we have used (i), (ii), and (iii).

We have verified that $T = \{T(\omega)\}_{\omega\in\Omega}$ determines a module map on $\mathcal{A}(\Omega, X)$. In fact, T is a module map on $\mathcal{A}_1(\Omega, X)$. Assume $\Omega = \{\omega : |\omega - \omega_\circ| \leqslant R\}$. Let Ω' be an open disk with center $\omega_\circ$ and such that $\Omega \subseteq \Omega' \subseteq \Gamma$. For each $x \in X$, the argument above shows that $T(\omega)x$ is holomorphic in Ω'. Thus using [7], p. 97, $T(\omega)x$ has a Taylor series representation in Ω', $T(\omega)x = \sum_{k=0}^{\infty} x_k(\omega - \omega_\circ)^k$. Using the Cauchy estimates ([7], p. 97), we have $\sum_{k=0}^{\infty} \|x_k\|R^k < \infty$. Therefore by Theorem 1.4 $\{T(w)\}_{\omega\in\Omega}$ determines a module map on $\mathcal{A}_1(\Omega, X)$. This allows us to study $\{S(\gamma)\}_{\gamma\in\Gamma}$ locally using results in this paper.

Many specific examples of families of operators which are holomorphic of type A can be found in [10], Chapter 7, Sections 2 and 3.

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